Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 267, pp. 1–27.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

METHODS IN HALF-LINEAR ASYMPTOTIC THEORY

PAVEL ŘEHÁK

ABSTRACT. We study the asymptotic behavior of eventually positive solutions of the second-order half-linear differential equation

$$(r(t)|y'|^{\alpha-1}\operatorname{sgn} y')' = p(t)|y|^{\alpha-1}\operatorname{sgn} y,$$

where $r(t)$ and $p(t)$ are positive continuous functions on $[a,\infty)$, $\alpha \in (1,\infty)$. The aim of this article is twofold. On the one hand, we show applications of a wide variety of tools, like the Karamata theory of regular variation, the de Haan theory, the Riccati technique, comparison theorems, the reciprocity principle, a certain transformation of dependent variable, and principal solutions. On the other hand, we solve open problems posed in the literature and generalize existing results. Most of our observations are new also in the linear case.

1. Introduction

We consider the second-order half-linear differential equation

$$(r(t)\Phi(y'))' = p(t)\Phi(y), \tag{1.1}$$

where r, p are positive continuous functions on $[a,\infty)$ and $\Phi(u) = |u|^{\alpha-1}\operatorname{sgn} u$ with $\alpha > 1$. Actually, the aim of this paper is twofold. First, we complete somehow the study of asymptotic properties of solutions to (1.1) given in [21] and solve some open problems posed there; see also [7, 9, 12, 13, 14, 15, 16, 19] for closely related works. Second, we present applications of various tools in the asymptotic theory of linear and half-linear differential equations. In particular, we deal with the Riccati technique, the Karamata theory of regular variation, the de Haan theory, the reciprocity principle, comparison results, a certain transformation of dependent variable, and principal solutions.

We give conditions guaranteeing regular variation of all positive solutions of equation (1.1) and establish asymptotic formulas for them. In several cases we offer more than one approach. Some of the results (or at least the methods) appear to be new even in the linear case. Our results can be understood as that they provide a more precise description of behavior of solutions in standard asymptotic classes. For the standard classification of nonoscillatory solutions to (1.1) and basic existence theorems see [2, 3, 4], [5, Chapter 4], and [17].

That the theory of regular variation is well suited for the study of asymptotic behavior of differential equations was shown in particular in the monograph [15] which summarizes the research up to 2000. A survey of recent progress is made

2010 *Mathematics Subject Classification.* 34C11, 34C41, 34E05, 26A12.
Key words and phrases. Half-linear differential equation; nonoscillatory solution; regular variation; asymptotic formula.

Submitted August 14, 2015. Published October 7, 2016.

in [20]. The already mentioned papers [7, 9, 16] present applications to linear differential equations. Half-linear differential equations in the framework of the Karamata theory and the de Haan theory are treated in the works [12, 13, 18, 19, 22, 21].

The article is organized as follows. In the next section we recall some information on the Karamata theory of regularly varying functions and on the de Haan theory. Basic classification of nonoscillatory solutions to (1.1) is given in Section 3. We will utilize results on slowly varying solutions to (1.1) established in [21]; they are recalled and bettered in Section 4. A theorem on non-slowly varying solutions under the same setting as in Section 4 forms the main part of Section 5. We offer two approaches. The first one is based on the result from Section 4, the reciprocity principle, and the Karamata theory. The second one uses the Riccati technique, the Karamata theory, and the de Haan theory. We discuss regularly varying solutions also under a different setting. Section 6 offers a summary, and incorporates the results into a broader context, namely the standard classification of nonoscillatory solutions. In Section 7 we discuss some methods that are not fully available in the half-linear case. Some directions for a future research are indicated in the last section. Sections 5–7 contain various examples and further comments including a comparison with existing results.

2. Regular variation and de Haan class Π

In this section we recall basic information on the Karamata theory of regularly varying functions and the de Haan theory; for a deeper study of this topic see the monographs [1, 8, 10].

A measurable function $f:[a,\infty)\to(0,\infty)$ is called *regularly varying (at infinity) of index* ϑ if

$$\lim_{t\to\infty}\frac{f(\lambda t)}{f(t)}=\lambda^{\vartheta}\quad\text{for every }\lambda>0;\tag{2.1}$$

we write $f\in\mathcal{RV}(\vartheta)$. If $\vartheta=0$, then we speak about *slowly varying* functions; we write $f\in\mathcal{SV}$, thus $\mathcal{SV}=\mathcal{RV}(0)$.

The so-called Uniform Convergence Theorem (see e.g. [1]) says that if $f\in\mathcal{RV}(\vartheta)$, then relation (2.1) holds uniformly on each compact λ-set in $(0,\infty)$.

It follows that $f\in\mathcal{RV}(\vartheta)$ if and only if there exists a function $L\in\mathcal{SV}$ such that $f(t)=t^{\vartheta}L(t)$ for every t. The slowly varying component of $f\in\mathcal{RV}(\vartheta)$ will be denoted by L_f, i.e.,

$$L_f(t):=\frac{f(t)}{t^{\vartheta}}.$$

The Representation Theorem (see e.g. [1]) says the following:

Theorem 2.1. *$f\in\mathcal{RV}(\vartheta)$ if and only if*

$$f(t)=\varphi(t)t^{\vartheta}\exp\Big\{\int_a^t\frac{\psi(s)}{s}\,\mathrm{d}s\Big\},\tag{2.2}$$

$t\ge a$, for some $a>0$, where φ,ψ are measurable with $\lim_{t\to\infty}\varphi(t)=C\in(0,\infty)$ and $\lim_{t\to\infty}\psi(t)=0$.

A function $f\in\mathcal{RV}(\vartheta)$ can alternatively be represented as

$$f(t)=\varphi(t)\exp\Big\{\int_a^t\frac{\omega(s)}{s}\,\mathrm{d}s\Big\},\tag{2.3}$$

$t \ge a$, for some $a > 0$, where φ, ω are measurable with $\lim_{t\to\infty} \varphi(t) = C \in (0,\infty)$ and $\lim_{t\to\infty} \omega(t) = \vartheta$.

A regularly varying function f is said to be *normalized regularly varying*, we write $f \in \mathcal{NRV}(\vartheta)$, if $\varphi(t) \equiv C$ in (2.2) or in (2.3). If (2.2) holds with $\vartheta = 0$ and $\varphi(t) \equiv C$, we say that f is *normalized slowly varying*, we write $f \in \mathcal{NSV}$. Clearly, if f is a C^1 function and $\lim_{t\to\infty} tf'(t)/f(t) = \vartheta$, then $f \in \mathcal{NRV}(\vartheta)$. Conversely, if $f \in \mathcal{NRV}(\vartheta) \cap C^1$, then $\lim_{t\to\infty} tf'(t)/f(t) = \vartheta$.

The following Karamata Integration Theorem (see e.g. [1, 8]) will be very helpful in the sequel. As usual, the relation $f(t) \sim g(t)$ (as $t \to \infty$) means $\lim_{t\to\infty} f(t)/g(t) = 1$.

Theorem 2.2. *If $L \in \mathcal{SV}$, then*

$$\int_t^\infty s^\vartheta L(s)\,\mathrm{d}s \sim \frac{1}{-\vartheta-1} t^{\vartheta+1} L(t) \quad \text{provided } \vartheta < -1,$$

$$\int_a^t s^\vartheta L(s)\,\mathrm{d}s \sim \frac{1}{\vartheta+1} t^{\vartheta+1} L(t) \quad \text{provided } \vartheta > -1$$

as $t \to \infty$. Moreover, if $\int_a^\infty L(s)/s\,\mathrm{d}s$ converges, then $\tilde{L}(t) = \int_t^\infty L(s)/s\,\mathrm{d}s$ is a $\mathcal{SV}$ function; if $\int_a^\infty L(s)/s\,\mathrm{d}s$ diverges, then $\tilde{L}(t) = \int_a^t L(s)/s\,\mathrm{d}s$ is a $\mathcal{SV}$ function; in both cases, $L(t)/\tilde{L}(t) \to 0$ as $t \to \infty$.

Here are further properties of $\mathcal{RV}$ functions that are useful in our theory.

Proposition 2.3.

(i) *If $f \in \mathcal{RV}(\vartheta)$, then $\ln f(t)/\ln t \to \vartheta$ as $t \to \infty$. It then clearly implies that $\lim_{t\to\infty} f(t) = 0$ provided $\vartheta < 0$, and $\lim_{t\to\infty} f(t) = \infty$ provided $\vartheta > 0$.*
(ii) *If $f \in \mathcal{RV}(\vartheta)$, then $f^\alpha \in \mathcal{RV}(\alpha\vartheta)$ for every $\alpha \in \mathbb{R}$.*
(iii) *If $f_i \in \mathcal{RV}(\vartheta_i)$, $i = 1, 2$, $f_2(t) \to \infty$ as $t \to \infty$, then $f_1 \circ f_2 \in \mathcal{RV}(\vartheta_1\vartheta_2)$.*
(iv) *If $f_i \in \mathcal{RV}(\vartheta_i)$, $i = 1, 2$, then $f_1 + f_2 \in \mathcal{RV}(\max\{\vartheta_1, \vartheta_2\})$.*
(v) *If $f_i \in \mathcal{RV}(\vartheta_i)$, $i = 1, 2$, then $f_1 f_2 \in \mathcal{RV}(\vartheta_1 + \vartheta_2)$.*
(vi) *If $f_1, \dots, f_n \in \mathcal{RV}$, $n \in \mathbb{N}$, and $R(x_1, \dots, x_n)$ is a rational function with nonnegative coefficients, then $R(f_1, \dots, f_n) \in \mathcal{RV}$.*
(vii) *If $L \in \mathcal{SV}$ and $\vartheta > 0$, then $t^\vartheta L(t) \to \infty$, $t^{-\vartheta} L(t) \to 0$ as $t \to \infty$.*
(viii) *If $f \in \mathcal{RV}(\vartheta)$, $\vartheta \ne 0$, then there exists $g \in C^1$ with $g(t) \sim f(t)$ as $t \to \infty$ and such that $tg'(t)/g(t) \to \vartheta$, whence $g \in \mathcal{NRV}(\vartheta)$. Moreover, g can be taken such that $|g'| \in \mathcal{NRV}(\vartheta - 1)$.*
(ix) *If $|f'| \in \mathcal{RV}(\vartheta)$, $\vartheta \ne -1$, with f' being eventually of one sign, then $f \in \mathcal{NRV}(\vartheta + 1)$.*

Proof. The proofs of (i)–(viii) are either easy or can be found in [1, 8].

(ix) By the Karamata theorem, $f(t) = f(a) + \int_a^t f'(s)\,\mathrm{d}s \sim \int_a^t f'(s)\,\mathrm{d}s \sim tf'(t)/(\vartheta+1)$ as $t \to \infty$ when $\vartheta > -1$, resp. $f(t) = -\int_t^\infty f'(s)\,\mathrm{d}s \sim tf'(t)/(\vartheta+1)$ as $t \to \infty$ when $\vartheta < -1$. Hence, $\lim_{t\to\infty} tf'(t)/f(t) = \vartheta + 1$. $\square$

A measurable function $f : [a,\infty) \to \mathbb{R}$ is said to belong to the class Π if there exists a function $w : (0,\infty) \to (0,\infty)$ such that for $\lambda > 0$

$$\lim_{t\to\infty} \frac{f(\lambda t) - f(t)}{w(t)} = \ln \lambda; \tag{2.4}$$

we write $f \in \Pi$ or $f \in \Pi(w)$. The function w is called an *auxiliary function* for f. The class Π, after taking absolute values, forms a proper subclass of $\mathcal{SV}$.

Proposition 2.4.

(i) *If $f \in \Pi$, then for $0 < c < d < \infty$ relation (2.4) holds uniformly for $\lambda \in [c,d]$.*

(ii) *Auxiliary function is unique up to asymptotic equivalence.*

(iii) *If $f \in \Pi(v)$, then*

$$v(t) \sim f(t) - \frac{1}{t}\int_a^t f(s)\,\mathrm{d}s \tag{2.5}$$

as $t \to \infty$.

(iv) *If $f \in \Pi$, then $\lim_{t\to\infty} f(t) =: f(\infty) \le \infty$ exists. If the limit is infinite, then $f \in \mathcal{SV}$. If the limit is finite, then $f(\infty) - f(t) \in \mathcal{SV}$.*

(v) *If $f' \in \mathcal{RV}(-1)$, then $f \in \Pi(tf'(t))$.*

Proof. The proofs of (i)–(iv) can be found in [8, 10].

(v) For every $\lambda > 0$, we have

$$\frac{f(\lambda t) - f(t)}{tf'(t)} = \int_t^{\lambda t} \frac{f'(u)}{tf'(t)}\,\mathrm{d}u = \int_1^{\lambda} \frac{f'(st)}{f'(t)}\,\mathrm{d}s \to \int_1^{\lambda} \frac{1}{s}\,\mathrm{d}s = \ln\lambda,$$

as $t \to \infty$, because $f'(st)/f'(t) \to 1/s$ uniformly as $t \to \infty$ in the interval $\big[\min\{1,\lambda\}, \max\{1,\lambda\}\big]$. $\square$

We write $f \in \Pi\mathcal{RV}(\vartheta; w)$ if $t^{-\vartheta} f \in \Pi(w)$. Then we speak about Π-regular variation; this concept was introduced in [6].

3. Basic information on nonoscillatory solutions

It is known (see [5, Chapter 4]) that (1.1) with positive r, p is nonoscillatory, i.e. all its solutions are eventually of constant sign. Without loss of generality, we work just with positive solutions, i.e. with the class

$$\mathcal{S} = \{y : y(t) \text{ is a positive solution of (1.1) for large } t\}.$$

Because of the sign conditions on the coefficients, all positive solutions of (1.1) are eventually monotone, therefore they belong to one of the following disjoint classes:

$$\mathcal{IS} = \{y \in \mathcal{S} : y'(t) > 0 \text{ for large } t\},$$
$$\mathcal{DS} = \{y \in \mathcal{S} : y'(t) < 0 \text{ for large } t\}.$$

It can be shown that both these classes are nonempty (see [5, Lemma 4.1.2]). The classes $\mathcal{IS}, \mathcal{DS}$ can be divided into four mutually disjoint subclasses:

$$\mathcal{IS}_\infty = \{y \in \mathcal{IS} : \lim_{t\to\infty} y(t) = \infty\}, \quad \mathcal{IS}_B = \{y \in \mathcal{IS} : \lim_{t\to\infty} y(t) = b \in \mathbb{R}\},$$
$$\mathcal{DS}_B = \{y \in \mathcal{DS} : \lim_{t\to\infty} y(t) = b > 0\}, \quad \mathcal{DS}_0 = \{y \in \mathcal{DS} : \lim_{t\to\infty} y(t) = 0\}.$$

Define the so-called quasiderivative of $y \in \mathcal{S}$ by $y^{[1]} = r\Phi(y')$. We introduce the following convention

$$\mathcal{IS}_{u,v} = \{y \in \mathcal{IS} : \lim_{t\to\infty} y(t) = u,\ \lim_{t\to\infty} y^{[1]}(t) = v\}$$
$$\mathcal{DS}_{u,v} = \{y \in \mathcal{DS} : \lim_{t\to\infty} y(t) = u,\ \lim_{t\to\infty} y^{[1]}(t) = v\}.$$

For subscripts of $\mathcal{IS}$ and $\mathcal{DS}$, by $u = B$ resp. $v = B$ we mean that the value of u resp. v is a real nonzero number. Using this convention we further distinguish the following types of solutions which form subclasses in $\mathcal{DS}_0, \mathcal{DS}_B, \mathcal{IS}_B$, and $\mathcal{IS}_\infty$:

$$\mathcal{DS}_{0,0}, \mathcal{DS}_{0,B}, \mathcal{DS}_{B,0}, \mathcal{DS}_{B,B}, \mathcal{IS}_{B,B}, \mathcal{IS}_{B,\infty}, \mathcal{IS}_{\infty,B}, \mathcal{IS}_{\infty,\infty}. \tag{3.1}$$

More information about (non)existence of solutions in these subclasses is recalled in Section 6, where we include our results into the framework of the standard classification of nonoscillatory solutions. We will also make a comparison with general existence conditions. Basic classification of nonoscillatory solutions and existence results can be found in [2, 3, 4, 17]. For a partial survey see [5, Chapter 4]. In some places we need to emphasize that the classes of eventually positive increasing resp. decreasing solutions resp. their subclasses are associated to a particular equation, say $(*)$. Then we write $\mathcal{IS}^{(*)}$, $\mathcal{DS}^{(*)}$, $\mathcal{IS}^{(*)}_\infty$, etc.

