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ANALYSIS AND SIMULATION OF RADIALLY SYMMETRIC SOLUTIONS FOR FREE BOUNDARY PROBLEMS WITH SUPERLINEAR REACTION TERM

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Abstract. This article concerns with the solution to a heat equation with a free boundary in n-dimensional space. By applying the energy inequality to the solutions that depend not only on the initial value but also on the dimension of space, we derive the sufficient conditions under which solutions blow up at finite time. We then explore the long-time behavior of global solutions. Results show that the solution is global and fast when initial value is small, and the solution is global but slow for suitable initial value. Numerical simulations are also given to illustrate the effect of the initial value on the free boundary.

1. Introduction

Free boundary problems have been attracting great attention [2, 3, 4, 5, 7, 11, 10, 15, 17, 18, 22, 25, 24, 26]. Recently, some works [6, 9] considered a heat-diffusive and chemically reactive substance in its liquid phase, and studied the free boundary problem

$$\begin{gathered} u_t - u_{xx} = u^p, \quad t > 0,\ 0 < x < h(t), \\ u_x(t,0) = 0, \quad u(t,h(t)) = 0, \quad t > 0, \\ h'(t) = -\mu u_x(t,h(t)), \quad t > 0, \\ h(0) = h_0, \quad u(0,x) = u_0(x), \quad 0 \le x \le h_0. \end{gathered} \tag{1.1}$$

If the left fixed boundary $x = 0$ in (1.1) is replaced by a free boundary $x = g(t)$ governed by $g'(t) = -\mu u_x(t, g(t))$, then (1.1) becomes a double-front free boundary problem [25]. Many previous investigations of the corresponding free boundary problem are restricted to one-dimensional space, and it remains unclear but really interesting what happens when spatial dimension increases, a question that is attempted to be addressed in the present paper. However, increasing of spatial dimension makes models more complicated and accordingly more difficulties are caused. As a starting point, we assume that both space and solution are radially

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symmetric. Under this assumption, model (1.1) can be rewritten as

$$
\begin{gathered}
u_t - d\Delta u = u^p, \quad t > 0,\ 0 < r < h(t), \\
u_r(t,0) = u(t,h(t)) = 0, \quad t > 0, \\
h'(t) = -\mu u_r(t,h(t)), \quad t > 0, \\
h(0) = h_0, \quad u(0,r) = u_0(r), \quad 0 \le r \le h_0,
\end{gathered} \tag{1.2}
$$

where $r = |x|$, $x \in \mathbb{R}^n (n \ge 2)$, $u \equiv u(t,r)$, $\Delta u = u_{rr} + \frac{n-1}{r} u_r$, and $r = h(t)$ is the moving boundary to be determined later together with the solution $u(t,r)$; h_0, d and μ are positive constants. We assume throughout this paper that $p > 1$ and the initial function u_0 satisfies

$$
\begin{gathered}
u_0 \in C^2([0,h_0]), \\
u_0'(0) = u_0(h_0) = 0, \quad u_0 > 0 \quad \text{in } [0,h_0).
\end{gathered} \tag{1.3}
$$

In the absence of free boundary, model (1.2) reduces to the Cauchy problem in $\mathbb{R}^n$; that is,

$$
\begin{gathered}
u_t - u_{xx} = u^p, \quad t > 0,\ x \in \mathbb{R}^n, \\
u(0,x) = u_0(x), \quad x \in \mathbb{R}^n.
\end{gathered} \tag{1.4}
$$

Such problem has been well studied [8, 12, 23], and some results are available: If $1 < p \le 1 + \frac{2}{n}$, then no non-negative global solution exists for any non-trivial initial value; If $p > 1 + \frac{2}{n}$, then the global solution does exist for any non-negative initial value dominated by a sufficiently small Gaussian. For the Cauchy problem with fixed domain, one can refer to [1, 13, 14, 19].

The rest of this article is organized as follows. Some basic results are presented in section 2. In section 3, we obtain some sufficient conditions, which depend on the initial value u_0 and the spatial dimension n, for the solution blows up. Section 4 deals with the existence of global fast and slow solutions. Though some of results and methods here are motivated from [6, 9], corresponding changes in the proofs are needed, due to the more general domain. Moreover, we try to illustrate the effect of the initial date on the free boundary by numerical tests, and a brief discussion is also presented in the last section.

2. Some basic results

In this section, we first prove the existence and uniqueness of local solution to (1.2) using the contraction mapping principle.

Theorem 2.1. *Under assumption* (1.3)*, for any $\alpha \in (0,1)$, there exists a $T > 0$ such that* (1.2) *admits a unique classic solution*

$$
(u,h) \in C^{(1+\alpha)/2,1+\alpha}(\overline{D}_T) \times C^{1+\alpha/2}([0,T]).
$$

Furthermore,

$$
\|u\|_{C^{(1+\alpha)/2,1+\alpha}(\overline{D}_T)} + \|h\|_{C^{1+\alpha/2}([0,T])} \le C, \tag{2.1}
$$

where $D_T = \{(t,r) \in \mathbb{R}^2 : t \in [0,T], r \in [0,h(t)]\}$, C and T depend only on h_0, α and $\|u_0\|_{C^2([0,h_0])}$.

