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INFINITELY MANY SOLUTIONS FOR p-LAPLACIAN BOUNDARY-VALUE PROBLEMS ON THE REAL LINE

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Abstract. Under appropriate oscillating behaviour of the nonlinear term, we prove the existence of multiple solutions for p-Laplacian parametric equations on unbounded intervals. These problems have a variational structure, so we use an abstract result for smooth functionals defined on reflexive Banach spaces.

1. Introduction

Boundary value problems (briefly BVPs) on infinite intervals frequently occur in mathematical modelling of various applied problems. Typically, these problems arise very frequently in fluid dynamics, aerodynamics, quantum mechanics, electronics, astrophysics, and other domains of science. As examples we have: the study of unsteady flow of a gas through a semi-infinite medium [19, 17], heat transfer in the radial flow between parallel circular disks [25], draining flows [1], circular membranes [3, 12, 13], plasma physics [16], non-Newtonian fluid flows [2], study of stellar structure, thermal behavior of a spherical cloud of gas, isothermal gas sphere, theory of thermionic currents [9, 11, 31], and modeling of vortex solitons [14, 26].

Motivated by this interest, the aim of this article is to study the following elliptic problem on the real line: Find $u \in W^{1,p}(\mathbb{R})$ satisfying

$$\begin{gathered} -\left(|u'(x)|^{p-2}u'(x)\right)' + B|u(x)|^{p-2}u(x) = \lambda\alpha(x)g(u(x)), \quad x \in \mathbb{R}, \\ u(-\infty) = u(+\infty) = 0, \end{gathered} \tag{1.1}$$

where λ is a real positive parameter, B is a real positive number, and $\alpha, g : \mathbb{R} \to \mathbb{R}$ are two functions such that

$$\alpha \in L^1(\mathbb{R}), \quad \alpha(x) \geq 0 \text{ for a.e. } x \in \mathbb{R},\ \alpha \not\equiv 0,$$

and g is continuous and non-negative.

Our goal in this paper is to obtain some sufficient conditions to guarantee that, for suitable values of λ, problem (1.1) has infinitely many solutions. To this end, we require that the potential G of g satisfies a suitable oscillatory behavior either at infinity (for obtaining unbounded solutions) or at the origin (for finding arbitrarily small solutions). Our analysis is mainly based on a general critical point theorem (see Lemma 2.5 below) contained in [4].

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We are motivated by the recent paper of Bonanno et al. [5] in which the existence and multiplicity of non-negative solutions for problem (1.1) was established.

For more information, we refer the reader to the papers [7, 8, 10, 15, 18, 21, 32] where the existence and multiplicity of solutions for BVPs (parametric or otherwise) on unbounded intervals using variational methods and critical point theory is proved. In conclusion, we cite a recent monograph by Kristály, Rădulescu and Varga [20] as a general reference on variational methods adopted here. Here, as an example, we state a special case of our results.

Theorem 1.1. *Let α be a continuous function on $\mathbb{R}$ and*

$$\liminf_{\xi\to+\infty}\frac{\int_0^\xi g(t)dt}{\xi^p}=0,\quad \limsup_{\xi\to+\infty}\frac{\int_0^\xi g(t)dt}{\xi^p}=+\infty.$$

Then, for each $\lambda>0$, problem (1.1) *admits an unbounded sequence of non-negative classical solutions.*

This article is organized as follows. In Section 2, we present some necessary preliminary facts that will be needed in the paper. In Section 3 our main result (see Theorem 3.1) and some significative consequences (see Corollaries 3.3, 3.5 and 3.6) are presented.

2. Preliminaries

In this section, we first introduce some necessary definitions and notation which will be used here.