No matter whether p is positive, if (1.1) is nonoscillatory, then there exists a nontrivial solution y of (1.1) such that for every nontrivial solution u of (1.1) with $u \neq \lambda y$, $\lambda \in \mathbb{R}$, we have $y'(t)/y(t) < u'(t)/u(t)$ for large t, see [5, Section 4.2]. Such a solution is said to be a principal solution. Solutions of (1.1) which are not principal, are called nonprincipal solutions. Principal solutions are unique up to a constant multiple.

Let $y \in \mathcal{S}$ and take $f \in C^1$ with $f(t) \neq 0$ for every (large) t. Denoted $w = fr\Phi(y'/y)$, it satisfies the generalized Riccati equation

$$w' - \frac{f'}{f}w - fp + (\alpha - 1)\frac{r^{1-\beta}}{\Phi^{-1}(f)}|w|^\beta = 0, \tag{3.2}$$

where Φ^{-1} stands for the inverse of Φ, i.e., $\Phi^{-1}(u) = |u|^{\beta-1}\operatorname{sgn} u$, and β denotes the conjugate number of α, i.e. $1/\alpha + 1/\beta = 1$. If $f(t) \equiv 1$, then (3.2) reduces to the usual generalized Riccati equation

$$w' - p(t) + (\alpha - 1)r^{1-\beta}(t)|w|^\beta = 0. \tag{3.3}$$

If $f(t) = t^{\alpha-1}/r(t)$, then (3.2) takes the form

$$tw' = \Big(\alpha - 1 - \frac{tr'(t)}{r(t)}\Big)w + \frac{t^\alpha p(t)}{r(t)} - (\alpha - 1)|w|^\beta; \tag{3.4}$$

note that $w(t) = \Phi(ty'(t)/y(t))$. A solution of the associated generalized Riccati equation which is generated by a principal solution is called an eventually minimal solution. According to [5, Theorem 4.2.2.], if $P(t) \le p(t)$ and $0 < R(t) \le r(t)$ for large t, then the eventually minimal solutions $w = r\Phi(y'/y)$ and $z = R\Phi(x'/x)$ of the generalized Riccati equations respectively associated to (1.1) and $(R(t)\Phi(x'))' = P(t)\Phi(x)$ satisfy

$$w(t) \le z(t) \tag{3.5}$$

for large t.

4. $\mathcal{SV}$ solutions

The following conditions appear frequently throughout this article:

$$p \in \mathcal{RV}(\delta), \quad r \in \mathcal{RV}(\delta + \alpha), \tag{4.1}$$

$$\frac{L_p(t)}{L_r(t)} \to 0 \quad \text{as } t \to \infty. \tag{4.2}$$

The relation between indices of regular variation of the coefficients in (4.1) appears to be quite natural when dealing with $\mathcal{SV}$ solutions of (1.1), see [21, Remark 11]. Set

$$G(t) = \Big(\frac{tp(t)}{r(t)}\Big)^{\frac{1}{\alpha-1}}.$$

If (4.1) holds, then

$$G(t) = \frac{1}{t}\Big(\frac{L_p(t)}{L_r(t)}\Big)^{\beta-1}.$$

A substantial part of the following statement follows from [21, Theorem 5, Theorem 6].

Theorem 4.1. *Assume that* (4.1) *and* (4.2) *hold. If* $\delta < -1$*, then* $\mathcal{DS} \subset \mathcal{NSV}$ *and* $-y(t) \in \Pi(-ty'(t))$ *for any* $y \in \mathcal{DS}$*. If* $\delta > -1$*, then* $\mathcal{IS} \subset \mathcal{NSV}$ *and* $y(t) \in \Pi(ty'(t))$ *for any* $y \in \mathcal{IS}$*. Moreover, for any* $y \in \mathcal{DS}$ *when* $\delta < -1$ *and any* $y \in \mathcal{IS}$ *when* $\delta > -1$ *the following hold:*

(i) If $\int_a^\infty G(s)\,\mathrm{d}s = \infty$*, then*

$$y(t) = \exp\Big\{\int_a^t (1+o(1))\frac{G(s)}{\Phi^{-1}(\delta+1)}\,\mathrm{d}s\Big\} \tag{4.3}$$

as $t \to \infty$*, with* $y \in \mathcal{DS}_{0,0}$ *provided* $y \in \mathcal{DS}$ *and* $\delta < -1$*, while* $y \in \mathcal{IS}_{\infty,\infty}$ *provided* $y \in \mathcal{IS}$ *and* $\delta > -1$*.*

(ii) If $\int_a^\infty G(s)\,\mathrm{d}s < \infty$*, then*

$$y(t) = N\exp\Big\{-\int_t^\infty (1+o(1))\frac{G(s)}{\Phi^{-1}(\delta+1)}\,\mathrm{d}s\Big\} \tag{4.4}$$

as $t \to \infty$*, where* $N = \lim_{t\to\infty} y(t) \in (0,\infty)$*, with* $y \in \mathcal{DS}_{B,0}$ *provided* $y \in \mathcal{DS}$ *and* $\delta < -1$*, while* $y \in \mathcal{IS}_{B,\infty}$ *provided* $y \in \mathcal{IS}$ *and* $\delta > -1$*. Moreover, in any case,* $|N-y| \in \mathcal{SV}$ *and*

$$\frac{L_p^{\beta-1}(t)}{L_r^{\beta-1}(t)(N-y(t))} = o(1) \tag{4.5}$$

as $t \to \infty$*.*

Proof. We will prove only the last part of the theorem. The fact that $|N-y| \in \mathcal{SV}$ follows from Proposition 2.4-(iv), since $y \in \Pi$. From the proofs of [21, Theorem 5, Theorem 6], we have that

$$y'(t) \sim \Phi^{-1}\Big(\frac{tp(t)}{(\delta+1)r(t)}\Big)y(t) \sim \frac{N}{\Phi^{-1}(\delta+1)}\cdot\frac{1}{t}\Big(\frac{L_p(t)}{L_r(t)}\Big)^{\beta-1}$$

as $t \to \infty$. Integrating from t to ∞, we obtain

$$N - y(t) \sim \frac{N}{\Phi^{-1}(\delta+1)}\int_t^\infty \frac{1}{s}\Big(\frac{L_p(s)}{L_r(s)}\Big)^{\beta-1}\mathrm{d}s$$

as $t \to \infty$. Formula (4.5) now follows from the latter part of Theorem 2.2. $\square$

Thanks to the next lemma, which follows from [21, Remark 8, Remark 11], we are dealing with all $\mathcal{SV}$ solutions of (1.1) in Theorem 4.1.

Lemma 4.2. *Assume that* (4.1) *holds. If* $\delta < -1$*, then* $\mathcal{S} \cap \mathcal{SV} \subseteq \mathcal{DS}$*. If* $\delta > -1$*, then* $\mathcal{S} \cap \mathcal{SV} \subseteq \mathcal{IS}$*.*

5. Non-$\mathcal{SV}$ solutions

This section discusses the complementary case with respect to Theorem 4.1; we study increasing solutions when $\delta < -1$ and decreasing solutions when $\delta > -1$. Under the same setting (i.e., (4.1) and (4.2)) we prove regular variation of these solutions where the index is equal to

$$\varrho := \frac{-1-\delta}{\alpha - 1}$$

and derive asymptotic formulas. Note that if $\delta = -\alpha$ (which happens, for instance, when $r(t) \equiv 1$ under condition (4.1)), then $\varrho = 1$. Set

$$H(t) = \frac{t^{\alpha-1}p(t)}{r(t)}.$$

If (4.1) holds, then

$$H(t) = \frac{1}{t} \cdot \frac{L_p(t)}{L_r(t)}.$$

5.1. **First approach.**

In this subsection, we use the existing results (presented in Section 4) in a combination with the reciprocity principle and the Karamata theory to study non-$\mathcal{SV}$ solutions.

The reciprocity principle is based on the following simple relation. If y is a solution of (1.1), then u defined by $u = Cr\Phi(y')$, $C \in \mathbb{R}$, is a solution of the reciprocal equation

$$(\widehat{r}(t)\widehat{\Phi}(u'))' = \widehat{p}(t)\widehat{\Phi}(u), \tag{5.1}$$

where $\widehat{r} = p^{1-\beta}$, $\widehat{p} = r^{1-\beta}$, and $\widehat{\Phi}(u) = |u|^{\widehat{\alpha}-1} \operatorname{sgn} u$ with $\widehat{\alpha} = \beta$. Note that $\widehat{\Phi} = \Phi^{-1}$.

Theorem 5.1. *Assume that* (4.1) *and* (4.2) *hold. If $\delta < -1$, then $\mathcal{IS} \subset \mathcal{NRV}(\varrho)$. If $\delta > -1$, then $\mathcal{DS} \subset \mathcal{NRV}(\varrho)$. Moreover, one has $y^{[1]}(t) \in \Pi(tp(t)\Phi(y(t)))$ for any $y \in \mathcal{S} \cap \mathcal{NRV}(\varrho)$. For any $y \in \mathcal{IS}$ when $\delta < -1$ and any $y \in \mathcal{DS}$ when $\delta > -1$ the following hold:*

(i) If $\int_a^\infty H(s)\,\mathrm{d}s = \infty$, then

$$y(t) = A + \int_a^t \frac{1}{r^{\beta-1}(s)} \exp\Big\{ \int_a^s (1+o(1)) \frac{\beta-1}{\varrho^{\alpha-1}} H(\tau)\,\mathrm{d}\tau \Big\}\,\mathrm{d}s \tag{5.2}$$

as $t \to \infty$, for some $A \in \mathbb{R}$, with $y \in \mathcal{IS}_{\infty,\infty}$ provided $y \in \mathcal{IS}$ and $\delta < -1$, while

$$y(t) = \int_t^\infty \frac{1}{r^{\beta-1}(s)} \exp\Big\{ -\int_a^s (1+o(1)) \frac{\beta-1}{|\varrho|^{\alpha-1}} H(\tau)\,\mathrm{d}\tau \Big\}\,\mathrm{d}s \tag{5.3}$$

as $t \to \infty$, with $y \in \mathcal{DS}_{0,0}$ provided $y \in \mathcal{DS}$ and $\delta > -1$.

(ii) If $\int_a^\infty H(s)\,\mathrm{d}s < \infty$, then

$$y(t) = A + \int_a^t \frac{M^{\beta-1}}{r^{\beta-1}(s)} \exp\Big\{ -\int_s^\infty (1+o(1)) \frac{\beta-1}{\varrho^{\alpha-1}} H(\tau)\,\mathrm{d}\tau \Big\}\,\mathrm{d}s \tag{5.4}$$

as $t \to \infty$, for some $A \in \mathbb{R}$, with $y \in \mathcal{IS}_{\infty,B}$ provided $y \in \mathcal{IS}$ and $\delta < -1$, while

$$y(t) = \int_t^\infty \frac{|M|^{\beta-1}}{r^{\beta-1}(s)} \exp\Big\{ \int_s^\infty (1+o(1)) \frac{\beta-1}{|\varrho|^{\alpha-1}} H(\tau)\,\mathrm{d}\tau \Big\}\,\mathrm{d}s \tag{5.5}$$

as $t \to \infty$, with $y \in \mathcal{DS}_{0,B}$ provided $y \in \mathcal{DS}$ and $\delta > -1$, where $M = \lim_{t\to\infty} y^{[1]}(t)$ in $\mathbb{R} \setminus \{0\}$. Moreover, in any case, $M - y^{[1]} \in \mathcal{SV}$ and

$$\frac{L_p(t)}{L_r(t)(M - y^{[1]}(t))} = o(1) \tag{5.6}$$

as $t \to \infty$.

Proof. Let $\widehat{\mathcal{S}}, \widehat{\mathcal{IS}}, \widehat{\mathcal{DS}}, \widehat{\mathcal{IS}}_\infty, \widehat{\mathcal{IS}}_B, \widehat{\mathcal{DS}}_0, \widehat{\mathcal{DS}}_B$ have the same meaning with respect to (5.1) as $\mathcal{S}, \mathcal{IS}, \mathcal{DS}, \mathcal{IS}_\infty, \mathcal{IS}_B, \mathcal{DS}_0, \mathcal{DS}_B$, respectively, have with respect to (1.1). We have that $\widehat{p} \in \mathcal{RV}(\widehat{\delta})$ and $\widehat{r} \in \mathcal{RV}(\widehat{\delta} + \widehat{\alpha})$, where $\widehat{\delta} := \delta(1-\beta) - \beta$, thanks to (4.1). Since $L_{\widehat{p}}/L_{\widehat{r}} = L_r^{1-\beta}/L_p^{1-\beta} = (L_p/L_r)^{\beta-1}$ and (4.2) holds, we have $\lim_{t\to\infty} L_{\widehat{p}}(t)/L_{\widehat{r}}(t) = 0$.