Proof. As in [2], we straighten the free boundary. Let

$$
x = y + \xi(|y|)(h(t) - h_0)y/|y|, \quad y \in \mathbb{R}^n,
$$

and $\xi(s) \in C^3([0,+\infty))$ satisfy

$$\xi(s) = \begin{cases} 1 & \text{if } |s-h_0| < \frac{h_0}{8}, \\ 0 & \text{if } |s-h_0| > \frac{h_0}{2}, \end{cases} \quad |\xi'(s)| < \frac{6}{h_0} \quad \text{for all } s.$$

Obviously the transformation $(t,y) \to (t,x)$ induces the following transformation

$$(t,s) \to (t,r) \quad \text{with } r = s + \xi(s)(h(t)-h_0),\ 0 \le s < +\infty.$$

For any fixed $t \ge 0$, as long as

$$|h(t) - h_0| \le \frac{h_0}{8},$$

the above transformation $y \to x$ is a diffeomorphism from $\mathbb{R}^n$ onto $\mathbb{R}^n$, and the induced transformation $s \to r$ is a diffeomorphism from $[0,+\infty)$ onto $[0,+\infty)$. If we define

$$u(t,r) = u(t, s+\xi(s)(h(t)-h_0)) = v(t,s),$$

then we obtain an equivalent system

$$\begin{gathered} v_t - Adv_{ss} - (Bd + h'C + Dd)v_s = v^p, \quad t>0,\ 0<s<h_0, \\ v = 0, \quad h'(t) = -\mu v_s, \quad t>0,\ s=h_0, \\ v_s(t,0) = 0, \quad t>0, \\ h(0) = h_0, \quad v(0,s) = u_0(s), \quad 0 \le s \le h_0, \end{gathered} \tag{2.2}$$

where

$$\begin{gathered} A \equiv A(h(t),s) = \frac{1}{[1+\xi'(s)(h(t)-h_0)]^2}, \\ B \equiv B(h(t),s) = -\frac{\xi''(s)(h(t)-h_0)}{[1+\xi'(s)(h(t)-h_0)]^3}, \\ C \equiv C(h(t),s) = \frac{\xi(s)}{1+\xi'(s)(h(t)-h_0)}, \\ D \equiv D(h(t),s) = \frac{(n-1)\sqrt{A}}{s+\xi(s)(h(t)-h_0)}. \end{gathered}$$

Denote $h_1 = -\mu u_0'(h_0)$, and for $0 < T \le \frac{h_0}{8(1+h_1)}$, take $\Delta_T = [0,T] \times [0,h_0]$,

$$\begin{gathered} U_T = \{v \in C(\Delta_T) : v(0,s) = u_0(s),\ \|v-u_0\|_{C(\Delta_T)} \le 1\}, \\ \mathcal{H}_T = \{h \in C^1([0,T]) : h(0) = h_0,\ h'(0) = h_1,\ |h'-h_1\|_{C([0,T])} \le 1\}. \end{gathered}$$

It is easy to see that $\Sigma_T := U_T \times \mathcal{H}_T$ is a complete metric space with the metric

$$d((v_1,h_1),(v_2,h_2)) = \|v_1 - v_2\|_{C(\Delta_T)} + \|h_1' - h_2'\|_{C([0,T])}.$$

The rest of the proof is similar as that of [4, Theorem 2.1], it follows from standard L^p theory and the Sobolev imbedding theorem [16] that there exists a $T>0$ such that $\mathcal{F}$ is a contraction. Hence we can apply the contraction mapping theorem to conclude that there is a unique fixed point (v,h) in Σ_T such that $\mathcal{F}(v,h) = (v,h)$. That is, (v,h) is the solution of (2.2), which implies that (u,h) is the solution of (1.2). Moreover, $(u(t,r),h(t))$ is the unique local classical solution of (1.2). This result together with the Schauder estimates proves additional regularity properties of the solution, $h(t) \in C^{1+\alpha/2}((0,T])$ and $u \in C^{1+\alpha/2,2+\alpha}((0,T]\times[0,h_0])$. $\square$

The next lemma states the property of the free boundary.

Lemma 2.2. *If u is a solution of* (1.2) *defined for $0 < t < T_0$ for some $T_0 \in (0, +\infty)$, and there exists a positive number M_1 such that $u(t, r) \le M_1$ for $(t, r) \in [0, T_0) \times (0, h(t))$, then there exists constant $M_2(M_1)$ independent of T_0, such that $0 < h'(t) \le M_2$.*

Proof. First, applying the Hopf lemma to the second equation of problem (1.2), we immediately obtain

$$u_r(t, h(t)) < 0 \quad \text{for } 0 < t \leqslant T_0,$$

then we have $h'(t) > 0$ by using the free boundary condition $h'(t) = -\mu u_r(t, h(t))$. The proof that $h'(t) \le M_2$ is almost identical to the argument of [4, Lemma 2.2]. So we omit the details. □

We now have the following comparison principle which can be used to estimate both $u(t, r)$ and the free boundary $r = h(t)$. Since the proof is similar to the one phase case [4, Lemma 3.5], we omit it here.

