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EXISTENCES AND UPPER SEMI-CONTINUITY OF PULLBACK ATTRACTORS IN $H^1(\mathbb{R}^N)$ FOR NON-AUTONOMOUS REACTION-DIFFUSION EQUATIONS PERTURBED BY MULTIPLICATIVE NOISE

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Abstract. In this article, we establish sufficient conditions on the existence and upper semi-continuity of pullback attractors in some *non-initial spaces* for non-autonomous random dynamical systems. As an application, we prove the existence and upper semi-continuity of pullback attractors in $H^1(\mathbb{R}^N)$ are proved for stochastic non-autonomous reaction-diffusion equation driven by a Wiener type multiplicative noise as well as a non-autonomous forcing. The asymptotic compactness of solutions in $H^1(\mathbb{R}^N)$ is proved by the well-known tail estimate technique and the estimate of the integral of L^{2p-2}-norm of truncation of solutions over a compact interval.

1. Introduction

In this paper, we consider the dynamics of solutions of the reaction-diffusion equation on $\mathbb{R}^N$ driven by a random noise as well as a deterministic non-autonomous forcing,

$$du + (\lambda u - \Delta u)dt = f(x, u)dt + g(t, x)dt + \varepsilon u \circ d\omega(t), \tag{1.1}$$

with the initial value

$$u(\tau, x) = u_0(x), \quad x \in \mathbb{R}^N, \tag{1.2}$$

where $u_0 \in L^2(\mathbb{R}^N)$, λ is a positive constant, ε is the intensity of noise, the unknown $u = u(x, t)$ is a real valued function of $x \in \mathbb{R}^N$ and $t > \tau$, $\omega(t)$ is a mutually independent two-sided real-valued Wiener process defined on a canonical Wiener probability space $(\Omega, \mathcal{F}, P)$.

The notion of random attractor of random dynamical system, which is introduced in [5, 6, 7, 15] and systematically developed in [1, 4], is an important tool to study the qualitative property of stochastic partial differential equations (SPDE) . We can find a large body of literature investigating the existence of random attractors in *an initial space* (the initial values located space) for some concrete SPDE, see [2, 9, 30, 18, 20, 22, 24, 25] and the references therein. In particular, [18, 19, 21] discussed the upper semi-continuity of a family of random attractors in the initial spaces.

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As we know, the solutions of SPDE may possess some regularities, for example, higher-order integrability or higher-order differentiability. In these cases, the the solutions may escape (or leave) the initial space and enter into another space, which we call *a non-initial space*. Thus it is interesting for us to further investigate the existence and upper semi-continuity of random attractors in a non-initial space, usually a higher-regularity space, e.g., $L^p(p>2)$ and H^1.

Recently in the case of bounded domain, Li *et al* [12, 10] discussed the existence of random attractor of stochastic reaction-diffusion equations in the non-initial spaces L^p, where p is the growth exponent of the nonlinearity. Zhao [28] investigated the existence of random attractor in H_0^1 for stochastic two dimensional micropolar fluid flows with coupled additive noises. When the state space is unbounded, Zhao and Li [27] proved the existence of random attractors for reaction-diffusion equations with additive noises in $L^p(\mathbb{R}^N)$, and for the same equation Li *et al* [11] obtained the upper semi-continuity of random attractor in $L^p(\mathbb{R}^N)$. Most recently Zhao [26, 29] proved the existence of random attractors for semi-linear degenerate parabolic equations in $L^{2p-2}(D)\cap H^1(D)$, where D is a unbounded domain. By using the notion of omega-limit compactness, Li [13] obtained the existence of random attractors in $L^q(\mathbb{R}^N)$ for semilinear Laplacian equations with multiplicative noise. Tang [16] considered the existence of random attractors for non-autonomous Fitzhugh-Nagumo system driven by additive noises in $H^1(\mathbb{R}^N)\times L^2(\mathbb{R}^N)$, and his work [17] investigated the random dynamics of stochastic reaction-diffusion equations with *additive noises* in $H^1(\mathbb{R}^N)$. However, it seems that the proofs in [16, 17] are essentially wrong, see Li and Yin [14] for the modified proof.

In this article, we study the existence and upper semi-continuity of pullback (random) attractors in $H^1(\mathbb{R}^N)$ for stochastic reaction diffusion equations with *multiplicative noise* as well as a non-autonomous forcing. The nonlinearity f and the deterministic non-autonomous function g satisfy almost the same conditions as in [18], in which the author obtained the existence and upper-continuity of pullback attractors in the initial space $L^2(\mathbb{R}^N)$. Here, we develop their results and show that such attractors are also compact and attracting in $H^1(\mathbb{R}^N)$. Furthermore, we find that the upper continuity can also happen in $H^1(\mathbb{R}^N)$. We recall that the existence of pullback attractors in an initial space for a non-autonomous SPDE is established in [19, 20], where the measurability of such attractors is proved. The applications we may see [9, 18, 19, 20] and so forth. For the theory of the upper semi-continuity of attractors, we may refer to [18, 19, 21] for the stochastic case and to [3, 8] for the deterministic case.

To solve our problem, we establish a sufficient criteria for the existence and upper semi-continuity of pullback attractors in a non-initial space. It is showed that a family of such attractors obtained in an initial space are compact, attracting and upper semi-continuous in a non-initial space if some compactness conditions of the cocycle are satisfied, see Theorems 2.6–2.8. This implies that the continuity (or quasi-continuity [12], norm-weak continuity [32]) and absorption in the non-initial space are unnecessary things. This result is a meaningful and convenient tool for us to consider the existence and upper semi-continuity of pullback attractors in some related non-initial spaces for SPDE with a non-autonomous forcing term.

Considering that the stochastic equation (1.1) is defined on unbounded domains, the asymptotic compactness of solution in $H^1(\mathbb{R}^N)$ can not be derived by the traditional technique. The reasons are as follows. On the one hand, the equation (1.1) is stochastic and the Wiener process ω is only continuous in t but not differentiable. This leads to some difficulties for us to estimate the norm of derivative u_t by the trick employed in [31, 32] in the deterministic case. Then the asymptotic compactness in $H^1(\mathbb{R}^N)$ can not be proved by estimate of the difference of ∇u as in [31]. On the other hand, the estimate of Δu is not available for our problem (To our knoledge, actually we do not know how to estimate the norm Δu of problem (1.1) and (1.2), although this can be achieved by estimate u_t in the deterministic case, see [32]). Hence the Sobolev compact embeddings of $H^2 \hookrightarrow H^1$ on bounded domains is unavailable.

Here we give a new method to prove the asymptotical compactness of solutions in $H^1(\mathbb{R}^N)$. We first prove that the solutions vanish outside a ball centred at zero in the state space $\mathbb{R}^N$ in the topology of H^1 when both the time and the radius of ball are large enough, see Proposition 4.4. Second by a new developed estimate (where the minus or plus sign of nonlinearity is not required) we show that the integral of L^{2p-2}-norm of truncation of solutions over a compact interval is small for a large time, see Proposition 4.5. From these facts and along with some spectral arguments the asymptotic compactness of solutions on bounded domains is followed, and then the obstacles encountered in [16, 17] are overcome. The technique used here (without assumption that $\psi_1 \in L^\infty$, see (3.1)) is different from that in [14] and thus is optimal.

In the next section, we recall some notions and prove a sufficient standard for the existence and upper semi-continuity of pullback attractors of non-autonomous system in a non-initial space. In section 3, we give the assumptions on g and f, and define a continuous cocycle for problem (1.1) and (1.2). In section 4 and 5, we prove the existence and upper semi-continuity for this cocycle in $H^1(\mathbb{R}^N)$.

2. Preliminaries and abstract results

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two completely separable Banach spaces with Borel sigma-algebras $\mathcal{B}(X)$ and $\mathcal{B}(Y)$, respectively. $X \cap Y \neq \emptyset$. For convenience, we call X *an initial space* (which contains all initial values of a SPDE) and Y the associated *non-initial space* (usually the regular solutions located space).

In this section, we give a sufficient standard for the existence and upper semi-continuity of pullback attractors in the non-initial space Y for random dynamical system (RDS) over two parametric spaces. The readers may refer to [26, 27, 28, 10, 11, 12, 13, 23] for the existence and semi-continuity of such type attractors in the non-initial space Y for a RDS over one parametric space. The existence of random attractors in the initial space X for the RDS over one parametric space, the good references are [1, 2, 5, 15, 7, 6]. However, here we recall from [20] some basic notions for RDS over two parametric spaces, one of which is a real numbers space and the other of which is a measurable probability space.

2.1. **Preliminaries.** The basic notion in RDS is a metric dynamical system (MDS) $\vartheta \equiv (\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t\in\mathbb{R}})$, which is a probability space $(\Omega, \mathcal{F}, P)$ incorporating a group $\vartheta_t, t \in \mathbb{R}$, of measure preserving transformations on $(\Omega, \mathcal{F}, P)$. Sometimes, we call $\vartheta \equiv (\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t\in\mathbb{R}})$ a parametric dynamical system, see [18].

A MDS ϑ is said to be ergodic under P if for any ϑ-invariant set $F \in \mathcal{F}$, we have either $P(F) = 0$ or $P(F) = 1$, where the ϑ-invariant set is in the sense that $\vartheta_t F = F$ for $F \in \mathcal{F}$ and all $t \in \mathbb{R}$.

Definition 2.1. Let $(\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t\in\mathbb{R}})$ be a metric dynamical system. A family of measurable mappings $\varphi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ is called a cocycle on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t\in\mathbb{R}})$ if for all $\tau \in \mathbb{R}, \omega \in \Omega$ and $t, s \in \mathbb{R}^+$, the following conditions are satisfied:

$$\varphi(0, \tau, \omega, \cdot) \quad \text{is the identity on X},$$
$$\varphi(t + s, \tau, \omega, \cdot) = \varphi(t, \tau + s, \vartheta_s\omega, \cdot) \circ \varphi(s, \tau, \omega, \cdot).$$

In addition, if $\varphi(t, \tau, \omega, \cdot) : X \to X$ is continuous for all $t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega$, then φ is called a continuous cocycle on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t\in\mathbb{R}})$.

Definition 2.2. Let 2^X be the collection of all subsets of X. A set-valued mapping $K : \mathbb{R} \times \Omega \to 2^X$ is called measurable in X with respect to $\mathcal{F}$ in Ω if the mapping $\omega \in \Omega \mapsto \operatorname{dist}_X(x, K(\tau, \omega))$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable for every fixed $x \in X$ and $\tau \in \mathbb{R}$, where dist_X is the Haustorff semi-metric in X. In this case, we also say the family $\{K(\tau, \omega); \tau \in \mathbb{R}, \omega \in \Omega\}$ is measurable in X with respect to $\mathcal{F}$ in Ω. Furthermore if the value $K(\tau, \omega)$ is a closed nonempty subset of X for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$, then $\{K(\tau, \omega); \tau \in \mathbb{R}, \omega \in \Omega\}$ is called a closed measurable set of X with respect to $\mathcal{F}$ in Ω.

In this article, the cocycle φ acting on X is further assumed to take its values into the non-initial space Y in the following sense:

(H1) For every fixed $t > 0, \tau \in \mathbb{R}$ and $\omega \in \Omega$, $\varphi(t, \tau, \omega, \cdot) : X \to Y$.

We use $\mathfrak{D}$ to denote a collection of some families of nonempty subsets of X parametrized by $\tau \in \mathbb{R}$ and $\omega \in \Omega$ such that

$$\begin{aligned}\mathfrak{D} = \big\{B = \{B(\tau, \omega) \in 2^X; B(\tau, \omega) \neq \emptyset, \tau \in \mathbb{R}, \omega \in \Omega\big\};\\ f_B \text{ satisfies certain conditions}\}.\end{aligned}$$

In particular, for $B_1, B_2 \in \mathfrak{D}$ we say that $B_1 = B_2$ if $B_1(\tau, \omega) = B_2(\tau, \omega)$ for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$. The collection $\mathfrak{D}$ is called inclusion closed if $\tilde{B}(\tau, \omega) \subset B(\tau, \omega)$ and $B \in \mathfrak{D}$ for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, then $\tilde{B} \in \mathfrak{D}$.

Definition 2.3. Let $\mathfrak{D}$ be a collection of some families of nonempty subsets of X and $K = \{K(\tau, \omega); \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$. Then K is called a $\mathfrak{D}$-pullback absorbing set for a cocycle φ in X if for every $\tau \in \mathbb{R}, \omega \in \Omega$ and $B \in \mathfrak{D}$ there exists a absorbing time $T = T(\tau, \omega, B) > 0$ such that

$$\varphi(t, \tau - t, \vartheta_{-t}\omega, B(\tau - t, \vartheta_{-t}\omega)) \subseteq K(\tau, \omega) \quad \text{for all } t \geq T.$$

If in addition K is measurable in X with respect to $\mathcal{F}$ in Ω, then K is said to a measurable pullback absorbing set for φ.

Definition 2.4. Let $\mathfrak{D}$ be a collection of some families of nonempty subsets of X. A cocycle φ is said to be $\mathfrak{D}$-pullback asymptotically compact in X (resp. in Y) if for each $\tau \in \mathbb{R}, \omega \in \Omega$

$$\{\varphi(t_n, \tau - t_n, \vartheta_{-t_n}\omega, x_n)\} \text{ has a convergent subsequence in } X (\text{resp. in } Y)$$

whenever $t_n \to \infty$ and $x_n \in B(\tau - t_n, \vartheta_{-t_n}\omega)$ with $B = \{B(\tau, \omega); \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$.

Definition 2.5. Let $\mathfrak{D}$ be a collection of some families of nonempty subsets of X and $\mathcal{A} = \{\mathcal{A}(\tau, \omega); \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$. $\mathcal{A}$ is called a $\mathfrak{D}$-pullback attractor for a cocycle φ in X (resp. in Y) over $\mathbb{R}$ and $(\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t\in\mathbb{R}})$ if

(i) $\mathcal{A}$ is measurable in X with respect to $\mathcal{F}$, and $\mathcal{A}(\tau,\omega)$ is compact in X (resp. in Y) for each $\tau\in\mathbb{R},\omega\in\Omega$;

(ii) $\mathcal{A}$ is invariant, that is, for each $\tau\in\mathbb{R},\omega\in\Omega$,
$$\varphi(t,\tau,\omega,\mathcal{A}(\tau,\omega))=\mathcal{A}(\tau+t,\vartheta_t\omega),\quad \forall\ t\geq 0;$$

(iii) $\mathcal{A}$ attracts every element $B=\{B(\tau,\omega);\tau\in\mathbb{R},\omega\in\Omega\}\in\mathfrak{D}$ in X (resp. in Y), that is, for each $\tau\in\mathbb{R},\omega\in\Omega$,
$$\lim_{t\to+\infty}\operatorname{dist}_X(\varphi(t,\tau-t,\vartheta_{-t}\omega,B(\tau-t,\vartheta_{-t}\omega)),\mathcal{A}(\tau,\omega))=0$$
$$(\text{resp. } \lim_{t\to+\infty}\operatorname{dist}_Y(\varphi(t,\tau-t,\vartheta_{-t}\omega,B(\tau-t,\vartheta_{-t}\omega)),\mathcal{A}(\tau,\omega))=0).$$

2.2. **Existence of random attractors in a non-initial space.** This subsection is concerned with the existence of $\mathfrak{D}$-pullback attractor of the cocycle φ in the non-initial space Y. The continuity of φ in Y is not clear, and the embedding relation of X and Y is also unknown except that the following hypothesis (H2) holds:

(H2) If $\{x_n\}_n\subset X\cap Y$ such that $x_n\to x$ in X and $x_n\to y$ in Y respectively, then $x=y$.

