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MULTIPLE SOLUTIONS FOR BIHARMONIC ELLIPTIC PROBLEMS WITH THE SECOND HESSIAN

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ABSTRACT. In this article, we study the biharmonic elliptic problem with the secondnd Hessian

$$\begin{gathered}\Delta^2 u = S_2(D^2 u) + \lambda f(x)|u|^{p-1}u, \quad \text{in } \Omega \subset \mathbb{R}^3,\\ u = \frac{\partial u}{\partial n} = 0, \quad \text{on } \partial\Omega,\end{gathered}$$

where $f(x) \in C(\bar{\Omega})$ is a sign-changing weight function. By using variational methods and some properties of the Nehari manifold, we prove that the biharmonic elliptic problem has at least two nontrivial solutions.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^3$, $0 < p < 1$. In this work, we consider the problem

$$\begin{gathered}\Delta^2 u = S_2(D^2 u) + \lambda f(x)|u|^{p-1}u, \quad \text{in } \Omega,\\ u = \frac{\partial u}{\partial n} = 0, \quad \text{on } \partial\Omega,\end{gathered} \tag{1.1}$$

where $f(x) \in C(\bar{\Omega})$ is a sign-changing weight function,

$$S_2(D^2u)(x) = \sum_{1 \le i < j \le N} \lambda_i(x)\lambda_j(x),$$

λ_i, $(i = 1, \cdots, N)$ are the solutions of the equation

$$\det(\lambda I - D^2 u(x)) = 0,$$

and Δ^2 the bi-Laplacian operator.

The case $N = 2$ appears as the stationary part of a model of epitaxial growth of crystals (see [6, 15]) initially studied in [7]. In dimension $N = 3$ the model can be seen as the stationary part of a 3-dimensional growth problem driven by the scalar curvature.

For the case $n = 2$, the equation is expressed by the formula

$$\Delta^2 u = \det(D^2 u) + \lambda f(x)u, \quad \text{in } \Omega \subset \mathbb{R}^2. \tag{1.2}$$

In this case, (1.2) was studied by Escudero and Peral [7]. For a Dirichlet boundary condition, they used variational methods to prove that (1.2) has at least two

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solutions. However under the Navier boundary condition, (1.2) does not have a variational characteristic, so the authors used fixed point arguments to obtain existence of solutions.

For the evolution formula of problem (1.2), Escudero, Gazzola, and Peral [9] proved existence of local solutions for arbitrary data and existence of global solutions for small data. Moreover, by exploiting the boundary conditions and the variational structure of the equation, according to the size of the data the authors proved finite time blow-up of the solution and (or) convergence to a stationary solution for global solutions.

For problem (1.1), Ferrari, Medina and Peral [12] obtained the following results for $f(x) \equiv 1$:

(1) If $p < 1$ there exists a $\lambda_0 > 0$ such that if $0 < \lambda < \lambda_0$, problem (1.1) has at least two nontrivial solutions.
(2) If $p > 1$ problem (1.1) has at least one nontrivial solution for every $\lambda \geq 0$.
(3) If $p = 1$ problem (1.1) has at least one nontrivial solution whenever $0 < \lambda < \lambda_1$, where λ_1 denotes the first eigenvalue of Δ^2 in Ω with Dirichlet boundary conditions.

In the high dimensional case, Escudero and Torres [11] proved the existence of radial solutions for the problem

$$\Delta^2 u = (-1)^k S_k[u] + \lambda f(x), \quad \text{in } B_1(0) \subset \mathbb{R}^N,$$

provided either with Dirichlet boundary conditions or Navier boundary conditions, where the k-Hessian $S_k[u]$ is the k-th elementary symmetric polynomial of eigenvalues of the Hessian matrix.

We can state now the following result.

Theorem 1.1. *Let $0 < p < 1$. There exists $\lambda_0 > 0$ such that for each $\lambda \in (0, \lambda_0)$, problem (1.1) has at least two nontrivial solutions.*

As in [12], we will use variational methods and some properties of the Nehari manifold to obtain two nontrivial solutions. For a study on variational methods and their applications, we refer the reader to [4, 17, 18, 20, 21]. The Nehari manifold was introduced by Nehari in [19] and has been widely used; see [1, 2, 3, 13, 14, 16, 22, 23, 24, 25].

The main idea for the proof or theorem 1.1 is dividing the Nehari manifold into two parts and then considering the minimum of the functional on each part. This article is organized as follows. In Section 2, we give some preliminary lemmas. In Section 3, we present the proof of Theorem 1.1.

2. Preliminaries

To use variational methods and some properties of the Nehari manifold, we firstly define the corresponding functional and Nehari manifold with respect to problem (1.1). The energy functional for problem (1.1) is

$$I(u) = \frac{1}{2}\int_\Omega |\Delta u|^2 dx - \int_\Omega \sum_{1\leq i<j\leq N} \partial_{ij} u \partial_i u \partial_j u \, dx - \frac{\lambda}{p+1}\int_\Omega f(x)|u|^{p+1} dx, \quad (2.1)$$

$u \in W_0^{2,2}(\Omega)$. From [12], we know that

$$
\begin{aligned}
(I'(u), v) = &\int_\Omega \Delta u \Delta v \, dx - \int_\Omega \sum_{1\le i<j\le N} (\partial_i u \partial_j u \partial_{ij} v + \partial_j u \partial_{ij} u \partial_i v + \partial_i u \partial_{ji} u \partial_j v) \, dx \\
&- \int_\Omega \lambda f(x)|u|^p v
\end{aligned}
$$

$$
J(u) = (I'(u), u) = \int_\Omega |\Delta u|^2 dx - 3 \int_\Omega \sum_{1\le i<j\le N} \partial_i u \partial_j u \partial_{ij} u \, dx - \int_\Omega \lambda f(x)|u|^{p+1} dx,
$$

$$
(J'(u), u) = 2\int_\Omega |\Delta u|^2 dx - 9 \int_\Omega \sum_{1\le i<j\le N} \partial_i u \partial_j u \partial_{ij} u \, dx - (p+1) \int_\Omega \lambda f(x)|u|^{p+1} dx.
$$

As the energy functional I is not bounded on $W_0^{2,2}(\Omega)$, it is useful to consider the functional on the Nehari manifold

$$
\mathcal{N} = \{u : (I'(u), u) = 0\}.
$$

Furthermore, we consider the minimization problem: for $\lambda > 0$

$$
\alpha = \inf\{I(u) : u \in \mathcal{N}\}.
$$

The Nehari manifold $\mathcal{N}$ can be split three parts:

$$
\mathcal{N}^+ = \{u : (J'(u), u) > 0\}, \quad \mathcal{N}^0 = \{u : (J'(u), u) = 0\} \mathcal{N}^- = \{u : (J'(u), u) < 0\}.
$$

Lemma 2.1. *There exists $\lambda_1 > 0$ such that for each $\lambda \in (0, \lambda_1)$, $\mathcal{N}^0 = \emptyset$*

Proof. We consider the following two cases.

