Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 286, pp. 1–19.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

WEIGHTED PSEUDO ALMOST AUTOMORPHIC SOLUTIONS TO FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

YONG-KUI CHANG, SHAN ZHENG

Abstract. In this article, we first establish some results on composition of Stepanov-like weighted pseudo almost automorphic functions so called class r and class infinity under a uniform continuity condition with respect to L^p-norm. And then, we study the existence and uniqueness of weighted pseudo almost automorphic solutions to an abstract partial neutral functional differential equation with infinite delay with a Stepanov-like nonlinear term.

1. Introduction

The concept of almost automorphy, which was introduced by Bochner [4], as a generalization of the classical almost periodicity in the sense of Bohr; see for example [13, 23, 25]. N'Guérékata and Pankov introduced the concept of Stepanov-like almost automorphy in [24]. Diagana [11] introduced the concept of Stepanov-like pseudo almost automorphy as a natural generalization of the pseudo almost automorphy and an implement of the Stepanov-like almost automorphy due to N'Guérékata and Pankov [24]. Blot et al [5] introduced the notion of weighted pseudo almost automorphic functions with values in a Banach space. Xia and Fan presented the notion of Stepanov-like (or S^p-) weighted pseudo almost automorphic function in [30]. To study differential equations with delay, Zhang, Chang and N'Guérékata [33] further studied new types of functions so called Stepanov-like weighted pseudo almost automorphic functions of class r, and Stepanov-like weighted pseudo almost automorphic functions of class infinity. The above mentioned concepts have been considerably investigated and applied to various differential equations, see [1, 2, 3, 7, 9, 10, 14, 15, 17, 19, 27, 28, 29] and the references therein.

The main purpose of this article is to study composition results for Stepanov-like weighted pseudo almost automorphic functions of class r and Stepanov-like weighted pseudo almost automorphic functions of class infinity [33]. Considering the space of Stepanov-like weighted pseudo almost automorphic functions of class r and Stepanov-like weighted pseudo almost automorphic functions of class infinity with an integral norm coming from L^p-norm, we first prove new composition theorems for Stepanov-like weighted pseudo almost automorphic functions of class r

2010 *Mathematics Subject Classification.* 43A60, 34G20, 47D03.
Key words and phrases. Stepanov-like weighted function; functional differential equation; pseudo almost automorphic function; infinite delay.

Submitted September 20, 2016. Published October 26, 2016.

under a uniform continuity condition with respect to the L^p-norm suggested by [16]. Similarly, we can arrive at new composition theorems for Stepanov-like weighted pseudo almost automorphic functions of class infinity. And then, we apply the obtained results to prove the existence and uniqueness of weighted pseudo almost automorphic solutions for the following abstract partial neutral functional-differential equation with infinite delay under Stepanov-like nonlinear forcing term

$$\frac{d}{dt}(u(t)+f(t,u_t)) = A(t)u(t)+g(t,u_t), \quad t\in\mathbb{R}, \tag{1.1}$$

where $A(t): D(A(t))\subset\mathbb{X}\to\mathbb{X}$ is a family of densely defined closed linear operators on the domain $D=D(A(t))$, which is independent of t, the history $u_t:(0,\infty]\to\mathbb{X}$ defined by $u_t(\theta)=u(t+\theta)$, belongs to some abstract phase space $\mathfrak{B}$ defined axiomatically, and $f,g:\mathbb{R}\times\mathfrak{B}\to\mathbb{X}$ are some suitable functions.

The rest of this paper is organized as follows. In Section 2, we recall some basic definitions, lemmas, and notation which will be used throughout this paper. In Section 3, we establish some new results on composition of Stepanov-like weighted pseudo almost automorphic functions of class r and Stepanov-like weighted pseudo almost automorphic functions of class infinity under L^p-norm uniform continuity condition. In Section 4, we prove the existence and uniqueness of weighted pseudo almost automorphic solutions to the equation (1.1) under Stepanov-like nonlinear forcing term. An example is also given to illustrate the main results.

2. Preliminaries

Let $(\mathbb{X},\|\cdot\|)$ and $(\mathbb{Y},\|\cdot\|_{\mathbb{Y}})$ be two Banach spaces and $\mathbb{N},\mathbb{R}$ stand for sets of natural numbers and real numbers, respectively. To facilitate discussions later, we introduce the following notation:

- $BC(\mathbb{R},\mathbb{X})$ (respectively, $BC(\mathbb{R}\times\mathbb{Y},\mathbb{X})$): The Banach spaces of bounded continuous function from $\mathbb{R}$ to $\mathbb{X}$ (respectively, from $\mathbb{R}\times\mathbb{Y}$ to $\mathbb{X}$) with the sup norm.
- $L^p(\mathbb{R},\mathbb{X})$: The space of all classes of equivalence (with respect to the equality almost everywhere on $\mathbb{R}$) of measurable function $f:\mathbb{R}\to\mathbb{X}$ such that $\|f\|\in L^p(\mathbb{R},\mathbb{R})$.
- $L^p_{\rm loc}(\mathbb{R},\mathbb{X})$: The space of all classes of equivalence of measurable function $f:\mathbb{R}\to\mathbb{X}$ such that the restriction of f to every bounded subinterval of $\mathbb{R}$ is in $L^p(\mathbb{R},\mathbb{X})$.
- $\mathfrak{L}(\mathbb{X},\mathbb{Y})$: Stands for the Banach space of bounded linear operators from $\mathbb{X}$ to $\mathbb{Y}$ equipped with its natural topology.

Definition 2.1 ([25]). A continuous function $f:\mathbb{R}\to\mathbb{X}$ is said to be almost automorphic if for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$ there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that

$$g(t):=\lim_{n\to\infty} f(t+s_n)$$

is well defined for each $t\in\mathbb{R}$, and

$$\lim_{n\to\infty} g(t-s_n)=f(t)$$

for each $t\in\mathbb{R}$. The collection of all such functions will be denoted by $AA(\mathbb{X})$.

Definition 2.2 ([22, 25]). A continuous function $f:\mathbb{R}\times\mathbb{X}\to\mathbb{X}$ is said to be almost automorphic if $f(t,x)$ is almost automorphic for each $t\in\mathbb{R}$ uniformly for all $x\in\mathbb{B}$, where $\mathbb{B}$ is any bounded subset of $\mathbb{X}$. The collection of all such functions will be denoted by $AA(\mathbb{R}\times\mathbb{X},\mathbb{X})$.

Definition 2.3 ([32])**.** A continuous function $f(t,x):\mathbb{R}\times\mathbb{R}\to\mathbb{X}$ is called bi-almost automorphic if for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that

$$g(t,s):=\lim_{n\to\infty} f(t+s_n,s+s_n)$$

is well-defined for each $t,s\in\mathbb{R}$, and

$$\lim_{n\to\infty} g(t-s_n,s-s_n)=f(t,s)$$

for each $t,s\in\mathbb{R}$. The collection of all functions will be denoted by $bAA(\mathbb{R}\times\mathbb{R},\mathbb{X})$.

Let $\mathbb{U}$ denote the set of all functions $\rho:\mathbb{R}\to(0,\infty)$, which are locally integrable over $\mathbb{R}$ such that $\rho>0$ almost everywhere. For a given $T>0$ and for each $\rho\in\mathbb{U}$, we set $m(T,\rho):=\int_{-T}^{T}\rho(t)dt$.

Thus the space of weights $\mathbb{U}_\infty$ is defined by

$$\mathbb{U}_\infty:=\{\rho\in\mathbb{U}:\lim_{T\to\infty} m(T,\rho)=\infty\}.$$

For a given $\rho\in\mathbb{U}_\infty$, we define

$$PAA_0(\mathbb{X},\rho):=\Big\{f\in BC(\mathbb{R},\mathbb{X}):\lim_{T\to\infty}\frac{1}{m(T,\rho)}\int_{-T}^{T}\|f(t)\|\rho(t)dt=0\Big\};$$

$$\begin{aligned}PAA_0(\mathbb{Y},\mathbb{X},\rho):=\Big\{&f\in C(\mathbb{R}\times\mathbb{Y},\mathbb{X}):f(\cdot,y)\text{ is bounded for each } y\in\mathbb{Y}\text{ and}\\ &\lim_{T\to\infty}\frac{1}{m(T,\rho)}\int_{-T}^{T}\|f(t,y)\|\rho(t)dt=0\text{ uniformly for } y\in\mathbb{Y}\Big\}.\end{aligned}$$

To study the delay case, we introduce spaces of functions defined for each $r>0$ by

$$\mathcal{W}(T,f,r,\rho)=\frac{1}{m(T,\rho)}\int_{-T}^{T}\Big(\sup_{\theta\in[t-r,t]}\|f(\theta)\|\Big)\rho(t)dt\,,$$

$$PAA_0(\mathbb{X},r,\rho):=\Big\{f\in BC(\mathbb{R},\mathbb{X}):\lim_{T\to\infty}\frac{1}{m(T,\rho)}\int_{-T}^{T}\Big(\sup_{\theta\in[t-r,t]}\|f(\theta)\|\Big)\rho(t)dt=0\Big\},$$

$$\begin{aligned}&PAA_0(\mathbb{Y},\mathbb{X},r,\rho)\\ &:=\Big\{f\in C(\mathbb{R}\times\mathbb{Y},\mathbb{X}):f(\cdot,y)\text{ is bounded for each } y\in\mathbb{Y}\text{ and}\\ &\lim_{T\to\infty}\frac{1}{m(T,\rho)}\int_{-T}^{T}\Big(\sup_{\theta\in[t-r,t]}\|f(\theta,y)\|\Big)\rho(t)dt=0\text{ uniformly for } y\in\mathbb{Y}\Big\}.\end{aligned}$$

Definition 2.4 ([5])**.** Let $\rho\in\mathbb{U}_\infty$. A function $f\in BC(\mathbb{R},\mathbb{X})$ (respectively, $f\in BC(\mathbb{R}\times\mathbb{Y},\mathbb{X})$) is called weighted pseudo almost automorphic if it can be expressed as $f=g+\phi$, where $g\in AA(\mathbb{X})$ (respectively, $AA(\mathbb{R}\times\mathbb{Y},\mathbb{X})$) and $\phi\in PAA_0(\mathbb{X},\rho)$ (respectively, $PAA_0(\mathbb{Y},\mathbb{X},\rho)$). We denote by $WPAA(\mathbb{X})$ (respectively, $WPAA(\mathbb{R}\times\mathbb{Y},\mathbb{X})$) the set of all such functions.

