INTEGRABILITY OF THE LIMIT MODEL FOR A VACUUM DIODE AND A SOLUTION TO THE SINGULAR BOUNDARY-VALUE PROBLEM

ALEXANDER A. KOSOV, EDWARD I. SEMENOV, ALEXANDER V. SINITSYN

ABSTRACT. This article concerns singular boundary-value problems for a vacuum diode model. We prove the integrability of a system of nonlinear differential equations and construct a complete system of the first integrals; thus developing a method for solving singular boundary value problems. Also we study the asymptotic behavior of the solution in a neighborhood of the singular point.

1. Introduction

Modeling a plasma as a flow of charged particles interacting in a vacuum usually needs the application of the Vlasov-Maxwell or Vlasov-Poisson equations [1, 3, 4, 6]. When solving these nonlinear systems of partial differential equations (PDEs) with initial and boundary conditions, it is necessary to find the solutions and ascertain their properties in a number of conditions (positiveness, monotonicity, singularity, etc.).

We study a simpler model described by a system of ordinary differential equations with boundary conditions that retains the principal physical properties of the initial model, and is a more efficient way to overcome possible mathematical difficulties. This approach allows us to obtain a limit model for plane vacuum diode magnetic insulation. This model is given by a system of two singular second-order ODEs [2].

Related results to this limit model and the boundary-value problem are found in [2, 8] where analytical and numerical methods are combined. In this article, we concentrate on exact analytical methods to integrate the corresponding nonlinear systems and transform the singular boundary value problem into a system of nonlinear equations.

This article is structured as follows. Section 2 presents the description of the model, the formulation of the corresponding singular boundary-value problem, and the definition of a solution taht differs from the classical solution. Section 3 presents the mathematical model in Hamiltonian form. Using Liouville theorem, we prove the system integrability and construct a complete system of four first integrals. Here the Hamiltonian form of equations allows us to apply the classic integration methods for nonlinear systems developed in analytical mechanics [10]. Section 4 describes the method of constructing a solution of the singular boundary value problem, which is

based on the use of the complete system of first integrals. Section 5 presents another method of integrating nonlinear systems, which is based on the replacement of the variables represented in a special form proposed by the authors in [7]. This method reduces the problem of integration to the same quadratures described in Section 3. The quadratures are represented by some combinations of elementary and elliptic functions. Section 6 describes a class of exact solutions for the nonlinear system in explicit form. The cases, when the explicit solutions imply exact solutions of non-singular boundary-value problems, and the cases, when a sequence of explicit solutions approximate a solution of the initial singular boundary-value problem (in some sense), are given. Section 7 shows an asymptotic representation of the solution for the boundary value problem in the vicinity of the singular point. In such representations, in particular, the electric potential approximation up to $O(x^{4/3})$, agree with the estimates of the upper and lower solutions (obtained by a different method) to the boundary-value problem formulated by the authors [8]. Section 8 gives examples illustrating the numerical results obtained.

2. Model and problem statement

The limit model of a plane vacuum diode was proposed in [2]. The model consists of the two second-order nonlinear ordinary differential equations:

$$\frac{d^2\varphi}{dx^2} = j\frac{(1+\varphi)}{\sqrt{(1+\varphi)^2 - a^2 - 1}}, \quad \frac{d^2a}{dx^2} = j\frac{a}{\sqrt{(1+\varphi)^2 - a^2 - 1}}, \qquad (2.1)$$

where the independent variable $x \in [0,1]$ denotes a relative distance from the cathode ($x = 1$ corresponds to the anode). The function $\varphi(x)$ describes the distribution of the electric potential in the process of moving from the cathode to anode; $a(x)$ is the potential of the magnetic field; the model's parameter j denotes the density of current through the diode. System (2.1) describes the electric and magnetic fields inside the diode, and its solution shall satisfy the following boundary conditions:

$$\varphi(0) = 0, \quad a(0) = 0, \quad \varphi'(0) = \frac{d\varphi}{dx}(0) = 0, \qquad (2.2)$$

$$\varphi(1) = \varphi_1, \quad a(1) = a_1. \qquad (2.3)$$

The boundary-value problem (2.1)–(2.3) is singular: after substituting conditions (2.2) into equations (2.1) for $x = 0$, the denominator vanishes. Therefore, the classical definition of the solution as a pair of functions $(\varphi(x), a(x))$ satisfying (2.2) and (2.3) and converting (2.1) into an identity on an interval $x \in [0,1]$ (the derivatives at the ends of the interval are considered as unilateral) cannot be applied to this problem. So, it is necessary to define the concept of a solution for (2.1)–(2.3). The parameter j is free, see [2], and must be found together with the solution of a boundary value problem (2.1)–(2.3).

Let $\Omega = \{(\varphi, a) : (1+\varphi)^2 - a^2 - 1 > 0\}$. On each compact subset Ω the right sides of (2.1) have bounded partial derivatives. Therefore, the conditions of existence and uniqueness of solutions for the initial-value problem (2.1) are satisfied. Furthermore, due to obvious symmetry we may confine our consideration to investigation of only the solutions with positive values of $1 + \varphi(x)$, $a(x)$, i.e., consider the problem only in domain $\Omega_+ = \Omega \cap \{(\varphi, a) : 1 + \varphi > 0, \ a > 0\}$.

Let the conditions of the theorem on existence and uniqueness be satisfied for the right end, i.e. $\Theta_1 = (1 + \varphi_1)^2 - 1 - a_1^2 > 0$.

Definition 2.1. A pair $(\varphi(x), a(x))$ of twice differentiable functions in $]0, 1]$ assuming values on Ω_+ is a solution of (2.1)–(2.3) when

(1) $\varphi(1) = \varphi_1$, $a(1) = a_1$;
(2) on each interval $x \in [\epsilon, 1]$, $0 < \epsilon < 1$ – after the substitution – (φ, a) satisfies (2.1);
(3) there are limits $\lim_{\epsilon \to +0} \varphi(\epsilon) = 0$, $\lim_{\epsilon \to +0} a(\epsilon) = 0$, $\lim_{\epsilon \to +0} \varphi'(\epsilon) = 0$.

At $x = 0$ this function is redefined by the first two relations in its property (3).