Let $(E,|\cdot|)$ be a real Banach space. We denote by E^* the dual space of E, while $\langle\cdot,\cdot\rangle$ stands for the duality pairing between E^* and E. We denote by $|\cdot|$ and by $|\cdot|_t$ the usual norms on $\mathbb{R}$ and on $L^t(\mathbb{R})$, for all $t\in[1,+\infty]$, while $W^{1,p}(\mathbb{R})$ indicates the closure of $C_0^\infty(\mathbb{R})$ with respect to the norm

$$\|u\|_{1,p}:=\left(|u'|_p^p+|u|_p^p\right)^{1/p}.$$

When $p=2$ the norm is induced by the scalar product

$$(u,v)=(u',v')_{L^2}+(u,v)_{L^2}.$$

It is well known that $W^{1,p}(\mathbb{R})\equiv W_0^{1,p}(\mathbb{R})$ and $W^{1,p}(\mathbb{R})$ is embedded in $L^t(\mathbb{R})$ for any $t\in[p,+\infty]$. Also, the embedding $W^{1,p}(\mathbb{R})\hookrightarrow C([-R,R])$, $R>0$, is compact and the embedding $W^{1,p}(\mathbb{R})\hookrightarrow L^\infty(\mathbb{R})$ is continuous.

In the following, we consider $E:=W^{1,p}(\mathbb{R})$ endowed with the norm

$$\|u\|:=\Big(\int_{\mathbb{R}}(|u'(x)|^p+B|u(x)|^p)\,dx\Big)^{1/p},$$

which is equivalent to the usual norm, that is, when $B=1$. The following proposition corresponds to [5, Proposition 2.2].

Proposition 2.1. *One has*

$$|u|_\infty\le c_B\|u\| \tag{2.1}$$

for all $u\in W^{1,p}(\mathbb{R})$, where

$$c_B:=2^{(p-2)/p}\Big(\frac{p-1}{p}\Big)^{(p-1)/p}\Big(\frac{1}{B}\Big)^{(p-1)/p^2}. \tag{2.2}$$

Definition 2.2. We say that a function $u \in E$ is a *(weak) solution* of problem (1.1) if

$$\int_{\mathbb{R}} \big(|u'(x)|^{p-2}u'(x)v'(x) + B|u(x)|^{p-2}u(x)v(x)\big)\,dx - \lambda \int_{\mathbb{R}} \alpha(x)g(u(x))v(x)dx = 0$$

for all $v \in E$. Moreover, when α is, in addition, a continuous function on $\mathbb{R}$, the (weak) solutions of (1.1) are actually classical, as standard computations show.

Definition 2.3. Let $\Phi, \Psi : E \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals. Put $I_\lambda := \Phi - \lambda\Psi$, $\lambda > 0$, and fix $r \in [-\infty, +\infty]$. We say that the functional I_λ satisfies the Palais-Smale condition cut off upper at r (in short, the $(\mathrm{PS})^{[r]}$-condition) if any sequence $\{u_n\} \subset E$ such that

- $\{I_\lambda(u_n)\}$ is bounded,
- $\lim_{n\to+\infty} \|I'_\lambda(u_n)\|_* = 0$,
- $\Phi(u_n) < r$ for all $n \in \mathbb{N}$,

has a convergent subsequence.

Remark 2.4. Clearly, if $r = +\infty$, then $(\mathrm{PS})^{[r]}$-condition coincides with the classical (PS)-condition. Moreover, if I_λ satisfies $(\mathrm{PS})^{[r]}$-condition, then it satisfies $(\mathrm{PS})^{[\rho]}$-condition for all $\rho \in [-\infty, +\infty]$ such that $\rho \le r$. So, in particular, if I_λ satisfies the classical (PS)-condition, then it satisfies $(\mathrm{PS})^{[\rho]}$-condition for all $\rho \in [-\infty, +\infty]$.

We shall prove our results applying the following lemma by Bonanno [4, Theorem 7.4], which improves [30, Theorem 2.5]. We point out that Ricceri's variational principle generalizes the celebrated three critical point theorem by Pucci and Serrin [27, 28] and is an useful result that gives alternatives for the multiplicity of critical points of certain functions depending on a parameter.

Lemma 2.5. *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuous Gâteaux differentiable functionals with Φ bounded from below. For $r > \inf_X \Phi$, let*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{\big(\sup_{v\in\Phi^{-1}(-\infty,r)} \Psi(v)\big) - \Psi(u)}{r - \Phi(u)},$$
$$\gamma := \liminf_{r\to+\infty} \varphi(r), \quad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Then:

(a) *If $\gamma < +\infty$ and for every $\lambda \in (0, 1/\gamma)$ the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies the $(\mathrm{PS})^{[r]}$-condition for all $r \in \mathbb{R}$, then, for each $\lambda \in (0, 1/\gamma)$, the following alternative holds: either*
  (1) *I_λ possesses a global minimum, or*
  (2) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that $\lim_{n\to+\infty} \Phi(u_n) = +\infty$.*