Definition 2.5 ([33])**.** Let $\rho\in\mathbb{U}_\infty$. A function $f\in BC(\mathbb{R},\mathbb{X})$ (respectively, $f\in BC(\mathbb{R}\times\mathbb{Y},\mathbb{X})$) is called weighted pseudo almost automorphic of class r if it can be expressed as $f=g+\phi$, where $g\in AA(\mathbb{X})$ (respectively, $AA(\mathbb{R}\times\mathbb{Y},\mathbb{X})$) and $\phi\in PAA_0(\mathbb{X},r,\rho)$ (respectively, $PAA_0(\mathbb{Y},\mathbb{X},r,\rho)$). We denote by $WPAA(\mathbb{X},r)$ (respectively, $WPAA(\mathbb{R}\times\mathbb{Y},\mathbb{X},r)$) the set of all such functions.

Definition 2.6 ([12, 24]). The Bochner transform $f^b(t,s), t\in\mathbb{R}, s\in[0,1]$, of a function $f:\mathbb{R}\to\mathbb{X}$ is defined by

$$f^b(t,s) := f(t+s).$$

Remark 2.7 ([12]). (i) A function $\varphi(t,s), t\in\mathbb{R}, s\in[0,1]$, is the Bochner transform of a certain function f, $\varphi(t,s)=f^b(t,s)$, if and only if $\varphi(t+\tau,s-\tau)=\varphi(s,t)$ for all $t\in\mathbb{R}, s\in[0,1]$ and $\tau\in[s-1,s]$.

(ii) Note that if $f=h+\varphi$, then $f^b=h^b+\varphi^b$. Moreover, $(\lambda f)^b=\lambda f^b$ for each scalar λ.

Definition 2.8 ([12]). The Bochner transform $f^b(t,s,u), t\in\mathbb{R}, s\in[0,1]$, $u\in\mathbb{X}$ of a function $f:\mathbb{R}\times\mathbb{X}\to\mathbb{X}$ is defined by

$$f^b(t,s,u) := f(t+s,u)\quad \text{for each } u\in\mathbb{X}.$$

We always denote by $\|\cdot\|_p$ the norm of space $L^p(0,1;\mathbb{X})$ for $p\in[1,\infty)$.

Definition 2.9 ([21, 24]). Let $p\in[1,\infty)$. The space $BS^p(\mathbb{X})$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions $f:\mathbb{R}\to\mathbb{X}$ such that $f^b\in L^\infty(\mathbb{R},L^p(0,1;\mathbb{X}))$. This is a Banach space with the norm

$$\|f\|_{S^p}=\|f^b\|_{L^\infty(\mathbb{R},L^p)}=\sup_{t\in\mathbb{R}}\Big(\int_t^{t+1}\|f(\tau)\|^p d\tau\Big)^{1/p}=\sup_{t\in\mathbb{R}}\|f(t+\cdot)\|_p.$$

Lemma 2.10 ([33]). *Let $\rho\in\mathbb{U}_\infty$. Suppose that $PAA_0(\mathbb{X},r,\rho)$ is translation invariant. Then the decomposition of weighted pseudo almost automorphic functions of class r is unique.*

Lemma 2.11 ([33]). *Let $\rho\in\mathbb{U}_\infty$ and $PAA_0(\mathbb{X},r,\rho)$ be translation invariant, then $WPAA(\mathbb{X},r)$ is a Banach space with norm $\|\cdot\|_\infty$.*

Definition 2.12 ([21, 24]). The space $AS^p(\mathbb{X})$ of Stepanov-like almost automorphic (or S^p-almost automorphic) functions consists of all $f\in BS^p(\mathbb{X})$ such that $f^b\in AA(L^p(0,1;\mathbb{X}))$. In other words, a function $f\in L^p_{\rm loc}(\mathbb{R},\mathbb{X})$ is said to be S^p-almost automorphic if its Bochner transform $f^b:\mathbb{R}\to L^p(0,1;\mathbb{X})$ is almost automorphic in the sense that for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exist a subsequence $\{s_n\}_{n\in\mathbb{N}}$ and a function $g\in L^p_{\rm loc}(\mathbb{R},\mathbb{X})$ such that

$$\lim_{n\to\infty}\Big(\int_t^{t+1}\|f(s+s_n)-g(s)\|^p ds\Big)^{1/p}=0,$$
$$\lim_{n\to\infty}\Big(\int_t^{t+1}\|g(s-s_n)-f(s)\|^p ds\Big)^{1/p}=0.$$

pointwise on $\mathbb{R}$.

Definition 2.13 ([21, 24]). A function $f:\mathbb{R}\times\mathbb{Y}\to\mathbb{X}$, $(t,u)\to f(t,u)$ with $f(\cdot,u)\in L^p_{\rm loc}(\mathbb{R},\mathbb{X})$ for each $u\in\mathbb{Y}$, is said to be S^p-almost automorphic in $t\in\mathbb{R}$ uniformly in $u\in\mathbb{Y}$ if $t\to f(t,u)$ is S^p-almost automorphic for each $u\in\mathbb{Y}$. That means, for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exist a subsequence $\{s_n\}_{n\in\mathbb{N}}$ and a function $g(\cdot,u)\in L^p_{\rm loc}(\mathbb{R},\mathbb{X})$ such that

$$\lim_{n\to\infty}\Big(\int_t^{t+1}\|f(s+s_n,u)-g(s,u)\|^p ds\Big)^{1/p}=0,$$

$$\lim_{n\to\infty}\left(\int_t^{t+1}\|g(s-s_n,u)-f(s,u)\|^p ds\right)^{1/p}=0,$$

pointwise on $\mathbb{R}$ and for each $u\in\mathbb{Y}$. We denote by $AS^p(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ the set of all such functions.

Definition 2.14 ([6])**.** Let $\rho\in\mathbb{U}_\infty$. A function $f\in BS^p(\mathbb{X})$ is said to be Stepanov-like weighted pseudo almost automorphic (or S^p-weighted pseudo almost automorphic) if it can be expressed as $f=g+\phi$, where $g\in AS^p(\mathbb{X})$ and $\phi^b\in PAA_0\left(L^p(0,1;\mathbb{X}),\rho\right)$. In other words, a function $f\in L^p_{\rm loc}(\mathbb{R},\mathbb{X})$ is said to be Stepanov-like weighted pseudo almost automorphic relatively to the weight $\rho\in\mathbb{U}_\infty$, if its Bochner transform $f^b:\mathbb{R}\to L^p(0,1;\mathbb{X})$ is weighted pseudo almost automorphic in the sense that there exist two functions $g,\phi:\mathbb{R}\to\mathbb{X}$ such that $f=g+\phi$, where $g\in AS^p(\mathbb{X})$ and $\phi^b\in PAA_0\left(L^p(0,1;\mathbb{X}),\rho\right)$. We denote by $WPAAS^p(\mathbb{X})$ the set of all such functions.

Definition 2.15 ([6])**.** Let $\rho\in\mathbb{U}_\infty$. A function $f:\mathbb{R}\times\mathbb{Y}\to\mathbb{X}$, $(t,u)\to f(t,u)$ with $f(\cdot,u)\in L^p_{\rm loc}(\mathbb{R},\mathbb{X})$ for each $u\in\mathbb{Y}$, is said to be Stepanov-like weighted pseudo almost automorphic (or S^p-weighted pseudo almost automorphic) if it can be expressed as $f=g+\phi$, where $g\in AS^p(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ and $\phi^b\in PAA_0\left(\mathbb{Y},L^p(0,1;\mathbb{X}),\rho\right)$. We denote by $WPAAS^p(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ the set of all such functions.

Definition 2.16 ([33])**.** Let $\rho\in\mathbb{U}_\infty$. A function $f\in BS^p(\mathbb{X})$ is said to be Stepanov-like weighted pseudo almost automorphic of class r (or S^p-weighted pseudo almost automorphic of class r) if it can be expressed as $f=g+\phi$, where $g\in AS^p(\mathbb{X})$ and $\phi^b\in PAA_0\left(L^p(0,1;\mathbb{X}),r,\rho\right)$. In other words, a function $f\in L^p_{\rm loc}(\mathbb{R},\mathbb{X},r)$ is said to be Stepanov-like weighted pseudo almost automorphic of class r relatively to the weight $\rho\in\mathbb{U}_\infty$, if its Bochner transform $f^b:\mathbb{R}\to L^p(0,1;\mathbb{X})$ is weighted pseudo almost automorphic of class r in the sense that there exist two functions $g,\phi:\mathbb{R}\to\mathbb{X}$ such that $f=g+\phi$, where $g\in AS^p(\mathbb{X})$ and $\phi^b\in PAA_0\left(L^p(0,1;\mathbb{X}),r,\rho\right)$. We denote by $WPAAS^p(\mathbb{X},r)$ the set of all such functions.

Definition 2.17 ([33])**.** Let $\rho\in\mathbb{U}_\infty$. A function $f:\mathbb{R}\times\mathbb{Y}\to\mathbb{X}$, $(t,u)\to f(t,u)$ with $f(\cdot,u)\in L^p_{\rm loc}(\mathbb{R},\mathbb{X})$ for each $u\in\mathbb{Y}$, is said to be Stepanov-like weighted pseudo almost automorphic of class r (or S^p-weighted pseudo almost automorphic of class r) if it can be expressed as $f=g+\phi$, where $g\in AS^p(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ and $\phi^b\in PAA_0\left(\mathbb{Y},L^p(0,1;\mathbb{X}),r,\rho\right)$. We denote by $WPAAS^p(\mathbb{R}\times\mathbb{Y},\mathbb{X},r)$ the set of all such functions.

