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A REGULARIZATION METHOD FOR TIME-FRACTIONAL LINEAR INVERSE DIFFUSION PROBLEMS

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Abstract. In this article, we consider an inverse problem for a time-fractional diffusion equation with a linear source in a one-dimensional semi-infinite domain. Such a problem is obtained from the classical diffusion equation by replacing the first-order time derivative by the Caputo fractional derivative. We show that the problem is ill-posed, then apply a regularization method to solve it based on the solution in the frequency domain. Convergence estimates are presented under the a priori bound assumptions for the exact solution. We also provide a numerical example to illustrate our results.

1. Introduction

In this article, we consider the following inverse problem for the time-fractional diffusion equation with a linear source in a one-dimensional semi-infinite domain,

$$\begin{gathered} -au_x(x,t) = D_t^\gamma u(x,t) + F(x,t,u(x,t)), \quad x>0,\ t>0, \\ u(1,t) = g(t), \quad t \ge 0, \\ \lim_{x\to+\infty} u(x,t) = u(x,0) = 0, \quad t\ge 0, \end{gathered} \tag{1.1}$$

where a is a constant diffusivity coefficient, F is the source function which defined later. The inverse problem here is of recovering $u(x,t)$, $0 \le x < 1$, from the given data u at $x=1$. The fractional derivative $D_t^\gamma u(x,t)$ is the Caputo fractional derivative of order $0<\gamma\le 1$ defined by [16]

$$\begin{gathered} D_t^\gamma u(x,t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^\alpha} \quad \text{for } 0<\gamma<1, \\ D_t^\gamma u(x,t) = \frac{\partial u(x,t)}{\partial t}, \quad \gamma = 1, \end{gathered}$$

where $\Gamma(\cdot)$ is the Gamma function. Problem (1.1) is an inverse problem and is ill-posed (see Lemma 2.1); that means the solution does not depend continuously on the given data and any small perturbation in the given data may cause a large change to the solution.

The homogeneous problem, i.e, $F=0$ has been considered by many authors. For example: In 2011, Zheng and Wei [26] considered a homogeneous time fractional

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diffusion problem, where the time fractional derivative is understood in sense of Dzerbayshan-Caputo also in a quarter plane, in the form

$$\begin{gathered} -au_x(x,t) = D_t^{\gamma} u(x,t), \quad x>0,\ t>0, \\ u(1,t) = g(t), \quad t \ge 0, \\ \lim_{x\to+\infty} u(x,t) = u(x,0) = 0, \quad t \ge 0, \end{gathered} \tag{1.2}$$

In 2012, Xiong et al [20] applied an optimal regularization method for solving this problem and obtained the optimal convergence estimate. In 2012, Hon et al [21] considered this problem in 2-dimensional case. In 2012, Fu et al [1] gave a new iteration regularization method to deal with this problem, and error estimates are obtained for a priori and a posteriori parameter choice rules. In 2014, MingLi et al [7] presented a new dynamic method for choosing a regularization parameter by using a spectral method.

Until now, to our knowledge, the time-fractional diffusion equation with a linear source term has not been studied. To solve the linear inhomogeneous problem, many techniques and new ideas to deal with the fractional terms and source term which can't be treated by using known ideas are required.

The techniques and methods in previous articles on the homogeneous case cannot be applied directly to solve the linear inhomogeneous problem. As is known, for a linear problem, the solution (exact solution) can be represented in an integral equation which contains some instability terms (See (2.9)). The main idea of this method is to find a suitable integral equation for approximating the exact solution. The working here is to replace instability terms by regularization terms and show that the solution of our regularized problem converges to the exact solution, if it exists as the regularization parameter tends to zero. In case of the homogeneous problem, we have many choices of stability term for regularization. However, in the case of a linear inhomogeneous problem, the main solution u is complex and defined by an integral equation on right hand side dependents of u. This leads to studying a linear inhomogeneous problem. In this paper, based on [5], we develop some new techniques to overcome this difficulty.

This article is divided into five sections. In Section 2, we present the ill-posedness of the problem and propose our new regularization method. In Section 3, convergence estimates for the temperature u are given based on the a priori assumptions for the exact solution. In section 4, a numerical example is proposed to show the effectiveness of the regularized method.

2. Ill-posedness of the nonlinear problem

To use the Fourier transform, we extend the functions $u(x,t)$, $g(t)$ to the whole line $-\infty < t < +\infty$ by defining them to be zero for $t<0$. The Fourier transform of the function $f \in L^2(\mathbb{R})$ is written as

$$\widehat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt, \qquad -\infty < \omega < +\infty. \tag{2.1}$$

Since the measurements usually contain an error, we assume that the measured data function $g_\alpha \in L^2(\mathbb{R})$ satisfies

$$\|g_\alpha - g\|_{L^2(\mathbb{R})} \le \alpha, \tag{2.2}$$

where $\alpha > 0$ represents a bound on the measurement error. Here, we assume that $F(x,t,u) = b(x,t)u(x,t) + H(x,t)$, where $b \in L^\infty(0,T;L^2(\mathbb{R}))$ satisfies

$$\|b\|_{L^\infty(0,1;L^2(\mathbb{R}))} \le K \tag{2.3}$$

for any real number $K \ge 0$ and $H \in L^2(0,1;L^2(\mathbb{R}))$. It is easy to see that F satisfies the global Lispchitz condition, i.e.,

$$\|F(x,.,u_1) - F(x,.,u_2)\|_{L^2(\mathbb{R})} \le K\|u_1(x,\cdot) - u_2(x,\cdot)\|_{L^2(\mathbb{R})} \tag{2.4}$$

Taking the Fourier transformation of (1.1) with respect to t, we obtain

$$\begin{gathered} \widehat{u_x}(x,\omega) + \frac{(i\omega)^\gamma}{a}\hat{u}(x,\omega) = \frac{1}{a}\widehat{F}(x,\omega,u(x,\omega)), \quad x>0,\ \omega\in\mathbb{R} \\ \hat{u}(1,\omega) = \hat{g}(\omega), \quad \omega \ge 0, \\ \lim_{x\to+\infty} \hat{u}(x,\omega) = 0, \quad \omega \ge 0, \end{gathered} \tag{2.5}$$

where

$$(i\omega)^\gamma = |\omega|^\gamma \cos\frac{\gamma\pi}{2} + i|\omega|^\gamma \operatorname{sign}(\omega)\sin\frac{\gamma\pi}{2}, \tag{2.6}$$

and $\widehat{F}(x,\omega,u(x,\omega))$ is

$$\widehat{F}(x,\omega,u(x,\omega)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} F(x,t,u(x,t))e^{-i\omega t}dt. \tag{2.7}$$

The solution to problem (1.1) is given by

$$\widehat{u}(x,\omega) = \exp\Big(\frac{(i\omega)^\gamma(1-x)}{a}\Big)\widehat{g}(\omega) - \frac{1}{a}\int_x^1 \exp\Big(\frac{(i\omega)^\gamma(z-x)}{a}\Big)\widehat{F}(z,\omega,u(z,\omega))dz. \tag{2.8}$$

Applying the inverse Fourier transform,

$$\begin{aligned} u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \Big[&\exp\Big(\frac{(i\omega)^\gamma(1-x)}{a}\Big)\widehat{g}(\omega) \\ &- \frac{1}{a}\int_x^1 \exp\Big(\frac{(i\omega)^\gamma(z-x)}{a}\Big)\widehat{F}(z,\omega,u(z,\omega))dz\Big]e^{i\omega t}\,d\omega. \end{aligned} \tag{2.9}$$

Note that $(i\omega)^\gamma$ has the positive real part $|\omega|^\gamma\cos\frac{\gamma\pi}{2}$ and therefore the factors $|\exp(\frac{(i\omega)^\gamma(1-x)}{a})|$ and $|\exp(\frac{(i\omega)^\gamma(z-x)}{a})|$ tend to ∞ for $0 \le z < x < 1$ as $\omega \to +\infty$. So the small perturbation for the data $g(t)$ will be amplified infinitely by this factor and lead to the integral (2.9) blow-up, therefore recovering the temperature $u(x,t)$ from the measured data $g_\alpha(t)$ is ill-posed.