Theorem 2.6. *Let $\mathfrak{D}$ be a collection of some families of nonempty subsets of X which is inclusion closed. Let φ be a continuous cocycle on X over $\mathbb{R}$ and $(\Omega,\mathcal{F},P,\{\vartheta_t\}_{t\in\mathbb{R}})$. Assume that*

(i) *φ has a closed and measurable $\mathfrak{D}$-pullback bounded absorbing set $K=\{K(\tau,\omega);\tau\in\mathbb{R},\omega\in\Omega\}\in\mathfrak{D}$ in X;*

(ii) *φ is $\mathfrak{D}$-pullback asymptotically compact in X.*

Then the cocycle φ has a unique $\mathfrak{D}$-pullback attractor $\mathcal{A}_X=\{\mathcal{A}_X(\tau,\omega);\tau\in\mathbb{R},\omega\in\Omega\}\in\mathfrak{D}$ in X, structured by
$$\mathcal{A}_X(\tau,\omega)=\cap_{s\geq 0}\overline{\cup_{t\geq s}\varphi(t,\tau-t,\vartheta_{-t}\omega,K(\tau-t,\vartheta_{-t}\omega))}^X,\quad \tau\in\mathbb{R},\omega\in\Omega,\tag{2.1}$$
where the closure is taken in X.

If further (H1), (H2) *hold and*

(iii) *φ is $\mathfrak{D}$-pullback asymptotically compact in Y,*

Then the cocycle φ has a unique $\mathfrak{D}$-pullback attractor $\mathcal{A}_Y=\{\mathcal{A}_Y(\tau,\omega);\tau\in\mathbb{R},\omega\in\Omega\}$ in Y, given by
$$\mathcal{A}_Y(\tau,\omega)=\cap_{s>0}\overline{\cup_{t\geq s}\varphi(t,\tau-t,\vartheta_{-t}\omega,K(\tau-t,\vartheta_{-t}\omega))}^Y,\quad \tau\in\mathbb{R},\omega\in\Omega.\tag{2.2}$$
In addition, we have $\mathcal{A}_Y=\mathcal{A}_X\subset X\cap Y$ in the sense of set inclusion, i.e., for each $\tau\in\mathbb{R},\omega\in\Omega$, $\mathcal{A}_Y(\tau,\omega)=\mathcal{A}_X(\tau,\omega)$.

Proof. The first result is well known and thus we are interested in the second result. Indeed, (2.2) makes sense by (H1) and $\mathcal{A}_Y\neq\emptyset$ by the asymptotic compactness of the cocycle φ in Y. In the following, we show that $\mathcal{A}_Y$ satisfies Definition 2.5 in the space Y.

Step 1. We claim that the set $\mathcal{A}_Y$ is measurable in X (with respect to $\mathcal{F}$ in Ω) and $\mathcal{A}_Y\in\mathfrak{D}$ is invariant by proving that $\mathcal{A}_Y=\mathcal{A}_X$ since $\mathcal{A}_X$ is measurable (w.r.t $\mathcal{F}$ in Ω) and $\mathcal{A}_X\in\mathfrak{D}$ is invariant (the measurability of $\mathcal{A}_X$ is proved by [19, Theorem 2.14]).

For each fixed $\tau \in \mathbb{R}$ and $\omega \in \Omega$, taking $x \in \mathcal{A}_X(\tau, \omega)$, by (2.1), there exist two sequences $t_n \to +\infty$ and $x_n \in K(\tau - t_n, \vartheta_{-t_n}\omega)$ such that

$$\varphi(t_n, \tau - t_n, \vartheta_{-t_n}\omega, x_n) \xrightarrow[n\to\infty]{\|\cdot\|_X} x. \tag{2.3}$$

Since φ is $\mathfrak{D}$-asymptotically compact in Y, then there is a $y \in Y$ such that up to a subsequence,

$$\varphi(t_n, \tau - t_n, \vartheta_{-t_n}\omega, x_n) \xrightarrow[n\to\infty]{\|\cdot\|_Y} y. \tag{2.4}$$

It implies from (2.2) that $y \in \mathcal{A}_Y(\tau, \omega)$. Then by (H2), along with (2.3) and (2.4), we have $x = y \in \mathcal{A}_X(\tau, \omega)$ and thus $\mathcal{A}_X(\tau, \omega) \subseteq \mathcal{A}_Y(\tau, \omega)$ for every fixed $\tau \in \mathbb{R}$ and $\omega \in \Omega$. The inverse inclusion can be proved in the same way then we omit it here. Thus $\mathcal{A}_X = \mathcal{A}_Y$ as required.

noindent**Step 2.** We prove the attraction of $\mathcal{A}_Y$ in Y by a contradiction argument. Indeed, if there exist $\delta > 0$, $x_n \in B(\tau - t_n, \vartheta_{-t_n}\omega)$ with $B \in \mathfrak{D}$ and $t_n \to +\infty$ such that

$$\mathrm{dist}_Y\Big(\varphi(t_n, \tau - t_n, \vartheta_{-t_n}\omega, x_n), \mathcal{A}_Y(\tau, \omega)\Big) \geq \delta. \tag{2.5}$$

By the asymptotic compactness of φ in Y, there exists $y_0 \in Y$ such that up to a subsequence,

$$\varphi(t_n, \tau - t_n, \vartheta_{-t_n}\omega, x_n) \xrightarrow[n\to\infty]{\|\cdot\|_Y} y_0. \tag{2.6}$$

On the other hand, by condition (i), there exists a large time $T > 0$ such that

$$\begin{aligned} y_n &= \varphi(T, \tau - t_n, \vartheta_{-t_n}\omega, x_n) \\ &= \varphi(T, (\tau - t_n + T) - T, \vartheta_{-T}\vartheta_{-(t_n - T)}\omega, x_n) \\ &\in K(\tau - t_n + T, \vartheta_{-(t_n - T)}\omega). \end{aligned} \tag{2.7}$$

Then by the cocycle property in Definition 2.1, with (2.6) and (2.7), we infer that as $t_n \to \infty$,

$$\varphi(t_n, \tau - t_n, \vartheta_{-t_n}\omega, x_n) = \varphi(t_n - T, \tau - (t_n - T), \vartheta_{-(t_n - T)}\omega, y_n) \to y_0 \quad \text{in } Y.$$

Therefore by (2.2), $y_0 \in \mathcal{A}_Y(\tau, \omega)$. This implies

$$\mathrm{dist}_Y\Big(\varphi(t_n, \tau - t_n, \vartheta_{-t_n}\omega, x_n), \mathcal{A}_Y(\tau, \omega)\Big) \to 0 \tag{2.8}$$

as $t_n \to \infty$, which is a contradiction to (2.5).

Step 3. It remains to prove the compactness of A_Y in Y. Let $\{y_n\}_{n=1}^{\infty}$ be a sequence in $A_Y(\tau, \omega)$. By the invariance of $A_Y(\tau, \omega)$ which is proved in Step 1, we have

$$\varphi(t, \tau - t, \vartheta_{-t}\omega, \mathcal{A}_Y(\tau - t, \vartheta_{-t}\omega)) = \mathcal{A}_Y(\tau, \omega).$$

Then it follows that there is a sequence $\{z_n\}_{n=1}^{\infty}$ with $z_n \in \mathcal{A}_Y(\tau - t_n, \vartheta_{-t_n}\omega)$ such that for every $n \in \mathbb{Z}^+$,

$$y_n = \varphi(t_n, \tau - t_n, \vartheta_{-t_n}\omega, z_n). \tag{2.9}$$

Note that $A_Y \in \mathfrak{D}$. Then by the asymptotic compactness of φ in Y, $\{y_n\}$ has a convergence subsequence in Y, *i.e.*, there is a $y_0 \in Y$ such that

$$\lim_{n\to\infty} y_n = y_0 \quad \text{in } Y.$$

But $A_Y(\tau, \omega)$ is closed in Y, so $y_0 \in A_Y(\tau, \omega)$.

The uniqueness is easily followed by the attraction property of φ and $A_Y \in \mathfrak{D}$. This completes the total proofs. $\square$

2.3. **Upper semi-continuity of random attractors in a non-initial space.** Assume that (H1) and (H2) hold. Given the indexed set $I \subset \mathbb{R}$, for every $\varepsilon \in I$, we use $\mathfrak{D}_\varepsilon$ to denote a a collection of some families of nonempty subsets of X. Let $\varphi_\varepsilon(\varepsilon \in I)$ be a continuous cocycle on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t\in\mathbb{R}})$. We now consider the upper semi-continuous of pullback attractors of a family of cocycle φ_ε in Y.

Suppose first that for every $t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega, \varepsilon_n, \varepsilon_0 \in I$ with $\varepsilon_n \to \varepsilon_0$, and $x_n, x \in X$ with $x_n \to x$, there holds

$$\lim_{n\to\infty} \varphi_{\varepsilon_n}(t, \tau, \omega, x_n) = \varphi_{\varepsilon_0}(t, \tau, \omega, x) \quad \text{in } X. \tag{2.10}$$

Suppose second that there exists a map $R_{\varepsilon_0} : \mathbb{R} \times \Omega \to \mathbb{R}^+$ such that the family

$$B_0 = \{B_0(\tau, \omega) = \{x \in X; \|x\|_X \le R_{\varepsilon_0}(\tau, \omega)\} : \tau \in \mathbb{R}, \omega \in \Omega\} \tag{2.11}$$

belongs to $\mathfrak{D}_{\varepsilon_0}$. And further for every $\varepsilon \in I$, φ_ε has $\mathfrak{D}_\varepsilon$-pullback attractor $\mathcal{A}_\varepsilon \in \mathfrak{D}_\varepsilon$ in $X \cap Y$ and a closed and measurable $\mathfrak{D}_\varepsilon$-pullback absorbing set $K_\varepsilon \in \mathfrak{D}_\varepsilon$ in X such that for every $\tau \in \mathbb{R}, \omega \in \Omega$,

$$\limsup_{\varepsilon\to\varepsilon_0} \|K_\varepsilon(\tau, \omega)\| \le R_{\varepsilon_0}(\tau, \omega), \tag{2.12}$$

where $\|S\|_X = \sup_{x\in S} \|x\|_X$ for a set S. We finally assume that for every $\tau \in \mathbb{R}$, $\omega \in \Omega$,

$$\cup_{\varepsilon\in I}\mathcal{A}_\varepsilon(\tau, \omega) \text{ is precompact in } X, \text{ and} \tag{2.13}$$

$$\cup_{\varepsilon\in I}\mathcal{A}_\varepsilon(\tau, \omega) \text{ is precompact in } Y. \tag{2.14}$$

Then we have the upper semi-continuity in Y.

Theorem 2.7. *If* (2.10)–(2.13) *hold, then for each* $\tau \in \mathbb{R}$*,* $\omega \in \Omega$*,*

$$\lim_{\varepsilon\to\varepsilon_0} \operatorname{dist}_X(\mathcal{A}_\varepsilon(\tau, \omega), \mathcal{A}_{\varepsilon_0}(\tau, \omega)) = 0.$$

If further (H1)-(H2) hold and conditions (2.10)-(2.14) *are satisfied. Then for each* $\tau \in \mathbb{R}$*,* $\omega \in \Omega$*,*

$$\lim_{\varepsilon\to\varepsilon_0} \operatorname{dist}_Y(\mathcal{A}_\varepsilon(\tau, \omega), \mathcal{A}_{\varepsilon_0}(\tau, \omega)) = 0.$$

Proof. If (2.10)-(2.13) hold, the upper-continuous in X is proved in [18]. We only need to prove the upper semi-continuity of $\mathcal{A}_\varepsilon$ at $\varepsilon = \varepsilon_0$ in Y.

Suppose that there exist $\delta > 0$, $\varepsilon_n \to \varepsilon_0$ and a sequence $\{y_n\}$ with $y_n \in \mathcal{A}_{\varepsilon_n}(\tau, \omega)$ such that for all $n \in \mathbb{N}$,

$$\lim_{\varepsilon\to\varepsilon_0} \operatorname{dist}_Y(y_n, \mathcal{A}_{\varepsilon_0}(\tau, \omega)) \ge 2\delta. \tag{2.15}$$

Note that $y_n \in \mathcal{A}_{\varepsilon_n}(\tau, \omega) \subset \mathbb{A}(\tau, \omega) = \cup_{\varepsilon\in I}\mathcal{A}_\varepsilon(\tau, \omega)$. Then by (2.13) and (2.14) and using (H2), there exists a $y_0 \in X \cap Y$ such that up to a subsequence,

$$\lim_{n\to\infty} y_n = y_0 \quad \text{in } X \cap Y. \tag{2.16}$$

It suffices to show that $\operatorname{dist}_Y(y_0, \mathcal{A}_{\varepsilon_0}(\tau, \omega)) < \delta$. Given a positive sequence $\{t_m\}$ with $t_m \uparrow +\infty$ as $m \to \infty$. For $m = 1$, by the invariance of $\mathcal{A}_{\varepsilon_n}$, there exists a sequence $\{y_{1,n}\}$ with $y_{1,n} \in \mathcal{A}_{\varepsilon_n}(\tau - t_1, \vartheta_{-t_1}\omega)$ such that

$$y_n = \varphi_{\varepsilon_n}(t_1, \tau - t_1, \vartheta_{-t_1}\omega, y_{1,n}), \tag{2.17}$$

for each $n \in \mathbb{N}$. Since $y_{1,n} \in \mathcal{A}_{\varepsilon_n}(\tau - t_1, \vartheta_{-t_1}\omega) \subset \mathbb{A}(\tau - t_1, \vartheta_{-t_1}\omega)$, then by by (2.13) and (2.14) and using (H2), there is a $z_1 \in X \cap Y$ and a subsequence of $\{y_{1,n}\}$ such that

$$\lim_{n\to\infty} y_{1,n} = z_1 \quad \text{in } X \cap Y. \tag{2.18}$$

Then (2.10) and (2.18) imply

$$\lim_{n\to\infty} \varphi_{\varepsilon_n}(t_1, \tau - t_1, \vartheta_{-t_1}\omega, y_{1,n}) = \varphi_{\varepsilon_0}(t_1, \tau - t_1, \vartheta_{-t_1}\omega, z_1) \quad \text{in } X. \tag{2.19}$$

Thus by combining (2.16), (2.17) and (2.19) we obtain

$$y_0 = \varphi_{\varepsilon_0}(t_1, \tau - t_1, \vartheta_{-t_1}\omega, z_1). \tag{2.20}$$

Note that K_{ε_n} as a $\mathfrak{D}_{\varepsilon_n}$-pullback absorbing set in X absorbs $\mathcal{A}_{\varepsilon_n} \in \mathfrak{D}_{\varepsilon_n}$, *i.e.*, there is a $T = T(\tau, \omega, \mathcal{A}_{\varepsilon_n})$ such that for all $t \geq T$,

$$\varphi(t, \tau - t, \vartheta_{-t}\omega, \mathcal{A}_{\varepsilon_n}(\tau - t, \vartheta_{-t}\omega)) \subseteq K_{\varepsilon_n}(\tau, \omega). \tag{2.21}$$

Then by the invariance of $\mathcal{A}_{\varepsilon_n}(\tau, \omega)$, it follows from (2.21) that

$$\mathcal{A}_{\varepsilon_n}(\tau, \omega) \subseteq K_{\varepsilon_n}(\tau, \omega). \tag{2.22}$$

Since $y_{1,n} \in \mathcal{A}_{\varepsilon_n}(\tau - t_1, \vartheta_{-t_1}\omega) \subseteq K_{\varepsilon_n}(\tau - t_1, \vartheta_{-t_1}\omega)$, then by (2.18) and (2.12), we obtain

$$\begin{aligned} \|z_1\|_X &= \limsup_{n\to\infty} \|y_{1,n}\|_X \\ &\leq \limsup_{n\to\infty} \|K_{\varepsilon_n}(\tau - t_1, \vartheta_{-t_1}\omega)\|_X \\ &\leq R_{\varepsilon_0}(\tau - t_1, \vartheta_{-t_1}\omega). \end{aligned} \tag{2.23}$$