Lemma 2.3 (Comparison Principle)**.** *Suppose that $T \in (0, +\infty)$, $\overline{h} \in C^1([0, T])$, $\overline{u} \in C(\overline{D}_T^*) \cap C^{1,2}(D_T^*)$ with $D_T^* = \{(t, r) \in \mathbb{R}^2 : 0 < t \le T,\ 0 \le r \le \overline{h}(t)\}$, and*

$$\begin{gathered}
\overline{u}_t - d\Delta\overline{u} \ge \overline{u}^p, \quad t > 0,\ 0 < r < \overline{h}(t), \\
\overline{u} = 0, \quad \overline{h}'(t) \ge -\mu\overline{u}_r, \quad t > 0,\ r = \overline{h}(t), \\
\overline{u}_r(t, 0) \le 0, \quad t > 0.
\end{gathered}$$

If $h_0 \le \overline{h}(0)$ and $u_0(r) \le \overline{u}(0, r)$ in $[0, h_0]$, then the solution (u, h) of the free boundary problem (1.2) *satisfies*

$$\begin{gathered}
h(t) \le \overline{h}(t) \quad \text{in } (0, T], \\
u(t, r) \le \overline{u}(t, r) \quad \text{for } (t, r) \in (0, T] \times (0, h(t)).
\end{gathered}$$

3. Blow-up solutions

By Theorem 2.1, the solution of (1.2) exists, is unique, and it can be extended to $[0, T^*)$, where $T^* = T^*(u_0) \in (0, +\infty]$ is its maximum existence time. To state our blow-up result, we need the following lemmas. We begin with a lemma modified from [25, Lemma 4.5].

Lemma 3.1. *If $T^* < \infty$, we have*

$$\limsup_{t \to T^*} \|u(t, r)\|_{L^\infty([0,t] \times [0,h(t)])} = \infty, \tag{3.1}$$

and we say that u blows up in finite time.

Next introducing the definition of "energy" of solution u at time t by

$$E(t) = \int_0^{h(t)} r^{n-1}\Big(\frac{d}{2}(u_r)^2 - \frac{u^{p+1}}{p+1}\Big)(t, r)\mathrm{d}r$$

and its L^1-norm by $|u(t)|_1 = \int_0^{h(t)} r^{n-1}u(t, r)\mathrm{d}r$, we have the following "energy identities".

Lemma 3.2. *If u is the solution of problem* (1.2)*, then we have*

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} = -\int_0^{h(t)} r^{n-1}u_t^2(t, r)\mathrm{d}r - \frac{d}{2\mu^2}h^{n-1}(t)h'^3(t). \tag{3.2}$$

Moreover,

$$|u(t)|_1 - |u_0|_1 = \frac{d}{n\mu}\left(h_0^n - h^n(t)\right) + \int_0^t \int_0^{h(\tau)} r^{n-1} u^p(\tau, r) \mathrm{d}r \mathrm{d}\tau. \tag{3.3}$$

Proof. Direct differentiation yields

$$\begin{aligned} \frac{\mathrm{d}E(t)}{\mathrm{d}t} &= \int_0^{h(t)} r^{n-1}\left(d u_r u_{rt} - u^p u_t\right)(t, r)\mathrm{d}r \\ &\quad + h'(t)h^{n-1}(t)\Big[\frac{d}{2}(u_r)^2(t, h(t)) - \frac{u^{p+1}}{p+1}(t, h(t))\Big]. \end{aligned} \tag{3.4}$$

To calculate $\int_0^{h(t)} r^{n-1} u_r u_{rt} \mathrm{d}r$, differentiate the relation $u(t, h(t)) = 0$ in t and use the Stefan condition, we obtain

$$(u_r u_t)(t, h(t)) = -h'(t) u_r^2(t, h(t)) = -\frac{h'^3(t)}{\mu^2}.$$

Integrating by parts yields

$$\int_0^{h(t)} r^{n-1} u_r u_{rt} \mathrm{d}r = -\int_0^{h(t)} \left(r^{n-1} u_r\right)_r u_t \mathrm{d}r + h^{n-1}(t)(u_r u_t)(t, h(t)).$$

By substituting the above identity in (3.4) and using also $u(t, h(t)) = 0$, we see that

$$\begin{aligned} \frac{\mathrm{d}E(t)}{\mathrm{d}t} &= -\int_0^{h(t)} [d(r^{n-1}u_r)_r u_t + r^{n-1} u^p u_t](t, r)\mathrm{d}r - \frac{d}{2\mu^2} h^{n-1}(t) h'^3(t) \\ &= -\int_0^{h(t)} r^{n-1} u_t^2(t, r)\mathrm{d}r - \frac{d}{2\mu^2} h^{n-1}(t) h'^3(t). \end{aligned}$$

This implies (3.2).

It remains to prove (3.3). We compute

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_0^{h(t)} r^{n-1} u(t, r)\mathrm{d}r &= \int_0^{h(t)} r^{n-1} u_t(t, r)\mathrm{d}r + h'(t)h^{n-1}(t)u(t, h(t)) \\ &= d\int_0^{h(t)} r^{n-1}\Delta u \mathrm{d}r + \int_0^{h(t)} r^{n-1} u^p(t, r)\mathrm{d}r \\ &= \int_0^{h(t)} d\left(r^{n-1}u_r\right)_r \mathrm{d}r + \int_0^{h(t)} r^{n-1} u^p(t, r)\mathrm{d}r \\ &= -\frac{d}{\mu} h^{n-1}(t)h'(t) + \int_0^{h(t)} r^{n-1} u^p(t, r)\mathrm{d}r. \end{aligned}$$

Integrating the above equation between 0 and t, we know that

$$|u(t)|_1 - |u_0|_1 = \frac{d}{n\mu}(h_0^n - h^n(t)) + \int_0^t \int_0^{h(\tau)} r^{n-1} u^p(\tau, r)\mathrm{d}r\mathrm{d}\tau.$$