(b) *If $\delta < +\infty$ and for every $\lambda \in (0, 1/\delta)$ the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies the $(\mathrm{PS})^{[r]}$-condition for all $r > \inf_X \Phi$, then, for each $\lambda \in (0, 1/\gamma)$, the following alternative holds: either*
  (1) *there is a global minimum of Φ which is a local minimum of I_λ, or*
  (2) *there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ such that $\lim_{n\to+\infty} \Phi(u_n) = \inf_X \Phi$.*

We also refer the interested reader to the papers [6, 23, 24], in which Ricceri's variational principle and its variants have been successfully used to obtain the existence of solutions for different boundary value problems; see also the related papers [22, 29].

3. Main results

In this section we establish the main abstract result of this article. We set

$$G(\xi) := \int_0^\xi g(t)dt, \quad \forall \xi \in \mathbb{R}.$$

Our hypotheses on g guarantee that $G \in C^1(\mathbb{R})$ and $G'(\xi) = g(\xi) \geq 0$ for all $\xi \in \mathbb{R}$, so G is non-decreasing. Now, we put

$$\alpha_0 := \int_{-1}^1 \alpha(x)dx,$$

$$l := c_B \left(2^{2p-1} + \frac{B}{2(p+1)} + 2B\right)^{1/p},$$

$$B^\infty := \limsup_{\xi \to +\infty} \frac{G(\xi)}{\xi^p}.$$

With the above notation, we are able to prove the following result.

Theorem 3.1. *Assume that there exist two sequences $\{a_n\}$ and $\{b_n\}$ in $]0, +\infty[$, with $\lim_{n\to+\infty} b_n = +\infty$, such that*

- (H1) $a_n < \frac{b_n}{l}$ *for each $n \in \mathbb{N}$;*
- (H2) $\mathcal{A}_\infty := \lim_{n\to+\infty} \frac{|\alpha|_1 G(b_n) - \alpha_0 G(a_n)}{b_n^p - (a_n l)^p} < \frac{\alpha_0}{l^p} B^\infty$.

Then, for each

$$\lambda \in \frac{1}{pc_B^p}\Big] \frac{l^p}{\alpha_0 B^\infty}, \frac{1}{\mathcal{A}_\infty}\Big[,$$

problem (1.1) *admits an unbounded sequence of non-negative solutions in E.*

Proof. Our aim is to apply Lemma 2.5(a) to problem (1.1). To this end, let the functionals $\Phi, \Psi : E \to \mathbb{R}$ be defined by

$$\Phi(u) := \frac{1}{p}\|u\|, \quad \Psi(u) := \int_{\mathbb{R}} \alpha(x) G(u(x))dx,$$

and put

$$I_\lambda(u) := \Phi(u) - \lambda\Psi(u),$$

for every $u \in E$.

It is clear that the assumptions on α and g guarantee that the functional Ψ is well defined.

It is well known that Φ and Ψ are continuous Gâteaux differentiable functionals whose Gâteaux derivatives at the point $u \in E$ are

$$\Phi'(u)(v) = \int_{\mathbb{R}} \left(|u'(x)|^{p-2}u'(x)v'(x) + B|u(x)|^{p-2}u(x)v(x)\right) dx,$$

$$\Psi'(u)(v) = \int_{\mathbb{R}} \alpha(x)g(u(x))v(x)dx,$$

for every $v \in E$. Thus, a critical point of the functional I_λ is a solution of (1.1). Moreover, it is proved in [5, Lemma 2.8] that the functional I_λ satisfies $(\mathrm{PS})^{[r]}$-condition for all $r \in \mathbb{R}$.