By an induction argument, for each $m \geq 1$, there is $z_m \in X \cap Y$ such that for all $m \in \mathbb{N}$,

$$y_0 = \varphi_{\varepsilon_0}(t_m, \tau - t_m, \vartheta_{-t_m}\omega, z_m), \tag{2.24}$$

$$\|z_m\|_X \leq R_{\varepsilon_0}(\tau - t_m, \vartheta_{-t_m}\omega). \tag{2.25}$$

Thus from (2.11) and (2.25), for each $m \in \mathbb{N}$,

$$z_m \in B_0(\tau - t_m, \vartheta_{-t_m}\omega). \tag{2.26}$$

We consider that the pullback attractor $\mathcal{A}_{\varepsilon_0}$ attracts every element in $\mathfrak{D}_{\varepsilon_0}$ in the topology of Y and connection with $B_0 \in \mathfrak{D}_{\varepsilon_0}$. Then $\mathcal{A}_{\varepsilon_0}$ attracts B_0 in the topology of Y. Therefore by (2.24) and (2.26) we have

$$\operatorname{dist}_Y(y_0, \mathcal{A}_{\varepsilon_0}(\tau, \omega)) = \operatorname{dist}_Y(\varphi_{\varepsilon_0}(t_m, \tau - t_m, \vartheta_{-t_m}\omega, z_m), \mathcal{A}_{\varepsilon_0}(\tau, \omega)) \to 0, \tag{2.27}$$

as $m \to \infty$. That is to say, $\operatorname{dist}_Y(y_0, \mathcal{A}_{\varepsilon_0}(\tau, \omega)) = \inf_{u \in \mathcal{A}_{\varepsilon_0}(\tau,\omega)} \|y_0 - u\|_Y = 0$ and thus we can choose a $u_0 \in \mathcal{A}_{\varepsilon_0}(\tau, \omega)$ such that

$$\|y_0 - u_0\|_Y \leq \delta. \tag{2.28}$$

Therefore, by (2.16) and (2.28), as $n \to \infty$,

$$\operatorname{dist}_Y(y_n, \mathcal{A}_{\varepsilon_0}(\tau, \omega)) \leq \|y_n - u_0\|_Y \leq \|y_n - y_0\|_Y + \delta \to \delta,$$

which is a contradiction to (2.15). This concludes the proof. $\square$

We next consider a special case of Theorem 2.7, in which case the limit cocycle φ_{ε_0} is independent of the parameter $\omega \in \Omega$. We call such φ_{ε_0} a deterministic non-autonomous cocycle on X over $\mathbb{R}$. That is to say, φ_{ε_0} satisfies the following two statements:

(i) $\varphi_0(0, \tau, \cdot)$ is the identity on X;

(ii) $\varphi_0(t+s,\tau,\cdot) = \varphi_0(t,\tau+s,\cdot)\circ\varphi_0(s,\tau,\cdot)$.

If $\varphi_0(t,\tau,.) : X \to X$ is continuous for every $t \in \mathbb{R}^+$ and $\tau \in \mathbb{R}$, then φ_{ε_0} is called a deterministic non-autonomous continuous cocycle on X over $\mathbb{R}$.

Let $\mathfrak{D}_{\varepsilon_0}$ be a collection of some families of nonempty subsets of X denoted by

$$\mathfrak{D}_{\varepsilon_0} = \{B = \{B(\tau) \neq \emptyset; B(\tau) \in 2^X, \tau \in \mathbb{R}\}; f_B \text{ satisfies certain conditions}\}.$$

A family $\mathcal{A}_{\varepsilon_0} \in \mathfrak{D}_{\varepsilon_0}$ is called a $\mathfrak{D}_{\varepsilon_0}$-pullback attractor of φ_{ε_0} in X (resp. in Y) if

(i) for each $\tau \in \mathbb{R}$, $\mathcal{A}_{\varepsilon_0}(\tau)$ is compact in X(resp. of Y);
(ii) $\varphi_{\varepsilon_0}(t,\tau,\mathcal{A}_{\varepsilon_0}(\tau)) = \mathcal{A}_{\varepsilon_0}(\tau+t)$ for all $t \in \mathbb{R}^+$ and $\tau \in \mathbb{R}$;
(iii) $\mathcal{A}_{\varepsilon_0}$ pullback attracts every element of $\mathfrak{D}_{\varepsilon_0}$ under the Hausdorff semi-metric of X (resp. of Y).

To obtain the convergence at $\varepsilon = \varepsilon_0$ in Y, we make some modifications of the conditions used in random case. We assume that for every $t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega, \varepsilon_n \in I$ with $\varepsilon_n \to \varepsilon_0$, and $x_n, x \in X$ with $x_n \to x$, it holds

$$\lim_{n\to\infty} \varphi_{\varepsilon_n}(t,\tau,\omega,x_n) = \varphi_{\varepsilon_0}(t,\tau,x) \quad \text{in } X. \tag{2.29}$$

There exists a map $R'_{\varepsilon_0} : \mathbb{R} \to \mathbb{R}$ such that the family

$$B'_0 = \{B'_0(\tau) = \{x \in X; \|x\|_X \le R'_{\varepsilon_0}(\tau)\}; \tau \in \mathbb{R}\} \text{ belongs to } \mathfrak{D}_{\varepsilon_0}. \tag{2.30}$$

For every $\varepsilon \in I$, φ_ε has a closed measurable $\mathfrak{D}_\varepsilon$-pullback absorbing set $K_\varepsilon = \{K_\varepsilon(\tau,\omega); \omega \in \Omega\} \in \mathfrak{D}_\varepsilon$ in X such that for every $\tau \in \mathbb{R}, \omega \in \Omega$,

$$\limsup_{\varepsilon\to\varepsilon_0} \|K_\varepsilon(\tau,\omega)\| \le R'_{\varepsilon_0}(\tau). \tag{2.31}$$

Then we have the following, which can be proved by a similar argument as Theorem 2.7 and so the proof is omitted.

Theorem 2.8. *If* (2.13) *and* (2.29)-(2.31) *hold, then for each* $\tau \in \mathbb{R}$, $\omega \in \Omega$,

$$\lim_{\varepsilon\to\varepsilon_0} \operatorname{dist}_X(\mathcal{A}_\varepsilon(\tau,\omega), \mathcal{A}_{\varepsilon_0}(\tau)) = 0.$$

If further (H1)-(H2) hold and conditions (2.14) *and* (2.29)-(2.31) *are satisfied, then for each* $\tau \in \mathbb{R}, \omega \in \Omega$,

$$\lim_{\varepsilon\to\varepsilon_0} \operatorname{dist}_Y(\mathcal{A}_\varepsilon(\tau,\omega), \mathcal{A}_{\varepsilon_0}(\tau)) = 0. \tag{2.32}$$

3. Non-autonomous reaction-diffusion equation on $\mathbb{R}^N$ with multiplicative noise

For the non-autonomous reaction-diffusion equations (1.1) and (1.2), the nonlinearity $f(x,s)$ satisfies almost the same assumptions as in [18], i.e., for $x \in \mathbb{R}^N$ and $s \in \mathbb{R}$,

$$f(x,s)s \le -\alpha_1|s|^p + \psi_1(x), \tag{3.1}$$

$$|f(x,s)| \le \alpha_2|s|^{p-1} + \psi_2(x), \tag{3.2}$$

$$\frac{\partial f}{\partial s} f(x,s) \le \alpha_3, \tag{3.3}$$

$$\Big|\frac{\partial f}{\partial x} f(x,s)\Big| \le \psi_3(x), \tag{3.4}$$

where $\alpha_i > 0$ $(i = 1, 2, 3)$ are determined constants, $p \geq 2$, $\psi_1 \in L^1(\mathbb{R}^N) \cap L^{p/2}(\mathbb{R}^N)$, $\psi_2 \in L^2(\mathbb{R}^N)$ and $\psi_3 \in L^2(\mathbb{R}^N)$. And the non-autonomous term g satisfies that for every $\tau \in \mathbb{R}$ and some $\delta \in [0, \lambda)$,

$$\int_{-\infty}^{\tau} e^{\delta s} \|g(s, \cdot)\|^2_{L^2(\mathbb{R}^N)} ds < +\infty, \tag{3.5}$$

where λ is as in (1.1), which implies that

$$\int_{-\infty}^{0} e^{\delta s} \|g(s + \tau, \cdot)\|^2_{L^2(\mathbb{R}^N)} ds < +\infty, \quad g \in L^2_{Loc}(\mathbb{R}, L^2(\mathbb{R}^N)). \tag{3.6}$$

For the probability space $(\Omega, \mathcal{F}, P)$, we write $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}); \omega(0) = 0\}$. Let $\mathcal{F}$ be the Borel σ-algebra induced by the compact-open topology of Ω and P be the corresponding Wiener measure on $(\Omega, \mathcal{F})$. We define a shift operator ϑ on Ω by

$$\vartheta_t \omega(s) = \omega(s + t) - \omega(t), \quad \text{for every } \omega \in \Omega, t, s \in \mathbb{R}.$$

Then $(\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t \in \mathbb{R}})$ which is the model for random noise is called a metric dynamical system. Furthermore $(\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t \in \mathbb{R}})$ is ergodic with respect to $\{\vartheta_t\}_{t \in \mathbb{R}}$ under P, which means that every ϑ_t-invariant set has measure zero or one, $t \in \mathbb{R}$. By the law of the iterated logarithm (see [5]), we know that

$$\frac{\omega(t)}{t} \to 0, \quad \text{as } |t| \to +\infty. \tag{3.7}$$

For $\omega \in \Omega$, put $z(t, \omega) = z_\varepsilon(t, \omega) = e^{-\varepsilon \omega(t)}$. Then we have $dz + \varepsilon z \circ d\omega(t) = 0$. Put $v(t, \tau, \omega, v_0) = z(t, \omega) u(t, \tau, \omega, u_0)$, where u is a solution of problem (1.1) and (1.2) with the initial value u_0. Then v solves the non-autonomous equation

$$\frac{dv}{dt} + \lambda v - \Delta v = z(t, \omega) f(x, z^{-1}(t, \omega) v) + z(t, \omega) g(t, x), \tag{3.8}$$

with the initial value

$$v(\tau, x) = v_0(x) = z(\tau, \omega) u_0(x). \tag{3.9}$$

As pointed out in [18], for every $v_0 \in L^2(\mathbb{R}^N)$ we may show that the problem (3.8)-(3.9) possesses a continuous solution $v(\cdot)$ on $L^2(\mathbb{R}^N)$ such that $v(\cdot) \in C([\tau, +\infty), L^2(\mathbb{R}^N)) \cap L^2\text{loc}((\tau, +\infty), H^1(\mathbb{R}^N)) \cap L^p\text{loc}((\tau, +\infty), L^p(\mathbb{R}^N))$. In addition, the solution v is $(\mathcal{F}, \mathcal{B}(L^2(\mathbb{R}^N)))$-measurable in Ω. Then formally $u(\cdot) = z^{-1}(., \omega) v(\cdot)$ is a $(\mathcal{F}, \mathcal{B}(L^2(\mathbb{R}^N)))$-measurable and continuous solution of problem (1.1) and (1.2) on $L^2(\mathbb{R}^N)$ with $u_0 = z^{-1}(\tau, \omega) v_0$.

Define the mapping $\varphi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ such that

$$\begin{aligned} \varphi(t, \tau, \omega, u_0) &= u(t + \tau, \tau, \vartheta_{-\tau} \omega, u_0) \\ &= z^{-1}(t + \tau, \vartheta_{-\tau} \omega) v(t + \tau, \tau, \vartheta_{-\tau} \omega, z(\tau, \vartheta_{-\tau} \omega) u_0), \end{aligned} \tag{3.10}$$

where $u_0 = u_\tau \in L^2(\mathbb{R}^N)$ and $t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega$. Then by the measurability and continuity of v in $v_0 \in L^2(\mathbb{R}^N)$ and $t \in \mathbb{R}^+$, we see that the mappings φ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(L^2(\mathbb{R}^N))) \to \mathcal{B}(L^2(\mathbb{R}^N))$-measurable. That is to say, the mappings φ defined by (3.10) is a continuous cocycle on $L^2(\mathbb{R}^N)$ over $\mathbb{R}$ and $(\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t \in \mathbb{R}})$. Furthermore, from (3.10) we infer that

$$\begin{aligned} \varphi(t, \tau - t, \vartheta_{-t} \omega, u_0) &= u(\tau, \tau - t, \vartheta_{-\tau} \omega, u_0) \\ &= z(-\tau, \omega) v(\tau, \tau - t, \vartheta_{-\tau} \omega, z(\tau - t, \vartheta_{-\tau} \omega) u_0), \end{aligned} \tag{3.11}$$

where $u_0 = u_{\tau - t}$.

We define the collection $\mathfrak{D}$ as

$$\begin{aligned}\mathfrak{D} = \{B = \{B(\tau,\omega) \subseteq L^2(\mathbb{R}^N); \tau \in \mathbb{R}, \omega \in \Omega\};\\ \lim_{t\to+\infty} e^{-\lambda t} z^2(-t,\omega)\|B(\tau - t, \vartheta_{-t}\omega)\|^2 = 0 \text{ for } \tau \in \mathbb{R}, \omega \in \Omega\}\end{aligned} \tag{3.12}$$

where $\|B\| = \sup_{v\in B} \|v\|_{L^2(\mathbb{R}^N)}$ and λ is in (3.8). Note that this collection $\mathfrak{D}$ is much larger that the collection defined by [18]. That is to say, the collection $\mathfrak{D}$ defined above includes all tempered families of bounded nonempty subsets of $L^2(\mathbb{R}^N)$.

We can show that all the results in [18] hold for the collection $\mathfrak{D}$ defined by (3.12). Thus, the existence and upper semi-continuous of $\mathfrak{D}$-pullback attractors for the cocycle φ_ε in *the initial space* $L^2(\mathbb{R}^N)$ have been proved by [18].

Theorem 3.1 ([18]). *Assume that* (3.1)-(3.5) *hold. Then the cocycle φ_ε has a unique $\mathfrak{D}$-pullback attractor $\mathcal{A}_\varepsilon = \{\mathcal{A}_\varepsilon(\tau,\omega), \tau \in \mathbb{R}, \omega \in \Omega\}$ in $L^2(\mathbb{R}^N)$, given by*

$$\mathcal{A}_\varepsilon(\tau,\omega) = \cap_{s\ge 0}\overline{\cup_{t\ge s}\varphi(t, \tau - t, \vartheta_{-t}\omega, K_\varepsilon(\tau - t, \vartheta_{-t}\omega))}^{L^2(\mathbb{R}^N)}, \tag{3.13}$$

for $\tau \in \mathbb{R}$ and $\omega \in \Omega$, where K_ε is a closed and measurable $\mathfrak{D}$-pullback bounded absorbing set of φ_ε in $L^2(\mathbb{R}^N)$. Furthermore, $\mathcal{A}_\varepsilon$ is upper semi-continuous in $L^2(\mathbb{R}^N)$ at $\varepsilon = 0$.

Note that in most cases, we write v (resp. φ and z) as the abbreviation of v_ε (resp. φ_ε and z_ε). Next, we consider some applications of Theorems 2.6–2.8 to the non-autonomous stochastic reaction-diffusions (1.1) and (1.2). We emphasize that the result of Theorem 3.1 holds in the smooth functions space $H^1(\mathbb{R}^N)$. In particular, we prove the upper semi-continuity of the obtained attractors $\mathcal{A}_\varepsilon$ in $H^1(\mathbb{R}^N)$.