Assume that $\delta < -1$ and take $y \in \mathcal{IS}$. Let $t_0 \ge a$ be such that $y(t) > 0$, $y'(t) > 0$ for $t \ge t_0$. Set $u = r\Phi(y')$. Then $u \in \widehat{\mathcal{IS}}$ since $u' = p\Phi(y)$. We have $\widehat{\delta} + \beta = -\delta(\beta - 1) > \beta - 1$, and so $\widehat{\delta} > -1$. Note that $\widehat{\delta} + 1 = \varrho$. Now we can apply Theorem 4.1 to obtain $u \in \mathcal{NSV}$ and $u \in \Pi(tu')$. Hence, $r\Phi(y') \in \mathcal{NSV}$, i.e., $\Phi(y') \in \mathcal{RV}(-\delta-\alpha)$, i.e., $y' \in \mathcal{RV}((-\delta-\alpha)(\beta-1))$ with $(-\delta-\alpha)(\beta-1) > -1$. Thus, $y \in \mathcal{NRV}(\varrho)$ by Proposition 2.3–(ix). Moreover, $y^{[1]} = u \in \Pi(tu') = \Pi(tp\Phi(y))$ and $y \in \mathcal{IS}_\infty$ since $\varrho > 0$. Next we derive an asymptotic formula for y. If

$$\int_a^\infty \widehat{G}(s)\mathrm{d}s = \infty, \quad \text{where } \widehat{G}(t) = \Big(\frac{t\widehat{p}(t)}{\widehat{r}(t)}\Big)^{\frac{1}{\widehat{\alpha}-1}}, \tag{5.7}$$

then

$$u(t) = \exp\Big\{\int_a^t (1+o(1))\widehat{G}(s)(\widehat{\delta}+1)^{\frac{-1}{\widehat{\alpha}-1}}\mathrm{d}s\Big\} \tag{5.8}$$

as $t \to \infty$ and $u \in \widehat{\mathcal{IS}}_\infty$ for every $u \in \widehat{\mathcal{IS}}$, thanks to Theorem 4.1. From $u \in \widehat{\mathcal{IS}}_\infty$ and $y \in \mathcal{IS}_\infty$ we obtain $y \in \mathcal{IS}_{\infty,\infty}$. Since

$$\widehat{G}(t) = \Big(\frac{t\widehat{p}(t)}{\widehat{r}(t)}\Big)^{\frac{1}{\widehat{\alpha}-1}} = \Big(\frac{tr^{1-\beta}(t)}{p^{1-\beta}(t)}\Big)^{\alpha-1} = \frac{t^{\alpha-1}p(t)}{r(t)} = H(t), \tag{5.9}$$

$\widehat{\delta} + 1 = \varrho$, and $u = r(y')^{\alpha-1}$, from (5.8) we obtain

$$y'(t) = \frac{1}{r^{\beta-1}(t)} \exp\Big\{\int_a^t (1+o(1))\frac{\beta-1}{\varrho^{\alpha-1}}H(s)\,\mathrm{d}s\Big\} \tag{5.10}$$

as $t \to \infty$. Note that thanks to (5.9), $\int_a^\infty H(s)\,\mathrm{d}s = \infty$ is the same as (5.7). Integrating (5.10) from t_0 to t and realizing that $y(t_0)$ can be replaced by some $A \in \mathbb{R}$ when t_0 is replaced by a, we obtain formula (5.2).

If the integral in (5.7) is convergent, i.e., $\int_a^\infty H(s)\,\mathrm{d}s < \infty$, then we again use Theorem 4.1 to get

$$r(t)\Phi(y'(t)) = u(t) = M\exp\Big\{-\int_t^\infty (1+o(1))\frac{H(s)}{\varrho^{\alpha-1}}\,\mathrm{d}s\Big\} \tag{5.11}$$

as $\to \infty$, where $M = \lim_{t\to\infty} u(t) = \lim_{t\to\infty} r(t)\Phi(y'(t))$. Since $u \in \widehat{\mathcal{IS}}_B$ and $y \in \mathcal{IS}_\infty$, we have $y \in \mathcal{IS}_{\infty,B}$. Formula (5.4) follows from (5.11), where we first extract y' and then integrate from t_0 to t, with replacing $y(t_0)$ by A and t_0 by a. The fact that $M - y^{[1]} \in \mathcal{SV}$ follows from Proposition 2.4–(iv). Formula (5.6) is obtained from (4.5) applied to u, in view of $u = y^{[1]}$, $(\alpha-1)(\beta-1) = 1$, and $L_{\widehat{p}}/L_{\widehat{r}} = (L_p/L_r)^{\beta-1}$.

Assume $\delta > -1$. Take $y \in \mathcal{DS}$. Now we set $u = -r\Phi(y')$. Then $u \in \widehat{\mathcal{DS}}$. Since $\widehat{\delta} < -1$, we may apply Theorem 4.1 to equation (5.1). We get that $-y^{[1]} = u \in \mathcal{NSV}$ and $y^{[1]} = -u \in \Pi(-tu') = \Pi(tp\Phi(y))$. Similarly as above, we have $y \in \mathcal{NRV}(\varrho)$ with $\varrho = \widehat{\delta} + 1 < 0$. If $\int_a^\infty H(s)\,\mathrm{d}s = \infty$, then (5.7) holds and

$$-r(t)\Phi(y'(t)) = u(t) = \exp\Big\{-\int_a^t (1+o(1))\frac{H(s)}{|\varrho|^{\alpha-1}}\,\mathrm{d}s\Big\} \tag{5.12}$$

as $t \to \infty$ for $u \in \widehat{\mathcal{DS}}$, by Theorem 4.1. The fact that $y \in \mathcal{DS}_{0,0}$ is implied by $y \in \mathcal{DS}_0$ and $u \in \widehat{\mathcal{DS}}_0$. Formula (5.3) follows from (5.12), where we first extract y' and then integrate from t to ∞. If $\int_a^\infty H(s)\,\mathrm{d}s < \infty$, then the integral in (5.7) is convergent and application of Theorem 4.1 yields

$$-r(t)\Phi(y'(t)) = u(t) = |M|\exp\Big\{\int_t^\infty (1+o(1))\frac{H(s)}{|\varrho|^{\alpha-1}}\,\mathrm{d}s\Big\} \tag{5.13}$$

as $t \to \infty$, where $M = \lim_{t\to\infty} -u(t) = \lim_{t\to\infty} r(t)\Phi(y'(t))$. From $y \in \mathcal{DS}_0$ and $u \in \widehat{\mathcal{DS}}_B$, we obtain that $y \in \mathcal{DS}_{0,B}$. Formula (5.5) easily follows from (5.13). The fact that y satisfies $M - y^{[1]} \in \mathcal{SV}$ and (5.6) can be proved similarly as in the case $\delta < -1$. □

Remark 5.2. A closer examination of the previous proof shows that the expression $\exp\{\cdot\}$ in all formulae (5.2) and (5.3) is slowly varying. Moreover, $r^{1-\beta} \in \mathcal{RV}(\gamma)$, $\gamma := (1-\beta)(\delta+\alpha) = \varrho - 1$, with $\gamma > -1$ when $\delta < -1$ resp. $\gamma < -1$ when $\delta > -1$. Hence, we can apply the Karamata theorem to formula (5.2) resp. (5.3) which in either case yields

$$y(t) = (1+o(1))\frac{tr^{1-\beta}(t)}{|\gamma+1|}\exp\Big\{\int_a^t (1+o(1))\frac{\beta-1}{\Phi(\varrho)}H(s)\,\mathrm{d}s\Big\}$$

as $t \to \infty$. Realizing that $(1+o(1))/|\gamma+1| = \exp\ln((1+o(1))/|\gamma+1|)$ and $\int_a^\infty H(s)\,\mathrm{d}s$ diverges, we obtain

$$y(t) = tr^{1-\beta}(t)\exp\Big\{\int_a^t (1+o(1))\frac{\beta-1}{\Phi(\varrho)}H(s)\,\mathrm{d}s\Big\} \tag{5.14}$$

as $t \to \infty$. Alternatively, formula (5.14) can easily be obtained from (5.10) and (5.12), if we realize that $y \in \mathcal{NRV}(\varrho)$ implies that $y'(t) \sim \varrho y(t)/t$ as $t \to \infty$.

Assume – under the conditions of Theorem 5.1 – that y is a solution such that $\lim_{t\to\infty} y^{[1]}(t) = M \in \mathbb{R}\setminus\{0\}$. Integrating the relation $y^{[1]}(t) \sim M$ and applying the Karamata integration theorem (cf. (5.35), (5.37) below), we obtain the simple formula

$$y(t) = (1+o(1))\frac{\Phi^{-1}(M)}{\varrho}tr^{1-\beta}(t) \tag{5.15}$$

as $t \to \infty$.

Example 5.3. Let $p(t) = t^\delta L_p(t)$, $\delta \neq -1$, with $L_p(t) = (\ln t)^{\nu_1} + g_1(t)$, and $r(t) = t^{\delta+\alpha}L_r(t)$ with $L_r(t) = (\ln t)^{\nu_2} + g_2(t)$, where $|g_i(t)| = o((\ln t)^{\nu_i})$ as $t \to \infty$, $i = 1, 2$, and $\nu_1 < \nu_2$. Then $L_p, L_r \in \mathcal{SV}$ and $\lim_{t\to\infty} L_p(t)/L_r(t) = 0$. For example, one can take $g_i(t) = \sin t$ or $g_i(t) = \ln(\ln t)$ provided $\nu_i > 0$. We have

$$H(t) = \frac{L_p(t)}{tL_r(t)} \sim \frac{1}{t}\cdot\frac{1+g_1(t)/(\ln t)^{\nu_1}}{(\ln t)^{\nu_2-\nu_1}(1+g_2(t)/(\ln t)^{\nu_2})} \sim \frac{1}{t}(\ln t)^{\nu_1-\nu_2}$$

as $t \to \infty$. Let y be an eventually positive solution of (1.1). If $\nu_1 - \nu_2 = -1$, then $\int_a^t H(s)\,\mathrm{d}s \sim \ln(\ln t)$ and

$$y(t) = t^{\varrho}(\ln t)^{\nu_2 + (\beta-1)(1+o(1))/\Phi(\varrho)}$$

as $t \to \infty$, in view of (5.14). If $\nu_1 - \nu_2 > -1$, then

$$\int_a^t H(s)\,\mathrm{d}s \sim \frac{1}{\nu_1 - \nu_2 + 1}(\ln t)^{\nu_1-\nu_2+1}$$

and

$$y(t) = t^{\varrho}(\ln t)^{\nu_2} \exp\Big\{\frac{(\beta-1)(1+o(1))}{\Phi(\varrho)(\nu_1-\nu_2+1)}(\ln t)^{\nu_1-\nu_2+1}\Big\}$$

as $t \to \infty$, in view of (5.14). If $\nu_1 - \nu_2 < -1$, then

$$\int_t^\infty H(s)\,\mathrm{d}s \sim \frac{1}{-\nu_1 + \nu_2 - 1}(\ln t)^{\nu_1-\nu_2+1}$$

and

$$y(t) = (1+o(1))\frac{\Phi^{-1}(M)}{\varrho} t^{\varrho}(\ln t)^{\nu_2}$$

as $t \to \infty$, in view of (5.15), where $M = \lim_{t\to\infty} r(t)\Phi(y'(t)) \in \mathbb{R}\setminus\{0\}$.

Remark 5.4. For decreasing solutions, if $\alpha = 2$ and $r(t) = 1$, then Theorem 4.1 reduces to [7, Theorem 0.1-A]. In this case – since we have a linear equation – we can obtain a linearly independent solution x which satisfies $x(t)/t \in \Pi$ by the reduction of order formula (see [7, Remark 3]), thus $x \in \mathcal{RV}(1)$. A representation can also be given. This tool however is not at disposal in the half-linear case; see also Section 7.3. The conclusion of Theorem 5.1 for $r(t) \not\equiv 1$ is new in the linear case. The both proofs (the above one as well as the one in Subsection 5.3) are also new in the linear case.

Certain asymptotic formulae for regularly varying solutions of the equation

$$(\Phi(y'))' + p(t)\Phi(y) = 0 \tag{5.16}$$

with no sign condition on p were established in very recent paper [14]. The approach is quite different; a crucial role there is played by the Banach fixed point theorem. Under somewhat weaker assumptions than ours, the existence of a couple of $\mathcal{RV}$ solutions is established and asymptotic formulae are derived. On the other hand, in our paper we work with all positive (eventually decreasing or increasing solutions). See also Remark 5.8-(ii).

Remark 5.5. From [21, Remark 6], we know that to show $\widehat{\mathcal{DS}} \subset \mathcal{NSV}$ it is sufficient to assume weaker conditions, namely $\int_a^\infty \widehat{r}^{1-\widehat{\beta}}(s)\,\mathrm{d}s = \infty$, $\int_a^\infty \widehat{p}(s)\,\mathrm{d}s < \infty$, and $\lim_{t\to\infty} \frac{t^{\widehat{\alpha}-1}}{\widehat{r}(t)} \int_t^\infty \widehat{p}(s)\,\mathrm{d}s = 0$. In terms of equation (1.1) these conditions read as $\int_a^\infty p(s)\,\mathrm{d}s = \infty$, $\int_a^\infty r^{1-\beta}(s)\,\mathrm{d}s < \infty$, and

$$\lim_{t\to\infty} (tp(t))^{\beta-1} \int_t^\infty r^{1-\beta}(s)\,\mathrm{d}s = 0, \tag{5.17}$$

respectively, and guarantee that $-r\Phi(y') \in \mathcal{NSV}$ for any $y \in \mathcal{DS}$. If, moreover, $r \in \mathcal{RV}(\delta+\alpha)$ with $\delta > -1$, then $y \in \mathcal{NRV}(\varrho)$; this follows from (ii), (v), and (ix) of Proposition 2.3. Note that condition (5.17) can be written as

$$\lim_{t\to\infty} \frac{t^{\alpha}p(t)}{r(t)} = 0 \tag{5.18}$$

provided $r \in \mathcal{RV}(\delta+\alpha)$, by the Karamata theorem. Similarly, from [21, Remark 9], we obtain that the conditions $\int_a^\infty r^{1-\beta}(s)\,\mathrm{d}s = \infty$ and

$$\lim_{t\to\infty}(tp(t))^{\beta-1}\int_a^t r^{1-\beta}(s)\,\mathrm{d}s = 0 \tag{5.19}$$

imply $r\Phi(y') \in \mathcal{NSV}$ for any $y \in \mathcal{IS}$. If, in addition, $r \in \mathcal{RV}(\delta+\alpha)$ with $\delta < -1$, then $y \in \mathcal{NRV}(\varrho)$, and (5.19) can be written as (5.18).

5.2. **Necessity.** Let (4.1) hold. We claim that condition (4.2) is necessary for the existence of $y \in \mathcal{IS}\cap\mathcal{NRV}(\varrho)$ (when $\delta < -1$) or $y \in \mathcal{DS}\cap\mathcal{NRV}(\varrho)$ (when $\delta > -1$).