Lemma 2.1. *Problem* (2.9) *is ill-posed.*

Proof. To show the instability of u in this case, we construct the functions g_n defined by the Fourier transform, as follows:

$$\widehat{g_n}(\omega) = \begin{cases} 0, & \text{if } \omega\in\mathbb{R}\backslash W_n, \\ \frac{1}{\sqrt{2n}}, & \text{if } \omega \in W_n, \end{cases} \tag{2.10}$$

$$\widehat{F_0}(z,\omega,u(z,\omega)) = \frac{a\widehat{u}(z,\omega)}{2\exp(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2})} \tag{2.11}$$

where $W_n \subset \mathbb{R}$ is

$$W_n := \big\{\omega\in\mathbb{R}|\ n-1 \le \omega \le n+1\big\}.$$

Let u_n satisfy the integral equation

$$\begin{aligned} &u_n(x,t)\\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\Big(\frac{(i\omega)^\gamma(1-x)}{a}\Big) \widehat{g}_n(\omega) e^{i\omega t}\, d\omega \\ &\quad - \frac{1}{a\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_x^1 \exp\Big(\frac{(i\omega)^\gamma(z-x)}{a}\Big) \widehat{F_0}(z,\omega,u_n(z,\omega)) e^{i\omega t} dz\, d\omega \\ &= \frac{1}{n\sqrt{\pi}} \int_{n-1}^{n+1} \exp\Big(\frac{(i\omega)^\gamma(1-x)}{a}\Big) e^{i\omega t}\, d\omega \\ &\quad - \frac{1}{a\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_x^1 \exp\Big(\frac{(i\omega)^\gamma(z-x)}{a}\Big) \widehat{F_0}(z,\omega,u_n(z,\omega)) e^{i\omega t} dz\, d\omega. \end{aligned} \tag{2.12}$$

First, we show that (2.12) has a unique solution $u_n \in C([0,1];L^2(\mathbb{R}))$. In fact, we consider the function

$$\begin{aligned} \tilde{Q}(w)(x,t) &= \frac{1}{n\sqrt{\pi}} \int_{n-1}^{n+1} \exp\Big(\frac{(i\omega)^\gamma(1-x)}{a}\Big) e^{i\omega t}\, d\omega \\ &\quad - \frac{1}{a\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_x^1 \exp\Big(\frac{(i\omega)^\gamma(z-x)}{a}\Big) \widehat{F_0}(z,\omega,w(z,\omega)) e^{i\omega t} dz\, d\omega. \end{aligned}$$

Then, for any $w_1, w_2 \in C([0,1];L^2(\mathbb{R}))$ we obtain

$$\begin{aligned} &\|\tilde{Q}(w_1)(x,\cdot) - \tilde{Q}(w_2)(x,\cdot)\|^2_{L^2(\mathbb{R})} \\ &= \|\widehat{\tilde{Q}}(w_1)(x,\cdot) - \widehat{\tilde{Q}}(w_2)(x,\cdot)\|^2_{L^2(\mathbb{R})} \\ &= \frac{1}{2} \int_{-\infty}^{+\infty} \Big| \int_x^1 \frac{\exp\big(\frac{(i\omega)^\gamma(z-x)}{a}\big)}{\exp\big(\frac{1}{a}|\omega|^\gamma \cos\frac{\gamma\pi}{2}\big)} (\widehat{w_1} - \widehat{w_2}) dz \Big|^2 d\omega \\ &\le \frac{1}{2} \int_{-\infty}^{+\infty} (1-x) \int_x^1 \Big| \frac{\exp\big(\frac{(i\omega)^\gamma(z-x)}{a}\big)}{\exp\big(\frac{1}{a}|\omega|^\gamma \cos\frac{\gamma\pi}{2}\big)} \Big(\widehat{w_1} - \widehat{w_2}\Big) \Big|^2 dz\, d\omega \\ &\le \frac{1}{2} |||w_1 - w_2|||^2, \end{aligned}$$

where $|||\cdot|||$ is the sup norm in $L^2(\mathbb{R})$. This implies that $\widetilde{Q}$ is a contraction. Using the Banach fixed-point theorem, we conclude that the equation $\widetilde{Q}(w) = w$ has a unique solution $u_n \in C([0,1];L^2(\mathbb{R}))$. The inequality $|c-d| \ge |c| - |d|$ implies

$$\begin{aligned} |\widehat{u_n}(x,\omega)| &\ge \Big| \exp\Big(\frac{(i\omega)^\gamma(1-x)}{a}\Big) \widehat{g}_n(\omega) \Big| \\ &\quad - \Big| \frac{1}{a} \int_x^1 \exp\Big(\frac{(i\omega)^\gamma(z-x)}{a}\Big) \widehat{F_0}(z,\omega,u_n(z,\omega)) dz \Big|. \end{aligned} \tag{2.13}$$

It holds

$$\begin{aligned} &\Big| \frac{1}{a} \int_x^1 \exp\Big(\frac{(i\omega)^\gamma(z-x)}{a}\Big) \widehat{F_0}(z,\omega,u_n(z,\omega)) dz \Big| \\ &\le \frac{1}{a} \int_x^1 \Big| \exp\Big(\frac{(i\omega)^\gamma(z-x)}{a}\Big) \Big| \Big| \frac{\widehat{u}_n(z,\omega)}{\exp\big(\frac{1}{a}|\omega|^\gamma \cos\frac{\gamma\pi}{2}\big)} \Big| dz \\ &\le \frac{1}{a} \int_x^1 |\widehat{u}_n(z,\omega)|\, dz. \end{aligned} \tag{2.14}$$

Combining (2.13), (2.14) and using the inequality $2(c^2+d^2) \ge (c+d)^2$, we obtain

$$2|\widehat{u_n}(x,\omega)|^2 + \frac{2}{a^2}\int_x^1 \Big|\widehat{u}_n(z,\omega)\Big|^2 dz \ge \exp\Big(\frac{2}{a}|\omega|^\gamma(1-x)\cos\frac{\gamma\pi}{2}\Big)|\widehat{g}_n(\omega)|^2.$$

By integrating over $\mathbb{R}$ with respect to the variable ω, we obtain

$$\begin{aligned} &2\|\widehat{u_n}(x,\omega)\|^2_{L^2(\mathbb{R})} + \frac{2}{a^2}\int_x^1 \|\widehat{u_n}(z,\omega)\|^2_{L^2(\mathbb{R})}\,dz \\ &\ge \frac{1}{2n^2}\int_{n-1}^{n+1} \exp\Big(\frac{2}{a}|\omega|^\gamma(1-x)\cos\frac{\gamma\pi}{2}\Big)\,d\omega. \end{aligned}$$

A simple computation gives

$$\begin{aligned} \Big(2+\frac{2}{a^2}\Big)\sup_{0\le x\le 1}\|\widehat{u_n}(x,\cdot)\|^2_{L^2(\mathbb{R})} &\ge \sup_{0\le x\le 1}\frac{\exp\big(2\frac{1}{a}(n-1)^\gamma(1-x)\cos\frac{\gamma\pi}{2}\big)}{n^2} \\ &= \frac{\exp\big(2\frac{1}{a}(n-1)^\gamma\cos\frac{\gamma\pi}{2}\big)}{n^2}. \end{aligned}$$

By Parseval's identity, it follows from (2.10) that

$$\|g_n\|^2_{L^2(\mathbb{R})} = \|\widehat{g}_n(\omega)\|^2_{L^2(\mathbb{R})} = \int_{\omega\in W_n}\frac{1}{2n^2}d\omega = \frac{1}{n^2}. \tag{2.15}$$

As $n \to +\infty$, we see that

$$\|g_n\|_{L^2(\mathbb{R})} \to 0, \qquad \sup_{0\le x\le 1}\|\widehat{u_n}(x,\cdot)\|_{L^2(\mathbb{R})} \to +\infty. \tag{2.16}$$

Thus, problem (2.9) is, in general, ill-posed in the Hadamard sense in L^2-norm. □

3. Regularization and error estimate

We must use some regularization methods to deal with this problem. To regularize the problem, we have to replace the terms $\exp\big(\frac{1-x}{a}(i\omega)^\gamma\big)$ and $\exp\big(\frac{z-x}{a}(i\omega)^\gamma\big)$ by some other terms.