4. Existence of pullback attractor in $H^1(\mathbb{R}^N)$

In this section, we apply Theorem 2.6 to prove the existence of $\mathfrak{D}$-pullback attractors in $H^1(\mathbb{R}^N)$ for the cocycle defined in (3.10). To this end, we need to prove the uniform smallness of solutions outside a large ball under $H^1(\mathbb{R}^N)$ norm (see Proposition 4.4), and in the bounded ball of $\mathbb{R}^N$ we will prove the asymptotic compactness of solutions by space-splitting and function-truncation techniques (see Proposition 4.5 and Lemma 4.6).

We consider that $e^{-|\omega(s)|} \le z(s,\omega) = e^{-\varepsilon\omega(s)} \le e^{|\omega(s)|}$ for $\varepsilon \in (0,1]$, and that $\omega(s)$ is continuous function in s. Then there exist two positive random constants $E = E(\omega)$ and $F = F(\omega)$ depending only on ω such that for all $s \in [-2,0]$ and $\varepsilon \in (0,1]$.

$$0 < E \le z(s,\omega) \le F, \quad \omega \in \Omega. \tag{4.1}$$

Hereafter, we denote by $\|\cdot\|, \|\cdot\|_p$ and $\|\cdot\|_{H^1}$ the norms in $L^2(\mathbb{R}^N), L^p(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$, respectively. The numbers c and $C(\tau,\omega)$ are two generic positive constants which may have different values in different places even in the same line. The first one depends only on p, λ and $\alpha_i (i = 1,2,3)$, and the second one depends on τ, ω, p, λ and $\alpha_i (i = 1,2,3)$. We always assume $p > 2$ in the following discussions.

4.1. **H^1-tail estimate of solutions.** This can be achieved by a series of previously proved lemmas. First we stress that [18, Lemma 5.1] holds on the compact interval $[\tau - 1, \tau]$, which is necessary for us to estimate of the tail of solutions in $H^1(\mathbb{R}^N)$.

Lemma 4.1. *Assume that* (3.1) *and* (3.3)-(3.5) *hold. Let* $\tau \in \mathbb{R}, \omega \in \Omega$, $B = \{B(\tau,\omega); \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$ *and* $u_0 \in B(\tau - t, \vartheta_{-t}\omega)$. *Then there exists a constant* $T = T(\tau,\omega,B) \geq 2$ *such that for all* $t \geq T$, *the solution* v *of problem* (3.8) *and* (3.9) *satisfies that for every* $\zeta \in [\tau - 1, \tau]$,

$$\|v(\zeta, \tau - t, \vartheta_{-\tau}\omega, v_0)\|^2_{H^1(\mathbb{R}^N)} \leq L_1(\tau,\omega,\varepsilon), \tag{4.2}$$

$$\int_{\tau-2}^{\tau} \|v(s, \tau - t, \vartheta_{-\tau}\omega, v_0)\|_p^p ds \leq L_1(\tau,\omega,\varepsilon), \tag{4.3}$$

where $v_0 = z(\tau - t, \vartheta_{-\tau}\omega)u_0$ *and* $L_1(\tau,\omega,\varepsilon) =: cz^{-2}(-\tau,\omega)\int_{-\infty}^0 e^{\lambda s} z^2(s,\omega)(\|g(s + \tau, \cdot)\|^2 + 1)ds$.

The proof of the above lemma is similar to that of [18, Lemma 5.1], with a small modification, using $\zeta \in [\tau - 1, \tau]$ instead of τ.

Lemma 4.2. *Assume that* (3.1) *and* (3.3)-(3.5) *hold. Let* $\tau \in \mathbb{R}, \omega \in \Omega$ *and* $B = \{B(\tau,\omega); \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$. *Then for every* $\eta > 0$, *there exist two constants* $T = T(\tau,\omega,\eta,B) \geq 2$ *and* $R = R(\tau,\omega,\eta) > 1$ *such that the weak solution* v *of* (3.8) *and* (3.9) *satisfies that for all* $t \geq T$ *and* $k \geq R$,

$$\begin{aligned}
&\int_{|x|\geq k} |v(\tau, \tau - t, \vartheta_{-\tau}\omega, z(\tau - t, \vartheta_{-\tau}\omega)u_0)|^2 dx \\
&+ \int_{\tau-1}^{\tau} \int_{|x|\geq k} |\nabla v(s, \tau - t, \vartheta_{-\tau}\omega, z(\tau - t, \vartheta_{-\tau}\omega)u_0)|^2 \, dx \, ds \leq \eta,
\end{aligned}$$

where $u_0 \in B(\tau - t, \vartheta_{-t}\omega)$, R *and* T *are independent of* ε.

The proof of the above lemma is a simple modification of the proof of [18, Lemma 5.5].

Lemma 4.3. *Assume that* (3.1) *and* (3.3)-(3.5) *hold. Let* $\tau \in \mathbb{R}, \omega \in \Omega$ *and* $B = \{B(\tau,\omega); \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$. *Then there exists* $T = T(\tau,\omega,B) \geq 2$ *such that the weak solution* v *of problem* (3.8)-(3.9) *satisfies that for all* $t \geq T$,

$$\int_{\tau-1}^{\tau} \|v(s, \tau - t, \vartheta_{-\tau}\omega, z(\tau - t, \vartheta_{-\tau}\omega)u_0)\|_{2p-2}^{2p-2} ds \leq L_2(\tau,\omega,\varepsilon), \tag{4.4}$$

$$\int_{\tau-1}^{\tau} \|v_s(s, \tau - t, \vartheta_{-\tau}\omega, z(\tau - t, \vartheta_{-\tau}\omega)u_0)\|^2 ds \leq L_2(\tau,\omega,\varepsilon), \tag{4.5}$$

where $v_s = \frac{\partial v}{\partial s}$, $u_0 \in B(\tau - t, \vartheta_{-t}\omega)$ *and*

$$L_2(\tau,\omega,\varepsilon) =: C(\tau,\omega)\int_{-\infty}^0 e^{\lambda s}(z^2(s,\omega) + z^p(s,\omega))(\|g(s + \tau, \cdot)\|^2 + 1)ds. \tag{4.6}$$

Proof. In the sequel, we always regard v as a solution at the time t with the initial value $v_0 = v_{\tau-t}$ at the initial time $\tau - t$. We multiply (3.8) by $|v|^{p-2}v$ and then integrate over $\mathbb{R}^N$ to yield that

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|v\|_p^p + \lambda\|v\|_p^p \\
&\leq z(t,\omega)\int_{\mathbb{R}^N} f(x, z^{-1}v)|v|^{p-2}v \, dx + z(t,\omega)\int_{\mathbb{R}^N} |v|^{p-2}vg dx.
\end{aligned} \tag{4.7}$$

By using (3.1), we see that

$$
\begin{aligned}
&z(t,\omega)\int_{\mathbb{R}^N} f(x,z^{-1}v)|v|^{p-2}v\,dx\\
&\le -\alpha_1 z^{2-p}(t,\omega)\int_{\mathbb{R}^N}|v|^{2p-2}dx + z^2(t,\omega)\int_{\mathbb{R}^N}\psi_1(x)|v|^{p-2}dx\\
&\le -\alpha_1 z^{2-p}(t,\omega)\int_{\mathbb{R}^N}|v|^{2p-2}dx + \frac{\lambda}{2}\|v\|_p^p + \Big(\frac{2}{\lambda}\Big)^{-\frac{p-2}{2}} z^p(t,\omega)\|\psi_1\|_{p/2}^{p/2},
\end{aligned}
\tag{4.8}
$$

where the ϵ-Young's inequality are repeatedly used:

$$
|ab|\le \epsilon|a|^m+\epsilon^{-q/p}|b|^n,\quad \epsilon>0,\ m>1,n>1,\ \frac{1}{m}+\frac{1}{n}=1. \tag{4.9}
$$

At the same time, the last term on the right hand side of (4.7) is bounded as

$$
\begin{aligned}
&z(t,\omega)\int_{\mathbb{R}^N}|v|^{p-2}vg dx\\
&\le \frac{1}{2}\alpha_1 z^{2-p}(t,\omega)\int_{\mathbb{R}^N}|v|^{2p-2}dx + \frac{1}{2\alpha_1}z^p(t,\omega)\|g(t,\cdot)\|^2.
\end{aligned}
\tag{4.10}
$$

By a combination of (4.7)-(4.10), noticing that $p>2$, we obtain that

$$
\frac{d}{dt}\|v\|_p^p+\lambda\|v\|_p^p+\alpha_1 z^{2-p}(t,\omega)\|v\|_{2p-2}^{2p-2}\le cz^p(t,\omega)(\|g(t,\cdot)\|^2+1), \tag{4.11}
$$

where c only depends p,λ and α_1. Applying [26, Lemma 5.1](or [29]) over the interval $[\tau-2,\zeta]$, $\zeta\in[\tau-1,\tau]$, along with ω being replaced by $\vartheta_{-\tau}\omega$, we deduce that

$$
\begin{aligned}
&\|v(\zeta,\tau-t,\vartheta_{-\tau}\omega,v_0)\|_p^p\\
&\le \frac{e^{\lambda}}{\zeta-\tau+2}\int_{\tau-2}^{\tau} e^{\lambda(s-\tau)}\|v(s,\tau-t,\vartheta_{-\tau}\omega,v_0)\|_p^p ds\\
&\quad + ce^{\lambda}z^{-p}(-\tau,\omega)\int_{-\infty}^{0} e^{\lambda s}z^p(s,\omega)(\|g(s+\tau,\cdot)\|^2+1)ds.
\end{aligned}
\tag{4.12}
$$

Since $\frac{e^{\lambda}}{\zeta-\tau+2}\le 1$ for $\zeta\in[\tau-1,\tau]$, then by (4.3) and (4.12) we find that there exists $T>2$ such that for all $t\ge T$,

$$
\begin{aligned}
&\|v(\zeta,\tau-t,\vartheta_{-\tau}\omega,v_0)\|_p^p\\
&\le C(\tau,\omega)\int_{-\infty}^{0} e^{\lambda s}(z^2(s,\omega)+z^p(s,\omega))(\|g(s+\tau,\cdot)\|^2+1)ds.
\end{aligned}
\tag{4.13}
$$

Integrating (4.11) over the interval $[\tau-1,\tau]$, with ω replaced by $\vartheta_{-\tau}\omega$, yields

$$
\begin{aligned}
&\alpha_1\int_{\tau-1}^{\tau} z^{2-p}(s,\vartheta_{-\tau}\omega)\|v(s,\tau-t,\vartheta_{-\tau}\omega,v_0)\|_{2p-2}^{2p-2}ds\\
&\le \|v(\tau-1,\tau-t,\vartheta_{-\tau}\omega,v_0)\|_p^p + c\int_{\tau-1}^{\tau} z^p(s,\vartheta_{-\tau}\omega)(\|g(s,\cdot)\|^2+1)ds\\
&\le \|v(\tau-1,\tau-t,\vartheta_{-\tau}\omega,v_0)\|_p^p\\
&\quad + ce^{-\lambda}\int_{\tau-1}^{\tau} e^{\lambda(s-\tau)}z^p(s,\vartheta_{-\tau}\omega)(\|g(s,\cdot)\|^2+1)ds.
\end{aligned}
\tag{4.14}
$$

Then from (4.1), (4.13) and (4.14) we deduce for all $t \geq T$,

$$
\begin{aligned}
&\int_{\tau-1}^{\tau} \|v(s, \tau-t, \vartheta_{-\tau}\omega, v_0)\|_{2p-2}^{2p-2} ds \\
&\leq C(\tau, \omega) \int_{-\infty}^{0} e^{\lambda s}(z^2(s, \omega) + z^p(s, \omega))(\|g(s+\tau, \cdot)\|^2 + 1) ds,
\end{aligned}
$$

which proves (4.4).

To estimate the derivative v_t in $L^2\text{loc}(\mathbb{R}, L^2(\mathbb{R}^N))$, we multiply (3.8) by v_t and integrate over $\mathbb{R}^N$ to produce

$$
\begin{aligned}
&\|v_t\|^2 + \frac{1}{2}\frac{d}{dt}(\lambda\|v\|^2 + \|\nabla v\|^2) \\
&= z(t, \omega) \int_{\mathbb{R}^N} f(x, z^{-1}v) v_t dx + z(t, \omega) \int_{\mathbb{R}^N} g v_t dx \\
&\leq \frac{1}{2}\|v_t\|^2 + c\alpha_2^2 z^{4-2p}(t, \omega)\|v\|_{2p-2}^{2p-2} + cz^2(t, \omega)\|\psi_2\|^2 + cz^2(t, \omega)\|g(t, \cdot)\|^2,
\end{aligned}
$$

i.e., we have

$$
\begin{aligned}
&\|v_t\|^2 + \frac{d}{dt}(\lambda\|v\|^2 + \|\nabla v\|^2) \\
&\leq cz^{4-2p}(t, \omega)\|v\|_{2p-2}^{2p-2} + cz^2(t, \omega)(\|g(t, \cdot)\|^2 + \|\psi_2\|^2).
\end{aligned} \tag{4.15}
$$

Integrate (4.15) over the interval $[\tau-1, \tau]$ to obtain

$$
\begin{aligned}
&\int_{\tau-1}^{\tau} \|v_s(s, \tau-t, \vartheta_{-\tau}\omega, v_0)\|^2 ds \\
&\leq c\int_{\tau-1}^{\tau} z^{4-2p}(s, \vartheta_{-\tau}\omega)\|v(s, \tau-t, \vartheta_{-\tau}\omega, v_0)\|_{2p-2}^{2p-2} ds \\
&\quad + c\int_{\tau-1}^{\tau} z^2(s, \vartheta_{-\tau})(\|g(s, \cdot)\|^2 + 1) ds + c\|v(\tau-1, \tau-t, \vartheta_{-\tau}\omega, v_0)\|_{H^1}^2.
\end{aligned} \tag{4.16}
$$

Then by (4.1), (4.2), (4.4) and (4.16) we get that for all $t \geq T$,

$$
\begin{aligned}
&\int_{\tau-1}^{\tau} \|v_s(s, \tau-t, \vartheta_{-\tau}\omega, v_0)\|^2 ds \\
&\leq C(\tau, \omega) \int_{-\infty}^{0} e^{\lambda s}(z^2(s, \omega) + z^p(s, \omega))(\|g(s+\tau, \cdot)\|^2 + 1) ds,
\end{aligned} \tag{4.17}
$$

where T is as in Lemma 4.1. This completes the proof. □

We now can give the H^1-tail estimate of solutions of problem (3.8) and (3.9), which is one crucial condition for proving the asymptotic compactness in $H^1(\mathbb{R}^N)$.

Proposition 4.4. *Assume that* (3.1)-(3.5) *hold. Let* $\tau \in \mathbb{R}, \omega \in \Omega$ *and* $B = \{B(\tau, \omega); \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$. *Then for every* $\eta > 0$, *there exist two constants* $T = T(\tau, \omega, \eta, B) \geq 2$ *and* $R = R(\tau, \omega, \eta) > 1$ *such that the weak solution* v *of* (3.8) *and* (3.9) *satisfies that for all* $t \geq T$,

$$
\begin{aligned}
&\int_{|x|\geq R} \Big(|v(\tau, \tau-t, \vartheta_{-\tau}\omega, z(\tau-t, \vartheta_{-\tau}\omega)u_0)|^2 \\
&+ |\nabla v(\tau, \tau-t, \vartheta_{-\tau}\omega, z(\tau-t, \vartheta_{-\tau}\omega)u_0)|^2\Big) dx \leq \eta,
\end{aligned}
$$

where $u_0 \in B(\tau-t, \vartheta_{-t}\omega)$ *and* R, T *are independent of* ε.