Case 1. Assume that $u \in \mathcal{N}$ and $\int_\Omega \lambda f(x)|u|^{p+1} dx = 0$. This implies

$$
(I'(u), u) = \int_\Omega |\Delta u|^2 dx - 3 \int_\Omega \sum_{1\le i<j\le N} \partial_i u \partial_j u \partial_{ij} u \, dx = 0.
$$

Hence,

$$
(J'(u), u) = -\int_\Omega |\Delta u|^2 dx < 0
$$

and so $u \notin \mathcal{N}^0$

Case 2. $u \in \mathcal{N}$ and $\int_\Omega \lambda f(x)|u|^{p+1} dx \neq 0$. Assume that $\mathcal{N}^0 \neq \emptyset$ for all $\lambda > 0$. If $u \in \mathcal{N}^0$, then

$$
\begin{aligned}
0 = (J'(u), u) &= 2\int_\Omega |\Delta u|^2 dx - 9 \int_\Omega \sum_{1\le i<j\le N} \partial_i u \partial_j u \partial_{ij} u \, dx \\
&\quad - (p+1) \int_\Omega \lambda f(x)|u|^{p+1} dx \qquad (2.2) \\
&= (1-p) \int_\Omega |\Delta u|^2 dx - (6-3p) \int_\Omega \sum_{1\le i<j\le N} \partial_i u \partial_j u \partial_{ij} u \, dx.
\end{aligned}
$$

Therefore,

$$
\int_\Omega |\Delta u|^2 dx = \frac{(6-3p)}{(1-p)} \int_\Omega \sum_{1\le i<j\le N} \partial_i u \partial_j u \partial_{ij} u \, dx, \qquad (2.3)
$$

$$
\begin{aligned}
\int_\Omega \lambda f(x)|u|^{p+1}dx &= \int_\Omega |\Delta u|^2 dx - 3\int_\Omega \sum_{1\le i<j\le N} \partial_i u \partial_j u \partial_{ij} u\, dx \\
&= \frac{3}{1-p}\int_\Omega \sum_{1\le i<j\le N} \partial_i u \partial_j u \partial_{ij} u\, dx.
\end{aligned} \tag{2.4}
$$

Moreover, using Hölder's inequality, one has

$$
\begin{aligned}
\frac{1}{(2-p)}\int_\Omega |\Delta u|^2 dx &= \int_\Omega |\Delta u|^2 dx - 3\int_\Omega \sum_{1\le i<j\le N} \partial_i u \partial_j u \partial_{ij} u\, dx \\
&= \lambda \int_\Omega f(x)|u|^{p+1}dx \le \lambda \|f\|_{L^m}\|u\|_{1+q}^{p+1} \\
&\le \lambda \|f\|_{L^m} S^{p+1}\Big(\int_\Omega |\Delta u|^2 dx\Big)^{\frac{p+1}{2}},
\end{aligned} \tag{2.5}
$$

where $m = \frac{1+q}{q-p}$ (so the conjugate index $m' = \frac{1+q}{p+1}$), $q+1 < \frac{2N}{N-4}$. By (2.5), we have

$$
\Big(\int_\Omega |\Delta u|^2 dx\Big)^{\frac{1-p}{2}} \le \lambda(2-p)\|f\|_{L^m} S^{p+1}. \tag{2.6}
$$

or

$$
\Big(\int_\Omega |\Delta u|^2 dx\Big) \le \big(\lambda(2-p)\|f\|_{L^m} S^{p+1}\big)^{\frac{2}{p-1}}.
$$

Define the following functional on $W_0^{2,2}(\Omega)$,

$$
A(u) = K(p,q)\Big[\frac{(\int_\Omega |\Delta u|^2 dx)^q}{\int_\Omega \sum_{1\le i<j\le N} \partial_{ij} u \partial_i u \partial_j u\, dx}\Big]^{\frac{1}{q-1}} - \int_\Omega \lambda f(x)|u|^{p+1}dx,
$$

where

$$
K(p,q) = \frac{3}{1-p}\Big(\frac{1-p}{6-3p}\Big)^{\frac{q}{q-1}}.
$$

Then by (2.3) and (2.4), we have $A(u) = 0$.

On the other hand, for $u \in W_0^{2,2}(\Omega)$, we have

$$
\begin{aligned}
\int_\Omega \sum_{1\le i<j\le N} \partial_{ij} u \partial_i u \partial_j u\, dx &\le C\Big(\int_\Omega |\Delta u|^2 dx\Big)^{1/2}\Big(\int_\Omega |\nabla u|^4 dx\Big)^{1/2} \\
&\le C\Big(\int_\Omega |\Delta u|^2 dx\Big)^{3/2}
\end{aligned} \tag{2.7}
$$

Then using (2.5), (2.6), the Holder inequality and Sobolev inequality, for $u \in \mathcal{N}^0$, we deduce

$$
\begin{aligned}
A(u) &\ge K(p,q)\Big[\frac{\big(\int_\Omega |\Delta u|^2 dx\big)^q}{\int_\Omega \sum_{1\le i<j\le N} \partial_{ij} u \partial_i u \partial_j u\, dx}\Big]^{\frac{1}{q-1}} - \lambda \|f\|_{L^m}\|u\|_{1+q}^{p+1} \\
&\ge K(p,q)\Big[\frac{\big(\int_\Omega |\Delta u|^2 dx\big)^q}{C\big(\int_\Omega |\Delta u|^2 dx\big)^{3/2}}\Big]^{\frac{1}{q-1}} - C\lambda\|f\|_{L^m}\Big(\int_\Omega |\Delta u|^2 dx\Big)^{\frac{p+1}{2}} \\
&\ge \Big(\int_\Omega |\Delta u|^2 dx\Big)^{\frac{p+1}{2}}\Big[K(p,q)\Big(\int_\Omega |\Delta u|^2 dx\Big)^{\frac{q-2-pq+p}{2(q-1)}} - C\lambda\|f\|_{L^m}\Big] \\
&\ge \Big(\int_\Omega |\Delta u|^2 dx\Big)^{\frac{p+1}{2}}\Big[K(p,q)\Big(\big(\lambda(2-p)\|f\|_{L^m} S^{p+1}\big)^{\frac{2}{p-1}}\Big)^{\frac{q(1-p)+p-2}{2(q-1)}}
\end{aligned}
$$

$$- C\lambda\|f\|_{L^m}\Big].$$

Since

$$\frac{q(1-p)+p-2}{2(q-1)}\cdot\frac{2}{p-1}<0,$$

for λ sufficiently small, we have $A(u)>0$. This contradicts $A(u)=0$. Hence we can conclude that there exits $\lambda_1>0$ such that for $\lambda\in(0,\lambda_1)$, $\mathcal{N}^0=\emptyset$. □

Lemma 2.2. *If $u\in\mathcal{N}^+$, then $\int_\Omega \lambda f(x)|u|^{p+1}dx>0$.*

Proof. For $u\in\mathcal{N}^+$, we have

$$\int_\Omega |\Delta u|^2dx - 3\int_\Omega \sum_{1\le i<j\le N}\partial_i u\partial_j u\partial_{ij}u\,dx - \int_\Omega \lambda f(x)|u|^{p+1}dx = 0,$$

$$2\int_\Omega |\Delta u|^2dx - 9\int_\Omega \sum_{1\le i<j\le N}\partial_i u\partial_j u\partial_{ij}u\,dx - (p+1)\int_\Omega \lambda f(x)|u|^{p+1}dx > 0.$$

Combining the above two formulas, we have

$$(2-p)\int_\Omega \lambda f(x)|u|^{p+1}dx > \int_\Omega |\Delta u|^2dx > 0.$$

This completes the proof. □

According to Lemma 2.2, for $\lambda\in(0,\lambda_1)$, we can write $\mathcal{N}=\mathcal{N}^+\cup\mathcal{N}^-$ and define

$$\alpha^+ = \inf_{u\in\mathcal{N}^+} I(u),\quad \alpha^- = \inf_{u\in\mathcal{N}^-} I(u).$$

Next we show that the minimizers on $\mathcal{N}$ are the critical points for I. We denote the dual space of $W_0^{2,2}(\Omega)$ by $\big(W_0^{2,2}(\Omega)\big)^*$.