Theorem 3.1. *Suppose that* (1.1) *has a unique solution* $u \in C([0,1];L^2(\mathbb{R}))$ *such that*

$$\|u(\cdot,0)\|_{L^2(\mathbb{R})} \le E. \tag{3.1}$$

Suppose that $2K < a$. *Choose* $\epsilon := \epsilon(\alpha) > 0$ *such that*

$$\lim_{\alpha\to 0}\epsilon(\alpha) = 0, \quad \lim_{\alpha\to 0}\alpha\epsilon^{-1}(\alpha) \text{is bounded.} \tag{3.2}$$

Then we construct a regularized solution U_ϵ^α *such that*

$$\|u(x,\cdot) - U_\epsilon^\alpha(x,\cdot)\|_{L^2(\mathbb{R})} \le \sqrt{\frac{2(1+p)}{1-2(1+\frac{1}{p})\frac{K^2}{a^2}}}\Big(\epsilon^{-1}\alpha + \|u(0,\cdot)\|_{L^2(\mathbb{R})}\Big)\epsilon^x, \tag{3.3}$$

for any $p > 1/(\frac{a^2}{2K^2} - 1)$. *Here* U_ϵ^α *is the function whose Fourier transform is*

$$\begin{aligned}\widehat{U_\epsilon^\alpha}(x,\omega) &= \frac{\exp\left(\frac{(i\omega)^\gamma(1-x)}{a}\right)}{1+\epsilon\exp\left(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\right)}\widehat{g_\alpha}(\omega) \\ &\quad - \frac{1}{a}\int_x^1 \frac{\exp\left(\frac{(i\omega)^\gamma(z-x)}{a}\right)}{1+\epsilon\exp\left(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\right)}\widehat{F}(z,\omega,U_\epsilon^\alpha(z,\omega))dz \\ &\quad + \frac{1}{a}\int_0^x \frac{\epsilon\exp\left(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\right)}{1+\epsilon\exp\left(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\right)} \\ &\quad \times \exp\left(\frac{(i\omega)^\gamma(z-x)}{a}\right)\widehat{F}(z,\omega,U_\epsilon^\alpha(z,\omega))dz.\end{aligned} \tag{3.4}$$

Remark 3.2. *In the above Theorem, we can choose* $\epsilon(\alpha) := \alpha$.

Proof. Let V_ϵ^α be the function whose Fourier transform is defined by

$$\begin{aligned}\widehat{V_\epsilon^\alpha}(x,\omega) &= \frac{\exp\left(\frac{(i\omega)^\gamma(1-x)}{a}\right)}{1+\epsilon\exp\left(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\right)}\widehat{g}(\omega) \\ &\quad - \frac{1}{a}\int_x^1 \frac{\exp\left(\frac{(i\omega)^\gamma(z-x)}{a}\right)}{1+\epsilon\exp\left(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\right)}\widehat{F}(z,\omega,V_\epsilon^\alpha(z,\omega))dz \\ &\quad + \frac{1}{a}\int_0^x \frac{\epsilon\exp\left(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\right)}{1+\epsilon\exp\left(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\right)} \\ &\quad \times \exp\left(\frac{(i\omega)^\gamma(z-x)}{a}\right)\widehat{F}(z,\omega,V_\epsilon^\alpha(z,\omega))dz.\end{aligned} \tag{3.5}$$

We divide the proof of Theorem 3.1 into three steps.

Step 1. The existence and the uniqueness of a solution of (3.4). Let us define the norm on $C([0,1];L^2(\mathbb{R}))$ as follows

$$\|h\|_1 = \sup_{0\le x\le 1}\epsilon^{-x}\|h(x)\|_{L^2(\mathbb{R})}, \quad \text{for all } h\in L^2(\mathbb{R}) \text{ and } \epsilon>0. \tag{3.6}$$

It is obvious that $\|\cdot\|_1$ is a norm of $C([0,1];L^2(\mathbb{R}))$.

For $V \in C([0,1];L^2(\mathbb{R}))$, we consider

$$\begin{aligned}\Phi V &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \frac{\exp\left(\frac{(i\omega)^\gamma(1-x)}{a}\right)}{1+\epsilon\exp\left(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\right)}\widehat{g_\alpha}(\omega)e^{i\omega t}d\omega \\ &\quad - \frac{1}{a\sqrt{2\pi}}\int_{-\infty}^{+\infty}\int_x^1 \frac{\exp\left(\frac{(i\omega)^\gamma(z-x)}{a}\right)}{1+\epsilon\exp\left(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\right)}\widehat{F}(z,\omega,V(z,\omega))e^{i\omega t}dz d\omega \\ &\quad + \frac{1}{a\sqrt{2\pi}}\int_{-\infty}^{+\infty}\int_0^x \frac{\epsilon\exp\left(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\right)}{1+\epsilon\exp\left(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\right)} \\ &\quad \times \exp\left(\frac{(i\omega)^\gamma(z-x)}{a}\right)\widehat{F}(z,\omega,V(z,\omega))e^{i\omega t}dz\, d\omega.\end{aligned}$$

We will prove that

$$\|\Phi V_1 - \Phi V_2\|_1 \le \frac{K}{a}\|V_1 - V_2\|_1, \tag{3.7}$$

for any $V_1, V_2 \in C([0,1]; L^2(\mathbb{R}))$.
In fact, we have

$$
\begin{aligned}
&\Phi V_1(x,t) - \Phi V_2(x,t) \\
&= \frac{1}{a\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_x^1 \frac{\exp\Big(\frac{(i\omega)^\gamma (z-x)}{a}\Big)}{1+\epsilon \exp\big(\frac{1}{a}|\omega|^\gamma \cos\frac{\gamma\pi}{2}\big)} \\
&\quad \times \Big(\widehat{F}(z,\omega,V_2(z,\omega)) - \widehat{F}(z,\omega,V_1(z,\omega))\Big) e^{i\omega t} dz d\omega \\
&\quad + \frac{1}{a\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_0^x \frac{\epsilon \exp\big(\frac{1}{a}|\omega|^\gamma \cos\frac{\gamma\pi}{2}\big)}{1+\epsilon \exp\big(\frac{1}{a}|\omega|^\gamma \cos\frac{\gamma\pi}{2}\big)} \exp\Big(\frac{(i\omega)^\gamma (z-x)}{a}\Big) \\
&\quad \times \Big(\widehat{F}(z,\omega,V_1(z,\omega)) - \widehat{F}(z,\omega,V_2(z,\omega))\Big) e^{i\omega t} dz d\omega \\
&:= A_1(x,t) + A_2(x,t).
\end{aligned}
$$

First, for all $0 \le x \le z \le 1$, we have the inequality

$$
\begin{aligned}
&\Big| \frac{\exp\big(\frac{(i\omega)^\gamma (z-x)}{a}\big)}{1+\epsilon \exp\big(\frac{1}{a}|\omega|^\gamma \cos\frac{\gamma\pi}{2}\big)} \Big| \\
&= \frac{\exp\big(\frac{z-x}{a}|\omega|^\gamma \cos\frac{\gamma\pi}{2}\big)}{1+\epsilon \exp\big(\frac{1}{a}|\omega|^\gamma \cos\frac{\gamma\pi}{2}\big)} \\
&= \frac{\exp\big(\frac{z-x-1}{a}|\omega|^\gamma \cos\frac{\gamma\pi}{2}\big)}{\exp\big(\frac{-1}{a}|\omega|^\gamma \cos\frac{\gamma\pi}{2}\big) + \epsilon} \\
&= \frac{\exp\big(\frac{z-x-1}{a}|\omega|^\gamma \cos\frac{\gamma\pi}{2}\big)}{\big[\epsilon + \exp\big(\frac{-1}{a}|\omega|^\gamma \cos\frac{\gamma\pi}{2}\big)\big]^{x-z+1} \big[\epsilon + \exp\big(\frac{-1}{a}|\omega|^\gamma \cos\frac{\gamma\pi}{2}\big)\big]^{z-x}} \\
&\le \frac{1}{\big[\epsilon + \exp\big(\frac{-1}{a}|\omega|^\gamma \cos\frac{\gamma\pi}{2}\big)\big]^{z-x}} \le \epsilon^{x-z}.
\end{aligned}
$$