Lemma 2.18 ([33])**.** *Let $\rho\in\mathbb{U}_\infty$. The space $WPAAS^p(\mathbb{X},r)$ equipped with the norm $\|\cdot\|_{S^p}$ is a Banach space.*

Concerning infinite delays, we introduce the following spaces of functions as in [33]:

$$\begin{gathered}
PAA_0(\mathbb{X},\infty,\rho):=\cap_{r>0}PAA_0(\mathbb{X},r,\rho),\\
PAA_0(\mathbb{X},\mathbb{Y},\infty,\rho):=\cap_{r>0}PAA_0(\mathbb{X},\mathbb{Y},r,\rho),\\
PAA_0(L^p(0,1;\mathbb{X}),\infty,\rho):=\cap_{r>0}PAA_0(L^p(0,1;\mathbb{X}),r,\rho),\\
PAA_0(\mathbb{Y},L^p(0,1;\mathbb{X}),\infty,\rho):=\cap_{r>0}PAA_0(\mathbb{Y},L^p(0,1;\mathbb{X}),r,\rho).
\end{gathered}$$

Obviously, $PAA_0(\mathbb{X},\infty,\rho)$ and $PAA_0(\mathbb{X},\mathbb{Y},\infty,\rho)$ are, respectively, closed subspaces of $PAA_0(\mathbb{X},r,\rho)$ and $PAA_0(\mathbb{X},\mathbb{Y},r,\rho)$, and hence both are Banach spaces. By the same way, $PAA_0(L^p(0,1;\mathbb{X}),\infty,\rho)$ and $PAA_0(\mathbb{Y},L^p(0,1;\mathbb{X}),\infty,\rho)$ are, respectively, closed subspaces of $PAA_0\left(L^p(0,1;\mathbb{X}),r,\rho\right)$ and $PAA_0(\mathbb{Y},L^p(0,1;\mathbb{X}),r,\rho)$, and thus both are Banach spaces.

Definition 2.19 ([33])**.** Let $\rho\in\mathbb{U}_\infty$. A function $f\in BC(\mathbb{R},\mathbb{X})$ (respectively, $f\in BC(\mathbb{R}\times\mathbb{Y},\mathbb{X})$) is called weighted pseudo almost automorphic of class infinity if it can be expressed as $f=g+\phi$, where $g\in AA(\mathbb{X})$ (respectively, $AA(\mathbb{R}\times\mathbb{Y},\mathbb{X})$) and $\phi\in PAA_0(\mathbb{X},\infty,\rho)$ (respectively, $PAA_0(\mathbb{Y},\mathbb{X},\infty,\rho)$). We denote by $WPAA(\mathbb{X},\infty)$ (respectively, $WPAA(\mathbb{R}\times\mathbb{Y},\mathbb{X},\infty)$) the set of all such functions.

Definition 2.20 ([33])**.** Let $\rho\in\mathbb{U}_\infty$. A function $f\in BS^p(\mathbb{X})$ is said to be Stepanov-like weighted pseudo almost automorphic of class infinity (or S^p-weighted pseudo almost automorphic of class infinity) if it can be expressed as $f=g+\phi$, where $g\in AS^p(\mathbb{X})$ and $\phi^b\in PAA_0\left(L^p(0,1;\mathbb{X}),\infty,\rho\right)$. In other words, a function $f\in L^p_{\rm loc}(\mathbb{R},\mathbb{X})$ is said to be Stepanov-like weighted pseudo almost automorphic of class infinity relatively to the weight $\rho\in\mathbb{U}_\infty$, if its Bochner transform $f^b:\mathbb{R}\to L^p(0,1;\mathbb{X})$ is weighted pseudo almost automorphic of class infinity in the sense that there exist two functions $g,\phi:\mathbb{R}\to\mathbb{X}$ such that $f=g+\phi$, where $g\in AS^p(\mathbb{X})$ and $\phi^b\in PAA_0\left(L^p(0,1;\mathbb{X}),\infty,\rho\right)$. We denote by $WPAAS^p(\mathbb{X},\infty)$ the set of all such functions.

Definition 2.21 ([33])**.** Let $\rho\in\mathbb{U}_\infty$. A function $f:\mathbb{R}\times\mathbb{Y}\to\mathbb{X}$, $(t,u)\to f(t,u)$ with $f(\cdot,u)\in L^p_{\rm loc}(\mathbb{R},\mathbb{X})$ for each $u\in\mathbb{Y}$, is said to be Stepanov-like weighted pseudo almost automorphic of class infinity (or S^p-weighted pseudo almost automorphic of class infinity) if it can be expressed as $f=g+\phi$, where $g\in AS^p(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ and $\phi^b\in PAA_0\left(\mathbb{Y},L^p(0,1;\mathbb{X}),\infty,\rho\right)$. We denote by $WPAAS^p(\mathbb{R}\times\mathbb{Y},\mathbb{X},\infty)$ the set of all such functions.

Using similar ideas to those in [16, Lemma2.7], we can easily show the following results.

Lemma 2.22. *(i) Assume $PAA_0(L^p(0,1,\mathbb{X}),r,\rho)$ is translation invariant, then the decomposition of an S^p-weighted pseudo almost automorphic function of class r is unique.*

(ii) The space $WPAAS^p(\mathbb{X},r)$ equipped with $\|\cdot\|_{S^p}$ is a Banach space.

(iii) $WPAA(\mathbb{X},r)$ is continuously embedded in $WPAAS^p(\mathbb{X},r)$.

Lemma 2.23. *(i) Assume that $PAA_0(L^p(0,1,\mathbb{X}),\infty,\rho)$ is translation invariant, then the decomposition of an S^p-weighted pseudo almost automorphic function of class infinity is unique.*

(ii) The space $WPAAS^p(\mathbb{X},\infty)$ equipped with $\|\cdot\|_{S^p}$ is a Banach space.

(iii) $WPAA(\mathbb{X},\infty)$ is continuously embedded in $WPAAS^p(\mathbb{X},\infty)$.

In this work we use an axiomatic definition of the phase space $\mathfrak{B}$, which is similar to the one introduce in ([20]). $\mathfrak{B}$ is a vector space of functions mapping $(-\infty,0]$ into $\mathbb{X}$ endowed with a seminorm $\|\cdot\|_{\mathfrak{B}}$ such that the next axioms hold by ([33]).

- (A1) If $x:(-\infty,\sigma+a)\mapsto\mathbb{X}$, $a>0$, $\sigma\in\mathbb{R}$, is continuous on $[\sigma,\sigma+a)$ and $x_\sigma\in\mathfrak{B}$, then for every $t\in[\sigma,\sigma+a)$ the following hold:
  - (i) x_t is in $\mathfrak{B}$;
  - (ii) $\|x(t)\|\le H\|x_t\|_{\mathfrak{B}}$;
  - (iii) $\|x_t\|_{\mathfrak{B}}\le\widetilde{K}(t-\sigma)\sup\{\|x(s)\|:\sigma\le s\le t\}+\widetilde{M}(t-\sigma)\|x_\sigma\|_{\mathfrak{B}}$, where $H>0$ is a constant; $\widetilde{K},\widetilde{M}:[0,\infty)\mapsto[1,\infty)$, $\widetilde{K}$ is continuous, $\widetilde{M}$ is locally bounded and $H,\widetilde{K},\widetilde{M}$ are independent of $x(\cdot)$.

(A1') For the function $x(\cdot)$ appearing in (A1), the function $t \to x_t$ is continuous from $[\sigma, \sigma + a)$ into $\mathfrak{B}$.

(A2) The space $\mathfrak{B}$ is complete.

(A3) If $(\varphi^n)_{n\in\mathbb{N}}$ is a bounded sequence in $BC((-\infty, 0], \mathbb{X})$ given by functions with compact support and $\varphi^n \to \varphi$ in the compact-open topology, then $\varphi \in \mathfrak{B}$ and $\|\varphi^n - \varphi\|_{\mathfrak{B}} \to 0$ as $n \to \infty$.

Definition 2.24. [18] Let $\mathfrak{B}_0 = \{\varphi \in \mathfrak{B} : \varphi(0) = 0\}$ and $S(t) : \mathfrak{B} \to \mathfrak{B}$ be the C_0-semigroup defined by $S(t)\varphi(\theta) = \varphi(0)$ on $[-t, 0]$ and $S(t)\varphi(\theta) = \varphi(t + \theta)$ on $(-\infty, -t]$. The phase space $\mathfrak{B}$ is called a fading memory space if $\|S(t)\varphi\|_{\mathfrak{B}} \to 0$ as $t \to \infty$ for every $\varphi \in \mathfrak{B}_0$. We said that $\mathfrak{B}$ is a uniform fading memory space if $\|S(t)\|_{\mathfrak{L}(\mathfrak{B}_0)} \to 0$ as $t \to \infty$.

Remark 2.25 ([18])**.** In this article we assume $\varsigma > 0$ and $\|\varphi\|_{\mathfrak{B}} \leq \varsigma \sup_{\theta\leq 0} \|\varphi(\theta)\|$ for each $\varphi \in \mathfrak{B} \cap BC((-\infty, 0], \mathbb{X})$, see [20] for details. Moreover, if $\mathfrak{B}$ is a fading memory, we assume that $\max\{\widetilde{K}(t), \widetilde{M}(t)\} \leq \Re$ for all $t \geq 0$, see[20].

Lemma 2.26 ([20])**.** *The phase $\mathfrak{B}$ is a uniform fading memory space if, and only if, axiom* (A3) *holds, the function $\widetilde{K}$ is bounded and* $\lim_{t\to\infty} \widetilde{M}(t) = 0$.

3. Results on composition theorems

The aim of this section is to establish some new results on composition of Stepanov-like weighted pseudo almost automorphic functions of class infinity. We first list the following "uniform continuity condition" with respect to the L^p-norm for a function $h : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ with $h(\cdot, u) \in L^p_{Loc}(\mathbb{R}, \mathbb{X})$ for each $u \in \mathbb{X}$, which was initially adopted in [16]:

(A4) For any $\varepsilon > 0$, there exists $\sigma > 0$ such that $x, y \in L^p(0, 1, \mathbb{X})$ and $\|x-y\|_p < \sigma$ imply that

$$\|h(t + \cdot, x(\cdot)) - h(t + \cdot, y(\cdot))\|_p < \varepsilon, \quad t \in \mathbb{R}.$$

In the sequel, we say that a function ψ satisfies (A4) if ψ replaces h in (A4).

Let $f \in AS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$, then for a sequence $\{s_n\} \subset \mathbb{R}$, there exist a subsequence $\{\tau_n\}$ and a function $g : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ with $g(\cdot, x) \in L^p_{\rm loc}(\mathbb{R}, \mathbb{X})$, $x \in \mathbb{X}$ such that for each $t \in \mathbb{R}$,

$$\lim_{n\to\infty} \|f(t + \tau_n + \cdot, x) - g(t + \cdot, x)\|_p = \lim_{n\to\infty} \|g(t - \tau_n + \cdot, x) - f(t + \cdot, x)\|_p = 0.$$

(A5) $f \in AS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ satisfies (A4), and for a sequence $\{s_n\} \subset \mathbb{R}$, there exist a subsequence $\{\tau_n\}$ and a function g given above such that g satisfies (A4).