Lemma 2.3. *For $\lambda\in(0,\lambda_1)$, if u_0 is a local minimizer for $I(u)$ on $\mathcal{N}$, then $I'(u_0)=0$ in $\big((W_0^{2,2}(\Omega)\big)^*$.*

Proof. If u_0 is a local minimizer for $I(u)$ on $\mathcal{N}$, then u_0 is a solution of the optimization problem

$$\text{minimize } I(u) \text{ subject to } J(u)=0.$$

Hence, by the theory of Lagrange multipliers, there exists $\theta\in\mathbb{R}$ such that

$$I'(u_0)=\theta J'(u_0)\quad\text{in } \big(W_0^{2,2}(\Omega)\big)^*$$

Thus,

$$(I'(u_0),u_0)=\theta(J'(u_0),u_0). \tag{2.8}$$

Since $u_0\in\mathcal{N}$, $(I'(u_0),u_0)=0$. Moreover, since $N=\emptyset$, $(J'(u_0),u_0)\neq 0$ and by (2.8), $\theta=0$. This completes the proof. □

For $u\in W_0^{2,2}(\Omega)$, we write

$$t_{\max} = \frac{(1-p)\int_\Omega |\Delta u|^2dx}{(6-3p)\int_\Omega \sum_{1\le i<j\le N}\partial_{ij}u\partial_i u\partial_j u\,dx}.$$

Lemma 2.4. (1) *If $\int_\Omega \sum_{1\le i<j\le N}\partial_{ij}u\partial_i u\partial_j u\,dx<0$ (≥ 0), then there exits a unique $t^->0$ $(t^+>0)$ such that $t^-u\in\mathcal{N}^+$ $(t^+u\in\mathcal{N}^-)$ and $I(t^-u)=\min_{t>0}I(tu)$ $(I(t^-u)=\max_{t>0}I(tu))$;*

(2) *$t^-(u)$ is a continuous function for nonzero u;*

(3)
$$\mathcal{N}^+ = \{u \in W_0^{2,2}(\Omega) \setminus \{0\} : t^-\Big(\frac{u}{\|u\|}\Big)\frac{1}{\|u\|} = 1\}.$$

Proof. (1) We firstly define
$$\begin{aligned} i(t) := I(tu) = \frac{t^2}{2}\int_\Omega |\Delta u|^2 dx - t^3 \int_\Omega \sum_{1\le i<j\le N} \partial_{ij}u\partial_i u\partial_j u\, dx \\ - t^{p+1}\int_\Omega \frac{\lambda f(x)}{p+1}|u|^{p+1}dx\,. \end{aligned} \tag{2.9}$$
We easily compute
$$\begin{aligned} i'(t) := I'(tu) = t\int_\Omega |\Delta u|^2 dx - 3t^2 \int_\Omega \sum_{1\le i<j\le N} \partial_{ij}u\partial_i u\partial_j u\, dx \\ - t^{p}\int_\Omega \lambda f(x)|u|^{p+1}dx \end{aligned} \tag{2.10}$$
and
$$\begin{aligned} &(I'(tu), tu) \\ &= t^2\int_\Omega |\Delta u|^2 dx - 3t^3 \int_\Omega \sum_{1\le i<j\le N} \partial_i u\partial_j u\partial_{ij} u\, dx - t^{p+1}\int_\Omega \lambda f(x)|u|^{p+1}dx \\ &= ti'(t) \end{aligned} \tag{2.11}$$
We distinguish the following two cases.

Case i. $\int_\Omega \sum_{1\le i<j\le N} \partial_{ij}u\partial_i u\partial_j u\, dx < 0$. In this case, $i(t)$ is convex and achieves its minimum at t^- and $t^- \neq 0$. Thus, using (2.9) and (2.11), we obtain $t^-u \in \mathcal{N}^+$ and
$$I''(t) > 0 \quad \text{for } t = t^-.$$

Case ii. $\int_\Omega \sum_{1\le i<j\le N} \partial_{ij}u\partial_i u\partial_j u\, dx > 0$. Let
$$s(t) = t^{1-p}\int_\Omega |\Delta u|^2 dx - 3t^{2-p}\int_\Omega \sum_{1\le i<j\le N} \partial_{ij}u\partial_i u\partial_j u\, dx.$$
It is easy to show that $s(0) = 0$, $s(t) \to -\infty$ as $t \to +\infty$ is convex and achieves its maximum at
$$t_{\max} = \frac{(1-p)\int_\Omega |\Delta u|^2 dx}{(6-3p)\int_\Omega \sum_{1\le i<j\le N} \partial_{ij}u\partial_i u\partial_j u\, dx}.$$
Then, using (2.7) we obtain
$$\begin{aligned} &s(t_{\max}) \\ &= s(t) \\ &= \Big(\frac{(1-p)\int_\Omega |\Delta u|^2 dx}{(6-3p)\int_\Omega \sum_{1\le i<j\le N} \partial_{ij}u\partial_i u\partial_j u\, dx}\Big)^{1-p}\int_\Omega |\Delta u|^2 dx \\ &\quad - 3\Big(\frac{(1-p)\int_\Omega |\Delta u|^2 dx}{(6-3p)\int_\Omega \sum_{1\le i<j\le N} \partial_{ij}u\partial_i u\partial_j u\, dx}\Big)^{2-p}\int_\Omega \sum_{1\le i<j\le N} \partial_{ij}u\partial_i u\partial_j u\, dx. \end{aligned}$$

$$
\begin{aligned}
&= (3p-2)\int_\Omega |\Delta u|^2dx\Big(\frac{(1-p)\int_\Omega |\Delta u|^2dx}{(6-3p)\int_\Omega \sum_{1\le i<j\le N}\partial_{ij}u\partial_i u\partial_j u\,dx}\Big)^{1-p}\\
&\ge C_1\Big(\int_\Omega |\Delta u|^2dx\Big)^{\frac{1+p}{2}}.
\end{aligned}
$$

From the above inequality, there exists a λ_0 such that for $\lambda\in(0,\lambda_0)$ small,

$$
\begin{aligned}
s(0)=0&<\lambda\int_\Omega f(x)|u|^{p+1}dx\\
&=\lambda\int_\Omega f(x)|u|^{p+1}dx\le\lambda\|f\|_{L^m}\|u\|_{1+q}^{p+1}\\
&\le\lambda\|f\|_{L^m}S^{p+1}\Big(\int_\Omega|\Delta u|^2dx\Big)^{\frac{p+1}{2}}\le s(t_{\max}).
\end{aligned}
\tag{2.12}
$$

where $m=\frac{1+q}{q-p}$ (so the conjugate index $m'=\frac{1+q}{p+1}$), $q+1<\frac{2N}{N-4}$.