Indeed, assume first that $\delta < -1$ and that there exists $y \in \mathcal{IS}\cap\mathcal{NRV}(\varrho)$. Set $w = r\Phi(y'/y)$. Then w satisfies (3.3) for large t, and $0 < t^{\alpha-1}w(t)/r(t) = (ty'(t)/y(t))^{\alpha-1} \to \varrho^{\alpha-1}$. Hence, there exists $M_1 > 0$ such that $w(t) \le M_1 r(t)t^{1-\alpha}$ which belongs to $\mathcal{RV}(\delta+1)$, and so $w(t) \to 0$ as $t \to \infty$. Further, there exists $M_2 > 0$ such that

$$r^{1-\beta}(t)w^\beta(t) = r(t)(y'(t)/y(t))^\alpha \le M_2 r(t)t^{-\alpha} \in \mathcal{RV}(\delta).$$

This implies $\int_a^\infty r^{1-\beta}(s)w^\beta(s)\,\mathrm{d}s < \infty$. Integrating (3.3) from t to ∞ and multiplying by $t^{\alpha-1}/r(t)$ we obtain

$$\begin{gathered} -\frac{t^{\alpha-1}}{r(t)}w(t) = \frac{t^{\alpha-1}}{r(t)}\int_t^\infty p(s)\,\mathrm{d}s - (\alpha-1)z(t), \\ z(t) := \frac{t^{\alpha-1}}{r(t)}\int_t^\infty r^{1-\beta}(s)w^\beta(s)\,\mathrm{d}s. \end{gathered} \tag{5.20}$$

We claim that $z(t) \to \varrho^{\alpha-1}/(\alpha-1)$ as $t \to \infty$. Without loss of generality we may assume $r \in \mathcal{NRV}(\delta+\alpha)\cap C^1$. Indeed, if r is not normalized or is not in C^1, then we can take $\widetilde{r} \in \mathcal{NRV}(\delta+\alpha)\cap C^1$ with $\widetilde{r}(t) \sim r(t)$ when $t \to \infty$ (which is possible thanks to Proposition 2.3–(viii)), and we have

$$z(t) \sim \frac{t^{\alpha-1}}{\widetilde{r}(t)}\int_t^\infty \widetilde{r}^{1-\beta}(s)\left(\widetilde{r}(s)\Phi\left(\frac{y'(s)}{y(s)}\right)\right)^\beta \mathrm{d}s$$

as $t \to \infty$. By the L'Hospital rule,

$$\begin{aligned} \lim_{t\to\infty} z(t) &= \lim_{t\to\infty}\frac{-r^{1-\beta}(t)w^\beta(t)}{r'(t)t^{1-\alpha} + (1-\alpha)r(t)t^{-\alpha}} \\ &= \lim_{t\to\infty}\frac{(ty'(t)/y(t))^\alpha}{-tr'(t)/r(t)+\alpha-1} \\ &= \frac{\varrho^\alpha}{-\delta-\alpha+\alpha-1} = \frac{\varrho^{\alpha-1}}{\alpha-1}. \end{aligned}$$

From (5.20) we obtain

$$\lim_{t\to\infty}\frac{t^{\alpha-1}}{r(t)}\int_t^\infty p(s)\,\mathrm{d}s = 0, \tag{5.21}$$

which, thanks to the Karamata theorem, yields (4.2)

Similarly we proceed in the case $\delta > -1$. We assume that there exists $y \in \mathcal{DS}\cap\mathcal{NRV}(\varrho)$ and again set $w = r\Phi(y'/y)$. Instead of (5.20) we work with the Riccati type integral equation of the form

$$\frac{t^{\alpha-1}w(t)}{r(t)} - \frac{t^{\alpha-1}w(a)}{r(t)} = \frac{t^{\alpha-1}}{r(t)}\int_a^t p(s)\,\mathrm{d}s - \frac{t^{\alpha-1}}{r(t)}(\alpha-1)\int_a^t r^{1-\beta}(s)|w(s)|^\beta \mathrm{d}s.$$

Here, $t^{\alpha-1}/r(t) \to 0$ and $t^{\alpha-1}w(t)/r(t) \to \Phi(\varrho)$ as $t \to 0$. We then get

$$\lim_{t\to\infty} \frac{t^{\alpha-1}}{r(t)} \int_a^t p(s)\,\mathrm{d}s = 0, \tag{5.22}$$

which yields again (4.2), by the Karamata theorem. This conclusion can be reached also in an alternative way, where we apply [21, Remark 6, Remark 9] and the reciprocity principle.

Observe that if we drop the condition $p \in \mathcal{RV}(\delta)$, then necessary conditions read as (5.21) resp. (5.22).

5.3. **Second approach.** Let us assume that the assumptions of Theorem 5.1 hold. Next we present an alternative approach to the proof. The Riccati technique in combination with the Karamata theory is directly used to show that all increasing resp. decreasing solutions of (1.1) are in $\mathcal{NRV}(\varrho)$. Regular variation and the de Haan theory are then utilized to obtain asymptotic formulas from Remark 5.2. In Section 7, which is devoted to linear equations, we offer another approaches to this problem.

Proof of $\mathcal{IS} \subset \mathcal{NRV}(\varrho)$. Let $\delta < -1$. Take $y \in \mathcal{IS}$. We want to show that $y \in \mathcal{NRV}(\varrho)$.

First assume that $r \in C^1 \cap \mathcal{NRV}(\delta+\alpha)$. Set $w(t) = \Phi(ty'(t)/y(t))$. Then w is positive and satisfies equation (3.4) for large t, which can be written as

$$tw' = \frac{L_p(t)}{L_r(t)} + w\Big(\alpha - 1 - \frac{tr'(t)}{r(t)} - (\alpha-1)w^{\beta-1}\Big). \tag{5.23}$$

We claim that $\lim_{t\to\infty} w(t) = \varrho^{\alpha-1}$. Let $w'(t) > 0$ for large t. Then we obtain $\lim_{t\to\infty} w(t) = A \in (0,\infty]$. If $A = \infty$, then the right-hand side of (5.23) tends to $-\infty$, and so $\lim_{t\to\infty} tw'(t) = -\infty$, which contradicts eventual positivity of w'. If $A \in (0,\infty) \setminus \{\varrho^{\alpha-1}\}$, then $tw'(t) \sim C := A(-1-\delta-(\alpha-1)A^{\beta-1}) \neq 0$ as $t \to \infty$. Thus, $w(t) - w(a) \sim C\int_a^t 1/s\,\mathrm{d}s = C\ln(t/a)$ as $t \to \infty$, which contradicts $A \in \mathbb{R}$. Now let $w'(t) < 0$ for large t, say $t \geq t_0$. Then $\lim_{t\to\infty} w(t) = B \in [0,\infty)$. If $B = 0$, then $\lim_{t\to\infty} ty'(t)/y(t) = 0$, and so $y \in \mathcal{NSV}$. This contradicts with the fact that $\mathcal{SV}$ solutions cannot increase (see Lemma 4.2). If $B \in (0,\infty) \setminus \{\varrho^{\alpha-1}\}$, then similarly as above we obtain a contradiction with $B \in \mathbb{R}$. Finally, assume that there exists a sequence $\{t_n\}_n$, with $\lim_{n\to\infty} t_n = \infty$, such that $w'(t_n) = 0$. We take here zeroes of w' being consecutive. From (5.23), we have

$$0 = \frac{L_p(t_n)}{L_r(t_n)} + w(t_n)\Big(\alpha - 1 - \frac{t_n r'(t_n)}{r(t_n)} - (\alpha-1)w^{\beta-1}(t_n)\Big). \tag{5.24}$$

Hence, $w(t)$ hits the real roots of

$$0 = \frac{L_p(t_n)}{L_r(t_n)} + \lambda\Big(\alpha - 1 - \frac{t_n r'(t_n)}{r(t_n)} - (\alpha-1)\lambda^{\beta-1}\Big) \tag{5.25}$$

at $t = t_n$. Observe that for large n, the roots are (arbitrarily) close to the roots of

$$0 = \mu(-1-\delta-(\alpha-1)\mu^{\beta-1}), \tag{5.26}$$

i.e., $\mu = 0$ and $\mu = \varrho^{\alpha-1}$. Hence, if $|\lambda|$, $\lambda = \lambda(n)$ being the root of (5.25), is very small (such root clearly corresponds to the root $\mu = 0$ of (5.26)), then $\alpha-1-t_n r'(t_n)/r(t_n)-(\alpha-1)|\lambda|^{\beta-1} > 0$, thanks to $\alpha-1-t_n r'(t_n)/r(t_n) \to \alpha-1-\delta-\alpha = -1-\delta > 0$ as $n \to \infty$. Because of positivity of L_p/L_r, we must then have $\lambda < 0$, which is impossible since $w(t) > 0$. Hence, $w(t)$ hits the positive roots of (5.25) at

$t = t_n$ which tends to $\varrho^{\alpha-1}$ as $n \to \infty$. Consequently, $\lim_{n\to\infty} w(t_n) = \varrho^{\alpha-1}$. The function w is monotone between zeroes of w', thus $\min\{w(t_n), w(t_{n+1})\} \le w(t) \le \max\{w(t_n), w(t_{n+1})\}$, $t_n \le t \le t_{n+1}$. Hence, $\lim_{t\to\infty} w(t) = \varrho^{\alpha-1}$ also in this case.

Altogether we have $\lim_{t\to\infty}(ty'(t)/y(t))^{\alpha-1} = \lim_{t\to\infty} w(t) = \varrho^{\alpha-1}$, which implies $y \in \mathcal{NRV}(\varrho)$.

Now we drop the assumption of continuous differentiability and normality of regular variation of r. From Proposition 2.3–(viii), there exists $\bar r$ such that $\bar r(t) \sim r(t)$ as $t \to \infty$ and $\bar r \in C^1 \cap \mathcal{NRV}(\delta + \alpha)$. For every $\varepsilon \in (0,1)$ there exists t_ε such that $(1-\varepsilon)\bar r(t) \le r(t) \le (1+\varepsilon)\bar r(t)$ for $t \ge t_\varepsilon$. According to [5, Lemma 4.1.2], there exist eventually positive increasing solutions u, v, respectively, of the problems

$$\big((1-\varepsilon)\bar r(t)\Phi(u')\big)' = p(t)\Phi(u), \quad u(t_\varepsilon) = u_0,\ u'(t_\varepsilon) = u_1, \tag{5.27}$$

and

$$\big((1+\varepsilon)\bar r(t)\Phi(v')\big)' = p(t)\Phi(v), \quad v(t_\varepsilon) = v_0,\ v'(t_\varepsilon) = v_1, \tag{5.28}$$

with u_0, u_1, v_0, v_1 positive and such that

$$(1-\varepsilon)\bar r(t_\varepsilon)\Big(\frac{u_1}{u_0}\Big)^{\alpha-1} \le r(t_\varepsilon)\Big(\frac{y'(t_\varepsilon)}{y(t_\varepsilon)}\Big)^{\alpha-1} \le (1+\varepsilon)\bar r(t_\varepsilon)\Big(\frac{v_1}{v_0}\Big)^{\alpha-1}.$$

Define $w_u = (1-\varepsilon)\bar r\Phi(u'/u)$, $w_y = r\Phi(y'/y)$, and $w_v = (1+\varepsilon)\bar r\Phi(v'/v)$. These functions satisfy respectively the generalized Riccati equations $w_u' = p(t) - (\alpha-1)(1-\varepsilon)^{1-\beta}\bar r^{1-\beta}(t)w_u^\beta$, $w_y' = p(t) - (\alpha-1)r^{1-\beta}(t)w_y^\beta$, and $w_v' = p(t) - (\alpha-1)(1+\varepsilon)^{1-\beta}\bar r^{1-\beta}(t)w_v^\beta$. Since we actually have $(1-\varepsilon)^{1-\beta}\bar r^{1-\beta} \le -r^{1-\beta} \le (1+\varepsilon)^{1-\beta}\bar r^{1-\beta}$ and $w_u(t_\varepsilon) \le w_y(t_\varepsilon) \le w_v(t_\varepsilon)$, from the classical result on differential inequalities (see [11, Chapter III, Section 4]), we obtain $w_u(t) \le w_y(t) \le w_v(t)$ for $t \ge t_\varepsilon$. Consequently,

$$(1-\varepsilon)\Big(\frac{tu'(t)}{u(t)}\Big)^{\alpha-1} \le \frac{r(t)}{\bar r(t)}\Big(\frac{ty'(t)}{y(t)}\Big)^{\alpha-1} \le (1+\varepsilon)\Big(\frac{tv'(t)}{v(t)}\Big)^{\alpha-1} \tag{5.29}$$

for $t \ge t_\varepsilon$. From the previous part we know that $\lim_{t\to\infty} tu'(t)/u(t) = \varrho = \lim_{t\to\infty} tv'(t)/v(t)$. Hence,

$$(1-\varepsilon)^{\beta-1}\varrho \le \liminf_{t\to\infty}\frac{ty'(t)}{y(t)} \le \limsup_{t\to\infty}\frac{ty'(t)}{y(t)} \le (1+\varepsilon)\varrho.$$

Since $\varepsilon \in (0,1)$ was arbitrary, $\lim_{t\to\infty} ty'(t)/y(t) = \varrho$ and so $y \in \mathcal{NRV}(\varrho)$. □

Proof of asymptotic formula in the case $\delta < -1$. Take $y \in \mathcal{NRV}(\varrho) \cap \mathcal{IS}$. Clearly, $y \in \mathcal{IS}_\infty$ since $\varrho > 0$. From (1.1), $(r\Phi(y'))' = p\Phi(y) \in \mathcal{RV}(\delta + (\alpha-1)\varrho) = \mathcal{RV}(-1)$. Hence, $r\Phi(y') \in \Pi(t(r\Phi(y'))') = \Pi(tp\Phi(y))$, in view of Proposition 2.4–(v). Thanks to relation (2.5), we now obtain

$$tp(t)\Phi(y(t)) \sim r(t)\Phi(y'(t)) - \frac{1}{t}\int_a^t r(s)\Phi(y'(s))\,\mathrm{d}s \tag{5.30}$$

as $t \to \infty$; without loss of generality, we may assume that $y(t) > 0$ and $y'(t) > 0$ for $t \ge a$. Since $r\Phi(y') \in \mathcal{SV}$, the Karamata integration theorem yields

$$\int_a^t r(s)\Phi(y'(s))\,\mathrm{d}s \sim tr(t)\Phi(y'(t))$$

as $t \to \infty$. Further, from $y \in \mathcal{NRV}(\varrho)$, $y'(t) \sim \varrho y(t)/t$ as $t \to \infty$. These two relations imply

$$\int_a^t r(s)\Phi(y'(s))\,\mathrm{d}s \sim t^{2-\alpha} r(t)\Phi(\varrho) y^{\alpha-1}(t) \tag{5.31}$$

as $t \to \infty$. From (5.30) and (5.31), we obtain

$$\frac{r(t)\Phi(y'(t))}{\int_a^t r(s)\Phi(y'(s))\,\mathrm{d}s} - \frac{1}{t} \sim \frac{tp(t)y^{\alpha-1}(t)}{t^{2-\alpha}r(t)\varrho^{\alpha-1}y^{\alpha-1}(t)} = \frac{H(t)}{\varrho^{\alpha-1}} \tag{5.32}$$

as $t \to \infty$. Relation (5.32) can be rewritten as

$$\Big(\ln \frac{\int_a^t r(s)\Phi(y'(s))\,\mathrm{d}s}{t}\Big)' = (1+o(1))\frac{H(t)}{\varrho^{\alpha-1}} \tag{5.33}$$

as $t \to \infty$. Integration from $t_0 > a$ to t yields

$$\int_{t_0}^t (1+o(1))\frac{H(s)}{\Phi(\varrho)}\,\mathrm{d}s = \ln \frac{\int_a^t r(s)\Phi(y'(s))\,\mathrm{d}s}{Dt}, \tag{5.34}$$

where $D = \frac{1}{t_0}\int_a^{t_0} r(s)\Phi(y'(s))\,\mathrm{d}s$.