This completes the proof of Lemma 3.2. □

Lemma 3.3. *Assume $T^* = \infty$ and let $A = \int_0^\infty h^{n-1}(t)h'^3(t)\mathrm{d}t$, then we have*

$$A \geqslant N(u_0, n) := \Big(\frac{3}{n+2}\Big)^3 \frac{2^{4-\frac{4}{n}} d^2 \pi^2}{(\frac{1}{n}h_0^n + \frac{\mu}{d}|u_0|_1)^{\frac{4}{n}}} \Big[\Big(\frac{n\mu|u_0|_1}{2d} + h_0^n\Big)^{\frac{n+2}{3n}} - h_0^{\frac{n+2}{3}}\Big]^3.$$

Proof. By the same argument as in Theorem 2.1, the solution v of the auxiliary free boundary problem

$$\begin{gathered}
v_t - d\Delta v = 0, \quad 0 < t < \infty,\ 0 < r < \sigma(t), \\
v_r(t,0) = v(t,\sigma(t)) = 0, \quad 0 < t < \infty, \\
\sigma'(t) = -\mu v_r(t,\sigma(t)), \quad 0 < t < \infty, \\
\sigma(0) = h_0, v(0,r) = u_0(r), \quad 0 \leqslant r \leqslant h_0
\end{gathered}$$

exists for all $t > 0$ because of the boundedness of the solution. Moreover, by Lemma 2.3, we easily have $u \geqslant v \geqslant 0$ and $h(t) \geqslant \sigma(t) \geqslant h_0$ on $(0, T^*)$. Denoting $|v(t)|_1 = \int_0^{\sigma(t)} r^{n-1} v(t,r)\mathrm{d}r$, we obtain from the same discussion in Lemma 3.2 that

$$\sigma^n(t) - h_0^n = \frac{n\mu}{d}(|u_0|_1 - |v(t)|_1). \tag{3.5}$$

On the other hand, assume that $\bar{v}$ is the solution of the Cauchy problem

$$\begin{gathered}
\bar{v}_t - d\Delta\bar{v} = 0, \quad t > 0,\ 0 \le r < \infty, \\
\bar{v}(0,r) = \bar{u}_0(r) = \begin{cases} u_0(r), & 0 \leqslant r \leqslant h_0, \\ 0, & r \in [0,+\infty)/[0,h_0]. \end{cases}
\end{gathered}$$

Then we have $v \leqslant \bar{v}$. Using $L^1 - L^\infty$ estimate for above equation, we find that

$$\|v(t)\|_\infty \leqslant \|\bar{v}(t)\|_\infty \leqslant (4d\pi t)^{-n/2}|\bar{v}(0)|_1 = (4d\pi t)^{-n/2}|u_0|_1.$$

Hence, by (3.5),

$$\begin{aligned}
|v(t)|_1 &\leqslant \frac{1}{n}\sigma^n(t)\|v(t)\|_\infty \leqslant \frac{1}{n}\sigma^n(t)(4d\pi t)^{-n/2}|u_0|_1 \\
&= \Big(\frac{1}{n}h_0^n + \frac{\mu}{d}|u_0|_1 - \frac{\mu}{d}|v(t)|_1\Big)(4d\pi t)^{-n/2}|u_0|_1 \\
&\leqslant \Big(\frac{1}{n}h_0^n + \frac{\mu}{d}|u_0|_1\Big)(4d\pi t)^{-n/2}|u_0|_1.
\end{aligned}$$

Clearly,

$$|v(t_0)|_1 \leqslant |u_0|_1/2, \quad \text{for} \quad t_0 = \frac{\sqrt[n]{4}}{4d\pi}(\frac{1}{n}h_0^n + \frac{\mu}{d}|u_0|_1)^{2/n}. \tag{3.6}$$

Now by Hölder's inequality and $h(t)$ being nondecreasing, we find that, for any $t \geqslant 0$,

$$\int_0^t h^{\frac{n-1}{3}}(\tau)h'(\tau)\mathrm{d}\tau \leqslant \Big(\int_0^t h^{n-1}(\tau)h'^3(\tau)\mathrm{d}\tau\Big)^{1/3}\Big(\int_0^t \mathrm{d}\tau\Big)^{2/3}.$$

Thus

$$\begin{aligned}
A &\geqslant \int_0^t h^{n-1}(\tau)h'^3(\tau)\mathrm{d}\tau \\
&\geqslant t^{-2}\Big(\int_0^t h^{\frac{n-1}{3}}(\tau)h'(\tau)\mathrm{d}\tau\Big)^3 \\
&= (\frac{3}{n+2})^3 t^{-2}\Big(h^{\frac{n+2}{3}}(t) - h_0^{\frac{n+2}{3}}\Big)^3.
\end{aligned}$$

Recall that $h(t) \geqslant \sigma(t) \geqslant h_0$, therefore, by (3.5),

$$A \geqslant \Big(\frac{3}{n+2}\Big)^3 t^{-2}\Big[\big(\frac{n\mu}{d}(|u_0|_1 - |v(t)|_1) + h_0^n\big)^{\frac{n+2}{3n}} - h_0^{\frac{n+2}{3}}\Big]^3. \tag{3.7}$$

Taking $t = t_0$ in (3.7), together with (3.6), we immediately complete the proof. $\square$

Theorem 3.4. *Let u be the solution of* (1.2)*. Then u blows up in finite time if*

$$E(0) < \frac{d}{2\mu^2} N(u_0, n), \tag{3.8}$$

where $N(u_0, n)$ is defined in Lemma 3.3.