Using (2.12), we easily deduce that there are unique values t^+ and t^- such that $0<t^+<t_{\max}<t^-$,

$$
\begin{gathered}
s(t^+)=\lambda\int_\Omega f(x)|u|^{p+1}dx=s(t^-),\\
s'(t^+)>0>s'(t^-).
\end{gathered}
$$

We have $t^+u\in\mathcal{N}^+$, $t^-u\in\mathcal{N}^-$, and $I(t^-u)\ge I(tu)\ge I(t^+u)$ for each $t\in[t^+,t^-]$ and $I(t^+u)\le I(tu)$ for each $t\in[0,t^+]$. Thus

$$
I(t^-u)=\max_{t\ge t_{\max}}I(tu),\quad I(t^+u)=\min_{0\le t\le t^-}I(tu).
$$

In this case, $i(t)$ is concave and achieves its maximum at t^+ and $t^+\neq 0$. Thus, using (2.9) and (2.11), we obtain $t^+u\in\mathcal{N}^-$ and

$$
I''(t)<0\quad\text{for } t=t^+.
$$

(2) By the uniqueness of $t^-(u)$ and the external property of $t^-(u)$, we have that $t^-(u)$ is a continuous function of $u\neq 0$.

(3) For $u\in\mathcal{N}^+$, let $v=\frac{u}{\|u\|}$. Using the discussion (1), there exists an $t^->0$ such that $t^-v\in\mathcal{N}^+$, that is $t^-\Big(\frac{u}{\|u\|}\Big)\frac{u}{\|u\|}\in\mathcal{N}^+$. Since $u\in\mathcal{N}^+$, we obtain $t^-\big(\frac{u}{\|u\|}\big)\frac{1}{\|u\|}=1$. This shows that

$$
\mathcal{N}^+\subset\big\{u\in W_0^{2,2}(\Omega)\setminus\{0\}:t^-\Big(\frac{u}{\|u\|}\Big)\frac{1}{\|u\|}=1\big\}.
$$

Conversely, let $u\in W_0^{2,2}(\Omega)\setminus\{0\}$ such that $t^-\Big(\frac{u}{\|u\|}\Big)\frac{1}{\|u\|}=1$, then

$$
t^-\Big(\frac{u}{\|u\|}\Big)\frac{u}{\|u\|}\in\mathcal{N}^+.
$$

Hence,

$$
\mathcal{N}^+=\big\{u\in W_0^{2,2}(\Omega)\setminus\{0\}:t^-\Big(\frac{u}{\|u\|}\Big)\frac{1}{\|u\|}=1\big\}.
$$

$\square$

Now we consider the degenerate equation

$$\begin{aligned} \Delta^2 u &= S_2(D^2 u), \quad \text{in } \Omega, \\ u &= \frac{\partial u}{\partial n} = 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{2.13}$$

The functional corresponding to (2.13) is

$$H(u) = \frac{1}{2}\int_\Omega |\Delta u|^2 dx - \int_\Omega \sum_{1\le i<j\le N} \partial_{ij} u \partial_i u \partial_j u \, dx \, .$$

We consider the minimization problem

$$\beta = \inf\{H(u) : u \in N\},$$

where $N = \{u : u \in W_0^{2,2}(\Omega) \setminus \{0\} : (H'(u), u) = 0\}$. Next we show that problem (2.13) has a nontrivial solution ω_0 such that $H(\omega_0) = \beta > 0$.

Lemma 2.5. *For any $u \in W_0^{2,2}(\Omega) \setminus \{0\}$, there exits an unique $t(u) > 0$ such $t(u)u \in N$. The maximum of $H(tu)$ for $t \ge 0$ is achieved at $t = t(u)$. The function*

$$W_0^{2,2}(\Omega)\backslash\{0\} \to (0, +\infty) : \; u \to t(u)$$

is continuous and defines a homeomorphism of the unit sphere of $W_0^{2,2}(\Omega)$ with N.

Proof. Let $u \in W_0^{2,2}(\Omega)\backslash\{0\}$ be fixed and define the function $g(t) := H(tu)$ on $[0, \infty)$. Obviously, we obtain

$$g'(t) = 0 \Leftrightarrow tu \in N \tag{2.14}$$

$$\Leftrightarrow \int_\Omega |\Delta u|^2 dx = 3t \int_\Omega \sum_{1\le i<j\le N} \partial_i u \partial_j u \partial_{ij} u \, dx = 0. \tag{2.15}$$

If for all $u \in W_0^{2,2}(\Omega)$, it holds $\int_\Omega \sum_{1\le i<j\le N} \partial_{ij} u \partial_i u \partial_j u \, dx \le 0$, then 0 is an unique critical point of $H(u)$. And if $\int_\Omega \sum_{1\le i<j\le N} \partial_{ij} u \partial_i u \partial_j u \, dx > 0$, using the mountain pass theorem, we can show that $H(u)$ has a nontrivial critical point. So for each $u \in W_0^{2,2}(\Omega)\backslash\{0\}$, it is easy to verify that $g(0) = 0$ and $g(t) > 0$ for $t > 0$ small and $g(t) < 0$ for $t > 0$ large. Therefore $\max_{[0,\infty)} g(t)$ is achieved at an unique $t = t(u)$ such that $g'(t(u)) = 0$ and $t(u)u \in N$. To prove the continuity of $t(u)$, assume that $u_n \to u$ in $W_0^{2,2}(\Omega)\backslash\{0\}$. It is easy to verify that $\{t(u_n)\}$ is bounded. If a subsequence of $\{t(u_n)\}$ converges to t_0, it follows from (2.14) that $t_0 = t(u)$, but then $t(u_n) \to t(u)$. Finally the continuous map from the unit sphere of $W_0^{2,2}(\Omega)\backslash\{0\} \to N$, $u \to t(u)u$, is inverse of the retraction $u \to \frac{u}{\|u\|_a}$. □

Define

$$c_1 := \inf_{u\in W_0^{2,2}(\Omega)\backslash\{0\}} \max_{t\ge 0} H(tu), \quad c := \inf_{r\in\Gamma} \max_{t\in[0,1]} H(\gamma(tu)),$$

where

$$\Gamma := \big\{\gamma \in C[0,1], W_0^{2,2}(\Omega) : \gamma(0) = 0, H(\gamma(1)) < 0\big\}.$$

Lemma 2.6. $c_1 = c = \beta > 0$ *and c is a critical value of $H(u)$.*

Proof. From Lemma 2.5, we easily know that $\beta = c_1$. Since $H(tu) < 0$ for $u \in W_0^{2,2}(\Omega)\backslash\{0\}$ and t large, we have $c \le c_1$. The manifold $\mathcal{N}$ separates $W_0^{2,2}(\Omega)$ into two components. The component containing the origin also contains a small ball around the origin. Moreover $H(u) \ge 0$ for all u in this component, because

$(H(tu),u) \geq 0$ for all $0 \geq t \geq t(u)$. Thus every $\gamma \in \Gamma$ has to cross N and $\beta \leq c$. Since the embedding $W_0^{2,2}(\Omega) \hookrightarrow L^m(\Omega)$ ($m < 2^*$) is compact, it is easy to prove that $c > 0$ is a critical value of $H(u)$ and ω_0 a nontrivial solution corresponding to c. $\square$

Lemma 2.7. (1) *There exist $\hat{t} > 0$ such that*
$$\alpha \leq \alpha^+ < \frac{p-1}{6p+6}\hat{t}^2\beta < 0.$$
(2) *$I(u)$ is coercive and bounded below on $\mathcal{N}$ for λ sufficiently small.*