Let $\int_a^\infty H(s)\,\mathrm{d}s = \infty$. Then $\lim_{t\to\infty} \frac{1}{t}\int_a^t r(s)\Phi(y'(s))\,\mathrm{d}s = \infty$ and since $r\Phi(y')$ is positive increasing, the L'Hospital rule implies that $\lim_{t\to\infty} r(t)\Phi(y'(t)) = \infty$ otherwise we would get a contradiction. Hence, $y \in \mathcal{IS}_{\infty,\infty}$. Using (5.31) in (5.34),

$$Dt \exp\Big\{\int_{t_0}^t (1+o(1))\frac{H(s)}{\varrho^{\alpha-1}}\,\mathrm{d}s\Big\} = (1+o(1))t^{2-\alpha}r(t)\varrho^{\alpha-1}y^{\alpha-1}(t)$$

as $t \to \infty$. Realizing that $(1+o(1))D/\varrho^{\alpha-1} = \exp\ln((1+o(1))D/\varrho^{\alpha-1})$ and $\int_a^\infty H(s)\,\mathrm{d}s$ diverges, we have

$$y^{\alpha-1}(t) = \frac{t^{\alpha-1}}{r(t)}\exp\Big\{\int_{t_0}^t (1+o(1))\frac{H(s)}{\varrho^{\alpha-1}}\,\mathrm{d}s\Big\}$$

as $t \to \infty$. Raising by $\beta - 1$ we obtain formula (5.14).

Let $\int_a^\infty H(s)\,\mathrm{d}s < \infty$. Then, from (5.34), $\lim_{t\to\infty} \frac{1}{t}\int_a^t r(s)\Phi(y'(s))\,\mathrm{d}s = M \in (0,\infty)$. For the positive increasing $y^{[1]} = r\Phi(y')$ we must then have $\lim_{t\to\infty} y^{[1]}(t) = M$, and so $y \in \mathcal{IS}_{\infty,B}$. This immediately implies $y'(t) \sim M^{\beta-1}r^{1-\beta}(t)$. Integrating this relation and using the Karamata theorem,

$$\begin{aligned} y(t) &\sim y(t) - y(t_0) \sim M^{\beta-1}\int_{t_0}^t r^{1-\beta}(s)\,\mathrm{d}s \\ &\sim \frac{M^{\beta-1}}{(\delta+1)(1-\beta)} t^{(\delta+1)(1-\beta)} L_r^{1-\beta}(t) \\ &\sim \frac{M^{\beta-1}}{(\delta+1)(1-\beta)} t r^{1-\beta}(t) \end{aligned} \tag{5.35}$$

as $t \to \infty$. This implies (5.15). □

Proof of $\mathcal{DS} \subset \mathcal{NRV}(\varrho)$. Let $\delta > -1$. Assume first that $r \in C^1 \cap \mathcal{NRV}(\delta+\alpha)$. Take $y \in \mathcal{DS}$. We want to show that $y \in \mathcal{NRV}(\varrho)$. Set again $w(t) = \Phi(ty'(t)/y(t))$. Then w is negative and satisfies equation (3.4) for large t, which can be written as

$$tw' = \frac{L_p(t)}{L_r(t)} + (-w)\Big(\frac{tr'(t)}{r(t)} + 1 - \alpha - (\alpha-1)(-w)^{\beta-1}\Big).$$

We claim that $\lim_{t\to\infty} w(t) = \Phi(\varrho)$. Some of the arguments are similar to those in the case $\delta < -1$, but some steps require a different approach. Indeed, if $w'(t) > 0$ for large t, then $\lim_{t\to\infty} w(t) = A^- \in (-\infty, 0]$. The case $A^- = 0$ leads to $y \in \mathcal{NSV}$, which contradicts with the fact that $\mathcal{SV}$ solutions cannot decrease (see Lemma 4.2). If $A^- \in (-\infty, 0) \setminus \{\Phi(\varrho)\}$, then we obtain a contradiction with $A^- \in \mathbb{R}$ similarly as above. Let $w'(t) < 0$ for large t. Then $\lim_{t\to\infty} w(t) = B^- \in [-\infty, 0)$. Assume that $B^- = -\infty$. This implies $\lim_{t\to\infty} y(t)/(ty'(t)) = 0$. From (1.1), we obtain $r'\Phi(y') + (\alpha-1)r(-y')^{\alpha-2}y'' = py^{\alpha-1}$. Hence,

$$\begin{aligned} \frac{y''(t)y(t)}{y'^2(t)} &= \frac{p(t)y^\alpha(t)}{(\alpha-1)r(t)(-y'(t))^\alpha} - \frac{r'(t)y(t)}{(\alpha-1)r(t)y'(t)} \\ &= \frac{p(t)t^\alpha}{(\alpha-1)r(t)}\Big(\frac{-y(t)}{ty'(t)}\Big)^\alpha - \frac{tr'(t)}{(\alpha-1)r(t)} \cdot \frac{y(t)}{ty'(t)} \to 0 \end{aligned} \tag{5.36}$$

as $t \to \infty$. It follows that $\big(\frac{y(t)}{y'(t)}\big)' = 1 - \frac{y''(t)y(t)}{y'^2(t)} \to 1$, hence $\frac{y(t)}{y'(t)} \to \infty$ as $t \to \infty$, which implies $y'(t) > 0$, contradiction with $y \in \mathcal{DS}$. If $B^- \in (-\infty, 0) \setminus \{\Phi(\varrho)\}$, then we obtain a contradiction with $B^- \in \mathbb{R}$ similarly as above. The case when w' changes its sign and $w(t_n) = 0$ can be treated using arguments analogous to those from the proof of $\mathcal{IS} \subset \mathcal{NRV}(\varrho)$.

Now we drop the assumption of continuous differentiability and normality of regular variation of r. We take $\bar r \in C^1 \cap \mathcal{NRV}(\delta+\alpha)$ such that $\bar r(t) \sim r(t)$ as $t \to \infty$ and $(1-\varepsilon)\bar r(t) \le r(t) \le (1+\varepsilon)\bar r(t)$ for large t. Because of a certain uniqueness in the class $\mathcal{DS}$ (see [5, Section 4.1.3]) we however cannot use the same approach as above for increasing solutions. We utilize the concept of principal solution. Along with (1.1), let us consider the equations $((1-\varepsilon)\bar r(t)\Phi(u'))' = p(t)\Phi(u)$ and $((1+\varepsilon)\bar r(t)\Phi(v'))' = p(t)\Phi(v)$. Let $w_u = (1-\varepsilon)\bar r\Phi(u'/u)$, $w_y = r\Phi(y'/y)$, and $w_v = (1+\varepsilon)\bar r\Phi(v'/v)$, where u, y, v are eventually positive decreasing solutions of respective half-linear equations. We have $\lim_{T\to\infty}\int_a^T r^{1-\beta}(t)\left(\int_t^T p(s)\,\mathrm{d}s\right)^{\beta-1}\mathrm{d}t = \infty$ since $\int_a^\infty p(s)\,\mathrm{d}s = \infty$. By (6.2), the sets of all decreasing solutions of the equations which are working with are in fact formed by principal solutions. Consequently w_u, w_y, w_v are eventually minimal solutions of the associated generalized Riccati differential equations. Since $(1-\varepsilon)\bar r(t) \le r(t) \le (1+\varepsilon)\bar r(t)$, according to (3.5) we obtain $w_u(t) \ge w_y(t) \ge w_v(t)$ for large t. Consequently, $-(1-\varepsilon)\Phi(tu'(t)/u(t)) \le -r(t)/\bar r(t)\Phi(ty'(y)/y(t)) \le -(1+\varepsilon)\Phi(tv'(t)/v(t))$, and the rest of the proof is similar to that after (5.29). $\square$

Proof of asymptotic formula in the case $\delta > -1$. Take $y \in \mathcal{NRV}(\varrho) \cap \mathcal{DS}$. Clearly, $y \in \mathcal{DS}_0$ since $\varrho < 0$. From (1.1), $(r\Phi(y'))' = p\Phi(y) \in \mathcal{RV}(\delta + (\alpha-1)\varrho) = \mathcal{RV}(-1)$. Hence, $r\Phi(y') \in \Pi(t(r\Phi(y'))') = \Pi(tp\Phi(y))$, in view of Proposition 2.4–(v). Without loss of generality we may assume that $y(t) > 0$ and $y'(t) < 0$ for $t \ge a$. Similarly as above – we again use (5.30) (which is true thanks to $r\Phi(y') \in \Pi(tp\Phi(y))$) and (5.31) – we obtain

$$(1+o(1))\frac{H(t)}{\Phi(\varrho)} = \Big(\ln\frac{-\int_a^t r(s)\Phi(y'(s))\,\mathrm{d}s}{t}\Big)'$$

as $t \to \infty$. Integrating from $t_0 > a$ to t, we obtain (5.34) as $t \to \infty$. Note that in this case $y^{[1]}$ and D are negative.

Let $\int_a^\infty H(s)\,\mathrm{d}s = \infty$. Then $\int_a^t y^{[1]}(s)\,\mathrm{d}s/t \to 0$ as $t \to \infty$ and since $-y^{[1]}$ is positive decreasing, the L'Hospital rule gives $r(t)\Phi(y'(t)) \to 0$ as $t \to \infty$. From

(5.31) and (5.34) we obtain

$$y^{\alpha-1}(t)=(1+o(1))\frac{|D|t^{\alpha-1}}{r(t)|\varrho|^{\alpha-1}}\exp\Big\{\int_{t_0}^t(1+o(1))\frac{H(s)}{\Phi(\varrho)}\,\mathrm{d}s\Big\}$$

as $t\to\infty$, and formula (5.14) easily follows.

Let $\int_a^\infty H(s)\,\mathrm{d}s<\infty$. Then $\lim_{t\to\infty}r(t)\Phi(y'(t))=M\in(-\infty,0)$, and so $y\in\mathcal{DS}_{0,B}$. Consequently,

$$y(t)\sim|M|^{\beta-1}\int_t^\infty r^{1-\beta}(s)\,\mathrm{d}s \tag{5.37}$$

as $t\to\infty$. The Karamata integration theorem now yields formula (5.15). □

Remark 5.6. A closer examination of the above observations shows that – under the assumptions of Theorem 5.1 – asymptotic formulas for (non-$\mathcal{SV}$) solutions of (1.1) can be expressed in the following forms. If $\int_a^\infty H(s)\,\mathrm{d}s=\infty$, then

$$\Big|\int_a^t y^{[1]}(s)\,\mathrm{d}s\Big|=t\exp\Big\{\int_a^t(1+o(1))\frac{H(s)}{\Phi(\varrho)}\,\mathrm{d}s\Big\} \tag{5.38}$$

as $t\to\infty$. This follows from (5.33) by taking exp and including $|D|$ and $\int_a^{t_0}$ into $(1+o(1))$ term. Note that (5.14) can be obtained by using (5.31) in (5.38). If $\int_a^\infty H(s)\,\mathrm{d}s<\infty$, then

$$\int_a^t y^{[1]}(s)\,\mathrm{d}s=Mt\exp\Big\{-\int_t^\infty(1+o(1))\frac{H(s)}{\Phi(\varrho)}\,\mathrm{d}s\Big\} \tag{5.39}$$

as $t\to\infty$, where $\lim_{t\to\infty}r(t)\Phi(y'(t))=M\in\mathbb{R}\setminus\{0\}$. This follows from (5.33) by replacing t_0 by t and t by ∞ and taking exp.

5.4. **$\mathcal{RV}$ solutions when $t^\alpha p(t)/r(t)\to C>0$.** Condition (4.2) can be understood as $\lim_{t\to\infty}t^\alpha p(t)/r(t)=0$. A logical step is to assume that this limit is nonzero, i.e.,

$$\lim_{t\to\infty}t^\alpha p(t)/r(t)=C>0. \tag{5.40}$$

As we shall see, a modification of the approach from the previous subsection leads to the claim that any solution of (1.1) belongs to $\mathcal{RV}(\Phi^{-1}(\lambda))$, where λ is a root of

$$|\lambda|^\beta+\frac{\gamma+1-\alpha}{\alpha-1}\lambda-\frac{C}{\alpha-1}=0, \tag{5.41}$$

$\gamma\in\mathbb{R}$ being the index of regular variation of r. More precisely, we have the following statement, where $\lambda_1,\lambda_2\in\mathbb{R}$, $\lambda_2<\lambda_1$, denote the (real) roots of (5.41). Imaging the graphs of $\lambda\mapsto|\lambda|^\beta-\frac{C}{\alpha-1}$ and $\lambda\mapsto\frac{\alpha-1-\gamma}{\alpha-1}\lambda$, it is easy to see that $\lambda_2<0<\lambda_1$.

Theorem 5.7. *Assume that $r\in\mathcal{RV}(\gamma)$ and condition* (5.40) *holds. Then $\mathcal{IS}\subset\mathcal{NRV}(\Phi^{-1}(\lambda_1))$ and $\mathcal{DS}\subset\mathcal{NRV}(\Phi^{-1}(\lambda_2))$.*

Proof. Assume first that $r\in\mathcal{NRV}(\gamma)\cap C^1$. Take $y\in\mathcal{IS}$ and set $w(t)=\Phi(ty'(t)/y(t))$. Then w satisfies

$$tw'=\frac{t^\alpha p(t)}{r(t)}+w\Big(\alpha-1-\frac{tr'(t)}{r(t)}-(\alpha-1)w^{\beta-1}\Big)$$

for large t. Assume that $w'(t)>0$ for large t and denote $\lim_{t\to\infty}w(t)=:A\in(0,\infty]$. If $A=\infty$, then $tw'(t)\to-\infty$ as $t\to\infty$, contradiction with $w'>0$. If $\mathbb{R}\ni A\neq\lambda_1$, then $tw'(t)\sim D$ as $t\to\infty$ for some $D\neq0$, contradiction with $A\in\mathbb{R}$. Assume that $w'(t) < 0$ for large t and denote $\lim_{t\to\infty} w(t) =: B \in [0, \infty)$. If $B = 0$, then $tw'(t) \to C$ as $t \to \infty$, contradiction with $B \in \mathbb{R}$. If $B \in \setminus\{\lambda_1\}$, then $tw'(t) \sim D$ as $t \to \infty$ for some $D \neq 0$, contradiction with $B \in \mathbb{R}$. The case when $w'(t_n) = 0$ with $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$, can be managed similarly as under the assumption $C = 0$. Note only that instead of (5.26), we consider $0 = C + \mu(\alpha - 1 - \gamma - (\alpha - 1)\mu^{\beta-1})$. Altogether we obtain $\lim_{t\to\infty} w(t) = \lambda_1$, thus $\lim_{t\to\infty} ty'(t)/y(t) = \Phi^{-1}(\lambda_1)$, i.e., $y \in \mathcal{NRV}(\Phi^{-1}(\lambda_1))$. If r is not in C^1 or its regular variation is not normalized, then we proceed similarly as in the proof of $\mathcal{IS} \subset \mathcal{NRV}(\varrho)$ in the previous section.