Lemma 3.1 ([16])**.** *Let h be the function in* (A4)*, and x: $\mathbb{R} \to \mathbb{X}$ with $\overline{x(\mathbb{R})}$ compact. For $\varepsilon > 0$, there exist a finite set $\{x_k\}_{k=1}^m \subset \overline{x(\mathbb{R})}$ such that*

$$\|h(t + \cdot, x(t + \cdot))\|_p < \varepsilon + m \sup_{1\leq k\leq m} \|h(t + \cdot, x_k)\|_p, \quad t \in \mathbb{R}.$$

Lemma 3.2 ([33])**.** *Let $\rho \in U_\infty$ and $f \in BS^p(\mathbb{X})$, then $f^b \in PAA_0(L^p(0, 1, \mathbb{X}), r, \rho)$ if and only if for any $\varepsilon > 0$,*

$$\lim_{T\to\infty} \frac{1}{m(T, \rho)} \int_{M_{T,\varepsilon(f)}} \rho(t)dt = 0,$$

where

$$M_{T,\varepsilon(f)} = \big\{t \in [-T,T] : \sup_{\theta\in[t-r,t]} \Big(\int_\theta^{\theta+1} \|f(s)\|^p ds\Big)^{1/p} \geq \varepsilon\big\}.$$

Lemma 3.3 ([16]). *Assume that f satisfies* (A5) *and $x \in AS^p(\mathbb{X})$ with $\overline{x}(\mathbb{R})$ compact. Then $f(\cdot, x(\cdot)) \in AS^p(\mathbb{X})$.*

Next, we give results in the compositions of S^p-weighted pseudo almost automorphic functions of class r.

Theorem 3.4. *Let $\rho \in U_\infty$, $f = g + \phi \in WPAAS^p(\mathbb{R}\times\mathbb{X}, \mathbb{X}, r)$, $u = u_1 + u_2 \in WPAAS^p(\mathbb{X}, r)$, $g \in AS^p(\mathbb{R}\times\mathbb{X}, \mathbb{X})$, $\phi^b \in PAA_0(\mathbb{X}, L^p(0,1,\mathbb{X}), r, \rho)$, $u_1 \in AS^p(\mathbb{X}), u_2^b \in PAA_0(L^p(0,1,\mathbb{X}), r, \rho)$, $Q = \overline{\{u_1(t) : t \in \mathbb{R}\}}$ compact and there exist a continuous function $\mathfrak{L}_f(\cdot) : \mathbb{R} \to [0,\infty)$ satisfying*

$$\Big(\int_t^{t+1} \|f(s,x_1) - f(s,x_2)\|^p ds\Big)^{1/p} \leq \mathfrak{L}_f(t)\|x_1 - x_2\|. \tag{3.1}$$

If $\xi^b \in PAA_0(\mathbb{R}, L^p(0,1,\mathbb{X}), \rho)$, then

$$\lim_{T\to\infty}\sup \frac{1}{m(T,\rho)} \int_{-T}^T \Big(\sup_{\theta\in[t-r,t]} \mathfrak{L}_f(\theta+\cdot)\Big)\rho(t)dt < \infty, \tag{3.2}$$

$$\lim_{T\to\infty} \frac{1}{m(T,\rho)} \int_{-T}^T \Big(\sup_{\theta\in[t-r,t]} \mathfrak{L}_f(\theta+\cdot)\Big)\xi(t)\rho(t)dt < \infty\,. \tag{3.3}$$

If

(i) *$g(t,x)$ satisfies* (A5) *and*
(ii) *ϕ satisfies (A4),*

Then $f(\cdot, u(\cdot)) \in WPAAS^p(\mathbb{X}, r)$.

Proof. Let $G(t) = g\,(t, u_1(t))$, $H(t) = f\,(t, u(t)) - f\,(t, u_1(t))$, $\Lambda(t) = \phi\,(t, u_1(t))$, $t \in \mathbb{R}$. Then

$$f(t,u(t)) = g\,(t,u_1(t)) + f\,(t,u(t)) - f\,(t,u_1(t)) + \phi\,(t,u_1(t)) = G(t) + H(t) + \Lambda(t).$$

We have $G(t) \in AS^p(\mathbb{X})$ by Lemma 3.3, then it remains to show that H^b, Λ^b is in $PAA_0(L^p(0,1,\mathbb{X}), r, \rho)$.

Indeed, for $T > 0$, using (3.1), we see that

$$\begin{aligned}
&\frac{1}{m(T,\rho)} \int_{-T}^T \Big(\sup_{\theta\in[t-r,t]} \Big(\int_\theta^{\theta+1} \|H(s)\|^p ds\Big)^{1/p}\Big)\rho(t)dt \\
&= \frac{1}{m(T,\rho)} \int_{-T}^T \Big(\sup_{\theta\in[t-r,t]} \Big(\int_\theta^{\theta+1} \|f(t,u(t)) - f(t.u_1(t))\|^p ds\Big)^{1/p}\Big)\rho(t)dt \\
&\leq \frac{1}{m(T,\rho)} \int_{-T}^T \Big(\sup_{\theta\in[t-r,t]} \mathfrak{L}_f(\theta)\|u_2(\theta)\|\Big)\rho(t)dt \\
&\leq \frac{1}{m(T,\rho)} \int_{-T}^T \Big(\sup_{\theta\in[t-r,t]} \mathfrak{L}_f(\theta)\Big)\Big(\sup_{\theta\in[t-r,t]} \Big(\int_\theta^{\theta+1} \|u_2(s)\|^p ds\Big)^{1/p}\Big)\rho(t)dt \\
&= \frac{1}{m(T,\rho)} \int_{-T}^T \Big(\sup_{\theta\in[t-r,t]} \mathfrak{L}_f(\theta)\Big)\Big(\sup_{\theta\in[t-r,t]} u_2(\theta+\cdot)\Big)\rho(t)dt.
\end{aligned}$$

This implies that $H^b(t) \in PAA_0(L^p(0,1,\mathbb{X}), r, \rho)$ by (3.3).

Next, we prove $\Lambda^b \in PAA_0(L^p(0,1,\mathbb{X}),r,\rho)$. For $\varepsilon > 0$, let σ be given by (A4) with ϕ in the place of h, by Lemma 3.1, there is a finite set $\{x_k\}_{k=1}^m \subset \overline{\{u_1(t), t \in \mathbb{R}\}}$ such that for $t \in \mathbb{R}$,

$$\|\phi(t+\cdot, u_1(t+\cdot))\|_p < \varepsilon + m \sup_{1\le k\le m} \|\phi(t+\cdot, x_k)\|_p.$$

Since $\phi^b \in PAA_0(\mathbb{X}, L^p(0,1,\mathbb{X}), r, \rho)$, for each $x \in \mathbb{X}$, there is $T > T_0$, $1 \le k \le m$,

$$\mathcal{W}(T, \phi^b(\cdot, x_k), r, \rho) = \frac{1}{m(T,\rho)} \int_{-T}^{T} \Big(\sup_{\theta\in[t-r,t]} \|\phi(\theta+\cdot, x_k)\|_p \Big) \rho(t) dt < \frac{\varepsilon}{m}.$$

Then for $T > T_0$,

$$\begin{aligned}
\mathcal{W}(T, \Lambda^b, r, \rho) &= \mathcal{W}(T, \Lambda^b(\cdot, u_1(\cdot)), r, \rho) \\
&= \frac{1}{m(T,\rho)} \int_{-T}^{T} \Big(\sup_{\theta\in[t-r,t]} \|\phi(\theta+\cdot, u_1(t+\cdot))\|_p \Big) \rho(t) dt \\
&\le \varepsilon + m \sup_{1\le k\le m} \frac{1}{m(T,\rho)} \int_{-T}^{T} \Big(\sup_{\theta\in[t-r,t]} \|\phi(\theta+\cdot, x_k)\|_p \Big) \rho(t) dt \\
&= \varepsilon + m \mathcal{W}(T, \phi^b(\cdot, x_k), r, \rho) \\
&= \varepsilon + m\frac{\varepsilon}{m} = 2\varepsilon.
\end{aligned}$$

This yields $\lim_{T\to\infty} \mathcal{W}(T, \Lambda^b, r, \rho) = 0$. That is $\Lambda^b \in PAA_0(L^p(0,1,\mathbb{X}), r, \rho)$. The proof is complete. $\square$

Theorem 3.5. Let $\rho \in U_\infty$, $F = F_1 + F_2 \in WPAAS^p(\mathbb{R}\times\mathbb{X}, \mathbb{X}, r)$, $\phi = \phi_1 + \phi_2 \in WPAAS^p(\mathbb{X}, r)$ with $Q = \overline{\{\phi_1(t) : t \in \mathbb{R}\}}$ compact, $F_1 \in AS^p(\mathbb{R}\times\mathbb{X}, \mathbb{X})$, $F_2^b \in PAA_0(\mathbb{X}, L^p(0,1,\mathbb{X}), r, \rho)$, $\phi_1 \in AS^p(\mathbb{X})$, $\phi_2^b \in PAA_0(L^p(0,1,\mathbb{X}), r, \rho)$. Assume that F_1 satisfies (A5), F_2 satisfies (A4) and $\{F(\cdot, z) : z \in \mathbb{J}\}$ is bounded in $WPAAS^p(\mathbb{X}, r)$ for any bounded $\mathbb{J} \subset \mathbb{X}$, then $t \to F(t, \phi(\cdot)) \in WPAAS^p(\mathbb{X}, r)$.

Proof. Let $\Upsilon(t) = F_1(t, \phi_1(t))$, $\Psi(t) = F(t, \phi(t)) - F(t, \phi_1(t))$, $\Phi(t) = F_2(t, \phi_1(t))$, $t \in \mathbb{R}$. Then

$$F(t,\phi(t)) = F_1(t,\phi_1(t)) + F(t,\phi(t)) - F(t,\phi_1(t)) + F_2(t,\phi_1(t)) = \Upsilon(t) + \Psi(t) + \Phi(t).$$

We have $\Upsilon(t) \in AS^p(\mathbb{X})$ by Lemma 3.3, so we need only to prove $\Psi^b, \Phi^b \in PAA_0(L^p(0,1,\mathbb{X}), r, \rho)$.