Then, using the latter inequality, we have an estimation for $\|A_1\|_{L^2(\mathbb{R})}$ as follows

$$
\begin{aligned}
&\|A_1(x,\cdot)\|^2_{L^2(\mathbb{R})} \\
&= \int_{-\infty}^{+\infty} \Big(\int_x^1 \frac{\exp\big(\frac{(i\omega)^\gamma (z-x)}{a}\big)}{1+\epsilon \exp\big(\frac{1}{a}|\omega|^\gamma \cos\frac{\gamma\pi}{2}\big)} \Big(\widehat{F}(z,\omega,V_2(z,\omega)) \\
&\quad - \widehat{F}(z,\omega,V_1(z,\omega))\Big) e^{i\omega t} dz \Big)^2 d\omega \\
&\le (1-x) \int_{-\infty}^{+\infty} \int_x^1 \Big| \frac{\exp\big(\frac{(i\omega)^\gamma (z-x)}{a}\big)}{1+\epsilon \exp\big(\frac{1}{a}|\omega|^\gamma \cos\frac{\gamma\pi}{2}\big)} \Big|^2 \\
&\quad \times \Big| \widehat{F}(z,\omega,V_1(z,\omega)) - \widehat{F}(z,\omega,V_2(z,\omega)) \Big|^2 dz\, d\omega \\
&\le (1-x) \int_{-\infty}^{+\infty} \int_x^1 \epsilon^{2x-2z} \Big| \widehat{F}(z,\omega,V_1(z,\omega)) - \widehat{F}(z,\omega,V_2(z,\omega)) \Big|^2 dz d\omega \\
&\le \frac{K^2}{a^2} (1-x) \int_x^1 \epsilon^{2x-2z} \|V_1 - V_2\|^2_{L^2(\mathbb{R})} dz \\
&= \frac{K^2}{a^2} \epsilon^{2x} (1-x) \int_x^1 \epsilon^{-2z} \|V_1 - V_2\|^2_{L^2(\mathbb{R})}\, dz
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{K^2}{a^2}\epsilon^{2x}(1-x)^2 \sup_{0\le z\le 1}\epsilon^{-2z}\|V_1(z)-V_2(z)\|^2_{L^2(\mathbb{R})}\\
&= \frac{K^2}{a^2}\epsilon^{2x}(1-x)^2\|V_1-V_2\|_1^2. \qquad (3.8)
\end{aligned}
$$

Next, similarly, we obtain the estimation for $\|A_2(x,\cdot)\|_{L^2(\mathbb{R})}$ as follows

$$
\|A_2(x,\cdot)\|^2_{L^2(\mathbb{R})} \le \frac{K^2}{a^2}\epsilon^{2x}x^2\|V_1-V_2\|_1^2. \tag{3.9}
$$

For $0<x<1$, by using above observations and using the inequality $(c+d)^2 \le (1+\frac{1}{m})c^2+(m+1)d^2$ for all real numbers c,d and $m>0$, we obtain

$$
\begin{aligned}
&\|\Phi V_1-\Phi V_2\|^2_{L^2(\mathbb{R})}\\
&\le \Big(\|A_1(x,\cdot)\|_{L^2(\mathbb{R})}+\|A_2(x,\cdot)\|_{L^2(\mathbb{R})}\Big)^2\\
&\le (1+\frac{1}{m})\|A_1(x,\cdot)\|^2_{L^2(\mathbb{R})}+(m+1)\|A_2(x,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\le \frac{K^2}{a^2}(1+\frac{1}{m})\epsilon^{2x}(1-x)^2\|V_1-V_2\|_1^2+(m+1)\frac{K^2}{a^2}\epsilon^{2x}K^2x^2\|V_1-V_2\|_1^2.
\end{aligned}
$$

By choosing $m=(1-x)/x$, we obtain

$$
\epsilon^{-2x}\|\Phi V_1-\Phi V_2\|^2_{L^2(\mathbb{R})} \le \frac{K^2}{a^2}\|V_1-V_2\|_1^2, \quad \text{for } 0<x<1. \tag{3.10}
$$

Combining (3.8), (3.9), (3.10), we obtain

$$
\|\Phi V_1-\Phi V_2\|_1^2 = \sup_{0\le x\le 1}\epsilon^{-2x}\|\Phi V_1-\Phi V_2\|^2_{L^2(\mathbb{R})} \le \frac{K^2}{a^2}\|V_1-V_2\|_1^2. \tag{3.11}
$$

This completes the proof of Step 1.

Step 2. Estimate for $\|U_\epsilon^\alpha(x,\cdot)-V_\epsilon^\alpha(x,\cdot)\|_{L^2(\mathbb{R})}$. Substituting $\widehat{U}_\alpha^\epsilon$ and $\widehat{V}_\epsilon^\alpha$, we obtain

$$
\begin{aligned}
&\widehat{U}_\epsilon^\alpha(x,\omega)-\widehat{V}_\epsilon^\alpha(x,\omega)\\
&= \frac{\exp\big(\frac{(i\omega)^\gamma(1-x)}{a}\big)}{1+\epsilon\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)}\big(\widehat{g_\alpha}(\omega)-\widehat{g}(\omega)\big)\\
&\quad+\frac{1}{a}\int_x^1 \frac{\exp\big(\frac{(i\omega)^\gamma(z-x)}{a}\big)}{1+\epsilon\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)}\big[\widehat{F}(z,\omega,U_\epsilon^\alpha(z,\omega))-\widehat{F}(z,\omega,V_\epsilon^\alpha(z,\omega))\big]dz\\
&\quad+\frac{1}{a}\int_0^x \frac{\epsilon\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)}{1+\epsilon\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)}\exp\Big(\frac{(i\omega)^\gamma(z-x)}{a}\Big)\\
&\quad\times\Big[\widehat{F}(z,\omega,U_\epsilon^\alpha(z,\omega))-\widehat{F}(z,\omega,V_\epsilon^\alpha(z,\omega))\Big]dz\\
&=: I_1(x,\omega)+I_2(x,\omega)+I_3(x,\omega).
\end{aligned}
$$

We have the estimates

$$
\begin{aligned}
\|I_1(x,\cdot)\|^2_{L^2(\mathbb{R})} &\le \int_{-\infty}^{+\infty} \Big|\frac{\exp\Big(\frac{(i\omega)^\gamma(1-x)}{a}\Big)}{1+\epsilon\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)}\Big|^2 \big(\widehat{g_\alpha}(\omega)-\widehat{g}(\omega)\big)^2\,d\omega \\
&= \int_{-\infty}^{+\infty} \epsilon^{2x-2}\big(\widehat{g_\alpha}(\omega)-\widehat{g}(\omega)\big)^2\,d\omega \\
&= \epsilon^{2x-2}\|g_\alpha-g\|^2_{L^2(\mathbb{R})} \le \epsilon^{2x-2}\alpha^2,
\end{aligned}
\tag{3.12}
$$

and

$$
\begin{aligned}
&\|I_2(x,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\le \int_{-\infty}^{+\infty}\Big(\frac{1}{a}\int_x^1 \frac{\exp\big(\frac{(i\omega)^\gamma(z-x)}{a}\big)}{1+\epsilon\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)}\Big[\widehat{F}(z,\omega,U_\epsilon^\alpha(z,\omega)) \\
&\quad -\widehat{F}(z,\omega,V_\epsilon^\alpha(z,\omega))\Big]dz\Big)^2 d\omega \\
&\le \int_{-\infty}^{+\infty}\frac{1}{a^2}(1-x)\Big(\int_x^1 \Big|\frac{\exp\big(\frac{(i\omega)^\gamma(z-x)}{a}\big)}{1+\epsilon\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)}\Big|^2\Big[\widehat{F}(z,\omega,U_\epsilon^\alpha(z,\omega)) \\
&\quad -\widehat{F}(z,\omega,V_\epsilon^\alpha(z,\omega))\Big]^2dz\Big)\,d\omega \\
&\le \frac{1}{a^2}\int_{-\infty}^{+\infty}\int_x^1 \epsilon^{2x-2z}\big[\widehat{F}(z,\omega,U_\epsilon^\alpha(z,\omega))-\widehat{F}(z,\omega,V_\epsilon^\alpha(z,\omega))\big]^2dz\,d\omega \\
&= \frac{1}{a^2}\epsilon^{2x}\int_x^1 \epsilon^{-2z}\|F(z,.,U_\epsilon^\alpha)(z,\cdot)-F(z,.,V_\epsilon^\alpha)(z,\cdot)\|^2_{L^2(\mathbb{R})}dz \\
&\le \frac{K^2}{a^2}\epsilon^{2x}\int_x^1 \epsilon^{-2z}\|U_\epsilon^\alpha(z,\cdot)-V_\epsilon^\alpha(z,\cdot)\|^2_{L^2(\mathbb{R})}dz,
\end{aligned}
\tag{3.13}
$$