Fix λ as in the statement of the theorem. First, we show that $\lambda < 1/\gamma$. To this end, write

$$r_n := \frac{1}{p}\left(\frac{b_n}{c_B}\right)^p, \quad \forall\, n \in \mathbb{N}. \tag{3.1}$$

Then, for all $u \in E$ with $\Phi(u) < r_n$, taking Proposition 2.1 into account, one has

$$|u|_\infty \le c_B\|u\| < c_B(pr_n)^{1/p} = b_n, \quad \forall\, n \in \mathbb{N}.$$

Then, for every $n \in \mathbb{N}$, it follows that

$$\begin{aligned}
\varphi(r_n) &\le \inf_{\Phi(u)<r_n} \frac{\int_{\mathbb{R}} \alpha(x)\sup_{|\xi|<b_n} G(\xi)dx - \int_{\mathbb{R}} \alpha(x)G(u(x))dx}{\frac{1}{p}\big(\frac{b_n}{c_B}\big)^p - \frac{1}{p}\|u\|^p} \\
&\le (pc_B^p) \inf_{\Phi(u)<r_n} \frac{|\alpha|_1 G(b_n) - \int_{\mathbb{R}} \alpha(x)G(u(x))dx}{b_n^p - (c_B\|u\|)^p}.
\end{aligned}$$

For $n \in \mathbb{N}$, let

$$w_n(x) := \begin{cases} 4a_n(x+1) + a_n & x \in [-\frac{5}{4}, -1[, \\ a_n & x \in [-1, 1], \\ 4a_n(1-x) + a_n & x \in]1, \frac{5}{4}], \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $w_n \in E$. Moreover, one has

$$\begin{aligned}
\|w_n\|^p &= \int_{\mathbb{R}} |w_n'(x)|^p dx + B\int_{\mathbb{R}} |w_n(x)|^p dx \\
&= \frac{(4a_n)^p}{2} + B\Big(\frac{1}{2(p+1)} + 2\Big)a_n^p \\
&= a_n^p\Big(2^{2p-1} + \frac{B}{2(p+1)} + 2B\Big) \\
&= \Big(\frac{a_n l}{c_B}\Big)^p.
\end{aligned}$$

Hence, by assumption (H1), one has $\Phi(w_n) < r_n$. Moreover,

$$\begin{aligned}
\Psi(w_n) &= \int_{-\frac{5}{4}}^{\frac{5}{4}} \alpha(x)G(w_n(x))dx \\
&\ge \int_{-1}^{1} \alpha(x)G(w_n(x))dx \\
&= \alpha_0 G(a_n),
\end{aligned}$$

for each $n \in \mathbb{N}$. Then, it follows that

$$\varphi(r_n) \le (pc_B^p)\,\frac{|\alpha|_1 G(b_n) - \alpha_0 G(a_n)}{b_n^p - (a_n l)^p},$$

for every $n \in \mathbb{N}$. Hence, bearing in mind assumption (H2), we can write

$$0 \le \gamma \le \lim_{n\to+\infty} \varphi(r_n) \le pc_B^p \mathcal{A}_\infty < +\infty.$$

Taking into account the above relation, since

$$\lambda < \frac{1}{pc_B^p \mathcal{A}_\infty},$$

we also have $\lambda < 1/\gamma$.

Now, we claim that the functional I_λ is unbounded from below. Since

$$\frac{1}{\lambda} < \frac{pc_B^p \alpha_0}{l^p} B^\infty,$$

there exist a sequence $\{\eta_n\}$ of positive numbers and $\tau > 0$ such that $\lim_{n\to+\infty} \eta_n = +\infty$ and

$$\frac{1}{\lambda} < \tau < \frac{pc_B^p \alpha_0}{l^p} \frac{G(\eta_n)}{\eta_n^p},$$

for each $n \in \mathbb{N}$ large enough. For $n \in \mathbb{N}$, let $s_n \in E$ defined by

$$s_n(x) := \begin{cases} 4\eta_n(x+1) + \eta_n & x \in [-\frac{5}{4}, -1[, \\ \eta_n & x \in [-1, 1], \\ 4\eta_n(1-x) + \eta_n & x \in]1, \frac{5}{4}], \\ 0 & \text{otherwise}. \end{cases}$$

Thus, we obtain

$$\begin{aligned} I_\lambda(s_n) &= \Phi(s_n) - \lambda\Psi(s_n) \\ &\le \frac{1}{p}\left(\frac{\eta_n l}{c_B}\right)^p - \lambda\alpha_0 G(\eta_n) \\ &< \frac{1}{p}\left(\frac{\eta_n l}{c_B}\right)^p (1 - \lambda\tau), \end{aligned}$$

for every $n \in \mathbb{N}$ large enough. Since $\lambda\tau > 1$ and $\lim_{n\to+\infty} \eta_n = +\infty$, we have

$$\lim_{n\to+\infty} I_\lambda(s_n) = -\infty.$$

Then, the functional I_λ is unbounded from below, and it follows that I_λ has no global minimum. Therefore, by Lemma 2.5(a), there exists a sequence $\{u_n\}$ of critical points of I_λ such that

$$\lim_{n\to+\infty} \|u_n\| = +\infty.$$

Finally, it is proved in [5, proof of Theorem 3.1] that the critical points of the energy are non-negative. The proof is complete. □

