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A FUJITA-TYPE THEOREM FOR A MULTITIME EVOLUTIONARY p-LAPLACE INEQUALITY IN THE HEISENBERG GROUP

MOHAMED JLELI, MOKHTAR KIRANE, BESSEM SAMET

Abstract. A nonexistence result of global nontrivial positive weak solutions to a multitime evolutionary p-Laplace differential inequality in the Heisenberg group is obtained. Our technique of proof is based on the test function method.

1. Introduction

The standard time-dependent partial differential equations of mathematical physics involve evolution in one-dimensional time. Space can be multidimensional, but time stayed one dimensional. The term multitime was introduced in physics by Dirac, Fock and Podolsky, in 1932, considering multi-temporal wave-functions via m-time evolution equation. It was used in mathematics by Friedman and Littman (1962, 1963). Multitime evolution equations arise for example in Brownian motion (diffusion process with inertia) [2], transport theory (Fokker-Planck-type equations) [21], biology (age-structured population dynamics) [10], wave and Maxwell's equations [5, 9], mechanics, physics and cosmology [20, 25]. Some interesting multitime developments of classical, single-time theories and principles from different fields of mathematical research, appeared in the last years (see [1, 4, 8, 12, 15, 24, 26] and references therein).

The study of nonexistence of global solutions to multitime evolutionary problems has begun recently (see [11, 13]). This paper deals with the nonexistence of global (nontrivial) positive solutions to a multitime evolutionary p-Laplace differential inequality in the Heisenberg group. More precisely, we consider the multitime evolutionary p-Laplace problem

$$\begin{gathered}\sum_{i=1}^{i=k} u_{t_i} - \operatorname{div}_{\mathbb{H}}\left(|\nabla_{\mathbb{H}} u|^{p-2}\nabla_{\mathbb{H}} u\right) \geq u^q, \quad \text{in } \mathcal{H},\\ u \geq 0, \quad \text{a.e. in } \mathcal{H},\\ u_{|_{t_i=0}} = u_i, \quad \text{in } \mathcal{E},\end{gathered} \tag{1.1}$$

where $\mathbb{H}$ is the $(2N+1)$-dimensional Heisenberg group, k is a positive integer ($k \geq 1$), $p, q > 1$, $\mathcal{H} = \mathbb{H} \times (0,\infty)^k$, $\mathcal{E} = \mathbb{H} \times (0,\infty)^{k-1}$, and $u_i \in L^1_{\rm loc}(\mathcal{E})$, $i =$

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$1, 2 \dots, k$. Using a duality argument [16, 17], we provide a sufficient condition for the nonexistence of global nontrivial positive weak solutions to the above problem.

2. Preliminaries

For the reader's convenience, we recall some background facts used here. The $(2N+1)$-dimensional Heisenberg group $\mathbb{H}$ is the space $\mathbb{R}^{2N+1}$ equipped with the group operation

$$\vartheta \diamond \vartheta' = (x + x', y + y', \tau + \tau' + 2(x \cdot y' - x' \cdot y)),$$

for all $\vartheta = (x, y, \tau), \vartheta' = (x', y', \tau') \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$, where $\cdot$ denotes the standard scalar product in $\mathbb{R}^N$. This group operation endows $\mathbb{H}$ with the structure of a Lie group.

The distance from un element $\vartheta = (x, y, \tau) \in \mathbb{H}$ to the origin is given by

$$|\vartheta|_{\mathbb{H}} = \Big(\tau^2 + \Big(\sum_{i=1}^N x_i^2 + y_i^2\Big)^2\Big)^{1/4},$$

where $x = (x_1, \dots, x_N)$ and $y = (y_1, \dots, y_N)$.

The Gradient $\nabla_{\mathbb{H}}$ over $\mathbb{H}$ is

$$\nabla_{\mathbb{H}} = (X_1, \dots, X_N, Y_1, \dots, Y_N),$$

where for $i = 1, \dots, N$,

$$X_i = \partial_{x_i} + 2y_i\partial_\tau \quad \text{and} \quad Y_i = \partial_{y_i} - 2x_i\partial_\tau.$$

Let

$$A = \begin{pmatrix} I_N & 0 & 2y \\ 0 & I_N & -2x \end{pmatrix},$$

where I_N is the identity matrix of size $N \times N$, then

$$\nabla_{\mathbb{H}} = A\nabla_{\mathbb{R}^{2N+1}}.$$

A simple computation gives the expression

$$|\nabla_{\mathbb{H}} u|^2 = 4(|x|^2 + |y|^2)(\partial_\tau u)^2 + \sum_{i=1}^N \Big((\partial_{x_i} u)^2 + (\partial_{y_i} u)^2 + 4\partial_\tau u(y_i\partial_{x_i} u - x_i\partial_{y_i} u)\Big).$$

The divergence operator in $\mathbb{H}$ is

$$\operatorname{div}_{\mathbb{H}}(u) = \operatorname{div}_{\mathbb{R}^{2N+1}}(Au).$$

For more details on Heisenberg groups and partial differential equations in Heisenberg groups, we refer to [3, 7, 14, 22, 23] and references therein.