Properties of the above definition:

(a) This definition does not necessitate substitution of the boundary conditions at the interval's left end into the system, what allows us to avoid the division by zero;
(b) It does not impose any restrictions on the behavior of the first derivative $a'(x)$ and the second derivatives at the interval's left end;
(c) It may obviously be upgraded also for the case, when $\Theta_1 = (1 + \varphi_1)^2 - 1 - a_1^2 = 0$ and when there is a singularity at the right end of the interval.
  Furthermore, this definition allows for $\lim_{\epsilon \to +0} a'(\epsilon)$ not to exist or to be infinity.

The principal objective of the work initiated by the authors is to develop a method for constructing a solution of the singular boundary-value problem (2.1)–(2.3) in terms of Definition 2.1. To this end it is necessary to show that system (2.1) is integrable in quadratures and to construct a complete system of the first integrals. Furthermore, formulas in explicit form, which approximate the solution of the boundary-value problem at $x = 0$, are obtained.

3. Representation of the problem in Hamiltonian form and its integrability

Let

$$t = x, \quad q_1 = \varphi(x), \quad q_2 = a(x), \quad p_1 = -\varphi'(x),$$

$$p_2 = a'(x), \quad q = col(q_1, q_2) \in R^2, \quad p = col(p_1, p_2) \in R^2,$$

$$H(q, p) = \frac{1}{2}(-p_1^2 + p_2^2) + j\sqrt{(1 + q_1)^2 - q_2^2 - 1}.$$

Hence (2.1) is equivalent to

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}. \tag{3.1}$$

In terms of the new variables the boundary conditions may be rewritten as

$$q_1(0) = 0, \quad q_2(0) = 0, \quad p_1(0) = 0, \tag{3.2}$$

$$q_1(1) = Q_1 \equiv \varphi_1, \quad q_2(1) = Q_2 \equiv a_1. \tag{3.3}$$

The new boundary-value problem (3.1)–(3.3) is equivalent to the initial problem (2.1)–(2.3). It differs from the initial system only by the Hamiltonian form of the system of differential equations. The Hamiltonian form of (3.1)–(3.3) allows one to apply the integration technique developed for solving problems of analytical mechanics [10].

The Hamiltonian system (3.1) has the energy integral

$$J_1 \equiv H(q,p) = \frac{1}{2}(-p_1^2 + p_2^2) + j\sqrt{(1+q_1)^2 - q_2^2 - 1} = c_1 =: \text{const.} \qquad (3.4)$$

It can readily be seen that another first integral of system (3.1) has the form

$$J_2 = (1+q_1)p_2 + q_2 p_1 = c_2 =: \text{const.} \qquad (3.5)$$

The first integrals J_1, J_2 do not depend explicitly on time. The rank of the Jacobi matrix for the first integrals J_1, J_2 in $\mathcal{D} = \{(q_1, q_2, p_1, p_2) : (q_1, q_2) \in \Omega_+, (p_1, p_2) \in \mathbb{R}^2\}$ becomes less than 2 only for the set

$$\mathcal{M} = \Big\{(q_1, q_2, p_1, p_2) : (q_1, q_2) \in \Omega_+, \ p_1 = \pm \frac{\sqrt{j} q_2}{((1+q_1)^2 - q_2^2 - 1)^{1/4}},$$
$$p_2 = \pm \frac{\sqrt{j}(1+q_1)}{((1+q_1)^2 - q_2^2 - 1)^{1/4}}\Big\}.$$

This set $\mathcal{M}$ is two-dimensional, and it does not separate the four-dimensional set $\mathcal{D}$ into sub-domains. The first integrals J_1, J_2 are functionally independent in $\mathcal{D} \setminus \mathcal{M}$. So, due to the Liouville theorem [10], the diode model (3.1) is integrable in this domain. To construct the two necessary first integrals, let us express the momentum from (3.4) and (3.5) in terms of the coordinates

$$p_1 = -\frac{c_2 q_2}{w^2} \mp \frac{\sqrt{c_2^2 - 2w^2(c_1 - j\sqrt{w^2 - 1})}}{w^2}(1 + q_1), \qquad (3.6)$$

$$p_2 = \frac{c_2(1 + q_1)}{w^2} \pm \frac{\sqrt{c_2^2 - 2w^2(c_1 - j\sqrt{w^2 - 1})}}{w^2} q_2, \qquad (3.7)$$

where $w^2 = (1+q_1)^2 - q_2^2$. These formulas may be written in the form $p_1 = \frac{\partial \Phi}{\partial q_1}$, $p_2 = \frac{\partial \Phi}{\partial q_2}$, where function $\Phi(q_1, q_2, c_1, c_2)$ is given by the formula

$$\Phi(q_1, q_2, c_1, c_2)$$
$$= \frac{c_2}{2} \ln\Big|\frac{1 + q_1 + q_2}{1 + q_1 - q_2}\Big| \pm \int_{\sqrt{(1+q_1)^2 - q_2^2}}^{\sqrt{(1+Q_1)^2 - Q_2^2}} \frac{\sqrt{c_2^2 - 2s^2(c_1 - j\sqrt{s^2 - 1})}}{s} ds. \qquad (3.8)$$

The integral in (3.8) may be reduced to elementary and elliptic functions. By the Liouville theorem, the two other necessary first integrals J_3, J_4 are expressed via function $\Phi(q_1, q_2, c_1, c_2)$:

$$J_3 \equiv \frac{\partial \Phi}{\partial c_2}$$
$$= \frac{1}{2} \ln\Big|\frac{1 + q_1 + q_2}{1 + q_1 - q_2}\Big| \pm \int_{\sqrt{(1+q_1)^2 - q_2^2}}^{\sqrt{(1+Q_1)^2 - Q_2^2}} \frac{c_2 ds}{s\sqrt{c_2^2 - 2s^2(c_1 - j\sqrt{s^2 - 1})}} = c_3, \qquad (3.9)$$

$$J_4 \equiv \frac{\partial \Phi}{\partial c_1} = \mp \int_{\sqrt{(1+q_1)^2 - q_2^2}}^{\sqrt{(1+Q_1)^2 - Q_2^2}} \frac{s ds}{\sqrt{c_2^2 - 2s^2(c_1 - j\sqrt{s^2 - 1})}} = t + c_4, \qquad (3.10)$$

where c_3 and c_4 are constants. Integrals in (3.9) and (3.10) are used only when the radicals in the denominators are nonzero. Since the identity $(1+q_1)p_1 + q_2 p_2 \equiv 0$

holds on the whole set $\mathcal{M}$, and equality

$$(1 + q_1)p_1 + q_2p_2 = \mp\sqrt{c_2^2 - 2w^2(c_1 - j\sqrt{w^2 - 1})}$$

follows from (3.6), (3.7), the nonzero radicals in the denominators of (3.9), (3.10) provide for independence of the first integrals J_1 and J_2. The system of four first integrals (3.4), (3.5), (3.9) and (3.10) obtained may be used in domain Ω_+ for the purpose of solving initial and boundary-value problems, while including problem (3.1)–(3.3).