Proof. Suppose that (1.2) has no blow-up solution, by Lemma 3.1, we have $T^* = \infty$. By Lemma 3.3, condition (3.8) implies that

$$E(0) < \frac{d}{2\mu^2} \int_0^t h^{n-1}(\tau) h'^3(\tau) \mathrm{d}\tau \tag{3.9}$$

for sufficiently large $t \geqslant t_0$.

As in [9], we define auxiliary function

$$W(t) = F^{-\frac{p-1}{4}}(t), \quad F(t) = \int_0^t \int_0^{h(\tau)} r^{n-1} u^2(\tau, r) \mathrm{d}r \mathrm{d}\tau.$$

Then

$$\begin{gathered} W'(t) = -\frac{p-1}{4} F'(t) F^{-\frac{p+3}{4}}(t), \\ W''(t) = -\frac{p-1}{4} F^{-\frac{p+7}{4}} \Big(F F'' - \frac{p+3}{4} F'^2 \Big)(t), \end{gathered}$$

where

$$\begin{aligned} F'(t) &= \int_0^{h(t)} r^{n-1} u^2(t, r) \mathrm{d}r, \\ F''(t) &= \int_0^{h(t)} 2 r^{n-1} u u_t(t, r) \mathrm{d}r + h'(t) h^{n-1}(t) u^2(t, h(t)) \\ &= 2 \int_0^{h(t)} r^{n-1} u u_t(t, r) \mathrm{d}r \\ &= 2 \int_0^{h(t)} r^{n-1} u \left(d\Delta u + u^p \right)(t, r) \mathrm{d}r \\ &= 2 \int_0^{h(t)} \left[d \left(r^{n-1} u_r \right)_r u + r^{n-1} u^{p+1} \right](t, r) \mathrm{d}r \\ &= 2 \int_0^{h(t)} r^{n-1} \left(u^{p+1} - d u_r^2 \right)(t, r) \mathrm{d}r + 2 d r^{n-1} u u_r(t, r) \big|_0^{h(t)} \\ &= -2(p+1) E(t) + d(p-1) \int_0^{h(t)} r^{n-1} u_r^2(t, r) \mathrm{d}r. \end{aligned} \tag{3.10}$$

Combining (3.2) with (3.9) gives

$$\begin{aligned} F''(t) &= 2(p+1) \int_0^t \int_0^{h(\tau)} r^{n-1} u_t^2(\tau, r) \mathrm{d}r \mathrm{d}\tau + \frac{d}{\mu^2}(p+1) \int_0^t h^{n-1}(\tau) h'^3(\tau) \mathrm{d}\tau \\ &\quad - 2(p+1) E(0) + d(p-1) \int_0^{h(t)} r^{n-1} u_r^2(t, r) \mathrm{d}r \\ &> 2(p+1) \int_0^t \int_0^{h(\tau)} r^{n-1} u_t^2(\tau, r) \mathrm{d}r \mathrm{d}\tau \end{aligned} \tag{3.11}$$

for any $t \geqslant t_0$. In view of the Cauchy-Schwarz inequality and (3.10), it follows that

$$\begin{aligned}F(t)F''(t) &\geqslant 2(p+1)\int_0^t\int_0^{h(\tau)} r^{n-1}u^2\mathrm{d}r\mathrm{d}\tau\int_0^t\int_0^{h(\tau)} r^{n-1}u_t^2\mathrm{d}r\mathrm{d}\tau\\ &\geqslant 2(p+1)\Big(\int_0^t\int_0^{h(\tau)} r^{n-1}uu_t\mathrm{d}r\mathrm{d}\tau\Big)^2\\ &= \frac{p+1}{2}(F'(t)-F'(0))^2.\end{aligned}$$

On the other hand, from (3.11), we obtain

$$F'(t) \geqslant F'(t_0+1) = \int_0^{h(t_0+1)} r^{n-1}u^2(t_0+1,r)\mathrm{d}r > 0, \quad t \geqslant t_0+1.$$

Hence, $F(t) \to \infty$ as $t \to \infty$. We then deduce

$$F(t)F''(t) \geqslant \frac{p+3}{4}F'^2(t), \quad t \geqslant t_1$$

for some large $t_1 > t_0 + 1$ (since $p > 1$).

We obtain from above discussion that $W'(t) < 0$, $W''(t) \leq 0$ for any $t \geqslant t_1$. It follows that W is concave, decreasing and positive for any $t \geqslant t_1$, which is impossible. Thus we immediately have the blowup result. □

Remark 3.5. Theorem 3.4 shows that conditions for blow-up not only depend on the initial value u_0, but also on the spatial dimension n. Along with the increasing of spatial dimension n, blow-up conditions become stronger. When we fix n, the solution of free boundary problem (1.2) blows up if the initial value u_0 is sufficiently large. If the initial value is of the form $u_0 = \lambda\phi(r)$, where $\phi \in C^1([0, h_0])$ satisfies $\phi \geq 0$ and $\phi \not\equiv 0$ with $\phi_r(0) = \phi(h_0) = 0$, then Theorem 3.4 also implies that the solution of problem (1.2) blows up when λ is large enough.

4. Global fast and slow solution

This section is devoted to the existence of global fast and slow solutions. We start with the classification of global solutions.

Definition 4.1 (Fast solution)**.** Suppose u is the solution of (1.2). If $T^* = \infty$, and the free boundary grows up to a finite limit, that is, $h_\infty := \lim_{t\to\infty} h(t) < \infty$, then u is called global fast solution of (1.2).