and

$$
\begin{aligned}
&\|I_3(x,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\le \int_{-\infty}^{+\infty}\Big(\frac{1}{a}\int_0^x \frac{\epsilon\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)}{1+\epsilon\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)}\exp\Big(\frac{(i\omega)^\gamma(z-x)}{a}\Big) \\
&\qquad\times\big[\widehat{F}(z,\omega,U_\epsilon^\alpha(z,\omega))-\widehat{F}(z,\omega,V_\epsilon^\alpha(z,\omega))\big]dz\Big)^2 d\omega \\
&\le \int_{-\infty}^{+\infty}\frac{x}{a^2}\int_0^x \Big|\frac{\epsilon\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)}{1+\epsilon\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)}\exp\Big(\frac{(i\omega)^\gamma(z-x)}{a}\Big)\Big|^2 \\
&\qquad\times\Big[\widehat{F}(z,\omega,U_\epsilon^\alpha(z,\omega))-\widehat{F}(z,\omega,V_\epsilon^\alpha(z,\omega))\Big]dz\,d\omega \\
&\le \int_{-\infty}^{+\infty}\frac{1}{a^2}\int_0^x \epsilon^{2x-2z}\Big[\widehat{F}(z,\omega,U_\epsilon^\alpha(z,\omega))-\widehat{F}(z,\omega,V_\epsilon^\alpha(z,\omega))\Big]^2dz\,d\omega \\
&= \frac{1}{a^2}\epsilon^{2x}\int_0^x \epsilon^{-2z}\|F(z,.,U_\epsilon^\alpha)(z,\cdot)-F(z,.,V_\epsilon^\alpha)(z,\cdot)\|^2_{L^2(\mathbb{R})}dz \\
&\le \frac{K^2}{a^2}\epsilon^{2x}\int_0^x \epsilon^{-2z}\|U_\epsilon^\alpha(z,\cdot)-V_\epsilon^\alpha(z,\cdot)\|^2_{L^2(\mathbb{R})}dz.
\end{aligned}
\tag{3.14}
$$

From the inequality

$$(b_1+b_2+b_3)^2 \le 2(1+p)b_1^2 + 2(1+\frac{1}{p})b_2^2 + 2(1+\frac{1}{p})b_3^2$$

for any real numbers b_1, b_2, b_3 and $p>0$, we obtain

$$\begin{aligned}
&\|U_\epsilon^\alpha(x,\cdot) - V_\epsilon^\alpha(x,\cdot)\|_{L^2(\mathbb{R})}^2 \\
&\le 2\Big(1+\frac{1}{p}\Big)\|I_1(x,\cdot)\|_{L^2(\mathbb{R})}^2 + 2\Big(1+\frac{1}{p}\Big)\|I_2(x,\cdot)\|_{L^2(\mathbb{R})}^2 + 2(1+p)\|I_3(x,\cdot)\|_{L^2(\mathbb{R})}^2 \\
&\le 2(1+p)\epsilon^{2x-2}\alpha^2 + 2(1+\frac{1}{p})\frac{K^2}{a^2}\epsilon^{2x}\int_x^1 \epsilon^{-2z}\|U_\epsilon^\alpha(z,\cdot) - V_\epsilon^\alpha(z,\cdot)\|_{L^2(\mathbb{R})}^2 dz \\
&\quad + 2(1+\frac{1}{p})\frac{K^2}{a^2}\epsilon^{2x}\int_0^x \epsilon^{-2z}\|U_\epsilon^\alpha(z,\cdot) - V_\epsilon^\alpha(z,\cdot)\|_{L^2(\mathbb{R})}^2 dz \\
&= 2(1+p)\epsilon^{2x-2}\alpha^2 + 2(1+\frac{1}{p})\frac{K^2}{a^2}\epsilon^{2x}\int_0^1 \epsilon^{-2z}\|U_\epsilon^\alpha(z,\cdot) - V_\epsilon^\alpha(z,\cdot)\|_{L^2(\mathbb{R})}^2 dz.
\end{aligned}$$

This implies

$$\begin{aligned}
&\epsilon^{-2x}\|U_\epsilon^\alpha(x,\cdot) - V_\epsilon^\alpha(x,\cdot)\|_{L^2(\mathbb{R})}^2 \\
&\le 2(1+p)\epsilon^{-2}\alpha^2 + 2(1+\frac{1}{p})\frac{K^2}{a^2}\int_0^1 \epsilon^{-2z}\|U_\epsilon^\alpha(z,\cdot) - V_\epsilon^\alpha(z,\cdot)\|_{L^2(\mathbb{R})}^2 dz.
\end{aligned} \tag{3.15}$$

Set $J(x) = \epsilon^{-2x}\|U_\epsilon^\alpha(x,\cdot) - V_\epsilon^\alpha(x,\cdot)\|_{L^2(\mathbb{R})}^2$. Since $U_\epsilon^\alpha(x,\cdot), V_\epsilon^\alpha(x,\cdot) \in C([0,1];L^2(\mathbb{R}))$, we see that J is the continuous function on $[0,1]$. Hence, J attains over there its maximum P at some point $x_0 \in [0,1]$. This implies

$$J(x_0) \le 2(1+p)\epsilon^{-2}\alpha^2 + 2(1+\frac{1}{p})\frac{K^2}{a^2}J(x_0). \tag{3.16}$$

Hence

$$\Big[1 - 2(1+\frac{1}{p})\frac{K^2}{a^2}\Big]J(x_0) \le 2(1+p)\epsilon^{-2}\alpha^2. \tag{3.17}$$

Since $p > 1/(\frac{a^2}{2K^2}-1)$, we know that $1-2(1+\frac{1}{p})\frac{K^2}{a^2} > 0$. We deduce that

$$\|U_\epsilon^\alpha(x,\cdot) - V_\epsilon^\alpha(x,\cdot)\|_{L^2(\mathbb{R})}^2 \le J(x_0)\epsilon^{2x} \le \frac{2+2p}{1-2(1+\frac{1}{p})\frac{K^2}{a^2}}\epsilon^{2x-2}\alpha^2. \tag{3.18}$$

Step 3. Estimate for $\|u(x,\cdot) - V_\epsilon^\alpha(x,\cdot)\|_{L^2(\mathbb{R})}$. First, we have

$$\begin{aligned}
&\widehat{u}(x,\omega) \\
&= \exp\Big(\frac{(i\omega)^\gamma(1-x)}{a}\Big)\Big[\widehat{g}(\omega) \\
&\quad - \frac{1}{a}\int_x^1 \exp\Big(\frac{(i\omega)^\gamma(z-1)}{a}\Big)\widehat{F}(z,\omega,u(z,\omega))dz\Big] \\
&= \frac{\exp\big(\frac{(i\omega)^\gamma(1-x)}{a}\big)}{1+\epsilon\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)}\Big[\widehat{g}(\omega) - \frac{1}{a}\int_x^1 \exp\Big(\frac{(i\omega)^\gamma(z-1)}{a}\Big) \\
&\quad \times \widehat{F}(z,\omega,u(z,\omega))dz\Big] + \exp\Big(\frac{(i\omega)^\gamma(1-x)}{a}\Big)\frac{\epsilon\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)}{1+\epsilon\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)} \\
&\quad \times \Big[\widehat{g}(\omega) - \frac{1}{a}\int_x^1 \exp\Big(\frac{(i\omega)^\gamma(z-1)}{a}\Big)\widehat{F}(z,\omega,u(z,\omega))dz\Big].
\end{aligned} \tag{3.19}$$

On the other hand,

$$\widehat{u}(1,\omega)=\widehat{g}(\omega)=e^{-\frac{1}{a}(i\omega)^\gamma}\Big[\widehat{u}(0,\omega)+\frac{1}{a}\int_0^1 e^{\frac{1}{a}(i\omega)^\gamma z}\widehat{F}(z,\omega,u(z,\omega))dz\Big]. \tag{3.20}$$