4. Solving the singular boundary-value problem

Taking $t = 1$ in (3.10) and applying conditions (3.3) to the right end of the interval, one obtains $c_4 = -1$. Similarly, from (3.9) we have $c_3 = \frac{1}{2}\ln\left|\frac{1+Q_1+Q_2}{1+Q_1-Q_2}\right|$. At $t = 0$, from conditions (2.2) and (3.2) for the right end of the interval and from integrals (3.4), (3.5) one obtains $p_1(0) = 0$, $p_2(0) = c_2$, $c_2^2 = 2c_1$. Thus, we introduce the following functions

$$F(u, v, Q_1, Q_2) = \int_1^{\sqrt{(1+Q_1)^2 - Q_2^2}} \frac{s\,ds}{\sqrt{u^2(1 - s^2) + 2vs^2\sqrt{s^2 - 1}}},$$

$$G(u, v, Q_1, Q_2) = \int_1^{\sqrt{(1+Q_1)^2 - Q_2^2}} \frac{u\,ds}{s\sqrt{u^2(1 - s^2) + 2vs^2\sqrt{s^2 - 1}}}.$$

These functions may be represented as combinations of elementary and elliptic functions of their arguments. So, basing on (3.9) and (3.10), by the relations between the arbitrary constants derived from the boundary conditions, we have the following statement.

Theorem 4.1. *If $c_2 = u_*$ and $j = v_*$ represent a solution of the system of two nonlinear equations*

$$F(u, v, Q_1, Q_2) = 1, \quad G(u, v, Q_1, Q_2) = \frac{1}{2}\ln\left|\frac{1 + Q_1 + Q_2}{1 + Q_1 - Q_2}\right|, \quad (4.1)$$

then the solution of the boundary-value problem (3.1)–(3.3) *exists and represents a solution of the initial-value problem for* (3.1) *with the initial conditions at the right end of the interval assigned by*

$$q_1(1) = Q_1, \quad p_1(1) = -\frac{u_* Q_2}{Z^2} - \frac{\sqrt{u_*^2(1 - Z^2) + 2v_* Z^2\sqrt{Z^2 - 1}}}{Z^2}(1 + Q_1), \quad (4.2)$$

$$q_2(1) = Q_2, \quad p_2(1) = \frac{u_*(1 + Q_1)}{Z^2} + \frac{\sqrt{u_*^2(1 - Z^2) + 2v_* Z^2\sqrt{Z^2 - 1}}}{Z^2}Q_2, \quad (4.3)$$

from (3.6) *and* (3.7)*, where $Z^2 = (1 + Q_1)^2 - Q_2^2$.*

If (4.1) is inconsistent, this does not mean that the boundary-value problem (3.1)–(3.3) does not possess any solution. There are situations when equalities (4.1) are violated. This takes place when the signs change in the expressions (3.6) and (3.7) describing the momentum expressed via the coordinates in the solution of boundary value problem.

5. Integration by replacement of the variables

System (2.1) has solutions of the form [7]:

$$\varphi(x) = \frac{1}{2\gamma} z(x)[e^{\omega(x)} + \gamma^2 e^{-\omega(x)}] - 1,$$
$$a(x) = \frac{1}{2\gamma} z(x)[e^{\omega(x)} - \gamma^2 e^{-\omega(x)}]. \tag{5.1}$$

Here, $z(x)$, $\omega(x)$ are functions to be defined; $\gamma \neq 0$ is an arbitrary constant. To this end we substitute the ansatz (5.1) into (2.1) and, after simple transformations, we obtain the system

$$A\left[\frac{1}{2\gamma}e^{\omega(x)} + \frac{\gamma}{2}e^{-\omega(x)}\right] + B\left[\frac{1}{2\gamma}e^{\omega(x)} - \frac{\gamma}{2}e^{-\omega(x)}\right] = 0,$$
$$A\left[\frac{1}{2\gamma}e^{\omega(x)} - \frac{\gamma}{2}e^{-\omega(x)}\right] + B\left[\frac{1}{2\gamma}e^{\omega(x)} + \frac{\gamma}{2}e^{-\omega(x)}\right] = 0, \tag{5.2}$$

where

$$A := z''(x) + z(x){\omega'}^2(x) - j\frac{z(x)}{\sqrt{z^2(x) - 1}}, \quad B := 2z'(x)\omega'(x) + z(x)\omega''(x).$$

Hence, $A = 0$, $B = 0$, i.e.

$$z''(x) + z(x){\omega'}^2(x) - j\frac{z(x)}{\sqrt{z^2(x) - 1}} = 0, \tag{5.3}$$

$$2z'(x)\omega'(x) + z(x)\omega''(x) = 0. \tag{5.4}$$

Integrating (5.4), we obtain

$$\omega'(x) = \frac{C_1}{z^2(x)}, \tag{5.5}$$

where $C_1 > 0$ is an arbitrary constant. Substituting (5.5) in (5.3), we obtain a single nonlinear second-order ODE for the function $z(x)$

$$z''(x) + \frac{C_1^2}{z^3(x)} - j\frac{z(x)}{\sqrt{z^2(x) - 1}} = 0, \tag{5.6}$$

with the initial-boundary conditions

$$\frac{1}{2\gamma} z(0)\left[e^{\omega(0)} + \gamma^2 e^{-\omega(0)}\right] = 1, \quad \frac{1}{2\gamma} z(0)\left[e^{\omega(0)} - \gamma^2 e^{-\omega(0)}\right] = 0, \tag{5.7}$$

$$\frac{1}{2\gamma} z'(0)\left[e^{\omega(0)} + \gamma^2 e^{-\omega(0)}\right] + \frac{1}{2\gamma} z(0)\left[e^{\omega(0)} - \gamma^2 e^{-\omega(0)}\right]\omega'(0) = 0, \tag{5.8}$$

$$\frac{1}{2\gamma} z(1)\left[e^{\omega(1)} + \gamma^2 e^{-\omega(1)}\right] = \varphi_1 + 1, \quad \frac{1}{2\gamma} z(1)\left[e^{\omega(1)} - \gamma^2 e^{-\omega(1)}\right] = a_1. \tag{5.9}$$