Proof. We first need to define a smooth function $\xi(\cdot)$ on $\mathbb{R}^+$ such that

$$\xi(s) = \begin{cases} 0, & \text{if } 0 \le s \le 1, \\ 0 \le \xi(s) \le 1, & \text{if } 1 \le s \le 2, \\ 1, & \text{if } s \ge 2, \end{cases}$$

which obviously implies that there is a positive constant C_1 such that the $|\xi'(s)| + |\xi''(s)| \le C_1$ for all $s \ge 0$. For convenience, we write $\xi = \xi(\frac{|x|^2}{k^2})$.

We multiply (3.8) by $-\xi\Delta v$ and integrate over $\mathbb{R}^N$ to find that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} \xi|\nabla v|^2 dx + \int_{\mathbb{R}^N} (\nabla\xi.\nabla v)v_t dx + \lambda\int_{\mathbb{R}^N} \xi|\nabla v|^2 dx \\ &+ \lambda\int_{\mathbb{R}^N} (\nabla\xi.\nabla v)v\, dx + \int_{\mathbb{R}^N} \xi|\Delta v|^2 dx \\ &= -z(t,\omega)\int_{\mathbb{R}^N} f(x, z^{-1}v)\xi\Delta v\, dx - z(t,\omega)\int_{\mathbb{R}^N} g\xi\Delta v\, dx. \end{aligned} \tag{4.18}$$

Now, we estimate each term in (4.18) as follows. First we have

$$\begin{aligned} \Big|\int_{\mathbb{R}^N} (\nabla\xi.\nabla v)v_t dx + \lambda\int_{\mathbb{R}^N} (\nabla\xi.\nabla v)v\, dx\Big| &= \Big|\int_{\mathbb{R}^N} (v_t + \lambda v)(\frac{2x}{k^2}.\nabla v)\xi' dx\Big| \\ &\le \frac{c}{k}(\|v_t\|^2 + \|v\|_{H^1}^2). \end{aligned} \tag{4.19}$$

For the nonlinearity in (4.18), we see that

$$\begin{aligned} &-z\int_{\mathbb{R}^N} f(x, z^{-1}v)\xi\Delta v\, dx \\ &= z\int_{\mathbb{R}^N} f(x, z^{-1}v)(\nabla\xi.\nabla v)dx + z\int_{\mathbb{R}^N} (\frac{\partial}{\partial x} f(x, z^{-1}v).\nabla v)\xi dx \\ &\quad + \int_{\mathbb{R}^N} \frac{\partial}{\partial u} f(x, z^{-1}v)|\nabla v|^2 \xi dx. \end{aligned} \tag{4.20}$$

On the other hand, by using (3.2), (3.3) and (3.4), respectively, we calculate that

$$\begin{aligned} \Big|z\int_{\mathbb{R}^N} f(x, z^{-1}v)(\nabla\xi.\nabla v)dx\Big| &\le \frac{2z\sqrt{2}C_1}{k}\int_{k\le|x|\le\sqrt{2}k} |f(x, z^{-1}v)||\nabla v|dx \\ &\le \frac{c}{k}(z^{4-2p}\|v\|_{2p-2}^{2p-2} + z^2\|\psi_2\|^2 + \|\nabla v\|^2), \end{aligned} \tag{4.21}$$

$$\int_{\mathbb{R}^N} \frac{\partial}{\partial u} f(x, z^{-1}v)|\nabla v|^2 \xi dx \le \alpha_3\int_{\mathbb{R}^N} \xi|\nabla v|^2 dx, \tag{4.22}$$

$$\begin{aligned} \Big|z\int_{\mathbb{R}^N} (\frac{\partial}{\partial x} f(x, z^{-1}v).\nabla v)\xi dx\Big| &\le \Big|z\int_{\mathbb{R}^N} |\psi_3||\nabla v|\xi dx\Big| \\ &\le \frac{\lambda}{2}\int_{\mathbb{R}^N} \xi|\nabla v|^2 dx + cz^2\int_{\mathbb{R}^N} \xi|\psi_3|^2 dx. \end{aligned} \tag{4.23}$$

Then from 4.20)-(4.23) it follows that

$$
\begin{aligned}
&- z\int_{\mathbb{R}^N} f(x,z^{-1}v)\xi\Delta v\,dx\\
&\le \frac{c}{k}(z^{4-2p}\|v\|_{2p-2}^{2p-2}+z^2\|\psi_2\|^2+\|\nabla v\|^2)\\
&+\frac{\lambda}{2}\int_{\mathbb{R}^N}\xi|\nabla v|^2dx+cz^2\int_{\mathbb{R}^N}\xi|\psi_3|^2dx+\alpha_3\int_{\mathbb{R}^N}\xi|\nabla v|^2dx.
\end{aligned}\tag{4.24}
$$

For the last term on the right-hand side of (4.18), we have

$$
\Big|z\int_{\mathbb{R}^N} g\xi\Delta v\,dx\Big|\le\frac{\lambda}{2}\int_{\mathbb{R}^N}\xi|\Delta v|^2dx+\frac{1}{2\lambda}z^2\int_{\mathbb{R}^N}\xi|g|^2dx.\tag{4.25}
$$

Then we use (4.19) and (4.24)–(4.25) in (4.18) to find that

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^N}\xi|\nabla v|^2dx+\lambda\int_{\mathbb{R}^N}\xi|\nabla v|^2dx\\
&\le\frac{c}{k}(\|v_t\|^2+\|v\|_{H^1}^2+z^{4-2p}\|v\|_{2p-2}^{2p-2}+z^2\|\psi_2\|^2)\\
&+2\alpha_3\int_{\mathbb{R}^N}\xi|\nabla v|^2dx+cz^2\int_{\mathbb{R}^N}\xi(|\psi_3|^2+|g|^2)dx.
\end{aligned}\tag{4.26}
$$

Applying [26, Lemma 5.1] to (4.26) over the interval $[\tau-1,\tau]$, along with ω being replaced by $\vartheta_{-\tau}\omega$, we deduce that

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\xi|\nabla v(\tau,\tau-t,\vartheta_{-\tau}\omega,v_0)|^2dx\\
&\le\frac{c}{k}\int_{\tau-1}^{\tau}e^{\lambda(s-\tau)}\Big(\|v_s(s)\|^2+\|v(s)\|_{H^1}^2+z^{4-2p}(s,\vartheta_{-\tau}\omega)\|v(s)\|_{2p-2}^{2p-2}\\
&\quad+z^2(s,\vartheta_{-\tau}\omega)\|\psi_2\|^2\Big)ds+c\int_{\tau-1}^{\tau}e^{\lambda(s-\tau)}\int_{|x|\ge k}|\nabla v(s)|^2\,dx\,ds\\
&\quad+cz^{-2}(\tau,\omega)\int_{-1}^{0}e^{\lambda s}z^2(s,\omega)\int_{|x|\ge k}(|\psi_3|^2+|g(s+\tau,x)|^2)\,dx\,ds,
\end{aligned}\tag{4.27}
$$

where $v(s)=v(s,\tau-t,\vartheta_{-\tau}\omega,z(\tau-t,\vartheta_{-\tau}\omega)u_0)$. Our task in the following is to show that each term on the right hand side of (4.27) vanishes when t and k are larger. First, by Lemma 4.2, there are two constants $T_1=T_1(\tau,\omega,B,\eta)\ge 2$ and $R_1=R_1(\tau,\omega,\eta)\ge 1$ such that for all $t\ge T_1$ and $k\ge R_1$,

$$
\begin{aligned}
&c\int_{\tau-1}^{\tau}e^{\lambda(s-\tau)}\int_{|x|\ge k}|\nabla v(s)|^2\,dx\,ds\\
&\le c\int_{\tau-1}^{\tau}\int_{|x|\ge k}|\nabla v(s,\tau-t,\vartheta_{-\tau}\omega,v_0)|^2\,dx\,ds\le\frac{\eta}{6}.
\end{aligned}\tag{4.28}
$$

By (4.2) in Lemma 4.1, there exist $T_2=T_2(\tau,\omega,B)\ge 2$ and $R_2=R_2(\tau,\omega,\eta)\ge 1$ such that for all $t\ge T_2$ and $k\ge R_2$,

$$
\begin{aligned}
&\frac{c}{k}\int_{\tau-1}^{\tau}e^{\lambda(s-\tau)}\|v(s,\tau-t,\vartheta_{-\tau}\omega,v_0)\|_{H^1}^2ds\\
&\le\frac{c}{k}\int_{\tau-1}^{\tau}\|v(s,\tau-t,\vartheta_{-\tau}\omega,v_0)\|_{H^1}^2ds\le\frac{\eta}{6}.
\end{aligned}\tag{4.29}
$$

By Lemma 4.3, there exist $T_3 = T_3(\tau, \omega, B) \geq 2$ and $R_3 = R_3(\tau, \omega, \eta) \geq 1$ such that for all $t \geq T_3$ and $k \geq R_3$,

$$\begin{aligned} &\frac{c}{k}\int_{\tau-1}^{\tau} e^{\lambda(s-\tau)} z^{4-2p}(s, \vartheta_{-\tau}\omega)\|v(s, \tau-t, \vartheta_{-\tau}\omega, v_0)\|_{2p-2}^{2p-2} ds \\ &\leq \frac{c}{k} z^{2p-4}(-\tau, \omega) E^{4-2p} L_2(\tau, \omega, \varepsilon) \leq \frac{\eta}{6}, \end{aligned} \tag{4.30}$$

and

$$\frac{c}{k}\int_{\tau-1}^{\tau} e^{\lambda(s-\tau)} \|v_s(s, \tau-t, \vartheta_{-\tau}\omega, v_0)\|^2 ds \leq \frac{c}{k} L_2(\tau, \omega, \varepsilon) \leq \frac{\eta}{6}. \tag{4.31}$$

By the assumptions on ψ_3 and g, we deduce that there exist $R_4 = R_4(\tau, \omega, \eta)$ such that for all $k \geq R_4$,

$$c z^{-2}(\tau, \omega)\int_{-1}^{0} e^{\lambda s} z^2(s, \omega)\int_{|x|\geq k} (|\psi_3|^2 + |g(s+\tau, x)|^2)\, dx\, ds \leq \frac{\eta}{6}. \tag{4.32}$$

Obviously, there exists $R_5 = R_5(\tau, \omega, \eta)$ such that for all $k \geq R_5$,

$$\begin{aligned} &\frac{c}{k}\int_{\tau-1}^{\tau} e^{\lambda(s-\tau)} z^2(s, \vartheta_{-\tau}\omega)\|\psi_2\|^2 ds \\ &\leq \frac{c}{k}\|\psi_2\|^2 z^{-2}(-\tau, \omega)\int_{-1}^{0} z^2(s, \omega) ds \leq \frac{\eta}{6}, \end{aligned} \tag{4.33}$$

where $\int_{-1}^{0} z^2(s, \omega) ds < +\infty$. Finally, take

$$T = \{T_1, T_2, T_3\}, \quad R = \max\{R_1, R_2, R_3, R_4, R_5\}.$$

It is obvious that R and T are independent of the intension ε. Then (4.28)-(4.33) are integrated into (4.27) to get that for all $t \geq T$ and $k \geq R$,

$$\int_{|x|\geq\sqrt{2}k} |\nabla v(\tau, \tau-t, \vartheta_{-\tau}\omega, v_0)|^2 dx \leq \eta. \tag{4.34}$$

Then in connection with Lemma 4.2, the desired result is achieved. □

4.2. **Estimate of the truncation of solutions in L^{2p-2}.** Given u the solution of problem (1.1) and (1.2), for each fixed $\tau \in \mathbb{R}, \omega \in \Omega$, we write $M = M(\tau, \omega) > 1$ and

$$\mathbb{R}^N(|u(\tau, \tau-t, \vartheta_{-\tau}\omega, u_0)| \geq M) = \{x \in \mathbb{R}^N; |u(\tau, \tau-t, \vartheta_{-\tau}\omega, u_0)| \geq M|\}. \tag{4.35}$$

We introduce the truncation version of solutions of problem (3.8)-(3.9). Let $(v-M)_+$ be the positive part of $v-M$, i.e.,

$$(v-M)_+ = \begin{cases} v-M, & \text{if } v > M; \\ 0, & \text{if } v \leq M. \end{cases}$$

The next lemma shows that the integral of L^{2p-2}-norm of $|u|$ over the interval $[\tau-1, \tau]$ vanishes on the state domain $\mathbb{R}^N(|u(\tau, \tau-t, \vartheta_{-\tau}\omega), u_0)| \geq M)$ for M large enough, which is the second crucial condition for proving the asymptotic compactness of solutions in $H^1(\mathbb{R}^N)$.

Proposition 4.5. *Assume that* (3.1)-(3.5) *hold. Let* $\tau \in \mathbb{R}, \omega \in \Omega$,

$$B = \{B(\tau,\omega); \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$$

and $u_0 \in B(\tau - t, \vartheta_{-t}\omega)$. *Then for any* $\eta > 0$, *there exist constants* $\tilde{M} = \tilde{M}(\tau,\omega,\eta) > 1$ *and* $T = T(\tau,\omega,B) \geq 2$ *such that the solution* u *of problem* (3.8) *and* (3.9) *satisfies that for all* $t \geq T$ *and all* $\varepsilon \in (0,1]$,

$$\int_{\tau-1}^{\tau} e^{\tilde{\varrho}(s-\tau)} \int_{\mathcal{O}} |v(s,\tau-t,\vartheta_{-\tau}\omega, z(\tau-t,\vartheta_{-\tau}\omega)u_0)|^{2p-2}\,dx\,ds \leq \eta,$$

where $p > 2$, $\tilde{M}$ *and* T *are independent of* ε,

$$\mathcal{O} = \mathbb{R}^N(|v(s,\tau-t,\vartheta_{-\tau}\omega, z(\tau-t,\vartheta_{-\tau}\omega)u_0)| \geq \tilde{M})$$

and

$$\tilde{\varrho} = \tilde{\varrho}(\tau,\omega,\tilde{M}) = \alpha_1 F^{2-p} e^{-(p-2)|\omega(-\tau)|}\tilde{M}^{p-2}.$$

Proof. First, we replace ω by $\vartheta_{-\tau}\omega$ in (3.8) to see that

$$v = v(s) =: v(s,\tau-t,\vartheta_{-\tau}\omega, v_0), \quad s \in [\tau-1,\tau],$$

is a solution of the SPDE

$$\frac{dv}{ds} + \lambda v - \Delta v = \frac{z(s-\tau,\omega)}{z(-\tau,\omega)} f(x,u) + \frac{z(s-\tau,\omega)}{z(-\tau,\omega)} g(s,x), \tag{4.36}$$

with the initial data $v_0 = z(\tau-t,\vartheta_{-\tau}\omega)u_0$, where we have used $z(s,\vartheta_{-\tau}\omega) = \frac{z(s-\tau,\omega)}{z(-\tau,\omega)} > 0$.