Proof. (1) Let ω_0 be a nontrivial solution of problem (2.13)) such that $H(\omega_0) = \beta > 0$. Then
$$\int_\Omega |\Delta\omega_0|^2dx - 3\int_\Omega \sum_{1\leq i<j\leq N} \partial_i\omega_0\partial_j\omega_0\partial_{ij}\omega_0 dx = 0.$$
Set $\hat{t} = t^+(\Omega)$ as defined by Lemma 2.4. Hence $\hat{t}\omega_0 \in \mathcal{N}^+$ and
$$\begin{aligned}
I(\hat{t}\omega_0) &= \frac{\hat{t}^2}{2}\int_\Omega |\Delta\omega_0|^2dx - \hat{t}^3\int_\Omega \sum_{1\leq i<j\leq N} \partial_{ij}\omega_0\partial_i\omega_0\partial_j\omega_0 dx \\
&\quad - \hat{t}^{p+1}\int_\Omega \frac{\lambda f(x)}{p+1}|\omega_0|^{p+1}dx \\
&= (\frac{1}{2} - \frac{1}{p+1})\hat{t}^2\int_\Omega |\Delta\omega_0|^2dx \\
&\quad + (\frac{3}{p+1} - 1)\hat{t}^3\int_\Omega \sum_{1\leq i<j\leq N} \partial_{ij}\omega_0\partial_i\omega_0\partial_j\omega_0 dx \\
&< \frac{p-1}{6p+6}\hat{t}^2\beta.
\end{aligned} \tag{2.16}$$
This yields
$$\alpha \leq \alpha^+ < \frac{p-1}{6p+6}\hat{t}^2\beta < 0.$$
(2) For $u \in \mathcal{N}$, we have
$$\begin{aligned}
&J(u) = (I'(u),u) \\
&= \int_\Omega |\Delta u|^2dx - 3\int_\Omega \sum_{1\leq i<j\leq N} \partial_i u\partial_j u\partial_{ij}u\,dx - \lambda\int_\Omega f(x)|u|^{p+1}dx = 0.
\end{aligned}$$
Then by Hölder and Young inequalities,
$$\begin{aligned}
I(u) &= \frac{1}{2}\int_\Omega |\Delta u|^2dx - \int_\Omega \sum_{1\leq i<j\leq N} \partial_{ij}u\partial_i u\partial_j u\,dx - \frac{\lambda}{p+1}\int_\Omega f(x)|u|^{p+1}dx \\
&= \frac{1}{6}\int_\Omega |\Delta u|^2dx - (\frac{\lambda}{p+1} - \frac{\lambda}{3})\int_\Omega f(x)|u|^{p+1}dx \\
&\geq \frac{1}{6}\int_\Omega |\Delta u|^2dx - (\frac{\lambda}{p+1} - \frac{\lambda}{3})\|f\|_{L^m}S^{p+1}\Big(\int_\Omega |\Delta u|^2dx\Big)^{\frac{p+1}{2}} \\
&\geq \Big(\frac{1}{6} - \frac{2\lambda(2-p)}{3(p+1)^2}\Big)\int_\Omega |\Delta u|^2dx - \frac{\lambda(2-p)}{3(p+1)}\big(\|f\|_{L^m}S^{p+1}\big)^{\frac{2}{1-p}}.
\end{aligned} \tag{2.17}$$

In (2.17), since $p<1$, for λ small, we have $I(u)>0$ on $\mathcal{N}$. So we easily know that $I(u)$ is coercive and bounded below on $\mathcal{N}$ for λ sufficiently small. □

3. Proof of Theorem 1.1

We need the following lemmas.

Lemma 3.1. *For each $u \in \mathcal{N}$, there exist $\varepsilon>0$ and a differentiable function $\xi : B(0,\varepsilon) \subset W_0^{2,2}(\Omega) \to \mathbb{R}^+$ such that $\xi(0)=1$, the function $\xi(v)(u-v) \in \mathcal{N}$ and*

$$
\begin{aligned}
&(\xi'(0), v)\\
&= \frac{2\int_\Omega |\Delta u|^2 dx - 9\int_\Omega \sum_{1\le i<j\le N} \partial_i u \partial_j u \partial_{ij} u\, dx - \lambda(p+1)\int_\Omega f(x)|u|^{p+1} dx}{(1-p)\int_\Omega |\Delta u|^2 dx - (6-3p)\int_\Omega \sum_{1\le i<j\le N} \partial_i u \partial_j u \partial_{ij} u\, dx},
\end{aligned}
$$

for all $v \in W_0^{2,2}(\Omega)$

Proof. For $u \in \mathcal{N}$, define a function by $F : \mathbb{R} \times W_0^{2,2}(\Omega) \to \mathbb{R}$ by

$$
\begin{aligned}
&F_u(\xi,\omega)\\
&= (I(\xi(u-\omega)), \xi(u-\omega)))\\
&= \xi^2 \int_\Omega |\Delta(u-\omega)|^2 dx - 3\xi^3 \int_\Omega \sum_{1\le i<j\le N} \partial_i(u-\omega)\partial_j(u-\omega)\partial_{ij}(u-\omega) dx \quad (3.1)\\
&\quad - \lambda \xi^{p+1} \int_\Omega f(x)|(u-\omega)|^{p+1} dx.
\end{aligned}
$$

Then $F_u(1,0) = (I'(u), u) = 0$ and

$$
\begin{aligned}
\frac{d}{dt}F_u(1,0) &= 2\int_\Omega |\Delta u|^2 dx - 9\int_\Omega \sum_{1\le i<j\le N} \partial_i u \partial_j u \partial_{ij} u\, dx\\
&\quad - \lambda(p+1)\int_\Omega f(x)|u|^{p+1} dx. \quad (3.2)\\
&= (1-p)\int_\Omega |\Delta u|^2 dx - (6-3p)\int_\Omega \sum_{1\le i<j\le N} \partial_i u \partial_j u \partial_{ij} u\, dx \neq 0.
\end{aligned}
$$

According to the implicit function theorem, there exist $\varepsilon>0$ and a differentiable function $\xi : B(0,\varepsilon) \subset W_0^{2,2}(\Omega) \to \mathbb{R}^+$ such that $\xi(0)=1$ and

$$
\begin{aligned}
&(\xi'(0), v)\\
&= \frac{2\int_\Omega |\Delta u|^2 dx - 9\int_\Omega \sum_{1\le i<j\le N} \partial_i u \partial_j u \partial_{ij} u\, dx - \lambda(p+1)\int_\Omega f(x)|u|^{p+1} dx}{(1-p)\int_\Omega |\Delta u|^2 dx - (6-3p)\int_\Omega \sum_{1\le i<j\le N} \partial_i u \partial_j u \partial_{ij} u\, dx}
\end{aligned}
$$

and

$$
F_u(\xi(v), v) = 0 \quad \text{for all } v \in B(0,\varepsilon);
$$

that is, $\xi(v)(u-v) \in \mathcal{N}$. □

Similarity, we have the following result.

Lemma 3.2. *For each $u \in \mathcal{N}^-$, there exist $\varepsilon>0$ and a differentiable function $\xi^- : B(0,\varepsilon) \subset W_0^{2,2}(\Omega) \to \mathbb{R}^+$ such that $\xi^-(0)=1$, the function $\xi^-(v)(u-v) \in \mathcal{N}^-$ and*

$$
(\xi'(0), v)
$$

$$= \frac{2\int_\Omega |\Delta u|^2dx - 9\int_\Omega \sum_{1\le i<j\le N} \partial_i u\partial_j u\partial_{ij}u\,dx - \lambda(p+1)\int_\Omega f(x)|u|^{p+1}dx}{(1-p)\int_\Omega |\Delta u|^2dx - (6-3p)\int_\Omega \sum_{1\le i<j\le N} \partial_i u\partial_j u\partial_{ij}u\,dx},$$

for all $v \in W_0^{2,2}(\Omega)$.