Conditions (5.7)–(5.9) have been borrowed from (2.2) and (2.3) for the ansatz (5.1). Now by (5.5) we have $\omega'(0) = \frac{C_1}{z^2(0)}$. Multiplication of equation (5.6) by the function $z'(x) \neq 0$ and subsequent integration allow one to obtain

$${z'}^2(x) - \frac{C_1^2}{z^2(x)} - 2j\sqrt{z^2(x) - 1} = C_2. \tag{5.10}$$

Here C_2 is an integration constant, which is chosen on account of the initial boundary conditions (5.7), (5.8). From conditions (5.7), one obtains $z(0) = 1$. Hence $\omega(0) = \ln\gamma$, $\gamma > 0$. Therefore, from (5.10) we have ${z'}^2(0) - C_1^2 = C_2$, and from

(5.8) we have $z'(0) = 0$, hence $C_2 = -C_1^2$. So, function $z(x)$, which satisfies the conditions $z(0) = 1$, $z'(0) = 0$, may represent a solution of equation

$$z'(x) = \pm \frac{\sqrt{C_1^2(1 - z^2(x)) + 2jz^2(x)\sqrt{z^2(x) - 1}}}{z(x)},$$

from which it follows that

$$x = \pm \int_1^z \frac{s\,ds}{\sqrt{C_1^2(1 - s^2) + 2js^2\sqrt{s^2 - 1}}}. \tag{5.11}$$

So, we obtain z as an implicit function of x. Now, using (5.5), (5.11) and condition $\omega(x) = \ln\gamma$ at x = 0, it is possible to find ω as a function of z:

$$\omega(z) = \ln\gamma \pm \int_1^z \frac{C_1\,ds}{s\sqrt{C_1^2(1 - s^2) + 2js^2\sqrt{s^2 - 1}}}.$$

Now it is time to substitute conditions (2.3) at the right end into this equality and into (5.11). Taking into account the inequality $\Theta(1) > 0$ to choose signs before integrals, we return to the system of equations (4.1), what allows to define the values C_1 and j.

Therefore, the integration by the technique of replacement of variables lead us to the solvability conditions for the boundary value problem similar to those obtained with the application of the Liouville theorem.

It should be noted that integration (2.1) by means of replacements of the variables similar (5.1), was already carried out earlier by a number of authors. In [5] replacement with a hyperbolic sine and a cosine is used for separation of variables and the solution for (2.1) singular initial value problem on the left end of $[0, 1]$

$$\varphi(0) = 0, \quad a(0) = 0, \quad \varphi'(0) \equiv \frac{d\varphi}{dx}(0) = 0, \quad a'(0) \equiv \frac{da}{dx}(0) = \beta \in \mathbb{R}. \tag{5.12}$$

J. Batt stated the original approach to the solution of a singular boundary value problem in several lectures at Irkutsk in August, 2009 (The authors do not know, whether these results were published or not). In a preprint [9] replacement (5.1) is used with $\gamma = 1$ and also the above-stated singular initial value problem on the left end of a piece $[0, 1]$ is considered.

This article differs from [9] and others by two main stands: We give strict definition of the solution of a singular BVP, and we use an IVP only on the right end of the interval, at $x = 1$ where conditions for existence and uniqueness are satisfied. We will show the fundamental difference of our approach from that of [9] on a simple example.

Example 5.1. Let us consider a singular BVP

$$\frac{d^2\varphi(x)}{dx^2} = \frac{2}{a(x)}, \quad \frac{d^2a(x)}{dx^2} = -\frac{1}{\varphi(x)},$$

$$\varphi(0) = 0, \quad a(0) = 0, \quad \varphi'(0) \equiv \frac{d\varphi}{dx}(0) = 0,$$

$$\varphi(1) = \varphi_1 = 3, \quad a(1) = a_1 = 3/2.$$

The exact solution of this BVP in terms of Definition 2.1 is given by the functions

$$\varphi(x) = 3x^{4/3}, \quad a(x) = \frac{3}{2}x^{2/3}. \tag{5.13}$$

Here we have $a'(x) = x^{-1/3}$, therefore $\lim_{x \to +0} a'(x) = +\infty$. So there is no number $\beta \in \mathbb{R}$ such that (5.13) would be the solution of corresponding [9] the IVP (5.12).

This example shows that transition from a BVP with singularity on the left end of a piece to an IVP on the left end of a piece isn't correct and can lead to loss of the solution of the BVP. We will note also that singular BVP can have not less than two solutions [5] in certain cases.

6. Constant solution $z(x)$

In this section we consider the solutions of the initial system (2.1), which are constant in terms of variable z. Substituting $z = z(x) \equiv$ const into (5.5) and (5.6), we obtain the following equations:

$$\omega'(x) = \frac{C_1}{z^2}, \quad \frac{C_1^2}{z} - j \frac{z}{\sqrt{z-1}} = 0, \quad z \equiv \text{const.} \tag{6.1}$$

Let us show that the assumption of equality of $z(x) \equiv$ const is not compatible with the boundary conditions at the interval's left end for any value of parameter γ; there is no compatibility already with the first two equalities of (2.2). Adding the first two equalities from (5.8), one obtains $\gamma^{-1} z(0) e^{\omega(0)} = 1$. When subtracting one of these equalities from the other, one obtains $\gamma z(0) e^{-\omega(0)} = 1$. The result of multiplication of the two equalities obtained yields $z^2(0) = 1$. The latter is the initial condition, and function $z(x)$, which is constant, shall satisfy this condition. Hence $z(x) \equiv 1$, what makes the denominator in (5.6) vanish. Therefore, constant solutions of the form $z(x) \equiv$ const are not compatible with the boundary conditions (2.2) at the interval's left end, when $x = 0$.