This implies

$$\begin{aligned}&\widehat{g}(\omega)-\frac{1}{a}\int_x^1 \exp\Big(\frac{(i\omega)^\gamma(z-1)}{a}\Big)\widehat{F}(z,\omega,u(z,\omega))dz\\&=e^{-\frac{1}{a}(i\omega)^\gamma}\widehat{u}(0,\omega)+\frac{1}{a}\int_0^x \exp\Big(\frac{(i\omega)^\gamma(1-x)}{a}\Big)\widehat{F}(z,\omega,u(z,\omega))dz.\end{aligned} \tag{3.21}$$

Combining (3.19), (3.20) and (3.21), we obtain

$$\begin{aligned}\widehat{u}(x,\omega)=&\frac{\exp\big(\frac{(i\omega)^\gamma(1-x)}{a}\big)}{1+\epsilon\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)}\Big[\widehat{g}(\omega)\\&-\frac{1}{a}\int_x^1 \exp\Big(\frac{(i\omega)^\gamma(z-1)}{a}\Big)\widehat{F}(z,\omega,u(z,\omega))dz\Big]\\&+\frac{\epsilon\exp\big(\frac{-(i\omega)^\gamma x}{a}\big)\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)}{1+\epsilon\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)}\widehat{u}(0,\omega)\\&+\frac{\epsilon\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)}{1+\epsilon\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)}\frac{1}{a}\int_0^x \exp\Big(\frac{(i\omega)^\gamma(z-x)}{a}\Big)\\&\times\widehat{F}(z,\omega,u(z,\omega))dz.\end{aligned} \tag{3.22}$$

It follows from (3.4) and (3.22) that

$$\begin{aligned}&\widehat{u}(x,\omega)-\widehat{V_\epsilon^\alpha}(x,\omega)\\&=\frac{1}{a}\int_x^1\frac{\exp\big(\frac{(i\omega)^\gamma(z-x)}{a}\big)}{1+\epsilon\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)}\Big[\widehat{F}(z,\omega,u(z,\omega))-\widehat{F}(z,\omega,V_\epsilon^\alpha(z,\omega))\Big]dz\\&\quad+\frac{\epsilon\exp\Big(\frac{-(i\omega)^\gamma x}{a}\Big)\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)}{1+\epsilon\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)}\widehat{u}(0,\omega)\\&\quad+\frac{\epsilon\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)}{1+\epsilon\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)}\frac{1}{a}\int_0^x \exp\Big(\frac{(i\omega)^\gamma(z-x)}{a}\Big)\\&\quad\times\Big[\widehat{F}(z,\omega,u(z,\omega))-\widehat{F}(z,\omega,V_\epsilon^\alpha(z,\omega))\Big]dz\\&=: I_4(x,\omega)+I_5(x,\omega)+I_6(x,\omega).\end{aligned}$$

The term $\|I_4(x,\cdot)\|_{L^2(\mathbb{R})}$ can be estimated as follows

$$\begin{aligned}&\|I_4(x,\cdot)\|_{L^2(\mathbb{R})}^2\\&\le\frac{1}{a^2}\int_{-\infty}^{\infty}\Big(\int_x^1\frac{\exp\Big(\frac{(i\omega)^\gamma(z-x)}{a}\Big)}{1+\epsilon\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)}\Big[\widehat{F}(z,\omega,u(z,\omega))\\&\quad-\widehat{F}(z,\omega,V_\epsilon^\alpha(z,\omega))\Big]dz\Big)^2d\omega\\&\le\frac{1}{a^2}\int_{-\infty}^{\infty}(1-x)\int_x^1\Big|\frac{\exp\Big(\frac{(i\omega)^\gamma(z-x)}{a}\Big)}{1+\epsilon\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)}\Big|^2\end{aligned}$$

$$
\begin{aligned}
&\times \Big[\widehat{F}(z,\omega,u(z,\omega)) - \widehat{F}(z,\omega,V_\epsilon^\alpha(z,\omega))\Big]^2 dz\, d\omega \\
&\le \frac{1}{a^2}\int_{-\infty}^{\infty}\int_x^1 \epsilon^{2x-2z}\Big[\widehat{F}(z,\omega,u(z,\omega)) - \widehat{F}(z,\omega,V_\epsilon^\alpha(z,\omega))\Big]^2 dz\, d\omega \\
&= \frac{\epsilon^{2x}}{a^2}\int_x^1 \epsilon^{-2z}\|\widehat{F}(z,\omega,u(z,\omega)) - \widehat{F}(z,\omega,V_\epsilon^\alpha(z,\omega))\|_{L^2(\mathbb{R})}^2 dz
\end{aligned}
$$

It follows from the Lipschitz property of F that

$$
\|I_4(x,\cdot)\|_{L^2(\mathbb{R})}^2 \le \frac{K^2\epsilon^{2x}}{a^2}\int_x^1 \epsilon^{-2z}\|u(z,\omega) - V_\epsilon^\alpha(z,\omega)\|_{L^2(\mathbb{R})}^2 dz. \tag{3.23}
$$

The term $\|I_5(x,\cdot)\|_{L^2(\mathbb{R})}$ is bounded by

$$
\begin{aligned}
\|I_5(x,\cdot)\|_{L^2(\mathbb{R})}^2 &= \int_{-\infty}^{\infty}\Big|\frac{\epsilon\exp\Big(\frac{-(i\omega)^\gamma x}{a}\Big)\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)}{1+\epsilon\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)}\widehat{u}(0,\omega)\Big|^2 d\omega \\
&\le \epsilon^2\int_{-\infty}^{\infty}\epsilon^{2x-2}|\widehat{u}(0,\omega)|^2\, d\omega \\
&= \epsilon^{2x}\|u(0,.)\|_{L^2(\mathbb{R})}^2.
\end{aligned} \tag{3.24}
$$

The term $\|I_6(x,\cdot)\|_{L^2(\mathbb{R})}$ is bounded by

$$
\begin{aligned}
&\|I_6(x,\cdot)\|_{L^2(\mathbb{R})}^2 \\
&\le \int_{-\infty}^{+\infty}\Big(\frac{1}{a}\int_0^x \frac{\epsilon\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)}{1+\epsilon\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)}\exp\Big(\frac{(i\omega)^\gamma(z-x)}{a}\Big) \\
&\quad\times\Big[\widehat{F}(z,\omega,u(z,\omega)) - \widehat{F}(z,\omega,V_\epsilon^\alpha(z,\omega))\Big]dz\Big)^2 d\omega \\
&\le \int_{-\infty}^{+\infty}\frac{x}{a^2}\int_0^x\Big|\frac{\epsilon\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)}{1+\epsilon\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)}\exp\Big(\frac{(i\omega)^\gamma(z-x)}{a}\Big)\Big|^2 \\
&\quad\times\Big[\widehat{F}(z,\omega,u(z,\omega)) - \widehat{F}(z,\omega,V_\epsilon^\alpha(z,\omega))\Big]dz\, d\omega \\
&\le \int_{-\infty}^{+\infty}\frac{1}{a^2}\int_0^x \epsilon^{2x-2z}\Big[\widehat{F}(z,\omega,u(z,\omega)) - \widehat{F}(z,\omega,V_\epsilon^\alpha(z,\omega))\Big]^2 dz\, d\omega \\
&= \frac{1}{a^2}\epsilon^{2x}\int_0^x \epsilon^{-2z}\|F(z,.,U_\epsilon^\alpha)(z,\cdot) - F(z,\cdot,V_\epsilon^\alpha)(z,\cdot)\|_{L^2(\mathbb{R})}^2 dz \\
&\le \frac{K^2}{a^2}\epsilon^{2x}\int_0^x \epsilon^{-2z}\|u(z,\cdot) - V_\epsilon^\alpha(z,\cdot)\|_{L^2(\mathbb{R})}^2 dz.
\end{aligned} \tag{3.25}
$$