It is easy to see that $\Psi \in BS^p(\mathbb{X})$ since ϕ and ϕ_1 are bounded and $\{F(\cdot, z) : z \in \mathbb{J}\}$ is bounded in $WPAAS^p(\mathbb{X}, r)$ for any bounded $\mathbb{J} \subset \mathbb{X}$. Noticing that F satisfies (A4) since F_1 and F_2 satisfies (A4), for $\varepsilon > 0$, let $\sigma > 0$ be given by (A4), then

$$\|\Psi(t+\cdot)\|_p = \|F(t+\cdot, \phi(t+\cdot)) - F(t+\cdot, \phi_1(t+\cdot))\|_p < \varepsilon,$$

for $\|\phi_2(t+\cdot)\|_p < \sigma$, where $\phi_2(s) = \phi(s) - \phi_1(s)$. Hence, for each $t \in \mathbb{R}$, $\|\phi_2(s)\| < \sigma$, $s \in [t, t+1]$ implies that

$$\begin{aligned}
\Big(\int_t^{t+1} \|\Psi(s)\|^p ds \Big)^{1/p} &= \Big(\int_t^{t+1} \|F(s,\phi(s)) - F(s,\phi_1(s))\|^p ds \Big)^{1/p} \\
&= \|F(t+\cdot, \phi(t+\cdot)) - F(t+\cdot, \phi_1(t+\cdot))\|_p < \varepsilon.
\end{aligned}$$

We can obtain

$$\sup_{\theta\in[t-r,t]} \Big(\int_\theta^{\theta+1} \|\Psi(s)\|^p ds \Big)^{1/p}$$

$$= \sup_{\theta\in[t-r,t]} \Big(\int_\theta^{\theta+1} \|F(s,\phi(s)) - F(s,\phi_1(s))\|^p ds\Big)^{1/p} < \varepsilon.$$

Let

$$M_{T,\sigma(\phi_2)} = \Big\{t \in [-T,T] : \sup_{\theta\in[t-r,t]} \Big(\int_\theta^{\theta+1} \|\phi_2(s)\|^p ds\Big)^{1/p} \geq \sigma\Big\}.$$

So we obtain

$$M_{T,\varepsilon(\phi_2)} = M_{T,\varepsilon(F(\cdot,\phi(\cdot))-F(\cdot,\phi_1(\cdot)))} \subseteq M_{T,\sigma(\phi_2)}.$$

Since $\phi_2^b \in PAA_0(L^p(0,1,\mathbb{X}), r, \rho)$ by Lemma 3.2, we obtain

$$\lim_{T\to\infty} \int_{M_{T,\sigma(\phi_2)}} \rho(t)dt = 0.$$

Thus

$$\lim_{T\to\infty} \int_{M_{T,\varepsilon(\Psi)}} \rho(t)dt = 0.$$

This shows that $\Psi^b \in PAA_0(L^p(0,1,\mathbb{X}), r, \rho)$.

For $\varepsilon > 0$, let σ be given by (A4) with F_2 in the place of h, by Lemma 3.1, there is a finite set $\{x_k\}_{k=1}^m \subset \overline{\{\phi_1(t) : t \in \mathbb{R}\}}$ such that for $t \in \mathbb{R}$

$$\|F_2(t+\cdot,\phi_1(t+\cdot))\|_p < \varepsilon + m \sup_{1\leq k\leq m} \|F_2(t+\cdot,x_k)\|_p.$$

Since $F_2^b \in PAA_0(\mathbb{X}, L^p(0,1,\mathbb{X}), r, \rho)$ for each $x \in \mathbb{X}$, there is $T > T_0$, $1 \leq k \leq m$,

$$\mathcal{W}(T, F_2^b(\cdot,x_k), r, \rho) = \frac{1}{m(T,\rho)} \int_{-T}^T \Big(\sup_{\theta\in[t-r,t]} \|F_2(\theta+\cdot,x_k)\|_p\Big)\rho(t)dt < \frac{\varepsilon}{m}.$$

Then for $T > T_0$,

$$\begin{aligned}
\mathcal{W}(T,\Phi^b,r,\rho) &= \mathcal{W}(T, F_2^b(\cdot,\phi_1(\cdot)), r, \rho) \\
&= \frac{1}{m(T,\rho)} \int_{-T}^T \Big(\sup_{\theta\in[t-r,t]} \|F_2(\theta+\cdot,\phi_1(t+\cdot))\|_p\Big)\rho(t)dt \\
&\leq \varepsilon + m \sup_{1\leq k\leq m} \frac{1}{m(T,\rho)} \int_{-T}^T \Big(\sup_{\theta\in[t-r,t]} \|\phi(\theta+\cdot,x_k)\|_p\Big)\rho(t)dt \\
&= \varepsilon + m\mathcal{W}(T, F_2^b(\cdot,x_k), r, \rho) \\
&= \varepsilon + m\frac{\varepsilon}{m} = 2\varepsilon.
\end{aligned}$$

This implies $\lim_{T\to\infty} \mathcal{W}(T,\Phi^b,r,\rho) = 0$. That is $\Phi^b \in PAA_0(L^p(0,1,\mathbb{X}), r, \rho)$. The proof is complete. □

Lemma 3.6 ([33]). *Let $u \in WPAA(\mathbb{X},\infty)$ where $\rho \in \mathbb{U}_\infty$. Assume that $\mathfrak{B}$ is a uniform fading memory space. Then the function $t \to u_t$ belongs to $WPAA(\mathfrak{B},\infty)$.*

Lemma 3.7 ([33]). *Let $\rho \in \mathbb{U}_\infty$, $u \in WPAAS^p(\mathbb{X},\infty)$ and assume that $\mathfrak{B}$ is a uniform fading memory space. Then the function $t \to u_t$ belongs to $WPAAS^p(\mathfrak{B},\infty)$.*

One of the consequence of Lemma 3.7 is the following modified version of Theorems 3.4 and 3.5.

Corollary 3.8. *Let $\rho \in U_\infty$, $f \in WPAAS^p(\mathbb{R}\times\mathbb{X},\mathbb{X},\infty)$ and $u \in WPAAS^p(\mathbb{X},\infty)$. Assume that the condition of (i) and (ii) in the Theorem 3.4 is satisfied and there exists a continuous function $\mathfrak{L}(\cdot) : \mathbb{R} \to [0,\infty)$, such that* (3.1) *holds. If conditions* (3.2) *and* (3.3) *hold for every $r > 0$, then the function $f(t,u(t)) \in WPAAS^p(\mathbb{X},\infty)$.*

Corollary 3.9. *Let $\rho \in U_\infty$, $F = F_1 + F_2 \in WPAAS^p(\mathbb{R}\times\mathbb{X},\mathbb{X},\infty)$, $\phi = \phi_1 + \phi_2 \in WPAAS^p(\mathbb{X},\infty)$ with $Q = \overline{\{\phi_1(t) : t \in \mathbb{R}\}}$ compact, $F_1 \in AS^p(\mathbb{R}\times\mathbb{X},\mathbb{X})$, $F_2^b \in PAA_0(\mathbb{X}, L^p(0,1,\mathbb{X}),\infty,\rho)$, $\phi_1 \in AS^p(\mathbb{X})$, $\phi_2^b \in PAA_0(L^p(0,1,\mathbb{X}),\infty,\rho)$. Assume that F_1 satisfies* (A5)*, F_2 satisfies* (A4) *and $\{F(\cdot,z) : z \in \mathbb{J}\}$ is bounded in $WPAAS^p(\mathbb{X},\infty)$ for any bounded $\mathbb{J} \subset \mathbb{X}$, then $t \to F(t,\phi(\cdot)) \in WPAAS^p(\mathbb{X},\infty)$.*

By Lemma 2.22 (iii) and Theorem 3.5, Lemma 2.23 (iii) and Corollary 3.9, we have the following corollaries:

Corollary 3.10. *Let $\rho \in U_\infty$, $F = F_1 + F_2 \in WPAAS^p(\mathbb{R}\times\mathbb{X},\mathbb{X},r)$ and $\phi \in WPAA(\mathbb{X},r)$. Assume that F_1 satisfies* (A5)*, F_2 satisfies* (A4) *and $\{F(\cdot,z) : z \in \mathbb{J}\}$ is bounded in $WPAAS^p(\mathbb{X},r)$ for any bounded $\mathbb{J} \subset \mathbb{X}$, then $t \to F(t,\phi(\cdot)) \in WPAAS^p(\mathbb{X},r)$.*

Corollary 3.11. *Let $\rho \in U_\infty$, $F = F_1 + F_2 \in WPAAS^p(\mathbb{R}\times\mathbb{X},\mathbb{X},\infty)$ and $\phi \in WPAA(\mathbb{X},\infty)$. Assume that F_1 satisfies* (A5)*, F_2 satisfies* (A4) *and $\{F(\cdot,z) : z \in \mathbb{J}\}$ is bounded in $WPAAS^p(\mathbb{X},\infty)$ for any bounded $\mathbb{J} \subset \mathbb{X}$, then $t \to F(t,\phi(\cdot)) \in WPAAS^p(\mathbb{X},\infty)$.*

Corollary 3.12 ([33])**.** *Let $\rho \in U_\infty$, $f \in WPAA(\mathbb{R}\times\mathbb{X},\mathbb{X},\infty)$ and $u \in WPAA(\mathbb{X},\infty)$. Assume that the following conditions are satisfied*

(i) *There exist a constant $L > 0$ such that $\|f(t,x) - f(t,y)\| \le L\|x-y\|$ for all $x,y \in \mathbb{X}$ and $t \in \mathbb{R}$.*
(ii) *$g(t,x)$ is uniformly continuous in any bounded subset $K' \subset \mathbb{X}$ uniformly for $t \in \mathbb{R}$.*

Then the function $t \to f(t,u(t))$ belongs to $WPAA(\mathbb{X},\infty)$.

4. Weighted pseudo almost automorphic mild solution

In this section, we study weighted pseudo almost automorphic mild solutions to the neutral equation (1.1). We list the following basic assumptions:

(A6) The system
$$u'(t) = A(t)u(t), \quad t \ge s, \quad u(s) = \phi \in \mathbb{X}.$$
has an associated evolution family of operators $\{U(t,s) : t \ge s$ with $t,s \in \mathbb{R}\}$. Further, we assume that the domains of operators $A(t)$ are constant in t, that is, $D(A(t)) = D = \mathcal{Y}$ for all $t \in \mathbb{R}$ and that the evolution family $U(t,s)$ is asymptotically stable in the sense that there exist some constants $M, \delta > 0$ such that
$$\|U(t,s)\| \le Me^{-\delta(t-s)}$$
for all $t,s \in \mathbb{R}$ with $t \ge s$.