Now introduce the condition $x_1 = 1$ and verify the condition of compatibility of constant solutions $z(x) \equiv$ const with the boundary conditions (2.3) at the right end of the interval, when $x = x_1$. Adding equalities (5.9), one obtains

$$\gamma^{-1} z(x_1) e^{\omega(x_1)} = 1 + \varphi_1 + a_1. \tag{6.2}$$

The result of subtraction of one of these equalities from the other yields $\gamma z(x_1) e^{-\omega(x_1)} = 1 + \varphi_1 - a_1$. Similarly, the product of the two equalities obtained gives $z^2(x_1) = (1 + \varphi_1)^2 - a_1^2 = \Theta(x_1) + 1$, and this is a positive number, which shall satisfy the condition $z^2(x_1) \neq 1$. Let $\Theta(x) = (1 + \varphi(x))^2 - a^2(x) - 1$. Since $(\varphi_1, a_1) \in \Omega_+$, it is obvious that $\Theta(x_1) > 0$, and condition $z^2(x_1) \neq 1$ holds.

It is also obvious from (5.12) that $C_1^2 \Theta^{1/2}(x_1) - j(\Theta(x_1) + 1)^2 = 0$. Therefore, on account of parity and oddness of, respectively, (5.5) and (5.6) with respect to argument z, an arbitrary constant C_1 must be fixed, i.e. $C_1 = \sqrt{j} \Theta^{-1/4}(x_1)(\Theta(x_1) + 1)$, and $z(x_1) = \sqrt{\Theta(x_1) + 1}$. Now, from (5.5), it is possible to obtain $\omega' = \sqrt{j} \Theta^{-1/4}(x_1)$, whence $\omega(x) = \sqrt{j} \Theta^{-1/4}(x_1)(x - x_1) + \omega_1$. The value of the arbitrary constant ω_1 is found from equality (5.13): $\omega_1 = \ln \frac{\gamma(1+\varphi_1+a_1)}{\sqrt{\Theta(x_1)+1}}$.

Now, substituting the obtained values of $z(x)$ and $\omega(x)$ in (5.1), we can formulate the solution of system (2.1) in terms of the initial variables

$$\begin{aligned}
\varphi(x) &= (1 + \varphi_1) \cosh \lambda(x) + a_1 \sinh \lambda(x) - 1, \\
a(x) &= (1 + \varphi_1) \sinh \lambda(x) + a_1 \cosh \lambda(x),
\end{aligned} \tag{6.3}$$

where $\lambda(x) = \sqrt{j} \Theta^{-1/4}(x_1)(x - x_1)$. Note that (6.1) gives an exact solution of (2.1). It satisfies conditions (2.3) at the right end of the interval, where $x = x_1 =$

1, and does not satisfy conditions (2.2) at the interval's left end, where $x = 0$. Furthermore, equality $x_1 = 1$ is not obligatory, so, solution (6.1) may be used also in assigning the boundary conditions at any point different from 1.

The parameter j remains arbitrary. So, we have to choose this parameter to satisfy not only (2.3) but also the following additional boundary conditions:

$$\varphi(x_0) = \varphi_0, \quad a(x_0) = a_0, \quad 0 < x_0 < x_1 = 1. \tag{6.4}$$

Substituting (6.1) into (6.2) and resolved the linear system of equations with respect to $\cosh \lambda(x_0)$ and $\sinh \lambda(x_0)$, one obtains

$$\cosh \lambda(x_0) = \frac{(1 + \varphi_0)(1 + \varphi_1) - a_1 a_0}{\Theta(x_1) + 1},$$

$$\sinh \lambda(x_0) = \frac{(1 + \varphi_1)a_0 - a_1(1 + \varphi_0)}{\Theta(x_1) + 1}.$$

By the procedures of addition and subtraction of these identities, one can find

$$e^{\lambda(x_0)} = \frac{1 + \varphi_0 + a_0}{1 + \varphi_1 + a_1}, \quad e^{-\lambda(x_0)} = \frac{1 + \varphi_0 - a_0}{1 + \varphi_1 - a_1}. \tag{6.5}$$

Multiplying (6.3), one obtains the condition of solvability for (6.2)

$$\Theta(x_0) \equiv (1 + \varphi_0)^2 - a_0^2 - 1 = (1 + \varphi_1)^2 - a_1^2 - 1 \equiv \Theta(x_1). \tag{6.6}$$

When condition (6.4) is satisfied, the value of the parameter j, which provides for satisfaction of the boundary condition (6.2), may be found from (6.3):

$$j = \frac{\Theta^{1/2}(x_1)}{(x_0 - x_1)^2} \ln^2 \left(\frac{1 + \varphi_0 + a_0}{1 + \varphi_1 + a_1} \right).$$

Therefore, the class of constant solutions (considered in this section) with $z(x) \equiv$ const allows us to obtain explicit solutions of non-singular boundary-value problems with the same values of effective potential ($\Theta(x_0) = \Theta(x_1)$) at the interval's endpoints.

The $\Theta(x)$, known as the effective potential [2], be the first integral for the solutions having the form (6.1), i.e.

$$\Theta(x)\Big|_{(6.1)} \equiv \left((1 + \varphi(x))^2 - a^2(x) - 1\right)\Big|_{(6.1)}$$
$$= (1 + \varphi_1)^2 - a_1^2 - 1 \equiv \Theta(x_1) = \text{const.}$$

Since $\Theta(1) > 0$, and conditions (2.2) correspond to $\Theta(0) = 0$, variables (6.1) do not represent a solution of the boundary-value problem (2.1)–(2.3). We intend to demonstrate that under some additional conditions imposed on the solution obtained not at the endpoints of interval $[0, 1]$ but rather on a countable set of its interior points one can obtain a solution of the boundary-value problem (in the sense of Definition 2.1) with the aid of (6.1). However, system (2.1) describes the functions $\varphi(x)$ and $a(x)$ not on the whole interval $(0, 1)$ but only on a sequence of subintervals.