We multiply (4.36) by $(v-M)_+^{p-1}$ and integrate over $\mathbb{R}^N$ to obtain that for every $s \in [\tau-1,\tau]$,

$$\begin{aligned}
&\frac{1}{p}\frac{d}{ds}\int_{\mathbb{R}^N}(v-M)_+^p dx + \lambda \int_{\mathbb{R}^N} v(v-M)_+^{p-1}dx - \int_{\mathbb{R}^N}\Delta v(v-M)_+^{p-1}dx \\
&= \frac{z(s-\tau,\omega)}{z(-\tau,\omega)}\int_{\mathbb{R}^N} f(x,u)(v-M)_+^{p-1}dx + \frac{z(s-\tau,\omega)}{z(-\tau,\omega)}\int_{\mathbb{R}^N} g(s,x)(v-M)_+^{p-1}dx.
\end{aligned} \tag{4.37}$$

We now need to estimate every term in (4.37). First, it is obvious that

$$-\int_{\mathbb{R}^N}\Delta v(v-M)_+^{p-1}dx = (p-1)\int_{\mathbb{R}^N}(v-M)_+^{p-2}|\nabla v|^2 dx \geq 0, \tag{4.38}$$

$$\lambda\int_{\mathbb{R}^N} v(v-M)_+^{p-1}dx \geq \lambda\int_{\mathbb{R}^N}(v-M)_+^p dx. \tag{4.39}$$

If $v > M$, then $u = z^{-1}(s,\vartheta_{-\tau}\omega)v > 0$. Therefore by assumption (3.1), we have

$$\begin{aligned}
f(x,u) &\leq -\alpha_1 u^{p-1} + \frac{\psi_1(x)}{u} \\
&= -\alpha_1\Big(\frac{z(s-\tau,\omega)}{z(-\tau,\omega)}\Big)^{1-p} v^{p-1} + \frac{z(s-\tau,\omega)}{z(-\tau,\omega)}\frac{\psi_1(x)}{v}.
\end{aligned} \tag{4.40}$$

Since $s \in [\tau-1,\tau]$ and $p > 2$, then by (4.1) we have

$$F^{2-p} \leq z^{2-p}(s-\tau,\omega) \leq E^{2-p},$$

from which and (4.40) it follows that

$$\frac{z(s-\tau,\omega)}{z(-\tau,\omega)} f(x,u)$$

$$
\begin{aligned}
&\le -\alpha_1\Big(\frac{z(s-\tau,\omega)}{z(-\tau,\omega)}\Big)^{2-p}v^{p-1}+\frac{z^2(s-\tau,\omega)}{z^2(-\tau,\omega)}\frac{\psi_1(x)}{v}\\
&= -\frac{\alpha_1}{2}\Big(\frac{z(s-\tau,\omega)}{z(-\tau,\omega)}\Big)^{2-p}v^{p-1}-\frac{\alpha_1}{2}\Big(\frac{z(s-\tau,\omega)}{z(-\tau,\omega)}\Big)^{2-p}v^{p-1}+\frac{z^2(s-\tau,\omega)}{z^2(-\tau,\omega)}\frac{\psi_1(x)}{v}\\
&\le -\frac{\alpha_1}{2}\frac{F^{2-p}}{z^{2-p}(-\tau,\omega)}M^{p-2}(v-M)-\frac{\alpha_1}{2}\frac{F^{2-p}}{z^{2-p}(-\tau,\omega)}(v-M)^{p-1}\\
&\quad+\frac{F^2}{z^2(-\tau,\omega)}|\psi_1(x)|(v-M)^{-1},
\end{aligned}
$$

which by the nonlinearity in (4.37) is estimated as

$$
\begin{aligned}
&\frac{z(s-\tau,\omega)}{z(-\tau,\omega)}\int_{\mathbb{R}^N} f(x,u)(v-M)_+^{p-1}dx\\
&\le -\frac{\alpha_1}{2}\frac{F^{2-p}}{z^{2-p}(-\tau,\omega)}M^{p-2}\int_{\mathbb{R}^N}(v-M)_+^p dx-\frac{\alpha_1}{2}\frac{F^{2-p}}{z^{2-p}(-\tau,\omega)}\\
&\quad\times\int_{\mathbb{R}^N}(v-M)_+^{2p-2}dx+\frac{F^2}{z^2(-\tau,\omega)}\int_{\mathbb{R}^N}|\psi_1(x)|(v-M)_+^{p-2}dx\\
&\le -\frac{\alpha_1}{2}\frac{F^{2-p}}{z^{2-p}(-\tau,\omega)}M^{p-2}\int_{\mathbb{R}^N}(v-M)_+^p dx-\frac{\alpha_1}{2}\frac{F^{2-p}}{z^{2-p}(-\tau,\omega)}\\
&\quad\times\int_{\mathbb{R}^N}(v-M)_+^{2p-2}dx+\frac{1}{2}\lambda\int_{\mathbb{R}^N}(v-M)_+^p dx\\
&\quad+\frac{cF^p}{z^p(-\tau,\omega)}\int_{\mathbb{R}^N(v\ge M)}|\psi_1(x)|^{p/2}dx,
\end{aligned}
\tag{4.41}
$$

where the last term the ϵ-Young's inequality (4.9) is used. The second term on the right-hand side of (4.37) is bounded as

$$
\begin{aligned}
&\frac{F}{z(-\tau,\omega)}\Big|\int_{\mathbb{R}^N} g(s,x)(v(s)-M)_+^{p-1}dx\Big|\\
&\le \frac{\alpha_1}{4}\frac{F^{2-p}}{z^{2-p}(-\tau,\omega)}\int_{\mathbb{R}^N}(v-M)_+^{2p-2}dx\\
&\quad+\frac{1}{\alpha_1}\frac{F^p}{z^p(-\tau,\omega)}\int_{\mathbb{R}^N(v(s)\ge M)}g^2(s,x)dx.
\end{aligned}
\tag{4.42}
$$

By a combination of (4.37)–(4.42), we obtain

$$
\begin{aligned}
&\frac{d}{ds}\int_{\mathbb{R}^N}(v(s)-M)_+^p dx+\frac{\alpha_1F^{2-p}}{z^{2-p}(-\tau,\omega)}M^{p-2}\int_{\mathbb{R}^N}(v(s)-M)_+^p dx\\
&+\frac{\alpha_1F^{2-p}}{z^{2-p}(-\tau,\omega)}\int_{\mathbb{R}^N}(v-M)_+^{2p-2}dx\\
&\le \frac{cF^p}{z^p(-\tau,\omega)}\Big(\|g(s,\cdot)\|^2+\|\psi_1\|_{p/2}^{p/2}\Big),
\end{aligned}
\tag{4.43}
$$

where the positive constant c is independent of ε,τ,ω and M. Note that for each $\tau\in\mathbb{R}$ and $\varepsilon\in(0,1]$,

$$
e^{-|\omega(-\tau)|}\le z(-\tau,\omega)=e^{-\varepsilon\omega(-\tau)}\le e^{|\omega(-\tau)|}. \tag{4.44}
$$

Here for convenience, we put

$$
\varrho=\varrho(\tau,\omega,M)=\alpha_1F^{2-p}e^{-(p-2)|\omega(-\tau)|}M^{p-2}>0,
$$

$$d = d(\tau,\omega) = \alpha_1 F^{2-p} e^{-(p-2)|\omega(-\tau)|} > 0,$$

where d is unchanged and $\varrho \to +\infty$ as $M \to +\infty$. Then from (4.43) and (4.44) we infer that

$$\begin{aligned} &\frac{d}{ds}\int_{\mathbb{R}^N} (v(s)-M)_+^p dx + \varrho \int_{\mathbb{R}^N} (v(s)-M)_+^p dx + d\int_{\mathbb{R}^N} (v-M)_+^{2p-2} dx \\ &\le cF^p e^{p|\omega(-\tau)|}\Big(\|g(s,\cdot)\|^2 + 1\Big), \end{aligned} \tag{4.45}$$

where $s \in [\tau-1,\tau]$ and ϱ, E, F are independent of ε and t. By using [26, Lemma 5.1] to (4.45) over the interval $[\tau-1,\tau]$, we find that

$$\begin{aligned} &\int_{\tau-1}^{\tau} e^{\varrho(s-\tau)} \int_{\mathbb{R}^N} (v(s)-M)_+^{2p-2}\, dx\, ds \\ &\le \frac{1}{d}\int_{\tau-1}^{\tau} e^{\varrho(s-\tau)} \int_{\mathbb{R}^N} \Big(v(s,\tau-t,\vartheta_{-\tau}\omega, v_0) - M\Big)_+^p\, dx\, ds \\ &\quad + \frac{cF^p e^{p|\omega(-\tau)|}}{d} \int_{\tau-1}^{\tau} e^{\varrho(s-\tau)}\Big(\|g(s,\cdot)\|^2 + 1\Big) ds. \end{aligned} \tag{4.46}$$

First by (4.13), there exists $T_1 = T_1(\tau,\omega,B) \ge 2$ such that for all $t \ge T_1$,

$$\begin{aligned} &\frac{1}{d}\int_{\tau-1}^{\tau} e^{\varrho(s-\tau)} \int_{\mathbb{R}^N} \Big(v(s,\tau-t,\vartheta_{-\tau}\omega, v_0) - M\Big)_+^p\, dx\, ds \\ &\le \frac{1}{d}\int_{\tau-1}^{\tau} e^{\varrho(s-\tau)} \|v(s,\tau-t,\vartheta_{-\tau}\omega, v_0)\|_p^p ds \\ &\le N(\tau,\omega)\frac{1}{d\varrho} \to 0, \end{aligned} \tag{4.47}$$

as $\varrho \to +\infty$, where $N(\tau,\omega)$ is the bound of the right hand side of (4.13). We then show that the second term on the right hand side of (4.46) is also small as $\varrho \to +\infty$. Indeed, choosing $\varrho > \delta$(where $\delta \in (0,\lambda)$ is in (3.5)) and taking $\varsigma \in (0,1)$, we have

$$\begin{aligned} &\int_{\tau-1}^{\tau} e^{\varrho(s-\tau)}\Big(\|g(s,\cdot)\|^2 + 1\Big) ds \\ &= \int_{\tau-1}^{\tau-\varsigma} e^{\varrho(s-\tau)}(\|g(s,\cdot)\|^2 + 1) ds + \int_{\tau-\varsigma}^{\tau} e^{\varrho(s-\tau)}(\|g(s,\cdot)\|^2 + 1) ds \\ &= e^{-\varrho\tau}\int_{\tau-1}^{\tau-\varsigma} e^{(\varrho-\delta)s} e^{\delta s}(\|g(s,\cdot)\|^2 + 1) ds + e^{-\varrho\tau}\int_{\tau-\varsigma}^{\tau} e^{\varrho s}(\|g(s,\cdot)\|^2 + 1) ds \\ &\le e^{-\varrho\varsigma} e^{\delta(\varsigma-\tau)}\int_{-\infty}^{\tau} e^{\delta s}(\|g(s,\cdot)\|^2 + 1) ds + \int_{\tau-\varsigma}^{\tau} (\|g(s,\cdot)\|^2 + 1) ds. \end{aligned}$$

By (3.5), the first term above vanishes as $\varrho \to +\infty$, and by $g \in L^2\mathrm{loc}(\mathbb{R}, L^2(\mathbb{R}^N))$ we can choose ς small enough such that the second term is small. Hence when $\varrho \to +\infty$, we have

$$\frac{cF^p e^{p|\omega(-\tau)|}}{d}\int_{\tau-1}^{\tau} e^{\varrho_2(s-\tau)}\Big(\|g(s,\cdot) + \|^2 + 1\Big) ds \to 0. \tag{4.48}$$

Then by (4.46)–(4.48), there exist two large positive constants $M_1 = M_1(\tau,\omega)$ and $T_1 = T_1(\tau,\omega,B) \ge 2$ such that all $t \ge T_1$,

$$\int_{\tau-1}^{\tau} e^{\varrho_1(s-\tau)} \int_{\mathbb{R}^N} (v(s)-M_1)_+^{2p-2}\, dx\, ds \le \eta, \tag{4.49}$$

where $\varrho_1 = \alpha_1 F^{2-p} e^{-(p-2)|\omega(-\tau)|} M_1$. Note that $v - M_1 \geq \frac{v}{2}$ for $v \geq 2M_1$. Then (4.49) gives that for all $t \geq T_1$,

$$\begin{aligned} &\int_{\tau-1}^{\tau} e^{\varrho_1(s-\tau)} \int_{\mathbb{R}^N(v(s)\geq 2M_1)} |v(s)|^{2p-2}\,dx\,ds \\ &\leq 2^{2p-2} \int_{\tau-1}^{\tau} e^{\varrho_1(s-\tau)} \int_{\mathbb{R}^N} (v(s)-M_1)_+^{2p-2}\,dx\,ds \leq 2^{2p-2}\eta. \end{aligned} \tag{4.50}$$

By a similar argument, we can show that there exist two large positive constants $M_2 = M_2(\tau,\omega)$ and $T_2 = T_2(\tau,\omega,B) \geq 2$ such that for all $t \geq T_2$,

$$\int_{\tau-1}^{\tau} e^{\varrho_2(s-\tau)} \int_{\mathbb{R}^N(v(s)\leq -2M_2)} |v(s)|^{2p-2}\,dx\,ds \leq 2^{2p-2}\eta, \tag{4.51}$$

where $\varrho_2 = \alpha_1 F^{2-p} e^{-(p-2)|\omega(-\tau)|} M_2$. Put $\tilde{M} = 2 \times \max\{M_1, M_2\}$ and $T = \max\{T_1, T_2\}$. Then (4.50) and (4.51) together imply the desired. $\square$

4.3. **Asymptotic compactness on bounded domains.** In this subsection, by using Proposition 4.5, we prove the asymptotic compactness of the cocyle φ defined by (3.10) in $H_0^1(\mathcal{O}_R)$ for any $R > 0$, where $\mathcal{O}_R = \{x \in \mathbb{R}^N; |x| \leq R\}$. For this purpose, we define $\phi(\cdot) = 1 - \xi(\cdot)$, where ξ is the cut-off function as in (4.16). Then we know that $0 \leq \phi(s) \leq 1$, and $\phi(s) = 1$ if $s \in [0,1]$ and $\phi(s) = 0$ if $s \geq 2$. Fix a positive constant k, we define

$$\tilde{v}(t,\tau,\omega,v_0) = \phi(\frac{x^2}{k^2}) v(t,\tau,\omega,v_0), \quad \tilde{u}(t,\tau,\omega,u_0) = \phi(\frac{x^2}{k^2}) u(t,\tau,\omega,u_0), \tag{4.52}$$

where v is the solution of problem (3.8)-(3.9) and u is the solution of problem (1.1)-(1.2) with $v = z(t,\omega)u$. Then we have

$$\tilde{u}(t,\tau,\omega,u_0) = z^{-1}(t,\omega)\tilde{v}(t,\tau,\omega,v_0). \tag{4.53}$$

It is obvious that $\tilde{v}$ solves the following equations:

$$\begin{gathered} \tilde{v}_t + \lambda\tilde{v} - \Delta\tilde{v} = \phi z f(x, z^{-1}v) + \phi z g - v\Delta\phi - 2\nabla\phi.\nabla v, \\ \tilde{v}|_{\partial\mathcal{O}_{k\sqrt{2}}} = 0, \\ \tilde{v}(\tau,x) = \tilde{v}_0(x) = \phi v_0(x), \end{gathered} \tag{4.54}$$

where $\phi = \phi(x^2/k^2)$.

It is well-known that the eigenvalue problem on bounded domains $\mathcal{O}_{k\sqrt{2}}$ with Dirichlet boundary condition:

$$\begin{gathered} -\Delta\tilde{v} = \lambda\tilde{v}, \\ \tilde{v}|_{\partial\mathcal{O}_{k\sqrt{2}}} = 0 \end{gathered}$$

has a family of orthogonal eigenfunctions $\{e_j\}_{=1}^{+\infty}$ in both $L^2(\mathcal{O}_{k\sqrt{2}})$ and $H_0^1(\mathcal{O}_{k\sqrt{2}})$ such that the corresponding eigenvalue $\{\lambda_j\}_{j=1}^{+\infty}$ is non-decreasing in j.