Definition 4.2 (Slow solution)**.** Suppose u is the solution of (1.2). If $T^* = \infty$ and the free boundary converges to infinity, that is, $h_\infty := \lim_{t\to\infty} h(t) = \infty$, then u is called global slow solution of (1.2).

The existence of global solutions of (1.2) is a consequence of the following two properties.

Proposition 4.3. *Let $p_S = +\infty$ for $n = 1, 2$ and $p_S = (n+2)/(n-2)$ for $n \geq 3$. If u is a global solution of problem* (1.2) *with $1 < p < p_S$, then there exists a constant $C > 0$ such that*

$$\sup_{t\geqslant 0}\|u(t,r)\|_{L^\infty(0,h(t))} \leqslant C,$$

where $C = C(\|u_0\|_{C^{1+\alpha}}, h_0, 1/h_0)$ is bounded for $\|u_0\|_{C^{1+\alpha}}$, h_0 and $1/h_0$ is bounded.

Proof. By the local theory for (1.2), for each $M > 1$, we can find a $\sigma > 0$ such that, $\|u(t,r)\|_{L^\infty} < 2M$ on $[0,\sigma]$ if $\|u_0\|_{C^{1+\alpha}} < M$ and $1/M < h_0 < M$.

Suppose that the above conclusion is not true, then it is easy to see that there exists some $M > 0$ and a sequence of global solutions (u_m, h_m) of (1.1) satisfying

$$\begin{gathered}1/M < h_m(0) < M, \quad \|u_m(0,r)\|_{C^{1+\alpha}([0,h_0])} < M, \\ \sup_{t \geqslant 0} \|u_m(t,r)\|_{L^\infty(0,h_m(t))} \to \infty \quad \text{as } m \to \infty.\end{gathered}$$

Thus for any large m, there exist $t_m \geqslant \sigma$ and $r_m \in (0, h_m(t_m))$ satisfying

$$\sup_{t \geqslant 0} \|u_m(t,r)\|_{L^\infty(0,h_m(t))} = u_m(t_m, r_m) =: \varrho_m.$$

Define $\lambda_m = \varrho_m^{-(p-1)/2}$, then we easily conclude that $\lambda_m \to 0$ as $m \to \infty$. By extending $u_m(t,\cdot)$ by 0 on $(h_m(t), \infty)$, then we define a rescaled function

$$v_m(\tau, s) = \lambda_m^{\frac{2}{p-1}} u_m(t_m + \lambda_m^2 \tau, r_m + \lambda_m s) \tag{4.1}$$

for $(\tau, s) \in \widetilde{D}_m = \{(\tau, s) : -\lambda_m^{-2} t_m \leqslant \tau \leqslant 0 \text{ and } -\lambda_m^{-1} r_m \leqslant s < \infty\}$.

Let us also define

$$D_m = \{(\tau, s) : -\lambda_m^{-2} t_m \leqslant \tau \leqslant 0 \text{ and } s_1(\tau) \leqslant s < s_2(\tau)\},$$

where $s_1(\tau) = -\lambda_m^{-1} r_m$, $s_2(\tau) = \lambda_m^{-1}(h(t_m + \lambda_m^2 \tau) - r_m)$. Clearly, the function v_m satisfies $v_m(0,0) = 1$, $0 \leqslant v_m \leqslant 1$ in $\widetilde{D}_m$, and

$$(v_m)_\tau - d\Delta v_m = v_m^p, \quad (\tau, s) \in D_m. \tag{4.2}$$

Similarly to [6, Lemmas 2.1-2.3], we can derive a function $w(s) \geqslant 0$, which is bounded, continuous on $[0,\infty)$, and satisfies that $-\Delta w = w^p$; hence w is concave. Since $1 < p < p_S$, it follows from [20, Theorem 8.1] that $w \equiv 0$, contradicting the fact $w(0) = 1$. This completes the proof. $\square$

Proposition 4.4. *If u is a global solution of (1.2) with $1 < p < p_S$, then*

$$\lim_{t \to +\infty} \|u(t,r)\|_{L^\infty(0,h(t))} = 0.$$

Proof. We shall prove the property by a different approach from the one in [6, Theorem A]. Suppose that $k := \limsup_{t\to+\infty} \|u(t,r)\|_{L^\infty(0,h(t))} > 0$ by contradiction, then we can find a sequence $(t_k, r_k) \in (0,\infty) \times (0, h(t))$ such that $u(t_k, r_k) \geq k/2$ for all $k \in N$, and $t_k \to \infty$ as $k \to \infty$, and can also find a subsequence of $\{r_k\}$ converges to $r_0 \in (0, h_\infty)$, since $-\infty < 0 \leq r_k < h_\infty < +\infty$. Without loss of generality, we suppose $r_k \to r_0$ as $k \to \infty$. Let

$$u_k(t,r) = u(t + t_k, r) \quad \text{for } (t,r) \in (-t_k, +\infty) \times (0, h(t + t_k)).$$

Then we can apply Proposition 4.3 and the parabolic regularity to conclude that there is a subsequence u_{k_i} of $\{u_k\}$ such that $u_{k_i} \to \overline{u}$ as $i \to \infty$ and $\overline{u}$ satisfies

$$\overline{u}_t - d\Delta \overline{u} = \overline{u}^p(t,r) \quad \text{for } (t,r) \in (-\infty, +\infty) \times (0, h_\infty).$$

Since $\overline{u}(0, r_0) \geq k/2$, by the strong maximum principle, we obtain $\overline{u} > 0$ in $(-\infty, +\infty) \times (0, h_\infty)$. Using Hopf lemma to the above equation $\overline{u}$ satisfying at the point $(0, h_\infty)$, we deduce that $\overline{u}_r(0, h_\infty) \leq -\sigma_0 < 0$.