Let there be given three numerical sequences x_k, α_{2k}, β_{2k} ($k = 1, 2, \ldots$) such that

$$1 = x_1 > x_2 > \cdots > x_k > x_{k+1} > \cdots > 0, \quad \lim_{k \to +\infty} x_k = 0,$$

$$0 < \alpha_{2k} < \alpha_* < 1, \quad 0 < \beta_{2k} < \beta_* < 1.$$

Consider system (2.1) with condition (2.3) on the interval $[x_2, x_1]$ for the case when $x = x_1$ (we try to obtain a solution of this problem in the form (6.1)). Compute the values $\varphi(x_2)$, $a(x_2)$ by formulas (6.1). With regard to the next interval $[x_3, x_2]$ let us assume that $\varphi(x)$, $a(x)$ satisfy the perturbed system (2.1) with some unknown additive perturbations in the right-hand side. This may be caused, for example, by the presence of some contamination inside the diode or by an external disturbed field.

Let functions $\varphi(x)$, $a(x)$ remain unknown within the interval $[x_3, x_2]$, but we know the relation between the values at the endpoints

$$\varphi(x_3) = \alpha_2 \varphi(x_2), \quad a(x_3) = \beta_2 a(x_2). \tag{6.7}$$

In accordance with the previous computations of $\varphi(x_2)$ and $a(x_2)$, it is possible to use (6.5) to find values of $\varphi(x_3)$, $a(x_3)$, and to employ (6.1) in order to construct functions $\varphi(x)$, $a(x)$ on the next interval $[x_4, x_3]$ (implying the replacement of x_1 with x_3). Continuing this process sequentially, it is possible to obtain exact solutions of (2.1) in the form (6.1) for the intervals $[x_{2k}, x_{2k-1}]$ ($k = 1, 2, \ldots$), while functions $\varphi(x)$, $a(x)$ remain unknown on the intervals (x_{2k-1}, x_{2k-2}) ($k = 2, 3, \ldots$).

Since functions $\cosh(\cdot)$, $\sinh(\cdot)$ are bounded on the restricted interval $x \in [0, 1]$, conditions $0 < \alpha_* < 1$ and $0 < \beta_* < 1$ presume that functions $\varphi(x)$, $a(x)$ tend to zero on intervals $[x_{2k}, x_{2k-1}]$ ($k = 1, 2, \ldots$), when $k \to +\infty$. This property is characterized by derivatives $\varphi'(x)$, $a'(x)$. If additional perturbations in the right-hand sides of (2.1), which are effective on the intervals (x_{2k-1}, x_{2k-2}) ($k = 2, 3, \ldots$), are bounded uniformly in all cases ($k = 2, 3 \ldots$), then functions $\varphi(x)$, $\varphi'(x)$ and $a(x)$ vanish, when $k \to +\infty$ also on the intervals (x_{2k-1}, x_{2k-2}). Consequently, functions $\varphi(x)$, $a(x)$ constructed in the process of successive application of formulas (6.1) give a solution of the boundary-value problem (2.2) in the sense of Definition 2.1. The diagrams representing the functions $\varphi(x)$, $a(x)$ in the case, when $j = \frac{1}{2}$, $\varphi_1 = a_1 = 1$, $\alpha_{2k} = \beta_{2k} = \frac{1}{2}$, $x_2 = \frac{1}{2}$, $x_3 = \frac{1}{4}$, $x_4 = \frac{1}{8}$, $x_5 = \frac{1}{16}, \ldots$, are shown below in Figures 1 and 2.
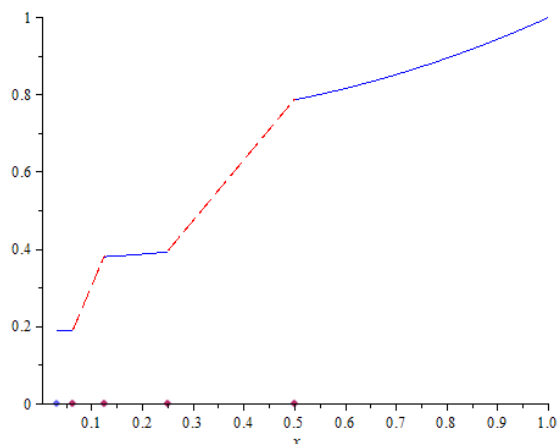

FIGURE 1. Function $\varphi(x)$. Solid line corresponds to formula (6.1). Dashed straight line connects the points where the exact behavior of the function is unknown
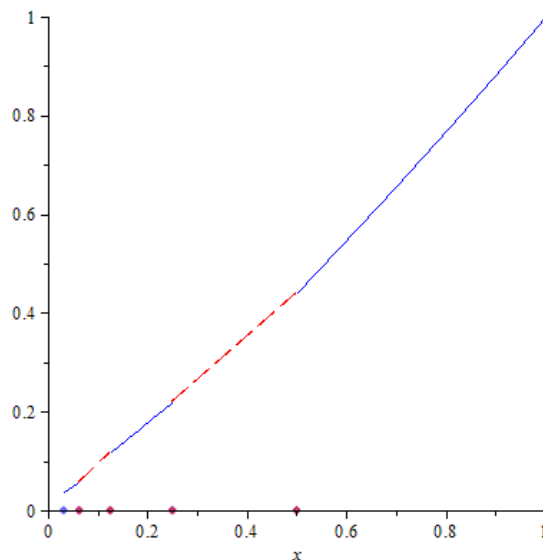
FIGURE 2. Function $a(x)$. Solid line shows the solution obtained by formula (6.1). Dashed shows where the exact form of the function is unknown

7. ASYMPTOTIC BEHAVIOR

Consider the function $\Theta(x) = (1+\varphi(x))^2 - a(x)^2 - 1$. Differentiating this function several times and using (2.1), we obtain the following identities

$$\Theta(x) \equiv (1 + \varphi(x))^2 - a(x)^2 - 1,$$

$$\Theta'(x) \equiv 2(1 + \varphi(x))\varphi'(x) - 2a(x)a'(x),$$

$$\Theta''(x) \equiv 2(\varphi'(x))^2 - 2(a'(x))^2 + 2j\big(\Theta^{1/2}(x) + \Theta^{-1/2}(x)\big), \qquad (7.1)$$

$$\Theta'''(x) \equiv j\big(3\Theta^{-1/2}(x) - \Theta^{-3/2}(x)\big)\Theta'(x). \qquad (7.2)$$