(A7) The function $s \to A(s)U(t,s)$ defined from $(-\infty,t)$ into $\mathfrak{L}(\mathbb{R}\times\mathcal{Y})$ is strongly measurable and there exist a nonincreasing function $H : [0,\infty) \to [0,\infty)$ and $\delta > 0$ with $e^{-\delta s}H(s) \in L^1([0,\infty))$ such that
$$\|A(s)U(t,s)\|_{\mathfrak{L}(\mathcal{Y},\mathbb{X})} \le e^{-\delta s}H(t-s), \quad t > s.$$

(A8) $g \in WPAAS^p(\mathbb{R}, \mathbb{X}, \infty)$ and there exist a positive constant L_g such that for $\psi_i \in \mathfrak{B}, i = 1, 2$, $\|g(t, \psi_1) - g(t, \psi_2)\|_p \le L_g \|\psi_1 - \psi_2\|_{\mathfrak{B}}$.

(A9) $f \in WPAA(\mathbb{R}, \mathbb{X}, \infty)$ and there exist a positive constant L_f such that for $\psi_i \in \mathfrak{B}, i = 1, 2$, $\|f(t, \psi_1) - f(t, \psi_2)\| \le L_f \|\psi_1 - \psi_2\|_{\mathfrak{B}}$.

(A10) The series

$$\sum_{k=1}^{\infty} \Big(\int_{t-k}^{t-k+1} e^{-\delta q(t-s)} H^q(t-s) ds \Big)^{1/q}$$

converges, $q > 1$, and let $K = \Big(\int_{-\infty}^{t} e^{-\delta q(t-s)} H^q(t-s) ds \Big)^{1/q}$.

Let $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Denote

$$\alpha_0 = M \Big(\frac{e^{q\delta - 1}}{q\delta} \Big)^{1/q}, \quad \alpha = \alpha_0 \sum_{k=1}^{\infty} e^{-\delta k}.$$

(A11) The function $\mathbb{R} \times \mathbb{R} \mapsto \mathbb{X}$, $(t, s) \mapsto U(t, s)x \in bAA(\mathbb{R} \times \mathbb{R}, \mathcal{Y})$ uniformly for $x \in \mathbb{X}$.

(A12) The function $\mathbb{R} \times \mathbb{R} \mapsto \mathbb{X}$, $(t, s) \mapsto A(s)U(t, s)x \in bAA(\mathbb{R} \times \mathbb{R}, \mathbb{X})$ uniformly for $x \in \mathcal{Y}$.

Definition 4.1 ([9]). A continuous function $u : [\sigma, \sigma + a) \to \mathbb{X}, a > 0$, is a mild solution of neutral system (1.1) on $[\sigma, \sigma + a)$, if the function $s \to A(s)U(t, s)f(s, u_s)$ is integrable on $[\sigma, t)$ for every $\sigma < t < \sigma + a$, $u_\sigma = \varphi$, and

$$\begin{aligned} u(t) &= U(t, \sigma)(\varphi(0) + f(\sigma, \varphi)) - f(t, u_t) - \int_{\sigma}^{t} A(s)U(t, s)f(s, u_s) ds \\ &\quad + \int_{\sigma}^{t} U(t, s)g(s, u_s) ds, \quad t \in [\sigma, \sigma + a). \end{aligned}$$

Under assumptions (A6) and (A7), it can be easily shown that the function

$$u(t) = -f(t, u_t) + \int_{-\infty}^{t} U(t, s)g(s, u_s) ds - \int_{-\infty}^{t} A(s)U(t, s)f(s, u_s) ds,$$

for each $t \in \mathbb{R}$ is a mild solution of (1.1).

Lemma 4.2 ([33]). *Assume that conditions* (A6), (A7), (A11) *hold. Let $\rho \in \mathbb{U}_\infty$ and $u \in WPAAS^p(\mathbb{X}, \infty)$, and*

$$v(t) := \int_{-\infty}^{t} U(t, s)u(s) ds, \quad t \in \mathbb{R}.$$

Then $v \in WPAA(\mathbb{X}, \infty)$.

Lemma 4.3. *Assume that conditions* (A6), (A7), (A12) *hold. Let $\rho \in \mathbb{U}_\infty$ and $u \in WPAAS^p(\mathbb{X}, \infty)$, and*

$$v(t) := \int_{-\infty}^{t} A(s)U(t, s)u(s) ds, \quad t \in \mathbb{R}.$$

Then $v \in WPAA(\mathbb{X}, \infty)$.

Proof. Since $u \in WPAAS^p(\mathbb{X}, \infty)$, there exist functions $\overline{\psi}$ in $AS^p(\mathbb{X})$ and $\overline{\omega}^b$ in $PAA_0(L^p(0,1;\mathbb{X}), \infty, \rho)$ such that $u = \overline{\psi} + \overline{\omega}$.

$$v(t) := \int_{-\infty}^{t} A(s)U(t,s)\overline{\psi}(s)ds + \int_{-\infty}^{t} A(s)U(t,s)\overline{\omega}(s)ds = \overline{x}(t) + \overline{y}(t).$$

To prove that v is a weighted pseudo almost automorphic function of class infinity, we only need to verify $\overline{x}(t) \in AA(\mathbb{X})$ and $\overline{y}(t) \in PAA_0(\mathbb{X}, \infty, \rho)$.

First, let us prove that $\overline{x}(t) \in AA(\mathbb{X})$. We consider for each $n = 1, 2, \cdots$, the integrals

$$\overline{x}_n(t) = \int_{t-n}^{t-n+1} A(\sigma)U(t,\sigma)\overline{\psi}(\sigma)d\sigma.$$

Now, let us show that each $\overline{x}_n(t) \in AA(\mathbb{X})$. Using the Hölder inequality and the exponential dissipation property of the evolution family $U(t,s)$, it follows that

$$\begin{aligned}
\|\overline{x}_n(t)\| &\le \int_{t-n}^{t-n+1} \|A(\sigma)U(t,\sigma)\overline{\psi}(\sigma)\| d\sigma \\
&\le \int_{t-n}^{t-n+1} \|A(\sigma)U(t,\sigma)\| \|\overline{\psi}(\sigma)\| d\sigma \\
&\le \int_{t-n}^{t-n+1} H(t-\sigma)e^{-\delta(t-\sigma)} \|\overline{\psi}(\sigma)\| d\sigma \\
&\le \Big(\int_{t-n}^{t-n+1} e^{-\delta q(t-\sigma)} H^q(t-\sigma)d\sigma \Big)^{1/q} \Big(\int_{t-n}^{t-n+1} \|\overline{\psi}(\sigma)\|^p \Big)^{1/p} d\sigma \\
&\le \Big(\int_{t-n}^{t-n+1} e^{-\delta q(t-\sigma)} H^q(t-\sigma)d\sigma \Big)^{1/q} \|\overline{\psi}(\sigma)\|_{S^p}.
\end{aligned}$$

Using the fact the series

$$\sum_{n=1}^{\infty} \Big(\int_{t-n}^{t-n+1} e^{-\delta q(t-\sigma)} H^q(t-\sigma)d\sigma \Big)^{1/q}$$

converges, we deduce from the well-known Weierstrass test that the series $\sum_{n=1}^{\infty} \overline{x}_n(t)$ is uniformly convergent on $\mathbb{R}$. Let

$$\overline{x}(t) = \sum_{n=1}^{\infty} \overline{x}_n(t) \quad \text{for each } t \in \mathbb{R}.$$

Observe that

$$\overline{x}(t) = \int_{-\infty}^{t} A(s)U(t,s)\overline{\psi}(s)ds \quad t \in \mathbb{R}.$$

Clearly, $\overline{x}(t) \in C(\mathbb{R}, \mathbb{X})$ and

$$\|\overline{x}(t)\| \le \sum_{n=1}^{\infty} \|\overline{x}_n(t)\| \le \sum_{n=1}^{\infty} \Big(\int_{t-n}^{t-n+1} e^{-\delta q(t-\sigma)} H^q(t-\sigma)d\sigma \Big)^{1/q} \|\overline{\psi}(\sigma)\|_{S^p}.$$

Since $\overline{\psi} \in AS^p(\mathbb{X})$ and $A(s)U(t,s)x \in bAA(\mathbb{R}, \mathbb{X})$, then for every sequence of real numbers $\{s_n\}_{n\in\mathbb{N}}$ there exist a subsequence $\{s_m\}_{m\in\mathbb{N}}$ and a function $\widetilde{\psi}(\cdot) \in L^p_{\rm loc}(\mathbb{R}, \mathbb{X})$ such that for each $t \in \mathbb{R}$,

$$\lim_{m\to\infty} \Big(\int_{t}^{t+1} \|\overline{\psi}(s+s_m) - \widetilde{\psi}(s)\|^p ds \Big)^{1/p} = 0, \tag{4.1}$$

$$\lim_{m\to\infty}\Big(\int_t^{t+1}\|\widetilde{\psi}(s-s_m)-\overline{\psi}(s)\|^p ds\Big)^{1/p}=0, \tag{4.2}$$
$$\lim_{n\to\infty}A(s+s_n)U(t+s_n,s+s_n)x=U'(t,s)x,\quad t,s\in\mathbb{R},\ x\in\mathbb{X}, \tag{4.3}$$
$$\lim_{n\to\infty}U'(t-s_n,s-s_n)x=A(s)U(t,s)x,\quad t,s\in\mathbb{R},\ x\in\mathbb{X}. \tag{4.4}$$

Let $\widetilde{x}_n=\int_{t-n}^{t-n+1}A(\sigma)U(t,\sigma)\widetilde{\psi}(\sigma)d\sigma$. Then using Hölder inequality, we have

$$\begin{aligned}
&\|\overline{x}_n(t+s_m)-\widetilde{x}_n(t)\|\\
&=\Big\|\int_{t-n}^{t-n+1}[A(\sigma+s_m)U(t+s_m,\sigma+s_m)\overline{\psi}(\sigma+s_m)-A(\sigma)U(t,\sigma)\widetilde{\psi}(\sigma)]d\sigma\Big\|\\
&\le\Big\|\int_{t-n}^{t-n+1}A(\sigma+s_m)U(t+s_m,\sigma+s_m)(\overline{\psi}(\sigma+s_m)-\widetilde{\psi}(\sigma))\Big\|\\
&\quad+\Big\|\int_{t-n}^{t-n+1}[A(\sigma+s_m)U(t+s_m,\sigma+s_m)\widetilde{\psi}(\sigma)-A(\sigma)U(t,\sigma)\widetilde{\psi}(\sigma)]d\sigma\Big\|\\
&=I_n(t)+J_n(t),
\end{aligned}$$

where

$$\begin{gathered}
I_n(t)=\Big\|\int_{t-n}^{t-n+1}A(\sigma+s_m)U(t+s_m,\sigma+s_m)(\overline{\psi}(\sigma+s_m)-\widetilde{\psi}(\sigma))\Big\|,\\
J_n(t)=\Big\|\int_{t-n}^{t-n+1}[A(\sigma+s_m)U(t+s_m,\sigma+s_m)-A(\sigma)U(t,\sigma)\widetilde{\psi}(\sigma)]d\sigma\Big\|.
\end{gathered}$$