From the inequality $(b_1+b_2+b_3)^2 \le 2(1+p)b_1^2 + 2(1+\frac{1}{p})b_2^2 + 2(1+\frac{1}{p})b_3^2$ for any real numbers b_1, b_2, b_3 and $p>0$, we obtain

$$
\begin{aligned}
&\|u(x,\cdot) - V_\epsilon^\alpha(x,\cdot)\|_{L^2(\mathbb{R})}^2 \\
&\le 2(1+p)\|I_5(x,\cdot)\|_{L^2(\mathbb{R})}^2 + 2(1+\frac{1}{p})\|I_4(x,\cdot)\|_{L^2(\mathbb{R})}^2 + 2(1+\frac{1}{p})\|I_6(x,\cdot)\|_{L^2(\mathbb{R})}^2 \\
&\le 2(1+p)\epsilon^{2x}\|u(0,.)\|_{L^2(\mathbb{R})}^2 + 2(1+\frac{1}{p})\frac{K^2}{a^2}\epsilon^{2x}\int_x^1 \epsilon^{-2z}\|u(z,\cdot) - V_\epsilon^\alpha(z,\cdot)\|_{L^2(\mathbb{R})}^2 dz \\
&\quad + 2(1+\frac{1}{p})\frac{K^2}{a^2}\epsilon^{2x}\int_0^x \epsilon^{-2z}\|u(z,\cdot) - V_\epsilon^\alpha(z,\cdot)\|_{L^2(\mathbb{R})}^2 dz
\end{aligned}
$$

$$= 2(1+p)\epsilon^{2x}\|u(0,.)\|^2_{L^2(\mathbb{R})} + 2(1+\frac{1}{p})\frac{K^2}{a^2}\epsilon^{2x}\int_0^1 \epsilon^{-2z}\|u(z,\cdot)-V_\epsilon^\alpha(z,\cdot)\|^2_{L^2(\mathbb{R})}dz.$$

This implies

$$\begin{aligned}&\epsilon^{-2x}\|u(x,\cdot)-V_\epsilon^\alpha(x,\cdot)\|^2_{L^2(\mathbb{R})}\\ &\le 2(1+p)\|u(0,\cdot)\|^2_{L^2(\mathbb{R})} + 2(1+\frac{1}{p})\frac{K^2}{a^2}\int_0^1 \epsilon^{-2z}\|u(z,\cdot)-V_\epsilon^\alpha(z,\cdot)\|^2_{L^2(\mathbb{R})}dz\end{aligned}$$

Set $\tilde{J}(x) = \epsilon^{-2x}\|u(x,\cdot)-V_\epsilon^\alpha(x,\cdot)\|^2_{L^2(\mathbb{R})}$. Since $u(x,\cdot), V_\epsilon^\alpha(x,\cdot) \in C([0,1];L^2(\mathbb{R}))$, we see the function $\tilde{J}$ is continous on $[0,1]$. and attains over there its maximum at some point $x_1 \in [0,1]$. It is obvious that

$$\tilde{J}(x_1) \le 2(1+p)\|u(0,.)\|^2_{L^2(\mathbb{R})} + 2(1+\frac{1}{p})\frac{K^2}{a^2}\tilde{J}(x_1).$$

Therefore,

$$\|u(x,\cdot)-V_\epsilon^\alpha(x,\cdot)\|^2_{L^2(\mathbb{R})} \le \tilde{J}(x_1)\epsilon^{2x} \le \frac{2(1+p)}{1-2(1+\frac{1}{p})\frac{K^2}{a^2}}\epsilon^{2x}\|u(0,\cdot)\|^2_{L^2(\mathbb{R})}. \tag{3.26}$$

Since (3.18), (3.26) and applying the triangle inequality, we obtain

$$\begin{aligned}\|u(x,\cdot)-U_\epsilon^\alpha(x,\cdot)\|_{L^2(\mathbb{R})} &\le \|u(x,\cdot)-V_\epsilon^\alpha(x,\cdot)\|_{L^2(\mathbb{R})} + \|U_\epsilon^\alpha(x,\cdot)-V_\epsilon^\alpha(x,\cdot)\|_{L^2(\mathbb{R})}\\ &\le \Big(\frac{2(1+p)}{1-2(1+\frac{1}{p})\frac{K^2}{a^2}}\Big)^{1/2}\Big(\epsilon^{-1}\alpha + \|u(0,.)\|_{L^2(\mathbb{R})}\Big)\epsilon^x.\end{aligned}$$

□

Remark 3.3. For the estimation in the case $x=0$, we can use the technique in [19].

4. Numerical experiments

To verify our proposed methods, we carry out the numerical experiment for the above regularization methods. The numerical example is implemented for $t \in (0,2\pi)$, $x \in (0,1)$. In order to illustrate the sensitivity of the computational accuracy to the noise of the measurement data, we use the random function to generate the noisy data similar to an observation data. The perturbation was defined as $\epsilon rand()$, where rand(size()) is a random number, and ϵ plays as an amplitude of the errors. The approximation of the regularization solution is computed by discrete Fourier algorithm. In the example, we consider an inverse problem for the time-fractional diffusion equation in a one-dimensional semi-infinite domain as follows:

$$\begin{gathered}-au_x(x,t) = D_t^\gamma u(x,t) + u + H(x,t), \quad x>0, t>0,\\ u(1,t) = g(t), \quad t \ge 0,\\ \lim_{x\to+\infty} u(x,t) = u(x,0) = 0, \quad t\ge 0,\end{gathered} \tag{4.1}$$

where

$$\begin{gathered}H(x,t) = \Big(\frac{a}{2}-1\Big)\exp(-\frac{x}{2})t^2 - 2\frac{t^{2-\gamma}}{\Gamma(3-\gamma)}\exp(-\frac{x}{2}),\\ g(t) = \exp(-0.5)t^2\,.\end{gathered} \tag{4.2}$$

The exact solution of (4.1) is $u(x,t)=\exp(-\frac{x}{2})t^2$. The measured data g_α is

$$g_\alpha(t)=g(t)+\alpha\,\mathrm{rand}(\mathrm{size}(g))\,, \tag{4.3}$$

where α indicates that the error level of g and the symbol rand(size()) is a random number in $[-1,1]$. Assume that the constant diffusivity coefficient a is 2.1. Now, we study the numerical results for t bounded. Let us choose $T=2\pi$. According to Theorem 3.1, the regularized solution is

$$\begin{aligned}
\widehat{U_\epsilon^\alpha}(x,\omega)&=\frac{\exp\big(\frac{(i\omega)^\gamma(1-x)}{a}\big)}{1+\alpha\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)}\widehat{g_\alpha}(\omega)\\
&-\frac{1}{a}\int_x^1\frac{\exp\big(\frac{(i\omega)^\gamma(z-x)}{a}\big)}{1+\alpha\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)}\widehat{F}(z,\omega,U_\epsilon^\alpha(z,\omega))dz\\
&+\frac{1}{a}\int_0^x\frac{\alpha\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)}{1+\alpha\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)}\exp\Big(\frac{(i\omega)^\gamma(z-x)}{a}\Big)\\
&\times\widehat{F}(z,\omega,U_\epsilon^\alpha(z,\omega))dz\,.
\end{aligned}\tag{4.4}$$

In general, the whole numerical procedure is proceeded in the following steps:

Step 1. Choose I and J to generate spatial and temporal discretizations as follows

$$\begin{gathered}
x_i=i\Delta x,\quad \Delta x=\frac{1}{I},\quad i=\overline{0,I},\\
t_j=j\Delta t,\quad \Delta t=\frac{2\pi}{J},\quad j=\overline{0,J}\,.
\end{gathered}$$

Of course, the higher value of I and J will provide more stable and accurate numerical calculation, however in the following examples $I=J=101$ are chosen.