Integrating the differential equation (7.1), we reduce its order by a unit

$$\Theta''(x) + c = 2j(3\Theta^{1/2}(x) + \Theta^{-1/2}(x)). \qquad (7.3)$$

Here c is an arbitrary constant. If we multiply equation (7.3) by $\Theta'(x) \neq 0$ and integrate the expression, we obtain

$$\Theta'(x) = \pm\sqrt{8j}\sqrt{\Theta^{3/2}(x) - \frac{c}{4j}\Theta(x) + \Theta^{1/2}(x) + \tilde{C}}.$$

Here $\tilde{C}$ is an arbitrary constant. It is possible to transform this equation by the replacement $H(x) = \Theta^{1/2}(x)$

$$H(x)H'(x) = \pm\sqrt{2j}\sqrt{H^3(x) - \frac{c}{4j}H^2(x) + H(x) + \tilde{C}}.$$

On account of the initial condition $\Theta(0) = 0$ we have $H(0) = 0$ and $\tilde{C} = 0$. And finally we obtain

$$\int_0^H \frac{\sqrt{s}\,ds}{\sqrt{s^2 - \frac{c}{4j}s + 1}} = \pm\sqrt{2j}x. \tag{7.4}$$

This formula determines $\Theta(x)$ as a function of argument x, where the integral in the left hand side of (7.2) is reduced to elliptic functions. And in the case, when $c^2 - 16j^2 = 0$, the integral in (7.2) is computed in terms of elementary functions. From (7.2) it is hardly ever possible to understand the nature and the behavior of function $H(x)$ (i.e. $\Theta(x)$). So, it is necessary to conduct asymptotic analysis. The arbitrary constant c in (7.1) may be taken according to the boundary conditions (2.2) and (2.3). To this end, we take an arbitrary small positive number $\epsilon > 0$ and consider the initial value problem for (2.1) under the following conditions:

$$\varphi(\epsilon) = \epsilon, \quad \varphi'(\epsilon) = \epsilon, \quad a(\epsilon) = \epsilon, \quad a'(\epsilon) = c_2$$

where $c_2 = p_2(0)$ is the value of the constant in the first integral J_2 in the solution of boundary-value problem (2.1)–(2.3). Let us denote the solution of this initial-value problem as $(\bar{\varphi}(x), \bar{a}(x))$, and function Θ taken along the solution is denoted as $\overline{\Theta}(x)$. Note, for any sufficiently small $\epsilon > 0$ the initial point lies in domain Ω_+, hence function $\overline{\Theta}(x)$ is defined at least on some interval $[\epsilon, \epsilon + \delta)$ and satisfies identities (6.6), (7.1). Here $\delta > 0$ is a positive number. From (6.6) we have

$$\overline{\Theta}''(\epsilon) = 2\epsilon^2 - 2c_2^2 + 2j(\sqrt{2\epsilon} + \frac{1}{\sqrt{2\epsilon}}).$$

Substituting this value into (7.1) one can find $c = 2c_2^2 - 2\epsilon^2 + 4j\sqrt{2\epsilon}$. It is important that this representation of constant c contains only positive powers of parameter ϵ.

Now, passing to the limit with $\epsilon \to +0$, we can state that function $\Theta(x)$ satisfies equation (7.1) on the solution of the boundary-value problem (2.1)–(2.3), when $c = 2c_2^2$ and $\Theta(0) = 0$.

So, we shall look for an approximate solution of (7.1) in the form $\Theta(x) = kx^\alpha$, where the positive parameters, i.e. α and k, are to be defined. By equating (after the substation into (7.1)) the main terms in the left-hand and right-hand sides, and next by separating the variables, we obtain $\alpha = \frac{4}{3}$ and $k = \left(\frac{9j}{2}\right)^{2/3}$. Therefore, when values of $x > 0$ are small, function $\Theta(x)$ taken along the solution of the boundary-value problem may be approximated by the formula

$$\tilde{\Theta}(x) = \left(\frac{9j}{2}\right)^{2/3} x^{4/3}. \tag{7.5}$$

Substituting (7.3) into (2.1), it is possible to decompose this system into two independent scalar equations, for which the following approximate solutions

$$\tilde{\varphi}(x) = \frac{1}{2}\left(\frac{9j}{2}\right)^{2/3} x^{4/3}, \tag{7.6}$$

$$\tilde{a}(x) = c_2 x\left(1 + \frac{1}{14}\left(\frac{9j}{2}\right)^{2/3} x^{4/3}\right). \tag{7.7}$$

may be obtained similarly, on account of the conditions at the left end of the interval, where $x = 0$.

As obvious from (7.4) and (7.5), the curve, which maps the solution of the boundary-value problem onto the plane (φ, a), may be approximated in the vicinity

of the origin by the function

$$a = c_2 2^{3/4} \left(\frac{9j}{2}\right)^{-1/2} \varphi^{3/4} \left(1 + \frac{1}{7}\varphi\right). \tag{7.8}$$

Given by equation (7.6), the curve passes from inside of domain Ω_+ to its boundary at the origin. It has a vertical tangent at this point, which touches also the boundary of domain Ω_+. It is incorrect to consider the initial-value problem (2.1) with the conditions (2.2) and $a'(0) = c_2$ because the conditions of the theorem on existence are violated under these initial conditions, and the substitution of these initial conditions into the equation implies division by zero. Meanwhile, using (7.4), (7.5), it is possible to correctly state the initial-value problem for an arbitrary small positive value of the independent variable $x = x_0 = \epsilon > 0$.