Proof. As in the proof in Lemma 3.1, there exist $\varepsilon > 0$ and a differentiable function $\xi^- : B(0,\varepsilon) \subset W_0^{2,2}(\Omega) \to \mathbb{R}^+$ such that $\xi^0 = 1$ and $\xi^-(v)(u-v) \in \mathcal{N}$ for all $v \in B(0,\varepsilon)$. Since

$$(J'(u), u) = (1-p)\int_\Omega |\Delta u|^2dx - (6-3p)\int_\Omega \sum_{1\le i<j\le N} \partial_i u\partial_j u\partial_{ij}u\,dx < 0.$$

Thus, by the continuity of the function $J'(u)$ and ξ^-, we have

$$\begin{aligned}
&(J'(\xi^-(v)(u-v)), \xi^-(v)(u-v)) \\
&= (1-p)\int_\Omega |\Delta\left(\xi^-(v)(u-v)\right)|^2dx \\
&\quad - (6-3p)\int_\Omega \sum_{1\le i<j\le N} \partial_i\left(\xi^-(v)(u-v)\right)\partial_j\left(\xi^-(v)(u-v)\right) \\
&\quad \times \partial_{ij}\left(\xi^-(v)(u-v)\right)dx < 0.
\end{aligned} \tag{3.3}$$

For ε sufficiently small, this implies $\xi^-(v)(u-v) \in \mathcal{N}^-$. □

Lemma 3.3. *Let* $\lambda_0 = \inf\{\lambda_1, \lambda_2\}$.

(1) *There exists a minimizing sequences* $\{u_n\} \subset \mathcal{N}$ *such that*

$$I(u_n) = \alpha + o(1), \quad I'(u_n) = o(1) \quad \text{for } \left(W_0^{2,2}(\Omega)\right)^*. \tag{3.4}$$

(2) *There exists a minimizing sequences* $\{u_n\} \subset \mathcal{N}^-$ *such that*

$$I(u_n) = \alpha^- + o(1), \quad I'(u_n) = o(1) \quad \text{for } \left(W_0^{2,2}(\Omega)\right)^*. \tag{3.5}$$

Proof. Using Lemma 2.7 and Ekeland variational principle [5], there exists a minimizing sequence $\{u_n\} \subset \mathcal{N}$ such that

$$I(u_n) < \alpha + \frac{1}{n}, \tag{3.6}$$

$$I(u_n) < I(\omega) + \frac{1}{n}\|\omega - u_n\| \quad \text{for each } \omega \in \mathcal{N}. \tag{3.7}$$

By taking n enough large, from Lemma 2.7 (1), we have

$$\begin{aligned}
I(u_n) &= \frac{1}{2}\int_\Omega |\Delta u_n|^2dx - \frac{(2-p)\lambda}{p+1}\int_\Omega f(x)|u_n|^{p+1}dx \\
&< \alpha + \frac{1}{n} < \frac{p-1}{6p+6}\hat{t}^2\beta < 0.
\end{aligned} \tag{3.8}$$

This implies

$$\|f\|_{L^m} S^{p+1}\Big(\int_\Omega |\Delta u_n|^2dx\Big)^{\frac{p+1}{2}} \ge \int_\Omega f(x)|u_n|^{p+1}dx > \frac{1-p}{6\lambda(2-p)}\hat{t}^2\beta. \tag{3.9}$$

Consequently $u_n \neq 0$ and combining the above two estimates and the Holder inequality, we obtain

$$\int_\Omega |\Delta u_n|^2dx > \Big[\frac{1-p}{6\lambda(2-p)}\hat{t}^2\beta\|f\|_{L^m}^{-1}S^{-p-1}\Big]^{\frac{2}{p+1}}, \tag{3.10}$$

$$\int_\Omega |\Delta u_n|^2 dx < \Big[\frac{(4-2p)\lambda}{(p+1)}\|f\|_{L^m} S^{p+1}\Big]^{\frac{2}{1-p}}. \tag{3.11}$$

Next we show that

$$\|I'(u_n)\|_{(W_0^{2,2}(\Omega))^*} \to 0 \quad \text{as } n \to +\infty.$$

Applying Lemma 3.1 with u_n to obtain the function $\xi_n : B(0,\varepsilon_n) \subset W_0^{2,2}(\Omega) \to \mathbb{R}^+$ for some $\varepsilon_n > 0$, such that $\xi_n(\omega)(u_n-\omega) \in \mathcal{N}$. Choose $0 < \rho < \varepsilon_n$. Let $u \in W_0^{2,2}(\Omega)$ with $u \not\equiv 0$ and let $\omega_\rho = \frac{\rho u}{\|u\|}$. We set $\eta_\rho = \xi_n(\xi_\rho)(u - \omega_\rho)$. Since $\eta_\rho \in \mathcal{N}$, we deduce that from (3.7) that

$$I(\eta_\rho) - I(u_n) \ge -\frac{1}{n}\|\eta_\rho - u_n\|,$$

and by the mean value theorem, we have

$$(I'(u_n), \eta_\rho - u_n) + o(\|\eta_\rho - u_n\| \ge \frac{-1}{n}\|\eta_\rho - u_n\|). \tag{3.12}$$

Thus

$$\begin{aligned} &(I'(u_n), -\omega_\rho) + (\xi_n(\omega_\rho) - 1)(I'(u_n), (u_n - \omega_\rho)) \\ &\ge -\frac{1}{n}\|\eta_\rho - u_n\| + o(\|\eta_\rho - u_n\|). \end{aligned} \tag{3.13}$$

It follows from $(\xi_n(\omega_\rho))(u_n - \omega_\rho) \in \mathcal{N}$ and (3.13) that

$$\begin{aligned} &-\rho\big(I'(u_n), \frac{u}{\|u\|}\big) + (\xi_n(\omega_\rho) - 1)(I'(u_n) - I'(\eta_\rho), (u_n - \omega_\rho)) \\ &\ge -\frac{1}{n}\|\eta_\rho - u_n\| + o(\|\eta_\rho - u_n\|). \end{aligned} \tag{3.14}$$

Thus

$$\begin{aligned} \big(I'(u_n), \frac{u}{\|u\|}\big) &\le \frac{(\xi_n(\omega_\rho) - 1)}{\rho}(I'(u_n) - I'(\eta_\rho), (u_n - \omega_\rho)) \\ &\quad + \frac{1}{n\rho}\|\eta_\rho - u_n\| + \frac{o(\|\eta_\rho - u_n\|)}{\rho}. \end{aligned} \tag{3.15}$$

Since $\|\eta_\rho - u_n\| \le |\xi_n(\omega_\rho - 1)|\|u_n\| + \rho|\xi_n(\omega_\rho)|$ and

$$\lim_{\rho\to 0} \frac{|\xi_n(\omega_\rho - 1)|}{\rho} \le \|\xi_n'(0)\|.$$

If we let $\rho \to 0$ in (3.15) for a fixed n, then by (3.11) we can find a constant $C > 0$, independent of ρ, such that

$$\big(I'(u_n), \frac{u}{\|u\|}\big) \le \frac{C}{n}(1 + \|\xi_n'(0)\|). \tag{3.16}$$