Take $y \in \mathcal{DS}$. Again, assume first that $r \in \mathcal{NRV}(\gamma) \cap C^1$ and set $w(t) = \Phi(ty'(t)/y(t))$. Then w satisfies

$$tw' = \frac{t^\alpha p(t)}{r(t)} + (-w)\Big(\frac{tr'(t)}{r(t)} + 1 - \alpha - (\alpha - 1)(-w)^{\beta-1}\Big)$$

for large t. Assume that $w'(t) > 0$ for large t and denote $\lim_{t\to\infty} w(t) =: A \in (-\infty, 0]$. If $A = 0$, then $\lim_{t\to\infty} tw'(t) = C$, whence $w(t) \sim C \ln t$ as $t \to \infty$, contradiction. The case $A \in (-\infty, 0) \setminus \{\lambda_2\}$ also leads to contradiction. Assume that $w'(t) < 0$ for large t and denote $\lim_{t\to\infty} w(t) =: B \in [-\infty, 0)$. If $B = -\infty$, then $\lim_{t\to\infty} y(t)/(ty'(t)) = 0$. From the identity in (5.36) we obtain $(y(t)/y'(t))' = 1 - y''(t)y(t)/y'^2(t) \to 1$ as $t \to \infty$, thus $y(t)/y'(t) \to \infty$ as $t \to \infty$, contradiction with $y \in \mathcal{DS}$. If $\mathbb{R} \ni B \neq \lambda_2$, then $tw'(t) \sim D$ as $t \to \infty$ for some $D \neq 0$, contradiction with $B \in \mathbb{R}$. The case when w' changes its sign and $w'(t_n) = 0$ can be treated similarly as when $y \in \mathcal{IS}$. Hence, $\lim_{t\to\infty} w(t) = \lambda_2$. Consequently, $y \in \mathcal{NRV}(\Phi^{-1}(\lambda_2))$. If r is not in C^1 or its regular variation is not normalized, then we proceed similarly as in the proof of $\mathcal{DS} \subset \mathcal{NRV}(\varrho)$ in the previous section. Indeed, it is not difficult to see that $\lim_{T\to\infty} \int_a^T r^{1-\beta}(t) \left(\int_t^\infty p(s)\, \mathrm{d}s\right)^{\beta-1} \mathrm{d}t = \infty$. Consequently the solutions which we are working with are all principal solutions, see (6.2). ☐

Remark 5.8. (i) A similar result as in Theorem 5.1 can be found in [21]; the approach is however somewhat different.

(ii) A fixed point approach was used in [12, 13] to derive conditions which guarantee the existence of a couple of $\mathcal{RV}$ solutions (with different indices which exactly correspond to the indices from Theorem 5.7) to equation (1.1). For $r(t) \not\equiv 1$, the concept of generalized regular variation was used. Similarly, the conditions for the existence of a $\mathcal{SV}$ solution and a non-$\mathcal{SV}$ solution are obtained in those works in the general case which can reduce to condition (4.2), i.e., $C = 0$. The conditions are in a more general integral form than (5.40) (for instance, that there exists the proper limit $\lim_{t\to\infty} t^\alpha \int_t^\infty p(s)\, \mathrm{d}s$ when $r(t) \equiv 1$) and are shown to be necessary, cf. Remark 5.5 and Subsection 5.2. On the other hand, here we prove regular variation of all eventually positive solutions.

(iii) Another approach which is based on a transformation of dependent variable – and is at disposal in the linear case – is presented in the proof of Theorem 7.1.

(iv) The extreme case where C from (7.4) is equal to ∞ corresponds, in a certain sense, with the setting in [19] and [21, Section 3]. Solutions in the class Γ, which forms a proper subset of rapidly varying functions, are studied there.

(v) A natural problem is to establish asymptotic formula for solutions from Theorem 5.7. This is done in Theorem 7.1 for linear equations.

6. Classification of nonoscillatory solutions in the framework of regular variation

Denote

$$\mathcal{S}_{\mathcal{SV}}=\mathcal{S}\cap\mathcal{SV},\quad \mathcal{S}_{\mathcal{RV}}(\vartheta)=\mathcal{S}\cap\mathcal{RV}(\vartheta),$$
$$\mathcal{S}_{\mathcal{NSV}}=\mathcal{S}\cap\mathcal{NSV},\quad \mathcal{S}_{\mathcal{NRV}}(\vartheta)=\mathcal{S}\cap\mathcal{NRV}(\vartheta).$$

If (4.1) holds, we set

$$J=\int_a^\infty \frac{1}{t}\Big(\frac{L_p(t)}{L_r(t)}\Big)^{\beta-1}\mathrm{d}t \quad\text{and}\quad R=\int_a^\infty \frac{1}{t}\cdot\frac{L_p(t)}{L_r(t)}\,\mathrm{d}t$$

We actually have

$$J=\int_a^\infty G(t)\,\mathrm{d}t \quad\text{and}\quad R=\int_a^\infty H(t)\,\mathrm{d}t$$

under condition (4.1). Further denote

$$\mathfrak{P}=\{y\in\mathcal{S}: y \text{ is principal}\}.$$

Theorem 4.1, Theorem 5.1, Lemma 4.2, Remark 5.2, Remark 5.6, and [4, Theorem B] (see also (6.2) below) yield the following corollary.

Corollary 6.1. *Let* (4.1) *and* (4.2) *hold.*
(i) *Assume that* $\delta<-1$.
- (i-a) *If* $J=\infty$, *then* $\mathcal{S}_{\mathcal{NSV}}=\mathcal{S}_{\mathcal{SV}}=\mathcal{DS}=\mathcal{DS}_{0,0}=\mathfrak{P}$. *For any* $y\in\mathcal{DS}$ *formula* (4.3) *holds.*
- (i-b) *If* $J<\infty$, *then* $\mathcal{S}_{\mathcal{NSV}}=\mathcal{S}_{\mathcal{SV}}=\mathcal{DS}=\mathcal{DS}_{B,0}=\mathfrak{P}$. *For any* $y\in\mathcal{DS}$ *formula* (4.4) *holds.*
- (i-c) *If* $R=\infty$, *then* $\mathcal{S}_{\mathcal{NRV}}(\varrho)=\mathcal{S}_{\mathcal{RV}}(\varrho)=\mathcal{IS}=\mathcal{IS}_{\infty,\infty}$. *For any* $y\in\mathcal{IS}$ *formulae* (5.2), (5.14), (5.38) *hold.*
- (i-d) *If* $R<\infty$, *then* $\mathcal{S}_{\mathcal{NRV}}(\varrho)=\mathcal{S}_{\mathcal{RV}}(\varrho)=\mathcal{IS}=\mathcal{IS}_{\infty,B}$. *For any* $y\in\mathcal{IS}$ *formulae* (5.4), (5.15), (5.39) *hold.*

(ii) *Assume that* $\delta>-1$.
- (ii-a) *If* $J=\infty$, *then* $\mathcal{S}_{\mathcal{NSV}}=\mathcal{S}_{\mathcal{SV}}=\mathcal{IS}=\mathcal{IS}_{\infty,\infty}$. *For any* $y\in\mathcal{IS}$ *formula* (4.3) *holds.*
- (ii-b) *If* $J<\infty$, *then* $\mathcal{S}_{\mathcal{NSV}}=\mathcal{S}_{\mathcal{SV}}=\mathcal{IS}=\mathcal{IS}_{B,\infty}$. *For any* $y\in\mathcal{IS}$ *formula* (4.4) *holds.*
- (ii-c) *If* $R=\infty$, *then* $\mathcal{S}_{\mathcal{NRV}}(\varrho)=\mathcal{S}_{\mathcal{RV}}(\varrho)=\mathcal{DS}=\mathcal{DS}_{0,0}=\mathfrak{P}$. *For any* $y\in\mathcal{DS}$ *formulae* (5.3), (5.14), (5.38) *hold.*
- (ii-d) *If* $R<\infty$, *then* $\mathcal{S}_{\mathcal{NRV}}(\varrho)=\mathcal{S}_{\mathcal{RV}}(\varrho)=\mathcal{DS}=\mathcal{DS}_{0,B}=\mathfrak{P}$. *For any* $y\in\mathcal{DS}$ *formulae* (5.5), (5.15), (5.39) *hold.*

The following example shows that the case, when one of the integrals J,R is convergent while the other one is divergent, may generally occur. Note that this is not possible in the linear case (where we always have $J=R$ since $\beta=2$). Thus we see that half-linear equations may exhibit more complex behavior than linear ones.

Example 6.2. Let r,p be such that $L_p(t)/L_r(t)=\ln^\gamma t$ with $\gamma\in(-\infty,0)$. Note that then (4.2) is satisfied. Let a in J,R be equal to 2. If $\alpha\in(1,2)$, then take γ such that $-1<\gamma<1-\alpha$, and we obtain

$$R=\lim_{t\to\infty}\Big[\frac{\ln^{\gamma+1}s}{\gamma+1}\Big]_2^t=\infty,\quad J=\lim_{t\to\infty}\Big[\frac{\ln^{\gamma(\beta-1)+1}s}{\gamma(\beta-1)+1}\Big]_2^t<\infty.$$

If $\alpha > 2$, then take γ such that $1-\alpha < \gamma < -1$, and we obtain

$$R < \infty, \quad J = \infty.$$

To include our results into a broader context, let us recall several existence results concerning asymptotic classes defined in Section 3. We already know that the classes $\mathcal{DS}$ and $\mathcal{IS}$ are nonempty. We set

$$J_1 = \lim_{T\to\infty} \int_a^T r^{1-\beta}(t)\Big(\int_a^t p(s)\,\mathrm{d}s\Big)^{\beta-1}\mathrm{d}t,$$
$$J_2 = \lim_{T\to\infty} \int_a^T r^{1-\beta}(t)\Big(\int_t^T p(s)\,\mathrm{d}s\Big)^{\beta-1}\mathrm{d}t.$$

The convergence or divergence of the above integrals fully characterize the classes $\mathcal{DS}_0, \mathcal{DS}_B, \mathcal{IS}_B$, and $\mathcal{IS}_\infty$. In particular, according to [5, Theorems 4.1.1–4.1.3]:

$$\begin{gathered}
\mathcal{IS} = \mathcal{IS}_\infty \;\Leftrightarrow\; J_1 = \infty, \\
\mathcal{IS} = \mathcal{IS}_B \;\Leftrightarrow\; J_1 < \infty, \\
\mathcal{DS} = \mathcal{DS}_B \;\Leftrightarrow\; J_1 = \infty \text{ and } J_2 < \infty, \\
\mathcal{DS} = \mathcal{DS}_0 \;\Leftrightarrow\; J_2 = \infty, \\
J_1 < \infty \text{ and } J_2 < \infty \;\Rightarrow\; \mathcal{DS}_0 \neq \emptyset \neq \mathcal{DS}_B.
\end{gathered}$$

Denote

$$J_r = \int_a^\infty r^{1-\beta}(s)\,\mathrm{d}s, \quad \text{and} \quad J_p = \int_a^\infty p(s)\,\mathrm{d}s.$$

It is useful to recall relations between J_1, J_2, J_r, J_p ([5, Lemma 4.1.5]): If $J_1 < \infty$, then $J_r < \infty$. If $J_2 < \infty$, then $J_p < \infty$. If $J_2 = \infty$, then $J_r = \infty$ or $J_p = \infty$. If $J_1 = \infty$, then $J_r = \infty$ or $J_p = \infty$. It holds $J_1 < \infty$ and $J_2 < \infty$ if and only if $J_r < \infty$ and $J_p < \infty$.

Further we set

$$R_1 = \lim_{T\to\infty} \int_a^T p(t)\Big(\int_a^t r^{1-\beta}(s)\,\mathrm{d}s\Big)^{\alpha-1}\mathrm{d}t,$$
$$R_2 = \lim_{T\to\infty} \int_a^T p(t)\Big(\int_t^T r^{1-\beta}(s)\,\mathrm{d}s\Big)^{\alpha-1}\mathrm{d}t.$$

Observe that the integral J_r (resp. J_p) for (1.1) plays the same role as J_p (resp. J_r) for the reciprocal equation (5.1). Similarly, the integrals J_1, J_2 become R_1, R_2, respectively, for the reciprocal equation.

The integrals J_1, J_2, R_1, R_2 characterize the subclasses defined in (3.1) in the following way (see [4, Theorem 1]):

$$\begin{array}{rl}
\text{(I)} & J_1 = J_2 = R_1 = R_2 = \infty \;\Rightarrow\; \mathcal{IS} = \mathcal{IS}_{\infty,\infty}, \mathcal{DS} = \mathcal{DS}_{0,0} \\
\text{(II)} & J_1 = R_2 = \infty, J_2 < \infty, R_1 < \infty \;\Rightarrow\; \mathcal{IS} = \mathcal{IS}_{\infty,B}, \mathcal{DS} = \mathcal{DS}_B \\
\text{(III)} & J_1 < \infty, R_2 < \infty, J_2 = R_1 = \infty \;\Rightarrow\; \mathcal{IS} = \mathcal{IS}_B, \mathcal{DS} = \mathcal{DS}_{0,B} \\
\text{(IV)} & J_1, J_2, R_1, R_2 < \infty \;\Rightarrow\; \mathcal{IS} = \mathcal{IS}_B, \mathcal{DS}_{0,0} = \emptyset, \mathcal{DS}_{0,B} \neq \emptyset \neq \mathcal{DS}_B \\
\text{(V)} & J_1 = J_2 = R_2 = \infty, R_1 < \infty \;\Rightarrow\; \mathcal{IS} = \mathcal{IS}_{\infty,B}, \mathcal{DS} = \mathcal{DS}_{0,0} \\
\text{(VI)} & J_1 = J_2 = R_1 = \infty, R_2 < \infty \;\Rightarrow\; \mathcal{IS} = \mathcal{IS}_{\infty,\infty}, \mathcal{DS} = \mathcal{DS}_{0,B} \\
\text{(VII)} & J_1 = R_1 = R_2 = \infty, J_2 < \infty \;\Rightarrow\; \mathcal{IS} = \mathcal{IS}_{\infty,\infty}, \mathcal{DS} = \mathcal{DS}_B \\
\text{(VIII)} & J_2 = R_1 = R_2 = \infty, J_1 < \infty \;\Rightarrow\; \mathcal{IS} = \mathcal{IS}_B, \mathcal{DS} = \mathcal{DS}_{0,0}.
\end{array} \tag{6.1}$$

Necessary conditions for non-emptiness in the subclasses can be found in [5, Theorem 4.1.5]. Note that the sufficient conditions in cases (V), (VI) may occur only if $\alpha > 2$, and the sufficient conditions in cases (VII), (VIII) may occur only if $\alpha < 2$.