Let $H_m = \mathrm{Span}\{e_1, e_2, \dots, e_m\} \subset H_0^1(\mathcal{O}_{k\sqrt{2}})$ and $P_m : H_0^1(\mathcal{O}_{k\sqrt{2}}) \to H_m$ be the canonical projector and I be the identity. Then for every $\tilde{u} \in H_0^1(\mathcal{O}_{k\sqrt{2}})$, $\tilde{u}$ has a unique decomposition: $\tilde{u} = \tilde{u}_1 + \tilde{u}_2$, where $\tilde{u}_1 = P_m\tilde{u} \in H_m$ and $\tilde{u}_2 = (I - P_m)\tilde{u} \in H_m^{\perp}$, i.e., $H_0^1(\mathcal{O}_{k\sqrt{2}}) = H_m \oplus H_m^{\perp}$.

Lemma 4.6. *Assume that* (3.1)–(3.5) *hold. Let* $\tau \in \mathbb{R}$, $\omega \in \Omega$ *and*

$$B = \{B(\tau,\omega); \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}.$$

Then for every $\eta > 0$, there are $N_0 = N_0(\tau, \omega, k, \eta) \in Z^+$ and $T = T(\tau, \omega, B, \eta) \geq 2$ such that for all $t \geq T$ and $m > N_0$,

$$\|(I - P_m)\tilde{u}(\tau, \tau - t, \vartheta_{-\tau}\omega, \tilde{u}_0)\|_{H_0^1(\mathcal{O}_{k\sqrt{2}})} \leq \eta,$$

where $\tilde{u}_0 = \phi u_0$ with $u_0 \in B(\tau - t, \vartheta_{-\tau}\omega)$. Here $\tilde{u}$ is as in (4.53) and N, T are independent of ε.

Proof. By (4.53), we start at the estimate of $\tilde{v}$. For $\tilde{v} \in H_0^1(\mathcal{O}_{k\sqrt{2}})$, we write $\tilde{v} = \tilde{v}_1 + \tilde{v}_2$ where $\tilde{v}_1 = P_m\tilde{v}$ and $\tilde{v}_2 = (I - P_m)\tilde{v}$. Then naturally, we have a splitting about $\tilde{u} = \tilde{u}_1 + \tilde{u}_2$ where $\tilde{u}_1 = P_m\tilde{u}$ and $\tilde{u}_2 = (I - P_m)\tilde{u}$. Multiplying (4.47) by $\Delta\tilde{v}_2$ we get that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\nabla\tilde{v}_2\|^2_{L^2(\mathcal{O}_{k\sqrt{2}})} + \lambda\|\nabla\tilde{v}_2\|^2_{L^2(\mathcal{O}_{k\sqrt{2}})} + \|\Delta\tilde{v}_2\|^2_{L^2(\mathcal{O}_{k\sqrt{2}})} \\ &= -z\int_{\mathcal{O}_{k\sqrt{2}}}\phi f(x, z^{-1}v)\Delta\tilde{v}_2 dx + \int_{\mathcal{O}_{k\sqrt{2}}}(\phi z g - v\Delta\phi - 2\nabla\phi.\nabla v)\Delta\tilde{v}_2 dx, \end{aligned} \tag{4.55}$$

where z is the abbreviation of $z(t, \omega)$. By (3.2), we deduce that

$$z\int_{\mathcal{O}_{k\sqrt{2}}}\phi f(x, z^{-1}v)\Delta\tilde{v}_2 dx \leq \frac{1}{4}\|\Delta\tilde{v}_2\|^2_{L^2(\mathcal{O}_{k\sqrt{2}})} + cz^{4-2p}\|v\|^{2p-2}_{L^{2p-2}(\mathcal{O}_{k\sqrt{2}})} + z^2\|\psi_2\|^2. \tag{4.56}$$

On the other hand,

$$\begin{aligned} &\int_{\mathcal{O}_{k\sqrt{2}}}(\phi z g - v\Delta\phi - 2\nabla\phi.\nabla v)\Delta\tilde{v}_2 dx \\ &\leq \frac{1}{4}\|\Delta\tilde{v}_2\|^2_{L^2(\mathcal{O}_{k\sqrt{2}})} + c(z^2\|g\|^2 + \|v\|^2 + \|\nabla v\|^2). \end{aligned} \tag{4.57}$$

Then by (4.55)–(4.57) we find that

$$\begin{aligned} &\frac{d}{dt}\|\nabla\tilde{v}_2\|^2_{L^2(\mathcal{O}_{k\sqrt{2}})} + \|\Delta\tilde{v}_2\|^2_{L^2(\mathcal{O}_{k\sqrt{2}})} \\ &\leq c(z^{4-2p}\|v\|^{2p-2}_{L^{2p-2}(\mathcal{O}_{k\sqrt{2}})} + z^2\|\psi_2\|^2 + z^2\|g\|^2 + \|v\|^2_{H^1}). \end{aligned}$$

from which and Poincaré's inequality

$$\|\Delta\tilde{v}_2\|^2_{L^2(\mathcal{O}_{k\sqrt{2}})} \geq \lambda_{m+1}\|\nabla\tilde{v}_2\|^2_{L^2(\mathcal{O}_{k\sqrt{2}})},$$

it follows that

$$\begin{aligned} &\frac{d}{dt}\|\nabla\tilde{v}_2\|^2_{L^2(\mathcal{O}_{k\sqrt{2}})} + \lambda_{m+1}\|\nabla\tilde{v}_2\|^2_{L^2(\mathcal{O}_{k\sqrt{2}})} \\ &\leq c(z^{4-2p}\|v\|^{2p-2}_{L^{2p-2}(\mathcal{O}_{k\sqrt{2}})} + z^2\|\psi_2\|^2 + z^2\|g\|^2 + \|v\|^2_{H^1}). \end{aligned} \tag{4.58}$$

Applying [26, Lemma 5.1] to (4.58) over the interval $[\tau - 1, \tau]$, along with ω being replaced by $\vartheta_{-\tau}\omega$, we find that

$$\begin{aligned} &\|\nabla\tilde{v}_2(\tau, \tau - t, \vartheta_{-\tau}\omega, \tilde{v}_0)\|^2_{L^2(\mathcal{O}_{k\sqrt{2}})} \\ &\leq \int_{\tau-1}^{\tau} e^{\lambda_{m+1}(s-\tau)}\|\nabla\tilde{v}_2(s, \tau - t, \vartheta_{-\tau}\omega, \tilde{v}_0)\|^2_{L^2(\mathcal{O}_{k\sqrt{2}})} ds \\ &\quad + c\int_{\tau-1}^{\tau} e^{\lambda_{m+1}(s-\tau)} z^{4-2p}(s, \vartheta_{-\tau}\omega)\|v(s, \tau - t, \vartheta_{-\tau}\omega, \tilde{v}_0)\|^{2p-2}_{L^{2p-2}(\mathcal{O}_{k\sqrt{2}})} ds \\ &\quad + c\int_{\tau-1}^{\tau} e^{\lambda_{m+1}(s-\tau)} z^2(s, \vartheta_{-\tau}\omega)(\|\psi_2\|^2 + \|g(s, \cdot)\|^2) ds \end{aligned}$$

$$
\begin{aligned}
&+ c\int_{\tau-1}^{\tau} e^{\lambda_{m+1}(s-\tau)}\|v(s,\tau-t,\vartheta_{-\tau}\omega,\tilde{v}_0)\|_{H^1}^2 ds\\
&\le c\int_{\tau-1}^{\tau} e^{\lambda_{m+1}(s-\tau)} z^{4-2p}(s,\vartheta_{-\tau}\omega)\|v(s,\tau-t,\vartheta_{-\tau}\omega,\tilde{v}_0)\|_{L^{2p-2}(\mathcal{O}_{k\sqrt{2}})}^{2p-2} ds\\
&\quad + c\int_{\tau-1}^{\tau} e^{\lambda_{m+1}(s-\tau)}\|v(s,\tau-t,\vartheta_{-\tau}\omega,\tilde{v}_0)\|_{H^1}^2 ds\\
&\quad + c\int_{\tau-1}^{\tau} e^{\lambda_{m+1}(s-\tau)} z^{2}(s,\vartheta_{-\tau}\omega)\Big(\|g(s,\cdot)\|^2+1\Big) ds\\
&= I_1 + I_2 + I_3.
\end{aligned}
\tag{4.59}
$$

We next to show that I_1, I_2 and I_3 converge to zero as m increases to infinite. First since by (4.1), $z^{4-2p}(s-\tau,\omega) \le E^{4-2p}$ for $s \in [-1,0]$, then we have

$$
\begin{aligned}
&I_1\\
&= z^{2p-4}(-\tau,\omega)\int_{\tau-1}^{\tau} e^{\lambda_{m+1}(s-\tau)} z^{4-2p}(s-\tau,\omega)\\
&\quad\times \|v(s,\tau-t,\vartheta_{-\tau}\omega,\tilde{v}_0)\|_{L^{2p-2}(\mathcal{O}_{k\sqrt{2}})}^{2p-2} ds\\
&\le z^{2p-4}(-\tau,\omega)E^{4-2p}\int_{\tau-1}^{\tau} e^{\lambda_{m+1}(s-\tau)}\|v(s,\tau-t,\vartheta_{-\tau}\omega,\tilde{v}_0)\|_{L^{2p-2}(\mathcal{O}_{k\sqrt{2}})}^{2p-2} ds\\
&\le z^{2p-4}(-\tau,\omega)E^{4-2p}\Big(\int_{\tau-1}^{\tau} e^{\lambda_{m+1}(s-\tau)}\\
&\quad\times \int_{\mathcal{O}_{k\sqrt{2}}(|v(s)|\ge M)} |v(s,\tau-t,\vartheta_{-\tau}\omega,\tilde{v}_0)|^{2p-2}\, dx\, ds\\
&\quad+ \int_{\tau-1}^{\tau} e^{\lambda_{m+1}(s-\tau)} \int_{\mathcal{O}_{k\sqrt{2}}(|v(s)|\le M)} |v(s,\tau-t,\vartheta_{-\tau}\omega,\tilde{v}_0)|^{2p-2}\, dx\, ds\Big).
\end{aligned}
\tag{4.60}
$$

By Proposition 4.5, there exist $T_1 = T_1(\tau,\omega,B,\eta) \ge 2$, $\tilde{M} = \tilde{M}(\tau,\omega,\eta)$ such that for all $t \ge T_1$,

$$
\begin{aligned}
&z^{2p-4}(-\tau,\omega)E^{4-2p}\int_{\tau-1}^{\tau} e^{\tilde{\varrho}(s-\tau)}\\
&\times \int_{\mathcal{O}_{k\sqrt{2}}(|v(s)|\ge \tilde{M})} |v(s,\tau-t,\vartheta_{-\tau}\omega,\tilde{v}_0)|^{2p-2}\, dx\, ds \le \eta.
\end{aligned}
\tag{4.61}
$$

But $\lambda_{m+1} \to +\infty$, then there exists $N' = N'(\tau,\omega,\eta) > 0$ such that for all $m > N'$, $\lambda_{m+1} > \tilde{\varrho}$. Hence by (4.61) it gives us that for all $t \ge T_1$ and $m > N'$ there holds

$$
\begin{aligned}
&z^{2p-4}(-\tau,\omega)E^{4-2p}\int_{\tau-1}^{\tau} e^{\lambda_{m+1}(s-\tau)}\\
&\times \int_{\mathcal{O}_{k\sqrt{2}}(|v(s)|\ge \tilde{M})} |v(s,\tau-t,\vartheta_{-\tau}\omega,\tilde{v}_0)|^{2p-2}\, dx\, ds \le \eta.
\end{aligned}
\tag{4.62}
$$

For the second term on the right hand side of (4.60), since $\mathcal{O}_{k\sqrt{2}}(|v(s)| \le \tilde{M})$ is a bounded domain, then there exists $N'' = N''(\tau,\omega,\eta) > 0$ such that for all $m > N''$,

$$\begin{aligned} & z^{2p-4}(-\tau,\omega)E^{4-2p}\int_{\tau-1}^{\tau} e^{\lambda_{m+1}(s-\tau)} \\ & \times \int_{\mathcal{O}_{k\sqrt{2}}(|v(s)|\le\tilde{M})} |v(s,\tau-t,\vartheta_{-\tau}\omega,\tilde{v}_0)|^{2p-2}\,dx\,ds \\ & \le z^{2p-4}(-\tau,\omega)E^{4-2p}\frac{\tilde{M}^{2p-2}}{\lambda_{m+1}}|(\mathcal{O}_{k\sqrt{2}}(|v(s)|\le\tilde{M}))|\le\eta, \end{aligned} \tag{4.63}$$

where $|(\mathcal{O}_{k\sqrt{2}}(|v(s)| \le \tilde{M}))|$ is the finite measure of the bounded domain $\mathcal{O}_{k\sqrt{2}}(|v(s)| \le \tilde{M})$. Put $N_1 = \max\{N', N''\}$. It follows from (4.60)-(4.63) that for all $m > N_1$ and $t \ge T_1$,

$$I_1 \le 2\eta. \tag{4.64}$$

By Lemma 4.1, there exists $T_2 = T_2(\tau,\omega,B)$ and $N_2 = N_2(\tau,\omega,\eta) > 0$ such that for all $m > N_2$ and $t \ge T_2$,

$$I_2 \le \frac{L_1(\tau,\omega,\varepsilon)}{\lambda_{m+1}} \le \eta. \tag{4.65}$$

By a same technique as (4.48), we can show that there exists $N_3 = N_3(\tau,\omega,\eta) > 0$ such that for all $m > N_3$,

$$I_3 = c\int_{\tau-1}^{\tau} e^{\lambda_{m+1}(s-\tau)} z^2(s,\vartheta_{-\tau}\omega)\Big(\|g(s,\cdot)\|^2+1\Big)ds \le \eta. \tag{4.66}$$

Let $N_0 = \max\{N_1,N_2,N_3\}$ and $T = \max\{T_1,T_2\}$. Then (4.64)–(4.66) are integrated into (4.59) to get that for all $m > N_0$ and $t \ge T$,

$$\|\nabla\tilde{v}_2(\tau,\tau-t,\vartheta_{-\tau}\omega,\tilde{v}_0)\|_{L^2(\mathcal{O}_{k\sqrt{2}})} \le 4\eta. \tag{4.67}$$

Then by (3.11) and (4.67), we have

$$\begin{aligned} & \|\nabla\tilde{u}_2(\tau,\tau-t,\vartheta_{-\tau}\omega,\tilde{u}_0)\|_{L^2(\mathcal{O}_{k\sqrt{2}})} \\ & = z(-\tau,\omega)\|\nabla\tilde{v}_2(\tau,\tau-t,\vartheta_{-\tau}\omega,\tilde{v}_0)\|_{L^2(\mathcal{O}_{k\sqrt{2}})} \\ & \le C(\tau,\omega)\eta, \end{aligned}$$

for all $m > N_0$ and $t \ge T$, which completes the proof. □

Lemma 4.7. *Assume that* (3.1)–(3.5) *hold. Let $\tau \in \mathbb{R}, \omega \in \Omega$. Then for every $k > 0$, the sequence $\{\tilde{u}(\tau,\tau-t_n,\vartheta_{-\tau}\omega,\phi(\frac{x^2}{k^2})u_{0,n})\}_{n=1}^{\infty}$ has a convergent subsequence in $H_0^1(\mathcal{O}_{k\sqrt{2}})$ whenever $t_n \to +\infty$ and $u_{0,n} \in B(\tau-t_n,\vartheta_{-t_n}\omega)$.*

Proof. Given $\eta > 0$, by Lemma 4.6, there exists $N_0 \in \mathbb{Z}^+$ such that as $t_n \to +\infty$,

$$\|(I-P_{N_0})\tilde{u}(\tau,\tau-t_n,\vartheta_{-\tau}\omega,\phi(\frac{x^2}{k^2})u_{0,n})\|_{H^1(\mathcal{O}_{k\sqrt{2}})} \le \eta. \tag{4.68}$$

By Lemma 4.1, we deduce that for t_n large enough,

$$\|P_{N_0}\tilde{u}(\tau,\tau-t_n,\vartheta_{-\tau}\omega,\phi(\frac{x^2}{k^2})u_{0,n})\|_{H^1(\mathcal{O}_{k\sqrt{2}})} \le L_1(\tau,\omega,\varepsilon). \tag{4.69}$$

Note that

$$H^1(\mathcal{O}_{k\sqrt{2}}) = P_{N_0}H^1(\mathcal{O}_{k\sqrt{2}}) + (I-P_{N_0})H^1(\mathcal{O}_{k\sqrt{2}}),$$

but $P_{N_0}H^1(\mathcal{O}_{k\sqrt{2}})$ is a finite dimensional space, which is compact. Then by (4.69), if n, m large enough,

$$\begin{aligned}&\big\|P_{N_0}\tilde{u}(\tau,\tau-t_n,\vartheta_{-\tau}\omega,\phi(\frac{x^2}{k^2})u_{0,n})\\&-P_{N_0}\tilde{u}(\tau,\tau-t_m,\vartheta_{-\tau}\omega,\phi(\frac{x^2}{k^2})u_{0,m}\big\|_{H^1(\mathcal{O}_{k\sqrt{2}})}\le\eta.\end{aligned}\tag{4.70}$$

Then it is easy to complete the proof using (4.68) and (4.70) and a standard argument. □

4.4. **Existence of pullback attractor in $H^1(\mathbb{R}^N)$.** In this subsection, we prove the existences of pullback attractors in $H^1(\mathbb{R}^N)$ for problem (1.1) and (1.2) for every $\varepsilon\in(0,1]$.