8. Examples

Consider problem (2.1)–(2.3) with the following conditions for the right end of the interval: $q_1(1) = q_2(1) = Q_1 = Q_2 = 1$. When solving the corresponding system of equations (4.1) with the aid of the iterative method, one finds its approximate solution $c_2 = u_* = 0.8798287042$, $j = v_* = 0.5337203307$. This solution corresponds to the following values of momentum $p_1(1) = -1.444231410$, $p_2(1) = 1.162030057$ computed at the right end of the interval by formulas (6.5) and (6.6). The Figures 3 and 4 represent diagrams demonstrating components of the boundary-value problem solution (bold red line) obtained by numerical right to left integration from $t = 1$ to $t = 0$, and the solution asymptotics in the vicinity of the interval's left end (thin blue line) computed by formulas (7.3)–(7.6) at $t = 1$.
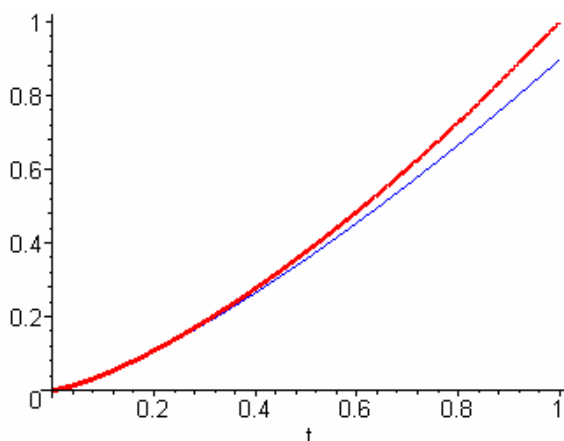

Figure 3. Solution of the boundary-value problem for the first coordinate $q_1(t)$ (bold red line) and its asymptotic computed by formula (7.4) (blue).

Consider now the boundary-value problem (2.1)–(2.3) with the values, which substantially differ at the right end of the interval, i.e. $q_1(1) = Q_1 = 10$, $q_2(1) = Q_2 = 1$. Solving system (4.1) by the iteration method, one can find its approximate solution $c_2 = u^* = 0.5404932672$, $j = v_* = 12.14221503$. It corresponds to the following values of momentum at the right end of the interval: $p_1(1) = -16.33935466$,
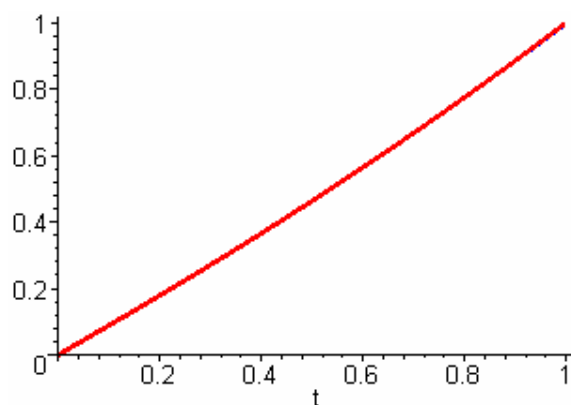
FIGURE 4. Solution of the boundary value problem on the second coordinate $q_2(t)$ (red) and it asymptotic by formula (7.5) (blue). The asymptotic behavior almost coincided with solution, the curves merge
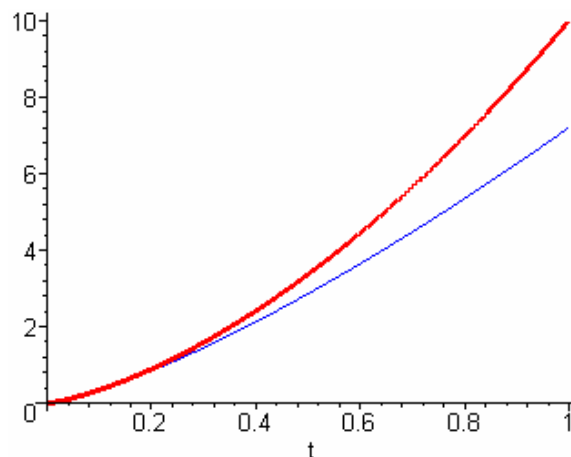

FIGURE 5. Solution of boundary value problem on the first coordinate $q_1(t)$ (red) and its asymptotic by formula (7.4) (blue) for $q_1(1) = 10, q_2(1) = 1$

$p_2(1) = 1.534531629$. Consider only the following two diagrams of coordinate variations for the given variant of initial data.

In this variant, the curves do not merge completely as we see in figure 4. Likewise for all other diagrams, we obviously see high precision of the solution for the boundary-value problem in the vicinity of the left end of the interval, represented by formulas (7.3)–(7.6).
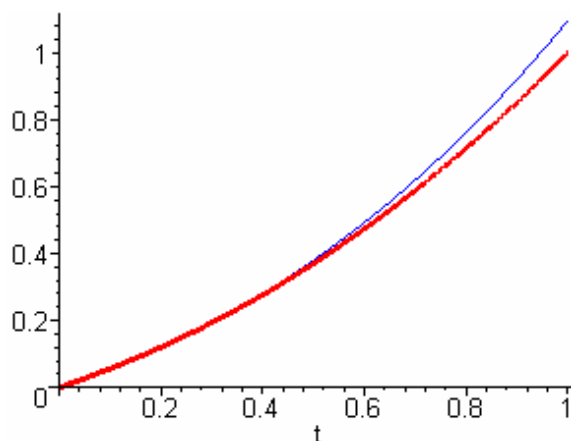
FIGURE 6. Solution of the boundary value problem on the second coordinate $q_2(t)$ (red) and its asymptotic by formula (7.5) (blue) for $q_1(1) = 10, q_2(1) = 1$

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ALEXANDER A. KOSOV

MATROSOV INSTITUTE FOR SYSTEM DYNAMICS AND CONTROL THEORY, SIBERIAN BRANCH OF RUSSIAN ACADEMY OF SCIENCES, ISDCT SB RAS LERMONTOV STR., 134, POST BOX 292 664033, IRKUTSK, RUSSIA
   E-mail address: `aakosov@yandex.ru`

EDWARD I. SEMENOV

MATROSOV INSTITUTE FOR SYSTEM DYNAMICS AND CONTROL THEORY, SIBERIAN BRANCH OF RUSSIAN ACADEMY OF SCIENCES, ISDCT SB RAS LERMONTOV STR., 134, POST BOX 292 664033, IRKUTSK, RUSSIA
   E-mail address: `edwseiz@gmail.com`

Alexander V. Sinitsyn
 Universidad Nacional de Colombia, Departamento de Matematicas Bogota, Colombia
    E-mail address: avsinitsyn@yahoo.com