We have already applied the following characterization of principal solutions ([4, Theorem B]) several times in our paper:

$$\mathfrak{P} = \begin{cases} \mathcal{DS}_B & \text{if } J_1 = \infty \text{ and } J_2 < \infty \\ \mathcal{DS}_0 & \text{otherwise.} \end{cases} \tag{6.2}$$

Let us now discuss our results in the just described framework of general existence conditions. Let us assume that (4.1) holds. First note that $J_p < \infty$ and $J_r = \infty$ when $\delta < -1$, while $J_p = \infty$ and $J_r < \infty$ when $\delta > -1$. In view of the Karamata theorem, $\int_t^\infty p(s)\,\mathrm{d}s \sim tp(t)/(-\delta - 1)$ as $t \to \infty$ and $\int_a^t r^{1-\beta}(s)\,\mathrm{d}s \sim tr^{1-\beta}(t)/\varrho$ when $\delta < -1$, while $\int_a^t p(s)\,\mathrm{d}s \sim tp(t)/(\delta + 1)$ as $t \to \infty$ and $\int_t^\infty r^{1-\beta}(s)\,\mathrm{d}s \sim -tr^{1-\beta}(t)/\varrho$ when $\delta > -1$. Hence, if $\delta < -1$, then

$$J_1 = \infty = R_2, \quad J_2 = \infty \Leftrightarrow J = \infty, \quad R_1 = \infty \Leftrightarrow R = \infty,$$

while if $\delta > -1$, then

$$J_2 = \infty = R_1, \quad J_1 = \infty \Leftrightarrow J = \infty, \quad R_2 = \infty \Leftrightarrow R = \infty.$$

Now we can easily see how the conditions in Corollary 6.1 match the general existence conditions (6.1). More precisely, the setting in (I) corresponds to (i-a), (ii-a), the setting in (II) corresponds to (i-b), (i-d), the setting in (III) corresponds to (ii-b), (ii-d), the setting in (V) corresponds to (i-a), (i-d), the setting in (VI) corresponds to (ii-a), (ii-d), the setting in (VII) corresponds to (i-b), (i-c), the setting in (VIII) corresponds to (ii-b), (ii-c). It is worth mentioning that the behavior of slowly varying components of r, p appear to be crucial. In view of relations among J, J_1, J_2, R, R_1, R_2, our conditions must naturally guarantee non-emptiness in the subclasses defined by (3.1) and yield the right-hand sides of (I)–(III) and of (V)–(VIII); arguments are however completely different from the general results in the previous literature. In addition, we claim that these subclasses are formed by (normalized) slowly varying functions or (normalized) regularly varying functions of index ϱ, and all their elements satisfy certain asymptotic formula. Note that the only case which is not included in Corollary 6.1 is (IV); the reason is that $\delta < -1$ excludes $J_1 < \infty$ while $\delta > -1$ excludes $R_1 < \infty$.

7. Some methods that are not fully available in the half-linear case

Consider the linear differential equation

$$(r(t)y')' = p(t)y, \tag{7.1}$$

where r, p are positive continuous functions on $[a, \infty)$. If $\alpha = 2$, then (1.1) reduces to (7.1). The main reasons why the below discussed tools cannot be directly used in the half-linear case are the lack of the additivity of the solution space and the absence of a reasonable Wronskian identity for (1.1).

7.1. **Transformation of dependent variable: the choice $h(t) = t^\vartheta$.** We use an argument based on transformation of dependent variable where Theorems 4.1 and 5.1 are applied to obtain asymptotic formulae for solutions of (7.1) in new situations.

Let $h \neq 0$ be a differentiable function such that rh' is also differentiable. Let us introduce a new independent variable $y = hu$. Then we have the identity

$$h[(ry')' - py] = (rh^2u')' - h[-(rh')' + ph]u.$$

In particular, if y is a solution of (7.1), then u is a solution of the equation

$$(\widetilde{r}(t)u')' - \widetilde{p}u = 0, \tag{7.2}$$

where $\widetilde{r} = rh^2$ and $\widetilde{p} = h[-(rh')' + p]$.

Let ϑ_1, ϑ_2, $\vartheta_2 < \vartheta_1$, denote the (real) roots of the equation

$$\vartheta^2 + \vartheta(\gamma - 1) - C = 0, \tag{7.3}$$

where $C \in (0, \infty), \gamma \in \mathbb{R}$. Clearly, $\vartheta_2 < 0 < \vartheta_1$.

Theorem 7.1. *Assume that $r \in \mathcal{NRV}(\gamma) \cap C^1$, $\gamma \in \mathbb{R}$,*

$$\lim_{t\to\infty} \frac{t^2 p(t)}{r(t)} = C \in (0, \infty), \tag{7.4}$$

$$L_i(t) := \frac{t^2 p(t)}{r(t)} - \vartheta_i \frac{tr'(t)}{r(t)} - \vartheta_i(\vartheta_i - 1) \in \mathcal{SV}, \tag{7.5}$$

$i = 1, 2$. Then $\mathcal{IS}^{(7.1)} \subset \mathcal{NRV}(\vartheta_1)$ and $\mathcal{DS}^{(7.1)} \subset \mathcal{NRV}(\vartheta_2)$. Moreover,

$$y_1(t) \in \Pi\mathcal{RV}\left(\vartheta_1; t^{1-\vartheta_1} y_1'(t) - \vartheta_1 t^{-\vartheta_1} y_1(t)\right) \text{ for every } y_1 \in \mathcal{IS}^{(7.1)},$$
$$-y_2(t) \in \Pi\mathcal{RV}\left(\vartheta_2; -t^{1-\vartheta_2} y_2'(t) + \vartheta_2 t^{-\vartheta_2} y_2(t)\right) \text{ for every } y_2 \in \mathcal{DS}^{(7.1)},$$

and

(i) if, for $i = 1$ or $i = 2$, $\int_a^\infty \frac{L_i(s)}{s}\,\mathrm{d}s = \infty$, then

$$y_i(t) = t^{\vartheta_i} \exp\Big\{ \int_a^t (1 + o(1)) \frac{L_i(s)}{(\gamma + 2\vartheta_1 - 1)s}\,\mathrm{d}s \Big\}, \tag{7.6}$$

with $t^{-\vartheta_1} y_1(t) \nearrow \infty$ and $t^{-\vartheta_2} y_2(t) \searrow 0$ as $t \to \infty$;

(ii) if, for $i = 1$ or $i = 2$, $\int_a^\infty \frac{L_i(s)}{s}\,\mathrm{d}s < \infty$, then

$$y_i(t) = D_i t^{\vartheta_i} \exp\Big\{ -\int_t^\infty (1 + o(1)) \frac{L_i(s)}{(\gamma + 2\vartheta_1 - 1)s}\,\mathrm{d}s \Big\}, \tag{7.7}$$

with $t^{-\vartheta_1} y_1(t) \nearrow D_1 = D_1(y_1) \in (0, \infty)$ and $t^{-\vartheta_2} y_2(t) \searrow D_2 = D_2(y_2) \in (0, \infty)$ as $t \to \infty$. Moreover, $|t^{\vartheta_i} y_i(t) - D_i| \in \mathcal{SV}$ and

$$\frac{L_i(t)}{D_i - t^{-\vartheta_i} y_i(t)} = o(1) \tag{7.8}$$

as $t \to \infty$.

Proof. Let $\widetilde{\mathcal{S}}, \widetilde{\mathcal{IS}}, \widetilde{\mathcal{DS}}$ have the same meaning with respect to (7.2) as the classes $\mathcal{S}^{(7.1)}, \mathcal{IS}^{(7.1)}, \mathcal{DS}^{(7.1)}$, respectively, have with respect to (7.1). Set $\delta = \gamma - 2$, $h = t^{\vartheta_i}$, and $\delta_i = \delta + 2\vartheta_i$, $i = 1, 2$. Then $\widetilde{r}$ becomes $\widetilde{r}(t) = r_i(t) := r(t)t^{2\vartheta_i} \in \mathcal{RV}(\delta_i + 2)$ and $\widetilde{p}(t) = p_i(t) := \frac{r_i(t)}{t^2} L_i(t) \in \mathcal{RV}(\delta_i)$, $i = 1, 2$. We have $\frac{L_{p_i}(t)}{L_{r_i}(t)} = \frac{t^2 p_i(t)}{r_i(t)} = L_i(t) \in \mathcal{SV}$ with $L_i(t) \to 0$ as $t \to \infty$, $i = 1, 2$, thanks to (7.4). Further, since $\vartheta_{1,2} = \frac{1}{2}\left(-\delta - 1 \pm \sqrt{(\delta + 1)^2 + C}\right)$, we obtain $\delta_1 > -1$ and $\delta_2 < -1$.

Take $y \in \mathcal{IS}^{(7.1)}$. Set $y = hu$ with $h(t) = t^{\vartheta_1}$. Then u solves (7.2) with $\widetilde{p} = p_1, \widetilde{r} = r_1$. Clearly, $u \in \widetilde{\mathcal{S}} = \widetilde{\mathcal{IS}} \cup \widetilde{\mathcal{DS}}$. If $u \in \widetilde{\mathcal{IS}}$, then $u \in \mathcal{NSV}$ and $u \in \Pi(tu'(t))$ by Theorem 4.1. Hence, $y \in \mathcal{RV}(\vartheta_1)$ and $y \in \Pi(\vartheta_1; t^{1-\vartheta_1} y'(t) - \vartheta_1 t^{-\vartheta_1} y(t))$. Denote

$g_i(t) = \frac{tp_i(t)}{|\delta_i+1|r_i(t)}$, $i = 1,2$. By Theorem 4.1, $u(t) = \exp\{\int_a^t (1+o(1))g_1(s)\,\mathrm{d}s\}$ with $u(t) \to \infty$ as $t \to \infty$, provided $\int_a^\infty g_1(s)\,\mathrm{d}s = \infty$. Formula (7.6) with $i = 1$ now easily follows if we realize that $y = t^{\vartheta_1}u$, $\frac{tp_1}{r_1} = \frac{L_1}{t}$, and $\delta_1 = \delta + 2\vartheta_1$. Similarly we obtain formula (7.7) with $i = 1$, since if $\int_a^\infty \frac{L_1(s)}{s}\,\mathrm{d}s < \infty$, then $u(t) = D_1 \exp\big\{-\int_t^\infty (1+o(1))g_1(s)\,\mathrm{d}s\big\}$ as $t \to \infty$ with $\lim_{t\to\infty} u(t) = D_1 \in (0,\infty)$. If $u \in \widetilde{\mathcal{DS}}$, then $u \in \mathcal{NRV}(-\delta_1 - 1)$ by Theorem 5.1. Hence, recalling that $\vartheta_1 + \vartheta_2 = -\delta - 1$, $y \in \mathcal{NRV}(\vartheta_2)$ which contradicts $y \in \mathcal{IS}^{(7.1)}$ since $\vartheta_2 < 0$.

Take $y \in \mathcal{DS}^{(7.1)}$. We proceed similarly as before. Now we set $y = hu$ with $h(t) = t^{\vartheta_2}$. Consequently, u is in $\widetilde{\mathcal{S}} = \widetilde{\mathcal{IS}} \cup \widetilde{\mathcal{DS}}$ where $\widetilde{p} = p_2 \in \mathcal{RV}(\delta_2)$ and $\widetilde{r} = r_2 \in \mathcal{RV}(\delta_2 + 2)$ with $\delta_2 < -1$. If $u \in \widetilde{\mathcal{DS}}$, then we apply Theorem 4.1 to obtain $u \in \mathcal{NSV}$, $-u \in \Pi(-tu(t))$, and $u(t) = \exp\{-\int_a^t (1+o(1))g_2(s)\,\mathrm{d}s\}$ as $t \to \infty$ provided $\int_a^\infty g_2(s)\,\mathrm{d}s = \infty$ while $u(t) = D_2 \exp\{\int_t^\infty (1+o(1))g_2(s)\,\mathrm{d}s\}$ as $t \to \infty$ provided $\int_a^\infty g_2(s)\,\mathrm{d}s < \infty$, where $\lim_{t\to\infty} u(t) = D_2 \in (0,\infty)$. Asymptotic formulae (7.6) with $i = 2$ and (7.7) with $i = 2$ then clearly follow. If $u \in \widetilde{\mathcal{IS}}$, then $u \in \mathcal{NRV}(-\delta_2 - 1)$ by Theorem 5.1. This implies $y \in \mathcal{NRV}(\vartheta_1)$, which contradicts $y \in \mathcal{DS}^{(7.1)}$.

By (4.5), we have $L_{p_i}(t)/(L_{r_i}(t)(D_i - u(t))) = o(1)$ as $t \to \infty$, and so (7.8) follows. $\square$

Remark 7.2. (i) If $r = 1$, then formula (7.6) with $i = 2$ (i.e., for decreasing solutions) reduces to the former formula in [7, Theorem 0.2]. The quoted result was however proved by a quite different method. For increasing solutions and with $r(t) \neq 1$, Theorem 7.1 is new.

(ii) Observe that – for linear equations – we have presented another method of the proof of $\mathcal{IS}^{(7.1)} \subset \mathcal{NRV}(\vartheta_1)$ and $\mathcal{DS}^{(7.1)} \subset \mathcal{NRV}(\vartheta_2)$, see the proof of Theorem 5.7 and Remark 5.8.

(iii) Assuming $r \in \mathcal{NRV}(\gamma) \cap C^1$ and (7.4), if $\vartheta_i tr'(t)/r(t) - t^2p(t)/r(t) \in \Pi$, then condition (7.5) is satisfied thanks to Proposition 2.4–(iv).

To obtain information about the behavior of solutions in $\mathcal{DS}^{(7.1)}$ and $\mathcal{IS}^{(7.1)}$, the previous theorem requires both the functions L_1 and L_2 to be $\mathcal{SV}$ and so they are necessarily positive. The next remark reveals that asymptotic formulas for all solutions in $\mathcal{S}^{(7.1)}$ can be obtained also in the case when one of these functions is not positive. First however we give an example which discusses various possibilities for behavior of the functions L_1 and L_2. Let ϑ_i, $i = 1,2$, denote the roots of (7.3). The functions L_1, L_2 can be written in the form

$$L_i(t) = \vartheta_i\Big(\gamma - \frac{tr'(t)}{r(t)}\Big) - C + \frac{t^2p(t)}{r(t)},$$

$i = 1,2$. For $r \in \mathcal{NRV}(\gamma) \cap C^1$, we have $tr'(t)/r(t) = \gamma + tL_r'(t)/L_r(t)$.

Example 7.3. (a) If $r(t) = t^\gamma$, then $L_1 = -C + t^2p/t = L_2$, and so the functions L_1 and L_2 coincide.

(b) Let $r \in \mathcal{NRV}(\gamma) \cap C^1$ and $p(t) = CL_r(t)t^{\gamma-2}$, $C \in (0,\infty)$. Then $t^2p(t)/r(t) = C$, and so $L_i = -\vartheta_i tL_r'(t)/L_r(t)$, $i = 1,2$. If, in addition, $L_r' > 0$, then $L_1 < 0$ and $L_2 > 0$. Moreover, L_r can be taken such that $L_2 \in \mathcal{SV}$. An example is $L_r(t) = \ln t$. Thus the situation where (7.4) is fulfilled and (7.5) holds for only one index can occur.