On the other hand, we introduce a transform to straighten the free boundary. Let

$$s = \frac{h_0 r}{h(t)}, \quad v(t,s) = u(t,r),$$

where $v(t,s)$ satisfies

$$\begin{gathered} v_t - \frac{dh_0^2}{h^2(t)}\Delta_s v - \frac{h'(t)}{h(t)} s v_s = v^p, \quad t>0,\ 0<s<h_0, \\ v_s(t,0) = v(t,h_0) = 0, \quad t>0, \\ v(0,s) = v_0(s) := u_0(s) \ge 0, \quad 0 \le s \le h_0. \end{gathered} \tag{4.3}$$

By the same argument as in Lemma 2.2, we can infer that $h'(t)$ is uniformly bounded for any $t>0$. Moreover, there exist constants M_3 and M_4, such that

$$\|v\|_{L^\infty} \le M_3, \quad \|\frac{h'(t)}{h(t)} s\|_{L^\infty} \le M_4.$$

Then by standard L^P theory and the Sobolev imbedding theorem, we conclude that

$$\|v\|_{C^{(1+\alpha)/2,1+\alpha}([0,\infty)\times(0,h_0))} \le M_5,$$

where M_5 depends on α, h_0, C, M_3, M_4, $\|u_0\|_{C^{1+\alpha}([0,h_0])}$ and h_∞. Hence for any $\alpha \in (0,1)$, there exists a constant M^* depending on α, h_0, $\|u_0\|_{C^{1+\alpha}([0,h_0])}$ and h_∞ such that

$$\|u\|_{C^{(1+\alpha)/2,1+\alpha}([0,\infty)\times(0,h(t)))} + \|h\|_{C^{1+\alpha/2}([0,\infty))} \le M^*. \tag{4.4}$$

Therefore $\|h\|_{C^{1+\alpha/2}([0,\infty))} \le M^*$. Recalling that $h(t)$ is monotonically bounded, we then have $h'(t) \to 0$, which implies $u_r(t_k, h(t_k)) \to 0$ by the Stefan condition. Furthermore, in view of the fact $\|u\|_{C^{(1+\alpha)/2,1+\alpha}([0,\infty)\times(0,h(t)))} \le M^*$, we have $u_r(t_k + 0, h(t_k)) = (u_k)_r(0, h(t_k)) \to \overline{u}_r(0,h_\infty)$ as $k \to \infty$ and therefore $\overline{u}_r(0,h_\infty) = 0$, contradicting the result $\overline{u}_r(0,h_\infty) \le -\sigma_0 < 0$. This completes the proof. $\square$

The following existence results of global fast and slow solutions can be proved by the same argument as [9, Theorem 3.2] and [6, Theorem 1] respectively, so we only state these results.

Theorem 4.5. *If u is a solution of (1.2), and u_0 satisfies*

$$\|u_0\|_\infty \leqslant \frac{1}{2}\min\Big\{\Big(\frac{d}{16h_0^2}\Big)^{\frac{1}{p-1}}, \frac{d}{8\mu}\Big\},$$

then (1.2) has a global fast solution and there exist some real numbers K, $\beta > 0$ depending on $u_0(r)$ such that

$$\|u(t)\|_\infty \leqslant K e^{-\beta t}, \quad t \geqslant 0.$$

Theorem 4.6. *If $\phi(r)$ satisfies the condition in Remark 3.5, then there exists $\lambda > 0$ such that the solution of (1.2) with $1 < p < p_S$ and initial value $u_0(r) = \lambda\phi(r)$ is a global slow solution.*

5. Numerical illustration and discussion

In this section, we first perform numerical simulations to illustrate the theoretical results given above. Because of the moving boundary, it is a little difficult to present the numerical solution compared to the problem in fixed boundary. For simplicity, we only consider the one-dimensional case, that is,

$$\begin{gathered} u_t - u_{xx} = u^p, \quad t>0,\ 0<x<h(t), \\ u_x(t,0) = 0, \quad u(t,h(t)) = 0, \quad t>0, \\ h'(t) = -\mu u_x(t,h(t)), \quad t>0, \\ h(0) = h_0, \quad u(0,x) = u_0(x), \quad 0 \le x \le h_0. \end{gathered} \tag{5.1}$$

We use an implicit scheme as in [21] and then obtain a nonlinear system of algebraic equations, which was solved with Newton-Raphson method. The numerical solution was performed by using Matlab software. Let us fix some coefficients. Assume that $p = 2$, $\mu = 10$ and $h_0 = 1.5$, then the asymptotic behaviors of the solution to (5.1) are shown by choosing different initial functions.