We are done once we show that $\|\xi_n'(0)\|$ is uniformly bounded in n. By (3.11) and Lemma 3.1 and Hölder inequality, we have

$$(\xi'(0), v) = \frac{b\|v\|}{\big|(1-p)\int_\Omega |\Delta u|^2 dx - (6-3p)\int_\Omega \sum_{1\le i<j\le N} \partial_i u \partial_j u \partial_{ij} u \, dx\big|},$$

for some $b > 0$. We only need to show that

$$\Big|(1-p)\int_\Omega |\Delta u|^2 dx - (6-3p)\int_\Omega \sum_{1\le i<j\le N} \partial_i u \partial_j u \partial_{ij} u \, dx\Big| > 0, \tag{3.17}$$

for some $c > 0$ and n large enough. We argue by contradiction. Assume that there exists a subsequence $\{u_n\}$ such that

$$(1-p)\int_\Omega |\Delta u|^2 dx - (6-3p)\int_\Omega \sum_{1\le i<j\le N} \partial_i u \partial_j u \partial_{ij} u\, dx = o(1). \tag{3.18}$$

Using (2.7), (3.18) and (3.10), we can find a constant $d > 0$ such that

$$\Big|\int_\Omega \sum_{1\le i<j\le N} \partial_i u_n \partial_j u_n \partial_{ij} u_n dx\Big| \ge d \tag{3.19}$$

for n sufficiently large. In addition (3.18), and the fact $\{u_n\} \subset \mathcal{N}$ also give

$$\lambda\int_\Omega f(x)|u_n|^{p+1}dx = \int_\Omega |\Delta u_n|^2 dx - 3\int_\Omega \sum_{1\le i<j\le N} \partial_i u_n \partial_j u_n \partial_{ij} u_n dx \tag{3.20}$$

$$= \frac{3}{1-p}\int_\Omega \sum_{1\le i<j\le N} \partial_i u_n \partial_j u_n \partial_{ij} u_n dx + o(1) \tag{3.21}$$

and

$$\int_\Omega |\Delta u_n|^2 dx < \Big[\frac{(4-2p)\lambda}{(p+1)}\|f\|_{L^m} S^{p+1}\Big]^{\frac{2}{1-p}} + o(1). \tag{3.22}$$

This implies

$$\begin{aligned} A(u) &= K(p,q)\Big[\frac{\big(\int_\Omega |\Delta u|^2 dx\big)^q}{\int_\Omega \sum_{1\le i<j\le N} \partial_{ij} u \partial_i u \partial_j u\, dx}\Big]^{\frac{1}{q-1}} - \int_\Omega \lambda f(x)|u|^{p+1}dx,\\ &\le \frac{3}{1-p}\big(\frac{1-p}{6-3p}\big)^{\frac{q}{q-1}}\Big[\frac{\big(\frac{6-3p}{1-p}\int_\Omega \sum_{1\le i<j\le N} \partial_i u \partial_j u \partial_{ij} u\, dx\big)^q}{\int_\Omega \sum_{1\le i<j\le N} \partial_{ij} u \partial_i u \partial_j u\, dx}\Big]^{\frac{1}{q-1}}\\ &\quad - \frac{3}{1-p}\int_\Omega \sum_{1\le i<j\le N} \partial_i u_n \partial_j u_n \partial_{ij} u_n dx + o(1) = o(1). \end{aligned} \tag{3.23}$$

However, from (3.19) and (3.22), for λ small, we have

$$\begin{aligned} A(u) &\ge K(p,q)\Big[\frac{\big(\int_\Omega |\Delta u|^2 dx\big)^q}{\int_\Omega \sum_{1\le i<j\le N} \partial_{ij} u \partial_i u \partial_j u\, dx}\Big]^{\frac{1}{q-1}} - \lambda\|f\|_{L^m}\|u\|_{1+q}^{p+1}\\ &\ge K(p,q)\Big[\frac{\big(\int_\Omega |\Delta u|^2 dx\big)^q}{C\big(\int_\Omega |\Delta u|^2 dx\big)^{3/2}}\Big]^{\frac{1}{q-1}} - C\lambda\|f\|_{L^m}\Big(\int_\Omega |\Delta u|^2 dx\Big)^{\frac{p+1}{2}}\\ &\ge \Big(\int_\Omega |\Delta u|^2 dx\Big)^{\frac{p+1}{2}}\Big[K(p,q)\Big(\int_\Omega |\Delta u|^2 dx\Big)^{\frac{q-2-pq+p}{2(q-1)}} - C\lambda\|f\|_{L^m}\Big]\\ &\ge \Big(\int_\Omega |\Delta u|^2 dx\Big)^{\frac{p+1}{2}}\Big[K(p,q)\Big(\big(\lambda(2-p)\|f\|_{L^m}S^{p+1}\big)^{\frac{2}{p-1}}\Big)^{\frac{q(1-p)+p-2}{2(q-1)}}\\ &\quad - C\lambda\|f\|_{L^m}\Big]. \end{aligned} \tag{3.24}$$

This contradicts (3.23). We deduce that

$$\Big(I'(u_n), \frac{u}{\|u\|}\Big) \le \frac{C}{n}. \tag{3.25}$$

The proof is complete.

(2) Similar to the proof of (1), we may prove (2). □

Now we establish the existence of a local minimum for I on $\mathcal{N}^+$.

Lemma 3.4. *For λ small, the functional I has a minimizer $u_0^+ \in \mathcal{N}^+$ and it satisfies*

(1) $I(u_0^+) = \alpha = \alpha^+$*;*
(2) u_0^+ *is a nontrivial nonnegative solution of problem* (1.1)*;*
(3) $I(u_0^+) \to 0$ *as* $\lambda \to 0$.

Proof. Let $\{u_n\} \subset \mathcal{N}$ be a minimizing sequence for I on $\mathcal{N}$ such that

$$I(u_n) = \alpha + o(1), \quad I'(u_n) = o(1), \quad \text{for } \big(W_0^{2,2}(\Omega)\big)^*. \tag{3.26}$$

Then by Lemma 2.7 and the compact embedding theorem, there exists a subsequence $\{u_n\}$ and $u_0^+ \in W_0^{2,2}(\Omega)$ such that

$$\begin{gathered} u_n \rightharpoonup u_0^+ \quad \text{in } W_0^{2,2}(\Omega), \\ u_n \to u_0^+ \quad \text{in } L^h(\Omega), \end{gathered}$$

where $1 < h < 2^*$. We now show that $\int_\Omega f(x)|u_0|^{p+1}dx \neq 0$. If not, by (3.26), we can conclude that

$$\begin{gathered} \int_\Omega f(x)|u_n|^{p+1}dx = 0, \\ \int_\Omega f(x)|u_n|^{p+1}dx \to 0 \quad \text{as} \quad n \to \infty. \end{gathered}$$

Thus,

$$\int_\Omega |\Delta u_n|^2 dx = 3\int_\Omega \sum_{1\le i<j\le N} \partial_i u_n \partial_j u_n \partial_{ij} u_n dx + o(1),$$

and

$$\begin{aligned} I(u_n) &= \frac{1}{2}\int_\Omega |\Delta u_n|^2 dx - \int_\Omega \sum_{1\le i<j\le N} \partial_{ij} u_n \partial_i u_n \partial_j u_n dx \\ &\quad - \frac{\lambda}{p+1}\int_\Omega f(x)|u_n|^{p+1}dx \\ &= \frac{1}{2}\int_\Omega \sum_{1\le i<j\le N} \partial_{ij} u_n \partial_i u_n \partial_j u_n dx + o(1) \\ &\to \frac{1}{2}\int_\Omega \sum_{1\le i<j\le N} \partial_{ij} u_0 \partial_i u_0 \partial_j u_0 dx \quad \text{as } n \to +\infty, \end{aligned} \tag{3.27}$$