Then using the Hölder inequality, we obtain

$$\begin{aligned}
I_n(t)&\le\int_{t-n}^{t-n+1}e^{-\delta(t-\sigma)}H(t-\sigma)\|\overline{\psi}(\sigma+s_m)-\widetilde{\psi}(\sigma)\|d\sigma\\
&\le\Big(\int_{t-n}^{t-n+1}e^{-\delta q(t-\sigma)}H^q(t-\sigma)d\sigma\Big)^{1/q}\Big(\int_{t-n}^{t-n+1}\|\overline{\psi}(\sigma+s_m)-\widetilde{\psi}(\sigma)\|^p d\sigma\Big)^{1/p}.
\end{aligned}$$

Now using (4.1) it follows that $I_n(t)\to 0$ as $m\to\infty$ for each $t\in\mathbb{R}$. Similarly, using the Lebesgue Dominated convergence Theorem and (4.3) it follows that $J_n(t)\to 0$ as $m\to\infty$ for each $t\in\mathbb{R}$. Now,

$$\|\overline{x}_n(t+s_m)-\widetilde{x}_n(t)\|\to 0,\quad\text{as } m\to\infty.$$

Similarly, using (4.2) and (4.4), it can be shown that

$$\|\widetilde{x}_n(t-s_m)-\overline{x}_n(t)\|\to 0,\quad\text{as } m\to\infty.$$

Thus, we can conclude that each $\overline{x}_n\in AA(\mathbb{X})$ and consequently their uniform limit $\overline{x}(t)\in AA(\mathbb{X})$.

Next we verify that $\overline{y}(t)\in PAA_0(\mathbb{X},\infty,\rho)$. For each n=1,2$\cdots$, we consider the integral

$$\overline{y}_n(t)=\int_{t-n}^{t-n+1}\|A(t,s)U(t,s)\|\,\|\overline{\omega}(s)\|ds.$$

For this we have the following estimates:

$$\begin{aligned}
&\sup_{\theta\in[t-r,t]}\|\overline{y}_n(\theta)\|\\
&\le\sup_{\theta\in[t-r,t]}\int_{\theta-n}^{\theta-n+1}\|A(\theta,s)U(\theta,s)\|\|\overline{\omega}(s)\|ds
\end{aligned}$$

$$
\begin{aligned}
&\le \sup_{\theta\in[t-r,t]} \int_{\theta-n}^{\theta-n+1} e^{-\delta(\theta-s)} H(\theta-s)\|\overline{\omega}(s)\|ds \\
&\le \Big(\int_{t-n}^{t-n+1} e^{-\delta q(t-s)} H^q(t-s)ds\Big)^{1/q} \Big(\sup_{\theta\in[t-r,t]} \int_{\theta-n}^{\theta-n+1} \|\overline{\omega}(s)\|^p ds)^{1/p}\Big).
\end{aligned}
$$

Then for $r>0$, we see that

$$
\begin{aligned}
&\frac{1}{m(T,\rho)} \int_{-T}^{T} \Big(\sup_{\theta\in[t-r,t]} \|\overline{y}_n((\theta))\|\Big)\rho(t)dt \\
&\le \mathcal{Z} \frac{1}{m(T,\rho)} \int_{-T}^{T} \Big(\sup_{\theta\in[t-r,t]} \Big(\int_{\theta-n}^{\theta-n+1} \|\overline{\omega}(s)\|^p ds\Big)^{1/p}\Big)\rho(t)dt,
\end{aligned}
$$

where $\mathcal{Z} = \Big(\int_{t-n}^{t-n+1} e^{-\delta q(t-s)} H^q(t-s)ds\Big)^{1/q}$.

Since $\overline{\omega}^b \in PAA_0(L^p(0,1,\mathbb{X}),\infty,\rho)$, we have $\overline{y}_n(t) \in PAA_0(\mathbb{X},\infty,\rho)$ from above inequality. Then we deduce from the Weierstrass test that the series $\sum_{n=1}^{\infty} \overline{y}_n(t)$ is uniformly convergent on $\mathbb{R}$, Moreover,

$$
\overline{y}(t) = \int_{-\infty}^{t} A(t,s)U(t,s)ds = \sum_{n=1}^{\infty} \overline{y}_n(t).
$$

and clearly $\overline{y}(t) \in C(\mathbb{R},\mathbb{X})$ and

$$
\|\overline{y}(t)\| \le \sum_{n=1}^{\infty} \|\overline{y}_n(t)\| \le \sum_{n=1}^{\infty} \mathcal{Z}\|\overline{\omega}\|_{S^p}.
$$

Applying $\overline{y}_n \in PAA_0(\mathbb{X},\infty,\rho)$ and the inequality

$$
\begin{aligned}
&\frac{1}{m(T,\rho)} \int_{-T}^{T} \Big(\sup_{\theta\in[t-r,t]} \|\overline{y}(\theta)\|\Big)\rho(t)dt \\
&\le \frac{1}{m(T,\rho)} \int_{-T}^{T} \Big(\sup_{\theta\in[t-r,t]} \|\overline{y}(\theta) - \sum_{k=1}^{n} \overline{y}_k(\theta)\|\Big)\rho(t)dt \\
&\quad + \sum_{k=1}^{\infty} \frac{1}{m(T,\rho)} \int_{-T}^{T} \Big(\sup_{\theta\in[t-r,t]} \|\overline{y}_k(\theta)\|\Big)\rho(t)dt.
\end{aligned}
$$

We deduce that the uniformly limit $\overline{y}(\cdot) = \sum_{n=1}^{\infty} \overline{y}_n(t) \in PAA_0(\mathbb{X},\infty,\rho)$. Therefore $v(t) = \overline{x}(t) + \overline{y}(t)$ is weighted pseudo almost automorphic of class infinity. □

Theorem 4.4. *If conditions* (A6)–(A12) *hold, then* (1.1) *admits a unique weighted pseudo almost automorphic mild solution of class infinity provided that*

$$
\Theta = \varsigma\Big(L_f + \sup_{t\in\mathbb{R}} \Big(\int_{-\infty}^{t} e^{-\delta q(t-s)} H^q(t-s)ds\Big)^{1/q} L_f + \alpha L_g\Big) < 1,
$$

where ς is defined as in Remark (2.25).

Proof. Define $\mathcal{F} : WPAA(\mathbb{X},\infty) \to WPAA(\mathbb{X},\infty)$ as

$$
(\mathcal{F}u)(t) = -f(t,u_t) + \int_{-\infty}^{t} U(t,s)g(s,u_s)ds - \int_{-\infty}^{t} A(s)U(t,s)f(s,u_s)ds.
$$

If $u \in WPAA(\mathbb{X},\infty)$ by Lemma 3.6 and Corollary 3.11, $g(s,u_s) \in WPAAS^p(\mathbb{X},\infty)$. By Lemma 3.6 and Corollary 3.12, $f(s,u_s) \in WPAA(\mathbb{X},\infty)$. Owing to Lemma 4.2

and Lemma 4.3, it is not difficult to see that $\mathcal{F}(WPAA(\mathbb{X},\infty)) \subseteq WPAA(\mathbb{X},\infty)$. For any $u, v \in WPAA(\mathbb{X},\infty)$, we have

$$
\begin{aligned}
&\|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\| \\
&= \Big\| - f(t,u_t) + \int_{-\infty}^t U(t,s)g(s,u_s)ds - \int_{-\infty}^t A(s)U(t,s)f(s,u_s)ds \\
&\quad + f(t,v_t) - \int_{-\infty}^t U(t,s)g(s,v_s)ds + \int_{-\infty}^t A(s)U(t,s)f(s,v_s)ds\Big\| \\
&\le L_f\|v_t - u_t\|_{\mathfrak{B}} + \Big\| \int_0^\infty U(t,t-s)(g(t-s,u_{t-s}) - g(t-s,v_{t-s}))ds\Big\| \\
&\quad + \Big(\int_{-\infty}^t e^{-\delta q(t-s)}H^q(t-s)ds\Big)^{1/q} L_f\|v_t - u_t\|_{\mathfrak{B}} \\
&\le L_f\|v_t - u_t\|_{\mathfrak{B}} + \Big(\int_{-\infty}^t e^{-\delta q(t-s)}H^q(t-s)ds\Big)^{1/q} L_f\|v_t - u_t\|_{\mathfrak{B}} \\
&\quad + M\sum_{k=1}^\infty \Big(\int_{k-1}^k e^{-\delta qs}ds\Big)^{1/q}\Big(\int_{k-1}^k \|g(s,u_s) - g(s,v_s)\|^p ds\Big)^{1/p} \\
&= L_f\|v_t - u_t\|_{\mathfrak{B}} + \Big(\int_{-\infty}^t e^{-\delta q(t-s)}H^q(t-s)ds\Big)^{1/q} L_f\|v_t - u_t\|_{\mathfrak{B}} \\
&\quad + \alpha_0\sum_{k=1}^\infty e^{-\delta k}\|g(t+k-2+\cdot, u_{t+k-2+\cdot}) - g(t+k-2+\cdot, v_{t+k-2+\cdot})\|_p \\
&= L_f\|v_t - u_t\|_{\mathfrak{B}} + \Big(\int_{-\infty}^t e^{-\delta q(t-s)}H^q(t-s)ds\Big)^{1/q} L_f\|v_t - u_t\|_{\mathfrak{B}} \\
&\quad + \alpha L_g\|u_{t+k-2+\cdot} - v_{t+k-2+\cdot}\|_{\mathfrak{B}} \\
&\le \varsigma\Big(L_f + \sup_{t\in\mathbb{R}}\Big(\int_{-\infty}^t e^{-\delta q(t-s)}H^q(t-s)ds\Big)^{1/q} L_f + \alpha L_g\Big)\|u - v\|_\infty \\
&= \Theta\|u - v\|_\infty.
\end{aligned}
$$

Consequently,

$$\|\mathcal{F}u - \mathcal{F}v\|_\infty \le \Theta\|u - v\|_\infty.$$

Then $\mathcal{F}$ is a contraction since $\Theta < 1$. By the Banach contraction mapping principle, $\mathcal{F}$ has a unique fixed point in $WPAA(\mathbb{X},\infty)$, which is the unique $WPAA$ mild solution to the problem. □

We end this paper with a simple example. Consider the first-order boundary-value problem which was used in [8],

$$
\begin{gathered}
\frac{\partial}{\partial t}\Big[u(t,\xi) + \int_{-\infty}^0\int_0^\pi b(s,\eta,\xi)u(t+s,\eta)d\eta ds\Big] \\
= \frac{\partial^2}{\partial\xi^2}u(t,\xi) + a_0(\xi)u(t,\xi) + \int_{-\infty}^0 a(s)u(t+s,\xi)ds, \\
u(0,t) = u(\pi,t) = 0,
\end{gathered}
\tag{4.5}
$$

for $t \in \mathbb{R}$ and $\xi \in \mathbb{I} = [0,\pi]$.