Step 2. We choose H^α as the observed data including the noise in the manner that

$$H^\alpha(\cdot,\cdot)=H(\cdot,\cdot)+\alpha\,\mathrm{rand}(\cdot). \tag{4.5}$$

Step 3. Errors between the exact and its regularized solutions are estimated by the relative error estimation

$$E(x)=\frac{\Big(\sum_{j=0}^J|U^\epsilon(:,t_j)-u(:,t_j)|^2\Big)^{1/2}}{\Big(\sum_{j=0}^J|u(:,t_j)|^2\Big)^{1/2}}\,. \tag{4.6}$$

From (4.1) and (4.2), we know that

$$F(x,t,U_\epsilon^\alpha(x,t))=U_\epsilon^\alpha(x,t)+H(x,t)\cdot \tag{4.7}$$

Combining (4.4) and (4.7), we can rewrite the regularized solution as follows

$$\widehat{U_\epsilon^\alpha}(x,\omega)=\Phi_1(\gamma,\alpha,a,\omega)\widehat{W}_{1,\omega}+\Phi_2(\gamma,\alpha,a,\omega)\widehat{W}_{2,\omega}\cdot \tag{4.8}$$

where

$$
\begin{aligned}
\Phi_1(\gamma,\alpha,a,\omega) &= \frac{\exp\big(\frac{(i\omega)^\gamma(1-x)}{a}\big)}{1+\alpha\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)},\\
\Phi_2(\gamma,\alpha,a,\omega) &= \frac{\alpha\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)\exp\big(\frac{1}{a}(i\omega)^\gamma(1-x)\big)}{1+\alpha\exp\big(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\big)},\\
\widehat{W}_{1,\omega} &= \Big(\widehat{g_\alpha}(\omega)-\frac{1}{a}\int_x^1\exp\Big(\frac{(i\omega)^\gamma(z-1)}{a}\Big)\Big(\widehat{U}_\epsilon^\alpha(z,\omega)+\widehat{H}(z,\omega)\Big)dz\Big),\\
\widehat{W}_{2,\omega} &= \int_0^x\exp\Big(\frac{(i\omega)^\gamma(z-1)}{a}\Big)\Big(\widehat{U}_\epsilon^\alpha(z,\omega)+\widehat{H}(z,\omega)\Big)dz\,.
\end{aligned}
\tag{4.9}
$$

Next, the integral equation (4.4) is calculated as follows:

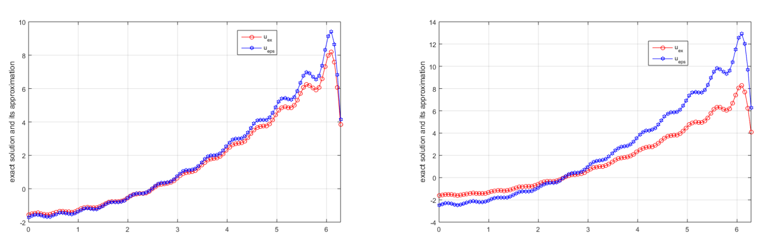
(1) We compute the Fourier transform of the function $g(t)$ and $H(x,t)$
(2) Next, we compute the Fourier transform of the function U_ϵ^α, and to control the nonlinear term, we use Gauss-Legendre quadrature method (see [17]).
(3) Finally, we integrate $\exp\big(\frac{(i\omega)^\gamma(z-1)}{den}\big)\big(\widehat{U}_\epsilon^\alpha(z,\omega)+\widehat{H}(z,\omega)\big)$ to obtain $\widehat{W}_{1,\omega}$ and $\widehat{W}_{2,\omega}$, then we multiply $\Phi_1(\gamma,\alpha,a,\omega)$ by $\widehat{W}_{1,\omega}$, and $\Phi_2(\gamma,\alpha,a,\omega)$ by $\widehat{W}_{2,\omega}$. Using (4.8), we have the result in equation (4.4).

TABLE 1. Relative error estimates between exact and regularized solutions at $\gamma = 0.1$.

	$\alpha = 0.1093$	$\alpha = 0.05$	$\alpha = 0.01$
$x = 0.5$	0.664	0.283	0.423
$x = 0.7$	0.443	0.671	0.759

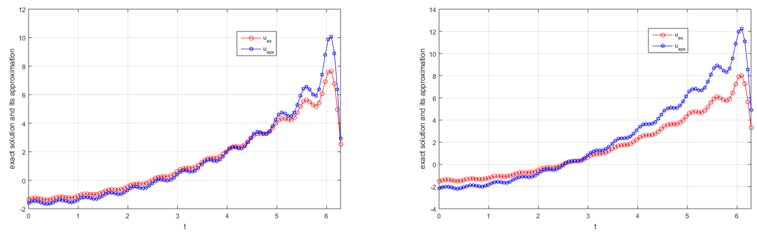
TABLE 2. Relative error estimates between exact and regularized solutions at $\gamma = 0.3$.

	$\alpha = 0.1093$	$\alpha = 0.05$	$\alpha = 0.01$
$x = 0.5$	0.459	0.095	0.186
$x = 0.7$	0.340	0.633	0.655



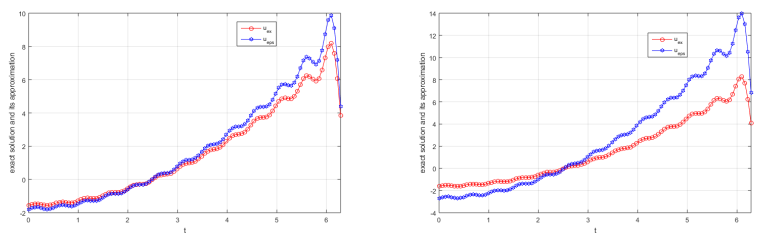
$x = 0.5$ $x = 0.7$

FIGURE 1. 2D graphs of exact and regularized solutions with $\gamma = 0.1$ and $\alpha = 0.1093$.



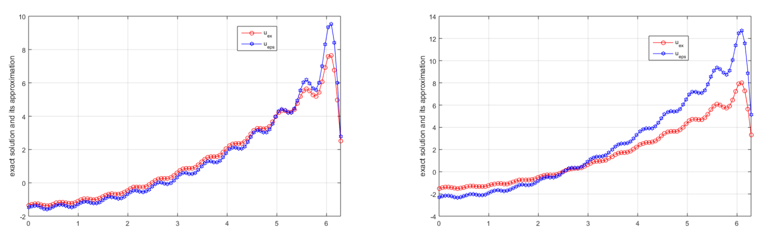
$x = 0.5$ $x = 0.7$

FIGURE 2. 2D graphs of exact and regularized solutions with $\gamma = 0.3$ and $\alpha = 0.1093$.



$x = 0.5$ $x = 0.7$

FIGURE 3. 2D graphs of exact and regularized solutions with $\gamma = 0.1$ and $\alpha = 0.05$.



$x = 0.5$ $x = 0.7$

FIGURE 4. 2D graphs of exact and regularized solutions with $\gamma = 0.3$ and $\alpha = 0.05$.

Tables 1 and 2 show the relative and absolute error estimates between the exact and regularized solutions with $\gamma = 0.1$ and $\gamma = 0.3$, respectively. They clearly show that the regularized solution converges to the exact solution with different values of γ. Figures 1–6 show a comparison between the exact and regularized solutions for several values of α and γ. We can see that numerical accuracy becomes worse as the order of the Caputo fractional derivative increases.

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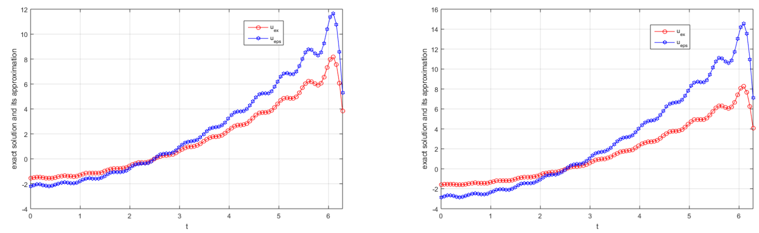
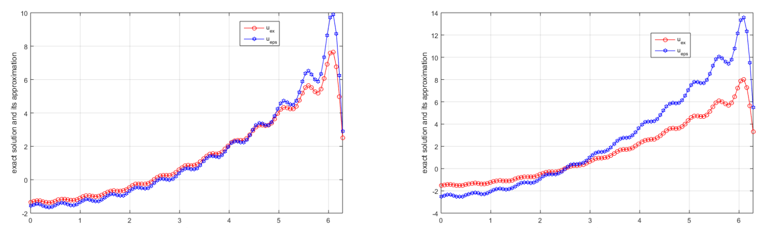


$x = 0.5$ $x = 0.7$

FIGURE 5. 2D graphs of exact solution and regularized solutions with $\gamma = 0.1$ and $\alpha = 0.01$.



$x = 0.5$ $x = 0.7$

FIGURE 6. 2D graphs of exact and regularized solutions with $\gamma = 0.3$ and $\alpha = 0.01$.

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