Put

$$B^0 := \limsup_{\xi\to 0^+} \frac{G(\xi)}{\xi^p}.$$

Arguing as in the proof of Theorem 3.1 and applying part Lemma 2.5 (b), we obtain the following result.

Theorem 3.2. *Assume that there exist two sequences $\{c_n\}$ and $\{d_n\}$ in $]0, +\infty[$, with $\lim_{n\to+\infty} d_n = 0$, such that*

- (H3) $c_n < \frac{d_n}{l}$ *for each* $n \in \mathbb{N}$;
- (H4) $\mathcal{A}_0 := \lim_{n\to+\infty} \frac{|\alpha|_1 G(d_n) - \alpha_0 G(c_n)}{d_n^p - (c_n l)^p} < \frac{\alpha_0}{l^p} B^0$.

Then, for each

$$\lambda \in \frac{1}{pc_B^p}\Big]\frac{l^p}{\alpha_0 B^0}, \frac{1}{\mathcal{A}_0}\Big[,$$

problem (1.1) *admits a sequence of non-trivial and non-negative solutions which converges strongly to zero in* E.

Now, we point out some consequences of Theorem 3.1. First, by setting

$$A_\infty := \liminf_{\xi\to+\infty} \frac{G(\xi)}{\xi^p},$$

we obtain the following result.

Corollary 3.3. *Assume that*

(H5) $A_\infty < \frac{\alpha_0}{|\alpha|_1 l^p} B^\infty$.

Then, for each

$$\lambda \in \frac{1}{pc_B^p}\Big]\frac{l^p}{\alpha_0 B^\infty}, \frac{1}{|\alpha|_1 A_\infty}\Big[,$$

problem (1.1) *admits an unbounded sequence of non-negative solutions in* E.

Proof. Let $\{b_n\}$ be a sequence of positive numbers which goes to infinity such that

$$\lim_{n\to+\infty} \frac{G(b_n)}{b_n^p} = A_\infty.$$

Taking $a_n = 0$ for every $n \in \mathbb{N}$, by Theorem 3.1 the conclusion follows. □

Remark 3.4. Theorem 1.1 immediately follows from Corollary 3.3.

A special case of Corollary 3.3 is the following.

Corollary 3.5. *Assume that*

(H6) $A_\infty < \frac{1}{pc_B^p|\alpha|_1}$ *and* $B^\infty > \frac{l^p}{pc_B^p\alpha_0}$.

Then, problem (1.1) *with* $\lambda = 1$ *admits an unbounded sequence of non-negative solutions in* E.

The next result is a consequence of Theorem 3.1 and guarantees the existence of infinitely many solutions to (1.1) for each λ which lies in a precise half-line.

Corollary 3.6. *Assume that there exist two sequences* $\{a_n\}$ *and* $\{b_n\}$ *in* $]0,+\infty[$, *with* $\lim_{n\to+\infty} b_n = +\infty$, *such that* (H1) *holds and*

(H7) $\alpha_0 G(a_n) = |\alpha|_1 G(b_n)$ *for each* $n \in \mathbb{N}$.

If $B^\infty > 0$, *then, for each*

$$\lambda > \frac{l^p}{pc_B^p\alpha_0|B^\infty},$$

problem (1.1) *admits an unbounded sequence of non-negative solutions in* E.

Proof. By (H7) we obtain $\mathcal{A}_\infty = 0$. Hence, since $B^\infty > 0$, condition (H2) of Theorem 3.1 holds and the proof is complete. □

Remark 3.7. From Theorem 3.2 we obtain the same consequences of Theorem 3.1. Namely, substituting $\xi \to +\infty$ with $\xi \to 0^+$, statements such as Corollaries 3.3, 3.5 and 3.6 can be established.