Note that equations of type (4.5) arise in control systems described by abstract retarded functional-differential equations with feedback control governed by proportional integro-diffential law, see [18] for details.

To analize (4.5), we let $\mathbb{X} = L^2([0,\pi])$ and $\mathfrak{B} = C((-\infty,0],\mathbb{X})$. In addition, we suppose that the function a, a_0, a_1 are continuous and satisfy the following conditions:

(i) The function $b(\cdot)$, $\frac{\partial^i}{\partial\varsigma^i} b(\tau,\, \eta, \varsigma)$, i=1,2, are (Lebesgue) measurable, $b(\tau,\eta,\pi) = 0, b(\tau,\, \eta, 0) = 0$ for every (τ,η) and

$$N_1 = \max\Big\{ \int_0^{\pi} \int_{-\infty}^{0} \int_0^{\pi} \Big(\frac{\partial^i}{\partial\varsigma^i} b(\tau,\eta,\varsigma)\Big)^2 d\eta d\tau d\varsigma : i = 0,1,2\Big\} < \infty.$$

Define $f,g : C((-\infty,0],\mathbb{X})$ by

$$f(t,\psi)(\xi) = \int_{-\infty}^{0} \int_0^{\pi} b(s,\eta,\xi)\psi(s,\eta) d\eta ds,$$

$$g(t,\psi)(\xi) = a_0(\xi)u(t,\xi) + \int_{-\infty}^{0} a(s)\psi(s,\xi)ds.$$

In view of above arguments, it is clear that (4.5) can be rewritten in the abstract form of (1.1). By direct estimation from (i), we can show that f takes values in $D(A)$ and that $f(t,\cdot) : C((-\infty,0] : \mathbb{X}) \to [D(A)]$ is a bounded linear operator with $\|Af(t,\cdot)\| \le (N_1 p)^{\frac{1}{2}}$ for each $t \in \mathbb{R}$. Furthermore, g is a bounded linear operator on $\mathbb{X}$ with

$$g(t,\cdot) \le \|a_0\|_\infty + \Big(p\Big(\int_{-\infty}^{0} a^2(s)ds\Big)\Big)^{1/2},$$

for every $t \in \mathbb{R}$.

As a consequence of Theorem 4.4, system (4.5) has a unique weighted pseudo almost automorphic mild solution of class infinity whenever

$$2\sqrt{N_1 p} + \|a_0\|_\infty + \Big(p\Big(\int_{-\infty}^{0} a^2(s)ds\Big)\Big)^{1/2} < 1.$$

Acknowledgments. The authors are grateful to the anonymous referee for carefully reading this manuscript and giving valuable suggestion for improvements. This work was partially supported by NSFC (11361032), and FRFCUC (JB160713).

References

[1] S. Abbas; *Pseudo almost automorphic solutions of some nonlinear integro-differential equations.* Comp. Math. Appl., 62 (2011), 2259–2272.

[2] S. Abbas, Y. Xia; *Existence and attractivity of k-almost automorphic solutions of a model of cellular neural network with delay.* Acta Mathematica Scientia, 1 (2013), 290–302.

[3] S. Abbas; *Weighted pseudo almost automorphic solutions of fractional functional differential equations*, Cubo (Temuco), 16 (2014), 21–36.

[4] S. Bochner; *Continuous mappings of almost automorphic and almost periodic functions.* Proc. Ncad. Sci. USA, 52 (1964), 907–910.

[5] J. Blot, G. M. Mophou, G. M. N'Guérékata, D. Pennequin; *Weighted pseudo almost automorphic functions and applications to abstract differential equations.* Nonlinear Anal., 71 (2009), 903–909.

[6] Y. K. Chang, R. Zhang, G. M. N'Guérékata; *Weighted pseudo almost automorphic solutions to nonautonomous semilinear evolution equations with delay and S^p-weighted pseudo almost automorphic coefficients.* Topol. Method Nonlinear Anal., 43 (2014), 69–88.

[7] Y. K. Chang, M. J. Zhang, R. Ponce; *Weighted pseudo almost automorphic mild solutions to semilinear fractionWeighted pseudo almost automorphic solutions to a semilinear fractional differential equation with Stepanov-like weighted pseudo almost automorphic nonlinear term.* Appl. Math. Comput., 257 (2015), 158–168.

[8] T. Diagana, E. M. Hernández; *Existence and uniqueness of pseudo almost periodic solutions to some abstract partial neutral functional-differential equations and applications.* J. Math. Anal. Appl., 327 (2007), 776–791.

[9] T. Diagana, E. Hernández, M. Rabello; *Pseudo almost periodic solutions to some non-autonomous neutral functional differential equations with unbounded delay.* Math. Comput. Modelling, 45 (2007), 1241–1252.

[10] T. Diagana, G. M. N'Guérékata; *Stepanov-like almost automorphic functions and applications to some semilinear equations*, Appl. Anal., 86 (2007), 723–733.

[11] T. Diagana; *Existence of pseudo-almost automorphic solutions to some abstract differential equations with S^p-pseudo-almost automorphic coefficients.* Nonlinear Anal., 70 (2009), 3781–3790.

[12] T. Diagana, G. M. Mophou, G. M. N'Guérékata; *Existence of weighted pseudo almost periodic solutions to some classes of differential equations with S^p-weighted pseudo almost periodic coefficients*, Nonlinear Anal., 72 (2010), 430–438.

[13] T. Diagana; *Almost Automorphic Type and Almost Periodic Type Functions in Abstract Spaces.* Springer, New York, 2013.

[14] H. S. Ding, J. Liang, T. J. Xiao; *Weighted pseudo almost automorphic functions and WPAA solutions to semilinear evolution equations.* J. Math. Anal. Appl., 409 (2014), 409–427.

[15] Z. B. Fan, J. Liang, T. J. Xiao; *On Stepanov-like (pseudo) almost automorphic functions.* Nonlinear Anal., 74 (2011), 2853–2861.

[16] C. H. Hua, H. X. Li;, *Composition of S^p-weighted pseudo almost automorphic function and applications.* Electron. J. Diff. Equ., 2014 (2014), 1–15.

[17] H. Henríquez, C. Cuevas; *Almost automorphy for abstract neutral differential equation via control theory.* Annali di Matematica, 192 (2013), 393–405.

[18] E. Hernández, H. Henriquez; *Pseudo-almost periodic solutions for non-autonomous neutral differential euqations with unbounded delay.* Nonlinear Anal. RWA, 9 (2008), 430–437.

[19] V. Kavitha, S. Abbas, R. Murugesu; *Asymptotically almost automorphic solutions of fractional order neutral integro-differential equations.* Bull. Malays. Math. Sci. Soc., 39 (2016), 1075–1088.

[20] Y. Hino, S. Murakami, T. Naito; *Functional-differential equations with infinite delay.* Springer, Berlin, 1991.

[21] H. Lee, H. Alkahby; *Stepanov-like almost automorphic solutions of nonautonomous semilinear evolution equations with delay.* Nonlinear Anal., 69 (2008), 2158–2166.

[22] J. Liang, G. M. N'Guérékata, T. J. Xiao, X. X. Zhu; *Some properties of pseudo almost automorphic functions and applications to abstract differential equations.* Nonlinear Anal., 70 (2009), 2731–2735.

[23] G. M. N'Guérékata; *Almost Automorphic and Almost Periodic Functions in Abstract Spaces.* Kluwer Academic, New York, 2001.

[24] G. M. N'Guérékata, A. Pankov; *Stepanov-like almost automorphic functions and monotone evolution equations.* Nonlinear Anal., 68 (2008), 2658–2667.

[25] G. M. N'Guérékata; *Topics in Almost Automorphy.* Springer, New York, 2005.

[26] A. Pazy; *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Springer, New York, 1983.

[27] R. Ponce; *Bounded mild solutions to fractional integro-differential equations in banach spaces.* Semigroup Forum, 87 (2013) 377–392.

[28] R. Sakthivel, P. Revathi, S. Marshal Anthoni; *Existence of pseudo almost automorphic mild solutions to stochastic fractional diffenertial equations.* Nonlinear Anal., 75 (2012), 3339–3347.

[29] R. Sakthivel, P. Revathi, Y. Ren; *Existence of solutions for nonlinear fractional stochastic differential equations.* Nonlinear Anal., 81 (2013), 70–86.

[30] Z. N. Xia, M. Fan; *Weighted Stepanov-like pseudo almost automorphy and applications.* Nonlinear Anal., 75 (2012), 2378–2397.

[31] Z. N. Xia; *Weighted pseudo periodic solutions of neutral functional differential equations.* Electron. J. Diff. Equ., 2014 (2014), 1–17.

[32] T. J. Xiao, X. X. Zhu, J. Zhang; *Pseudo-almost automorphic mild solutions to nonautonomous differential equations and applications.* Nonlinear Anal., 70 (2009), 4079–4085.

[33] R. Zhang, Y. K. Chang, G. M. N'Guérékata; *Weighted pseudo almost automorphic solutions for non-autonomous neutral functional differential equations with infinite delay(in Chinese).* Sci. Sin. Math., 43 (2013), 273–292.

Yong-Kui Chang
School of Mathematics and Statistics, Xidian University, Xi'an 710071, China
E-mail address: `lzchangyk@163.com`

Shan Zheng
Department of Mathematics, Lanzhou Jiaotong University, Lanzhou 730070, China
E-mail address: `1318434313@qq.com`