(c) Let $p(t) = (C \ln t + \vartheta_1 + C/\ln t)\, t^{\gamma-2}$, $C \in (0, \infty)$, and $r(t) = t^\gamma \ln t$. Then $r \in \mathcal{NRV}(\gamma) \cap C^1$ and condition (7.4) is fulfilled. Further, $L_1(t) = C/\ln^2 t \in \mathcal{SV}$ and $L_2(t) = (\vartheta_1 - \vartheta_2)/\ln t + C/\ln^2 t \in \mathcal{SV}$. Since $(-1/\ln t)' = L_1(t)/(Ct)$, we obtain $\int_a^\infty L_1(t)/t\,\mathrm{d}t < \infty$. On the other hand, we have $(\vartheta_1 - \vartheta_2)(\ln(\ln t))' < L_2(t)/t$, hence $\int_a^\infty L_2(t)/t\,\mathrm{d}t = \infty$. Consequently, the case when $\int_a^\infty L_i(t)/t\,\mathrm{d}t = \infty$ while $\int_a^\infty L_{3-i}(t)/t\,\mathrm{d}t < \infty$ for one of $i \in \{1, 2\}$ can generally occur in Theorem 7.1.

Remark 7.4. (i) Assume that $L_2 \in \mathcal{SV}$, $r \in \mathcal{NRV}(\gamma) \cap C^1$, and (7.4) holds. There is no assumption on L_1; in particular, L_1 might not be positive. As we could see in Example 7.3-(b), such a case can occur. Take $y \in \mathcal{S}^{(7.1)}$. Set $y = hu$, where $h(t) = t^{\vartheta_2}$. Then $u \in \widetilde{\mathcal{S}} = \widetilde{\mathcal{IS}} \cup \widetilde{\mathcal{DS}}$ where $\widetilde{p} = p_2 \in \mathcal{RV}(\delta_2)$ and $\widetilde{r} = r_2 \in \mathcal{RV}(\delta_2 + 2)$ with $\delta_2 < -1$. If $u \in \widetilde{\mathcal{DS}}$, then – as in the proof of the previous theorem – we obtain that y is in $\mathcal{NRV}(\vartheta_2)$ (and so in $\mathcal{DS}$) and satisfies formula (7.6) with $i = 2$ or formula (7.7) with $i = 2$. If $u \in \widetilde{\mathcal{IS}}$, then $u \in \mathcal{NRV}(-\delta_2 - 1)$ by Theorem 5.1, and so $y \in \mathcal{NRV}(\vartheta_1)$ (which yields $y \in \mathcal{IS}^{(7.1)}$). To obtain an asymptotic formula, we use Remark 5.2. For instance, if $\int_a^\infty \frac{L_2(s)}{s}\,\mathrm{d}s = \infty$, then

$$
\begin{aligned}
y(t) &= t^{\vartheta_2} \frac{t}{r_2(t)} \exp\Big\{ \int_a^t (1 + o(1)) \frac{s p_2(s)}{(-1 - \delta_2) r_2(s)}\,\mathrm{d}s \Big\} \\
&= \frac{t^{\vartheta_1}}{L_{r_2}(t)} \exp\Big\{ \int_a^t (1 + o(1)) \frac{L_2(s)}{(1 - \gamma - 2\vartheta_2)s}\,\mathrm{d}s \Big\}.
\end{aligned} \tag{7.9}
$$

Similarly we proceed if $\int_a^\infty \frac{L_2(s)}{s}\,\mathrm{d}s < \infty$. The case when $L_1 \in \mathcal{SV}$ (with L_2 not being necessarily positive) can be treated analogously.

(ii) A closer examination of the previous observation shows that the case may happen such that a formula for solutions of equation (7.2) where the coefficient $\widetilde{p}$ is not positive is obtained. Indeed, assume, for instance, that $L_2 \in \mathcal{SV}$ with $\int_a^\infty \frac{L_2(s)}{s}\,\mathrm{d}s = \infty$, and $L_1(t) \not> 0$. We already know that $y \in \mathcal{IS}^{(7.1)}$ is in $\mathcal{NRV}(\vartheta_1)$ and satisfies formula (7.9). Take this y and set $y(t) = t^{\vartheta_1} v(t)$. Then v satisfies equation (7.2) with $\widetilde{r} = r_1$ and $\widetilde{p} = p_1 = \frac{r_1}{t^2} L_1 \not> 0$. The formula for v can easily be obtained from $v = t^{-\vartheta_1} y$. Note that monotonicity for v is not guaranteed.

Remark 7.5. Similar arguments to those in the proof of Theorem 7.1 can be used to obtain a variant of Theorem 5.1 in the linear case. In fact, the following setting leads to $C = 0$ and $\gamma = \delta + 2$. Thus, as for the roots of (7.3), we obtain $\vartheta_1 = -\delta - 1$, $\vartheta_2 = 0$ when $\delta < -1$, resp. $\vartheta_1 = 0$, $\vartheta_2 = -\delta - 1$ when $\delta > -1$. Hence, in both cases we set $h(t) = t^{-\delta-1}$. Other details are left to the reader. The statement reads as follows:

Let $p \in \mathcal{RV}(\delta)$, $r \in \mathcal{NRV}(\delta + 2) \cap C^1$, $\delta \neq -1$, and $\lim_{t\to\infty} t^2 p(t)/r(t) = 0$. Assume that $L \in \mathcal{SV}$, where

$$
L(t) = \frac{L_p(t)}{L_r(t)} + \Phi(\varrho)\Big(\delta + 2 - \frac{t r'(t)}{r(t)}\Big).
$$

If $\delta < -1$, then $\mathcal{IS}^{(7.1)} \subset \mathcal{NRV}(-1 - \delta)$. If $\delta > -1$, then $\mathcal{DS}^{(7.1)} \subset \mathcal{NRV}(-1 - \delta)$. Moreover, for any $y \in \mathcal{IS}^{(7.1)}$ when $\delta < -1$ and any $y \in \mathcal{DS}^{(7.1)}$ when $\delta > -1$, the following hold:

(a) If $\int_a^\infty L(s)/s\,\mathrm{d}s = \infty$, then

$$
y(t) = t^{-\delta-1} \exp\Big\{ \int_a^t \frac{1 + o(1)}{-\delta - 1} \cdot \frac{L(s)}{s}\,\mathrm{d}s \Big\}
$$

with $y(t)t^{\delta+1} \nearrow \infty$ as $t \to \infty$ provided $y \in \mathcal{IS}^{(7.1)}$ and $\delta < -1$, and $y(t)t^{\delta+1} \searrow 0$ as $t \to \infty$ provided $y \in \mathcal{DS}^{(7.1)}$ and $\delta > -1$.

(b) If $\int_a^\infty L(s)/s \,\mathrm{d}s < \infty$, then

$$y(t) = Dt^{-\delta-1} \exp\Big\{ -\int_t^\infty \frac{1+o(1)}{-\delta-1} \cdot \frac{L(s)}{s} \,\mathrm{d}s \Big\}$$

with $y(t)t^{\delta+1} \nearrow D = D(y) \in (0,\infty)$ as $t \to \infty$ provided $y \in \mathcal{IS}^{(7.1)}$ and $\delta < -1$, resp. $y(t)t^{\delta+1} \searrow D = D(y) \in (0,\infty)$ as $t \to \infty$ provided $y \in \mathcal{DS}^{(7.1)}$ and $\delta > -1$. Moreover, $|y(t)t^{\delta+1} - D| \in \mathcal{SV}$ and

$$\frac{L(t)}{D - y(t)t^{\delta+1}} = o(1)$$

as $t \to \infty$.

7.2. **Transformation of dependent variable: the choice** $h(t) = \int_t^\infty 1/r(s)\,\mathrm{d}s$ **or** $h(t) = \int_a^t 1/r(s)\,\mathrm{d}s$**.** We have seen how the results of Theorem 4.1 and Theorem 5.1 can be related by the reciprocity principle. In this subsection we show that for linear equations the results in these theorems can be linked via other tool, namely a suitable transformation of dependent variable. We do not discuss all possibilities for the setting. Rather we illustrate the method on one selected case. Assume that $p \in \mathcal{RV}(\delta)$, $r \in \mathcal{RV}(\delta+2)$ with (4.2), and $\delta > -1$. Take, for instance, $y \in \mathcal{IS}^{(7.2)}$. Set $y = hu$ where $h(t) = \int_t^\infty 1/r(s)\,\mathrm{d}s$; this integral converges thanks to $\delta > -1$. Then u solves (7.1), where $\widetilde{r} = rh^2 \in \mathcal{RV}(\widetilde{\delta}+2)$ and $\widetilde{p} = ph^2 \in \mathcal{RV}(\widetilde{\delta})$ with $\widetilde{\delta} = -\delta - 2 < -1$. Moreover, $L_{\widetilde{p}}(t)/L_{\widetilde{r}}(t) = L_p(t)L_{h^2}(t)/(L_r(t)L_{h^2}(t)) = L_p(t)/L_r(t) \to 0$ as $t \to \infty$. Further, $\widetilde{H}(t) := t\widetilde{p}(t)/\widetilde{r}(t) = tp(t)/r(t)$. Since $y \in \mathcal{IS}$ and $1/h$ increases, $u \in \widetilde{\mathcal{IS}}$. Application of Theorem 5.1 yields $u \in \mathcal{NRV}(\varrho)$, where $\varrho = -1 - \widetilde{\delta} = \delta + 1$. Hence, $y/h \in \mathcal{NRV}(\varrho)$. Since $-h' = 1/r \in \mathcal{RV}(-\delta-2)$ implies $h \in \mathcal{NRV}(-\delta-1)$ by Proposition 2.3-(ix), we obtain $y \in \mathcal{NSV}$. Assume, for instance, $\int_a^\infty tp(t)/r(t)\,\mathrm{d}t = \infty$. The Karamata theorem yields $(\delta+1)h(t) \sim t/r(t)$ as $t \to \infty$. Hence, see (5.14),

$$\begin{aligned}
y(t) &= \frac{th(t)}{\widetilde{r}(t)} \exp\Big\{ \int_a^t (1+o(1)) \frac{\widetilde{H}(s)}{\varrho} \,\mathrm{d}s \Big\} \\
&= \frac{th(t)}{r(t)h^2(t)} \exp\Big\{ \int_a^t (1+o(1)) \frac{sp(s)}{(\delta+1)r(s)} \,\mathrm{d}s \Big\} \\
&= (1+o(1))(\delta+1) \exp\Big\{ \int_a^t (1+o(1)) \frac{sp(s)}{(\delta+1)r(s)} \,\mathrm{d}s \Big\} \\
&= \exp\Big\{ \int_a^t (1+o(1)) \frac{sp(s)}{(\delta+1)r(s)} \,\mathrm{d}s \Big\}
\end{aligned}$$

as $t \to \infty$, cf. (4.3). Similarly we treat other cases.

7.3. **Reduction of order formula.** Another tool which is not at disposal in the half-linear case is the reduction of order formula. If y is a solution of (7.1) such that $y(t) \neq 0$ on $[b,\infty)$, then any other solution x of (7.1) can be expressed as

$$x(t) = c_1 y(t) + c_2 y(t) \int_b^t \frac{\mathrm{d}s}{r(s)y^2(s)},$$

$c_1, c_2 \in \mathbb{R}$. In particular, $u(t) = y(t)\int_b^t 1/(r(s)y^2(s))\,\mathrm{d}s$ is a linearly independent solution (w.r.t. y). If $r \in \mathcal{RV}(\delta+2)$ and $y \in \mathcal{SV}$ resp. $y \in \mathcal{RV}(-\delta-1)$, then $1/(ry^2)$ is $\mathcal{RV}$ of index different form -1, and hence the Karamata theorem can be applied to get $u \in \mathcal{RV}(-\delta-1)$ resp. $u \in \mathcal{SV}$ with

$$u(t) \sim \frac{t}{|\delta+1|r(t)y(t)}$$

as $t \to \infty$. Similarly, under the setting of Theorem 7.1, if $y \in \mathcal{RV}(\vartheta_i)$, then $u \in \mathcal{RV}(\vartheta_{3-i})$ with

$$u(t) \sim \frac{t}{|\gamma+2\vartheta_i+1|r(t)y(t)}$$

as $t \to \infty$, $i = 1, 2$.

8. Conclusion, further research

We have presented several methods for the study of asymptotic properties of linear and half-linear differential equations in the framework of regular variation. We believe that these ideas and their modifications will be useful also in other settings, for example:

- (half-)linear equation of the form (1.1) with $p(t) < 0$ or with $p(t)$ which may change its sign;
- nearly (half-)linear differential equations (i.e., the equations of the form (1.1) where Φ in both terms is replaced by a regularly varying function at infinity or at zero of index α);
- (half-)linear differential equations with deviated arguments;
- first order (half-)linear systems or higher order equations;
- (half-)linear difference equations;
- (half-)linear dynamic equations on time scales.

Even though some results for linear or half-linear differential equations can be established via more approaches, not all these methods can be applicable in other settings. For instance, the reciprocity principle cannot be used in dynamic equations on time scales unless the graininess is constant; thus the approach from Section 5.3 or Section 7.1 might be more suitable for such an extension. Further, the facts like the absence of a chain rule (and, consequently, a substitution in the integral) in a discrete case or a time scale case might substantially affect availableness of some approaches. For half-linear differential equations with deviated argument, the Riccati type substitution does not lead to a "pure" generalized Riccati equation, which might be a serious problem in a delicate asymptotic analysis.

Of course, there is also some space for improving the presented results. In particular:

• Establish a half-linear extension of Theorem 7.1 – the part concerning asymptotic formulas. One of the proper tools is the transformation into a modified generalized Riccati differential equation. This tool can somehow substitute the transformation $y = hu$ used in Section 7.1 and it actually linearizes the problem, cf. [22], where all positive solutions of (1.1) are treated under this setting. A different approach, based on the Banach fixed point theorem, is used in [14] where the existence of regularly varying solutions of (5.16) along with asymptotic formulae is derived under the condition $\lim_{t\to\infty} t^{\alpha-1}\int_t^\infty p(s)\,\mathrm{d}s = C$ and some additional assumptions.

• Examine whether some "purely linear" techniques (e.g. the use of the Wronskian identity or the transformation of dependent variable) can directly be applied to half-linear equation at least in some "asymptotic sense".

• To obtain asymptotic formulae under relaxation of some conditions, such as $\lim_{t\to\infty} t^\alpha p(t)/r(t) = C$, into an integral form. See [9] for the linear case.

• Examine the borderline case $\delta = -1$. A suitable transformation of independent variable and utilization of Theorems 4.1 and 5.1 can quite satisfactorily solve this problem, cf. [22].

• Use the theory of regular variation and the de Haan theory to find more precise asymptotic formulae for solutions of (1.1) (and to find estimations for remainders), see e.g. [8, Theorem 0.1-B] for the linear case.

Acknowledgments. This research was supported by grant RVO 67985840.

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Pavel Řehák
Institute of Mathematics, Czech Academy of Sciences, Žižkova 22, CZ-61662 Brno, Czech Republic
E-mail address: rehak@math.cas.cz