Next we present an example which is an application of Corollary 3.3.

Example 3.8. Put

$$a_n := \frac{2n!(n+2)!-1}{4(n+1)!}, \quad b_n := \frac{2n!(n+2)!+1}{4(n+1)!}.$$

for every $n \in \mathbb{N}$, and define the non-negative continuous function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(\xi) := \begin{cases} \frac{32(n+1)!^2[(n+1)!^2-n!^2]}{\pi}\sqrt{\frac{1}{16(n+1)!^2} - \left(\xi - \frac{n!(n+2)}{2}\right)^2}, & \text{if } \xi \in \cup_{n\in\mathbb{N}}[a_n, b_n], \\ 0, & \text{otherwise.} \end{cases}$$

One has

$$\int_{n!}^{(n+1)!} g(t)dt = \int_{a_n}^{b_n} g(t)dt = (n+1)!^2 - n!^2$$

for every $n \in \mathbb{N}$. Then, one has

$$\lim_{n\to+\infty} \frac{G(a_n)}{a_n^2} = 0 \quad \text{and} \quad \lim_{n\to+\infty} \frac{G(b_n)}{b_n^2} = 4.$$

Therefore, by a simple computation, we obtain

$$\liminf_{\xi\to+\infty} \frac{G(\xi)}{\xi^2} = 0 \quad \text{and} \quad \limsup_{\xi\to+\infty} \frac{G(\xi)}{\xi^2} = 4.$$

Also, let $\alpha(x) := 1/(1+x^2)$ for all $x \in \mathbb{R}$. Then, α is a non-negative continuous function with

$$\alpha_0 = \frac{\pi}{2} \quad \text{and} \quad |\alpha|_1 = \pi.$$

We have

$$0 = \liminf_{\xi\to+\infty} \frac{G(\xi)}{\xi^2} < \frac{\alpha_0}{|\alpha|_1 l^2} \limsup_{\xi\to+\infty} \frac{G(\xi)}{\xi^2} = \frac{24}{61}.$$

So, from Corollary 3.3, for each $\lambda > 61/(24\pi)$, the problem

$$\begin{gathered} -u'' + u = \lambda \frac{g(u)}{1+x^2}, \quad x \in \mathbb{R}, \\ u(-\infty) = u(+\infty) = 0, \end{gathered}$$

admits an unbounded sequence of non-negative classical solutions in $W^{1,2}(\mathbb{R})$. In particular, since $1 > 61/(24\pi)$, the problem

$$\begin{gathered} -u'' + u = \frac{g(u)}{1+x^2}, \quad x \in \mathbb{R}, \\ u(-\infty) = u(+\infty) = 0, \end{gathered}$$

admits an unbounded sequence of non-negative classical solutions.

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References

[1] R. P. Agarwal, D. O'Regan; Singular problems on the infinite interval modelling phenomena in draining flows, *IMA J. Appl. Math.*, **66** (2001), 621–635.

[2] R. P. Agarwal, D. O'Regan; Infinite interval problems arising in non-linear mechanics and non-Newtonian fluid flows, *Int. J. Nonlin. Mech.* **38** (2003), 1369–1376.

[3] R. P. Agarwal, D. O'Regan; An infinite interval problem arising in circularly symmetric deformations of shallow membrane caps, *Int. J. Nonlin. Mech.* **39** (2004), 779–784.

[4] G. Bonanno; A critical point theorem via the Ekeland variational principle, *Nonlinear Anal.* **75** (2012), 2992–3007.

[5] G. Bonanno, G. Barletta, D. O'Regan; A variational approach to multiplicity results for boundary-value problems on the real line, *Proc. Roy. Soc. Edinburgh Sect. A* **145** (2015), 13–29.

[6] G. Bonanno, G. Molica Biscia, V. Rădulescu; Infinitely many solutions for a class of nonlinear eigenvalue problem in Orlicz-Sobolev spaces, *C. R. Math. Acad. Sci. Paris* **349** (2011), 263–268.

[7] G. Bonanno, D. O'Regan; A boundary value problem on the half-line via critical point methods, *Dynam. Systems Appl.* **15** (2006), 395–408.

[8] P. Candito, G. Molica Bisci; Radially symmetric weak solutions for elliptic problems in $\mathbb{R}^N$, *Differential Integral Equations* **26** (2013), 1009–1026.