Example 5.1. *Let $u_0(x) = 0.2\cos(\pi x/3)$, it is easy to see from Figure 1 that the free boundary $x = h(t)$ increases fast.*

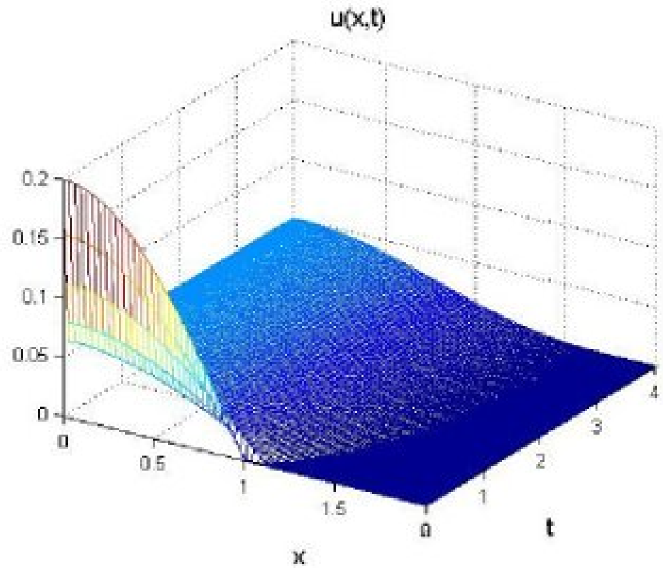

FIGURE 1. When $u_0(x) = 0.2\cos(\pi x/3)$, the free boundary increases fast.

Example 5.2. Let $u_0(x) = 0.1\cos(\pi x/3)$, compared the free boundary in Figure 2 with that in Figure 1, the free boundary $x = h(t)$ in Figure 2 increases slower than that in Figure 1.

Example 5.3. Let $u_0(x) = 0.05\cos(\pi x/3)$. Clearly, Figure 3 shows that $u(t, x)$ goes to 0 and the free boundary $x = h(t)$ increases slowly. Then the solution $(u(t, x), h(t))$ is called the global slow solution.

In this article, we considered the free boundary problem (1.2) in a higher dimensional space. The long time behaviors of the solution has been discussed. It is shown in Theorem 3.4 that the solution will blow up if the initial value is big enough. For the global solution, Theorem 4.5 shows that the solution is global and fast for small initial value, and Theorem 4.6 shows that there exists a $\lambda_0 > 0$ such that the solution of (1.2) with initial data $u_0 = \lambda_0\phi$ is a global slow solution. We remark here that the uniqueness of λ_0 is still not clear.

Compared with existing works, which considered usually the corresponding problem in the fixed bounded domain or the cauchy problem in the whole space, the free boundary problem (1.2) can be though of as a sort of intermediate between the

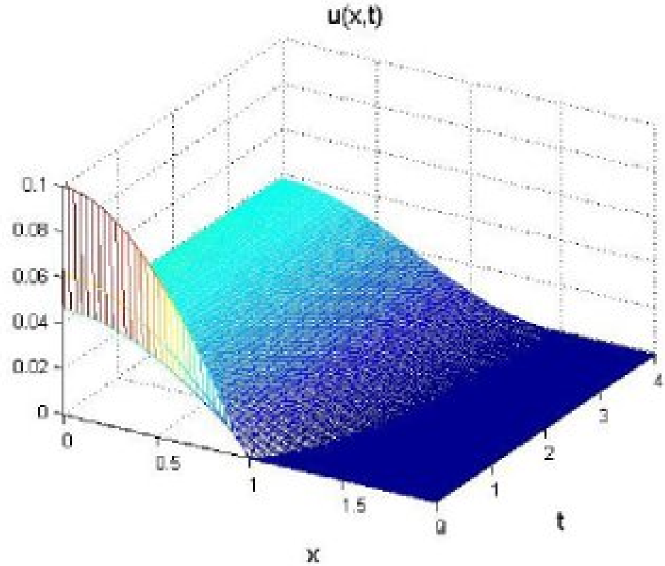

FIGURE 2. When $u_0(x) = 0.1\cos(\pi x/3)$, the free boundary increases slowly.

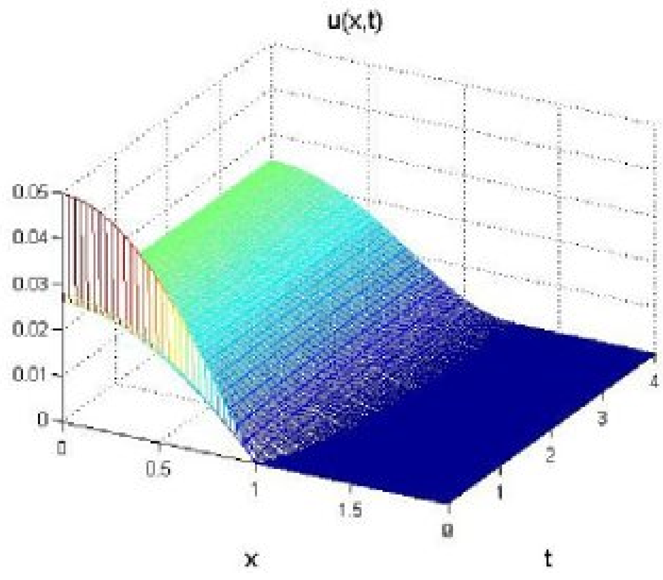

FIGURE 3. When $u_0(x) = 0.05\cos(\pi x/3)$, the free boundary increases slowly.

cases of bounded and unbounded intervals [9]. Besides the long time behavior of the solution $u(t,r)$, our results also present the expanding process of the domain. All theoretical results shows that the initial value determines the long time behaviors of the solution. Moreover, numerical simulations illustrate the effect of the initial value on the moving trend of the free boundary $x = h(t)$.

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