Proposition 4.8. *Assume that* (3.1)-(3.5) *hold. Then the cocycle φ defined by* (3.10) *is asymptotically compact in $H^1(\mathbb{R}^N)$; i.e., for every $\tau\in\mathbb{R}$ and $\omega\in\Omega$, the sequence $\{\varphi(t,\tau-t_n,\vartheta_{-t}\omega,u_{0,n})\}_{n=1}^{\infty}$ has a convergent subsequence in $H^1(\mathbb{R}^N)$ whenever $t_n\to+\infty$ and $u_{0,n}\in B=B(\tau-t_n,\vartheta_{-t_n}\omega)$ with $B\in\mathfrak{D}$.*

Proof. Given $R>0$, we denote $\mathcal{O}_R^c=\mathbb{R}^N-\mathcal{O}_R$, where $\mathcal{O}_R=\{x\in\mathbb{R}^N;|x|\le R\}$. By Proposition 4.4, for any $\eta>0$, there exist $R=R(\tau,\omega,\eta)>0$ and $N_1=N_1(\tau,\omega,B,\eta)\in\mathbb{Z}^+$ such that for all $n\ge N_1$,

$$\|v(\tau,\tau-t_n,\vartheta_{-\tau}\omega,z(\tau-t_n,\vartheta_{-\tau}\omega)u_{0,n})\|_{H^1(\mathcal{O}_R^c)}\le\frac{\eta}{8}e^{-|\omega(-\tau)|},\tag{4.71}$$

for every $u_{0,n}\in B=B(\tau-t_n,\vartheta_{-t_n}\omega)$. By (3.11) and (4.71), we have

$$\|u(\tau,\tau-t_n,\vartheta_{-\tau}\omega,z(\tau-t_n,\vartheta_{-\tau}\omega)u_{0,n})\|_{H^1(\mathcal{O}_R^c)}\le\frac{\eta}{8}.\tag{4.72}$$

On the other hand, for this R, by Lemma 4.7, there exists $N_2=N_2(\tau,\omega,B,\eta)\ge N_1$ such that for all $m,n\ge N_2$,

$$\begin{aligned}&\big\|u(\tau,\tau-t_n,\vartheta_{-\tau}\omega,\phi(\frac{x^2}{R^2})u_{0,n})\\&\quad-u(\tau,\tau-t_m,\vartheta_{-\tau}\omega,\phi(\frac{x^2}{R^2})u_{0,m})\big\|_{H_0^1(\mathcal{O}_{R\sqrt{2}})}\\&\le\frac{\eta}{8}.\end{aligned}\tag{4.73}$$

Then the desired result follows from (4.72) and (4.73) by a standard argument. □

Given $\varepsilon\in(0,1]$, by Lemma 4.1, we deduce that the $\mathfrak{D}$-pullback absorbing set K_ε of φ_ε in $L^2(\mathbb{R}^N)$ is defined by

$$K_\varepsilon=\{K_\varepsilon(\tau,\omega)=\{u\in L^2(\mathbb{R}^N);\|u\|\le L_\varepsilon(\tau,\omega)\};\tau\in\mathbb{R},\omega\in\Omega\},\tag{4.74}$$

where

$$L_\varepsilon(\tau,\omega)=\Big(c\int_{-\infty}^0e^{\lambda s}e^{-2\varepsilon\omega(s)}(\|g(s+\tau,\cdot)\|^2+1)\Big)^{1/2}.$$

By Proposition 4.8 and Theorem 2.6, we have the following result.

Theorem 4.9. *Assume that* (3.1)–(3.5) *hold. Then for every fixed $\varepsilon\in(0,1]$, the cocycle φ_ε defined by* (3.10) *possesses a unique $\mathfrak{D}$-pullback attractor $\mathcal{A}_{\varepsilon,H^1}=\{\mathcal{A}_{\varepsilon,H^1}(\tau,\omega);\tau\in\mathbb{R},\omega\in\Omega\}$ in $H^1(\mathbb{R}^N)$, given by*

$$\mathcal{A}_{\varepsilon,H^1}(\tau,\omega)=\cap_{s>0}\overline{\cup_{t\ge s}\varphi_\varepsilon(t,\tau-t,\vartheta_{-t}\omega,K_\varepsilon(\tau-t,\vartheta_{-t}\omega))}^{H^1(\mathbb{R}^N)},\quad\tau\in\mathbb{R},\ \omega\in\Omega.$$

Furthermore, $\mathcal{A}_{\varepsilon,H^1}$ is consistent with the $\mathfrak{D}$-pullback random attractor $\mathcal{A}_\varepsilon$ in the space $L^2(\mathbb{R}^N)$, which is defined as in (3.13).

5. Upper semi-continuity of pullback attractor in $H^1(\mathbb{R}^N)$

From Theorem 4.9, for every $\varepsilon \in (0,1]$, the cocycle φ_ε admits a common $\mathfrak{D}$-pullback attractor $\mathcal{A}_\varepsilon$ in both $L^2(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$, where $\mathfrak{D}$ is defined by (3.11). Then we may investigate the upper semi-continuity of $\mathcal{A}_\varepsilon$ in both $L^2(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$. Note that [18] only proved the upper semi-continuity in $L^2(\mathbb{R}^N)$ at $\varepsilon = 0$. In this section, we strengthen this study and prove that the upper semi-continuity of $\mathcal{A}_\varepsilon$ may happen in the topology of $H^1(\mathbb{R}^N)$ at $\varepsilon = 0$.

For the upper semi-continuity, we also give a further assumption as in [18], that is, f satisfies that for all $x \in \mathbb{R}^N$ and $s \in \mathbb{R}$,

$$\big|\frac{\partial}{\partial s} f(x,s)\big| \le \alpha_4 |s|^{p-2} + \psi_4(x), \tag{5.1}$$

where $\alpha_4 > 0$, $\psi_4 \in L^\infty(\mathbb{R}^N)$ if $p = 2$ and $\psi_4 \in L^{\frac{p}{p-2}}(\mathbb{R}^N)$ if $p > 2$.

Let φ_0 be the continuous cocycle associated with the problem (1.1) and (1.2) for $\varepsilon = 0$. That is to say, φ_0 is a deterministic non-autonomous cocycle over $\mathbb{R}$. Denote by $\mathfrak{D}_0$ the collection of some families of deterministic nonempty subsets of $L^2(\mathbb{R}^N)$:

$$\mathfrak{D}_0 = \{B = \{B(\tau) \subseteq L^2(\mathbb{R}^N); \tau \in \mathbb{R}\}; \lim_{t\to+\infty} e^{-\delta t}\|B(\tau - t)\| = 0, \tau \in \mathbb{R}, \delta < \lambda\},$$

where λ is as in (3.8). As a special case of Theorem 4.9, under the assumptions (3.1)-(3.5), φ_0 has a common $\mathfrak{D}_0$-pullback attractor $\mathcal{A}_0 = \{\mathcal{A}_0(\tau); \tau \in \mathbb{R}\}$ in both $L^2(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$.

To prove the upper semi-continuity of $\mathcal{A}_\varepsilon$ at $\varepsilon = 0$, we have to check that the conditions (2.10)-(2.14) in Theorem 2.8 hold in $L^2(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$ point by point. But (2.10)-(2.13) have been achieved, see [18, Corollary 7.2, Lemma 7.5 and equality (7.31)]. We only need to prove the condition (2.14) holds in $H^1(\mathbb{R}^N)$.

Lemma 5.1. *Assume that* (3.1)-(3.5) *hold. Then for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the union $\cup_{\varepsilon\in(0,1]}\mathcal{A}_\varepsilon(\tau,\omega)$ is precompact in $H^1(\mathbb{R}^N)$.*

Proof. For any $\eta > 0$, it suffices to show that for every fixed $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the set $\cup_{\varepsilon\in(0,1]}\mathcal{A}_\varepsilon(\tau,\omega)$ has finite η-nets in $H^1(\mathbb{R}^N)$. Let $\chi = \chi(\tau,\omega) \in \cup_{\varepsilon\in(0,1]}\mathcal{A}_\varepsilon(\tau,\omega)$. Then there exists a $\varepsilon \in (0,1]$ such that $\chi(\tau,\omega) \in \mathcal{A}_\varepsilon(\tau,\omega)$. By the invariance of $\mathcal{A}_\varepsilon(\tau,\omega)$, it follows that there is a $u_0 \in \mathcal{A}_\varepsilon(\tau - t, \vartheta_{-t}\omega)$ such that (by (3.11))

$$\chi(\tau,\omega) = \varphi_\varepsilon(t, \tau - t, \vartheta_{-t}\omega, u_0) = u_\varepsilon(\tau, \tau - t, \vartheta_{-\tau}\omega, u_0) \quad \forall t \ge 0\,. \tag{5.2}$$

Given $R > 0$, denote $\mathcal{O}_R^c = \mathbb{R}^N - \mathcal{O}_R$, where $\mathcal{O}_R = \{x \in \mathbb{R}^N; |x| \le R\}$. Note that $\mathcal{A}_\varepsilon(\tau,\omega) \in \mathfrak{D}$. Then by Proposition 4.4, for every $\eta > 0$, there exist $T = T(\tau,\omega,\eta) \ge 2$ and $R = R(\tau,\omega,\eta) > 1$ such that the solution u of problem (1.1) and (1.2) satisfies

$$\|u_\varepsilon(\tau, \tau - t, \vartheta_{-\tau}\omega, u_0)\|_{H^1(\mathcal{O}_R^c)} \le \eta, \quad \forall t \ge T\,. \tag{5.3}$$

Then by (5.2)-(5.3), we have

$$\|\chi(\tau,\omega)\|_{H^1(\mathcal{O}_R^c)} \le \eta, \quad \text{for all } \chi \in \cup_{\varepsilon\in(0,1]}\mathcal{A}_\varepsilon(\tau,\omega). \tag{5.4}$$

On the other hand, by Lemma 4.6, there exist a projector P_{N_0} and $T = T(\tau,\omega,\eta) \ge 2$ such that for all $t \ge T$,

$$\|(I - P_{N_0})\tilde{u}_\varepsilon(\tau, \tau - t, \vartheta_{-\tau}\omega, \tilde{u}_0)\|_{H_0^1(\mathcal{O}_{R\sqrt{2}})} \le \eta, \tag{5.5}$$

where $\tilde{u}_\varepsilon$ is the cut-off of u_ε on the domain $\mathcal{O}_{R\sqrt{2}}$, by (4.52). Because $P_{N_0}\tilde{u}_\varepsilon \in H_{N_0}$, where $H_{N_0} = \operatorname{span}\{e_{1,2}, \dots, e_{N_0}\}$ is a finite dimension space and $P_{N_0}\tilde{u}_\varepsilon(\tau, \tau - t, \vartheta_{-\tau}\omega, \tilde{u}_0)$ is bounded in H_{N_0} which is compact. Therefore there exist finite points $v_1, v_2, \dots, v_s \in H_{N_0}$ such that

$$\|P_{N_0}\tilde{u}_\varepsilon(\tau, \tau - t, \vartheta_{-\tau}\omega, \tilde{u}_0) - v_i\|_{H_0^1(\mathcal{O}_{R\sqrt{2}})} \le \eta. \tag{5.6}$$

Thus by (5.2), the inequalities (5.5) and (5.6) are rewritten as

$$\|(I - P_{N_0})\chi(\tau, \omega)\|_{H_0^1(\mathcal{O}_{R\sqrt{2}})} \le \eta, \quad \|P_{N_0}\chi(\tau, \omega) - v_i\|_{H_0^1(\mathcal{O}_{R\sqrt{2}})} \le \eta, \tag{5.7}$$

for all $\chi \in \cup_{\varepsilon \in (0,1]}\mathcal{A}_\varepsilon(\tau, \omega)$. We now define $\tilde{v}_i = \tilde{v}_i(x) = 0$ if $x \in \mathcal{O}^c_{R\sqrt{2}}$ and $\tilde{v}_i = v_i$ if $x \in \mathcal{O}_{R\sqrt{2}}$. Then for every $i = 1, 2, \dots, s$, $\tilde{v}_i \in H^1(\mathbb{R}^N)$. Furthermore, by (5.4) and (5.7), we have

$$\begin{aligned}
\|\chi(\tau, \omega) - \tilde{v}_i\|_{H^1(\mathbb{R}^N)} &\le \|\chi(\tau, \omega) - \tilde{v}_i\|_{H^1(\mathcal{O}^c_{R\sqrt{2}})} + \|\chi(\tau, \omega) - \tilde{v}_i\|_{H_0^1(\mathcal{O}_{R\sqrt{2}})} \\
&\le \|\chi(\tau, \omega)\|_{H^1(\mathcal{O}^c_{R\sqrt{2}})} + \|P_{N_0}\chi(\tau, \omega) - \tilde{v}_i\|_{H_0^1(\mathcal{O}_{R\sqrt{2}})} \\
&\quad + \|(I - P_{N_0})\chi(\tau, \omega)\|_{H_0^1(\mathcal{O}_{R\sqrt{2}})} \le 3\eta,
\end{aligned}$$

for all $\chi \in \cup_{\varepsilon \in (0,1]}\mathcal{A}_\varepsilon(\tau, \omega)$. Thus $\cup_{\varepsilon \in (0,1]}\mathcal{A}_\varepsilon(\tau, \omega)$ has finite η-nets in $H^1(\mathbb{R}^N)$, which implies that the union $\cup_{\varepsilon \in (0,1]}\mathcal{A}_\varepsilon(\tau, \omega)$ is precompact in $H^1(\mathbb{R}^N)$. $\square$

Then we obtain that the family of random attractors $\mathcal{A}_\varepsilon$ indexed by ε converges to the deterministic $\mathcal{A}_0$ in $H^1(\mathbb{R}^N)$ in the following sense.

Theorem 5.2. *Assume that* (3.1)-(3.5) *and* (5.1) *hold. Then for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$,*

$$\lim_{\varepsilon \downarrow 0} \operatorname{dist}_{H^1}(\mathcal{A}_\varepsilon(\tau, \omega), \mathcal{A}_0(\tau)) = 0,$$

where $\operatorname{dist}_{H^1}$ *is the Haustorff semi-metric in* $H^1(\mathbb{R}^N)$.

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