[9] S. Chandrasekhar; *An Introduction to the Study of Stellar Structure*, Dover, New York, NY, USA, 1957.

[10] E. B. Choi, Y.-H. Kim; Three solutions for equations involving nonhomogeneous operators of p-Laplace type in $\mathbb{R}^N$, *J. Inequal. Appl.* **427** (2014), 1–15.

[11] H. T. Davis; *Introduction to Nonlinear Differential and Integral Equations*, Dover, New York, NY, USA, 1962.

[12] R. W. Dickey; Membrane caps under hydrostatic pressure, *Quart. Appl. Math.* **46** (1988), 95–104.

[13] R. W. Dickey; Rotationally symmetric solutions for shallow membrane caps, *Quart. Appl. Math.* **47** (1989), 571–581.

[14] G. R. Flierl; Baroclinic solitary waves with radial symmetry, *Dynam. Atmos. Oceans* **183** (1979), 15–38.

[15] F. Gazzola, V. Rădulescu; A nonsmooth critical point theory approach to some nonlinear elliptic equations in $\mathbb{R}^N$, *Differential Integral Equations* **13** (2000), 47–60.

[16] M. Greguš; On a special boundary value problem, *Acta Math.* **40** (1982), 161–168.

[17] I. Hashim, S. K. Wilson; The onset of oscillatory Marangoni convection in a semi-infinitely deep layer of fluid, *Z. Angew. Math. Phys.* **50** (1999), 546–558.

[18] S. Heidarkhani, F. Gharehgazlouei, A. Solimaninia; Existence of infinitely many symmetric solutions to perturbed elliptic equations with discontinuous nonlinearities in $\mathbb{R}^N$, *Electron. J. Differential Equations* **123** (2015), 1–17.

[19] R. E. Kidder; Unsteady flow of gas through a semi-infinite porous medium, *J. Appl. Mech.* **24** (1957), 329–332.

[20] A. Kristály, V. Rădulescu, Cs. Varga; *Variational Principles in Mathematical Physics, Geometry, and Economics: Qualitative Analysis of Nonlinear Equations and Unilateral Problems*, Encyclopedia of Mathematics and its Applications, No. **136**, Cambridge University Press, Cambridge, 2010.

[21] Y. Li; Multiple solutions for perturbed p-Laplacian problems on $\mathbb{R}^N$, *Electron. J. Differential Equations* **23** (2015), 1–9.

[22] C. Li, R. P. Agarwal and C.-L. Tang; Infinitely many periodic solutions for ordinary p-Laplacian systems, *Adv. Nonlinear Anal.* **4** (2015), 251–261.

[23] G. Molica Bisci, V. Rădulescu; Bifurcation analysis of a singular elliptic problem modelling the equilibrium of anisotropic continuous media, *Topol. Methods Nonlinear Anal.* **45** (2015), 493–508.

[24] G. Molica Bisci, D. Repovš; Algebraic systems with Lipschitz perturbations, *J. Elliptic Parabol. Equ.* **1** (2015), 189–199.

[25] T. Y. Na; *Computational Methods in Engineering Boundary Value Problems*, Academic Press, New York, NY, USA, 1979.

[26] V. I. Petviashvili; Red spot of Jupiter and the drift soliton in a plasma, *Jetp Lett.* **32** (1981), 619–622.

[27] P. Pucci, J. Serrin; Extensions of the mountain pass theorem, *J. Funct. Anal.* **59** (1984), 185–210.

[28] P. Pucci, J. Serrin; A mountain pass theorem, *J. Differential Equations* **60** (1985), 142–149.

[29] D. Repovš; Stationary waves of Schrödinger-type equations with variable exponent, *Anal. Appl. (Singap.)* **13** (2015), 645–661.

[30] B. Ricceri; A general variational principle and some of its applications, *J. Comput. Appl. Math.* **113** (2000), 401–410.

[31] O. U. Richardson; *The Emission of Electricity from Hot Bodies*, London, UK, 1921.

[32] G. Zhang, S. Liu; Three symmetric solutions for a class of elliptic equations involving the p-Laplacian with discontinuous nonlinearities in $\mathbb{R}^N$, *Nonlinear Anal.* **67** (2007), 2232–2239.

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