Similar to Lemma 2.5, we can see that $\int_\Omega \sum_{1\le i<j\le N} \partial_{ij} u_0 \partial_i u_0 \partial_j u_0 dx > 0$. So (3.27) contradicts $I(u_n) \to \alpha < 0$ as $n \to +\infty$. In particular, $u_0^+ \in \mathcal{N}^+$ is a nontrivial solution of problem (1.1) and $I(u_0^+) \ge \alpha$. Similar to the proof of [12, Lemma 3.1], we can prove that $u_n \to u_0^+$ strongly in $W_0^{2,2}\Omega)$. In fact, if $u_0^+ \in \mathcal{N}^-$, by Lemma 2.4, there are unique t_0^+ and t_0^- such that $t_0^+ u_0^+ \in \mathcal{N}^+$ and $t_0^- u_0^+ \in \mathcal{N}^-$, we have $t_0^+ < t_0^- = 1$. Since

$$\frac{d}{dt} I(t_0^+ u_0^+) = 0 \quad \text{and} \quad \frac{d^2}{dt^2} I(t_0^+ u_0^+) > 0,$$

there exists $t_0^+ < \bar{t} \le t_0^-$ such that $I(t_0^+ u_0^+) < I(\bar{t} u_0^+)$. By

$$I(t_0^+ u_0^+) < I(\bar{t} u_0^+) \le I(t_0^- u_0^+) = I(u_0^+),$$

which is a contradiction. By Lemma 2.3, we know that u_0^+ is a nontrivial solution. Moreover, from (2.17), we know that

$$0 > I(u_0^+) \geq -\frac{\lambda(2-p)}{3(p+1)} \left(\|f\|_{L^m} S^{p+1} \right)^{\frac{2}{1-p}} .$$

It is clear that $I(u_0^+) \to 0$ as $\lambda \to 0$. $\square$

As in the proof of Lemma 3.4, we establish the existence of a local minimum for I on $\mathcal{N}^-$.

Lemma 3.5. *For λ small, the functional I has a minimizer $u_0^- \in \mathcal{N}^-$ and it satisfies*

(1) $I(u_0^-) = \alpha^-$*;*
(2) u_0^- *is a nontrivial nonnegative solution of problem* (1.1).

Combining Lemma 3.4 and 3.5, for problem (1.1), there exist two nontrivial solutions u_0^+ and u_-^0 such that $u_0^+ \in \mathcal{N}^+$, $u_0^- \in \mathcal{N}^-$. Since $\mathcal{N}^+ \cap \mathcal{N}^- = \emptyset$, this shows that u_0^+ and u_-^0 are different.

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References

[1] K. J. Brown; *The Nehari manifold for a semilinear elliptic equation involving a sublinear term*, Calc. Var. Partial Differential Equations 22 (2005) 483-494.
[2] K. J. Brown, Y. P. Zhang; *The Nehari manifold for a semilinear elliptic equation with a sign-changing weight function*, J. Differential Equations 193 (2003) 481-499.
[3] J. Chabrowski, D. G. Costa; *On a class of Schrodinger-type equations with indefinite weight functions*, Comm. Partial Differential Equations 33 (2008) 1368-1394.
[4] L. M. Del Pezzo, J. D. Rossi, N. Saintier, A. Salort; *An optimal mass transport approach for limits of eigenvalue problems for the fractional p-Laplacian*, Adv. Nonlinear Anal. 4 (3) (2015) 235-249.
[5] I. Ekeland; *On the variational principle*, J. Math. Anal. Appl. 17 (1974) 324-353.
[6] C. Escudero; *Geometric principles of surface growth.* Phys. Rev. Lett. 101, 196102 (2008)
[7] C. Escudero, I. Peral; *Some fourth order nonlinear elliptic problems related to epitaxial growth*, J. Differential Equations 254 (2013) 2515-2531.
[8] C. Escudero, R. Hakl, I. Peral, P. J. Torres; *On radial stationary solutions to a model of nonequilibrium growth*, Eur. J. Appl. Math., 24 (2013), 437-453.
[9] C. Escudero, F. Gazzola, I. Peral; *Global existence versus blow-up results for a fourth order parabolic PDE involving the Hessian*, J. Math. Pures Appl. 103 (2015), 924-957.
[10] C. Escudero, R. Hakl, I. Peral, P. J. Torres; *Existence and nonexistence results for a singular boundary value problem arising in the theory of epitaxial growth*, Math. Methods Appl. Sci. 37 (2014), 793-807.
[11] C. Escudero, P. J. Torres; *Existence of radial solutions to biharmonic k-Hessian equations*, J. Differential Equations 259 (7) (2015), 2732-2761.
[12] F. Ferrari, M. Medina, I. Peral; *Biharmonic elliptic problems involving the 2-nd Hessian operator*, Calc. Var. Partial Differential Equations 51 (2014), 867-886.
[13] C. Ji; *The Nehari manifold for a degenerate elliptic equation involving a sign-changing weight function*, Nonlinear Anal. 75 (2012) 806-818.
[14] C. Ji, W. H. Wang; *On the p-biharmonic equation involving concave-convex nonlinearities and sign-changing weight function*, Electron. J. Qual. Theorty Differ. Equ. No.2, (2012), 1-17.

[15] M. Marsili, A. Maritan, F. Toigo, J. R. Banavar; *Stochastic growth equations and reparametrization invariance*, Rev. Mod. Phys. 68 (1996), 963-983.
[16] G. Molica Bisci, V. D. Rădulescu; *Ground state solutions of scalar field fractional for Schrödinger equations*, Calc. Var. Partial Differential Equations 54 (2015), 2985-3008.
[17] G. Molica Bisci, D. Repovs; *On doubly nonlocal fractional elliptic equations*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. 26 (2) (2015), 161-176.
[18] G. Molica Bisci, V. D. Rădulescu, R. Servadei; *Variational methods for nonlocal fractional equations*, Encyclopedia of Mathematics and its Applications, 162, Cambridge University Press, Cambridge, 2016.
[19] Z. Nehari; *On a class of nonlinear second-order differential equations*, Trans. Amer. Math. Soc. 95 (1960), 101-123
[20] P. Pucci, M. Q. Xiang, B. L. Zhang; *Existence and multiplicity of entire solutions for fractional p-Kirchhoff equations*, Adv. Nonlinear Anal. 5 (1) (2016) 235-249.
[21] V.D. Rădulescu, D. Repovs; *Partial Differential Equations with Variable Exponents: Variational Methods and Qualitative Analysis*, CRC Press, Taylor & Francis Group, Boca Raton FL, 2015.
[22] G. Tarantello; *On nonhomogeneous elliptic equations involving critical Sobolev exponent*, Ann. Inst. H. Poincare Anal. Non Lineaire 9 (1992), 243-261.
[23] T. F. Wu; *Multiple positive solutions for a class of concave-convex elliptic problems in involving sign-changing*, J. Funct. Anal. 258 (2010) 99-131.
[24] T. F. Wu; *Three positive solutions for Dirichlet problems involving critical Sobolev exponent and sign-changing weight*, J. Differential Equations 249 (2010) 1549-1578.
[25] M. Q. Xiang, B. L. Zhang, V. D. Rădulescu; *Multiplicity of solutions for a class of quasilinear Kirchhoff system involving the fractional p-Laplacian*, Nonlinearity 29 (2016) 3186-3205.

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