In the proof of our main result, the following inequality will be used several times.

Lemma 2.1 (ε-Young inequality)**.** *Let $a, b, \varepsilon > 0$. Then*

$$ab \le \varepsilon a^p + c_\varepsilon b^{p'},$$

where $p > 1$, p' is its corresponding conjugate exponent, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$; and $c_\varepsilon = \left(\frac{1}{\varepsilon p}\right)^{p'/p} \frac{1}{p'}$.

3. Main result

In this paper, we use the notation:

$$
\begin{gathered}
d\mathcal{H} = dt_1 \dots dt_k d\vartheta, \\
d\mathcal{E}_1 = dt_2 \dots dt_k d\vartheta, \\
d\mathcal{E}_i = dt_1 \dots dt_{i-1} dt_{i+1} \dots dt_k d\vartheta \quad (i \neq 1).
\end{gathered}
$$

Moreover, for a given function $\varphi : \mathcal{H} \to \mathbb{R}$, we denote

$$
\varphi_i = \varphi\big|_{t_i=0}, \quad i = 1, 2, \dots, k.
$$

For $r > 1$, we denote by r' its corresponding conjugate exponent. Now, let us define the class of solutions under consideration.

Definition 3.1. Let $u \in W^{1,p'}_{\text{loc}}(\mathcal{H}; \mathbb{R}_+) \cap L^q_{\text{loc}}(\mathcal{H}; \mathbb{R}_+)$ and $u_i \in L^1_{\text{loc}}(\mathcal{E}; \mathbb{R}_+)$, $i = 1, 2, \dots, k$, with $p' = \frac{p}{p-1}$. We say that u is a global weak solution to problem (1.1) if the following conditions are satisfied:

(i) $|\nabla_{\mathbb{H}} u|^{p-2} \nabla_{\mathbb{H}} u \in L^p_{\text{loc}}(\mathcal{H}; \mathbb{R}^{2N+1})$;
(ii) For any $\varphi \in W^{1,p'}_{\text{loc}}(\mathcal{H}; \mathbb{R}_+)$ with compact support,

$$
\int_{\mathcal{H}} u^q \varphi \, d\mathcal{H} \le \int_{\mathcal{H}} |\nabla_{\mathbb{H}} u|^{p-2} \nabla_{\mathbb{H}} u \cdot \nabla_{\mathbb{H}} \varphi \, d\mathcal{H} - \sum_{i=1}^k \int_{\mathcal{H}} u \varphi_{t_i} \, d\mathcal{H} - \sum_{i=1}^k \int_{\mathcal{E}} u_i \varphi_i \, d\mathcal{E}_i. \tag{3.1}
$$

Observe that all the integrals in (3.1) are well defined. Our main result is given in the following theorem.

Theorem 3.2. *Let $p > 1$. If*

$$
\max\{1, p-1\} < q \le \frac{pk + Q(p-1)}{Q + (k-1)p}, \tag{3.2}
$$

where $Q = 2N + 2$ is the homogeneous dimension of $\mathbb{H}$, then (1.1) has no global nontrivial weak solutions.

To prove Theorem 3.2, we need the following lemma, which provides a preliminary estimate of possible solutions.

Lemma 3.3. *Let $p > 1$, $q > \max\{1, p-1\}$ and $\alpha \in (\delta, 0)$, where*

$$
\delta = \max\{-1, 1-p, \}.
$$

Let u be a global weak solution to (1.1). Then for any $\varphi \in W^{1,\infty}(\mathcal{H}; \mathbb{R}_+)$ with a compact support, we have

$$
\begin{aligned}
&\int_{\mathcal{H}} u^{q+\alpha} \varphi \, d\mathcal{H} + \int_{\mathcal{H}} |\nabla_{\mathbb{H}} u|^p u^{\alpha-1} \varphi \, d\mathcal{H} + \sum_{i=1}^k \int_{\mathcal{E}} {u_i}^{\alpha+1} \varphi_i \, d\mathcal{E}_i \\
&\le C\Big(\sum_{i=1}^k \int_{\mathcal{H}} \big(\frac{|\varphi_{t_i}|^r}{\varphi} \big)^{\frac{1}{r-1}} \, d\mathcal{H} + \int_{\mathcal{H}} \varphi^{1-ps'} |\nabla_{\mathbb{H}} \varphi|^{ps'} \, d\mathcal{H} \Big),
\end{aligned} \tag{3.3}
$$

for some constant $C > 0$, where $r = \frac{q+\alpha}{1+\alpha}$, $s = \frac{q+\alpha}{p+\alpha-1}$ and s' is the conjugate exponent of s.

Proof. Let $\varepsilon > 0$ be fixed and $\alpha \in (\delta, 0)$. Suppose that u is a global weak solution to (1.1). Let

$$u_\varepsilon(\vartheta, t_1, \dots, t_k) = u(\vartheta, t_1, \dots, t_k) + \varepsilon, \quad (\vartheta, t_1, \dots, t_k) \in \mathcal{H}.$$

Define φ_ε as

$$\varphi_\varepsilon(\vartheta, t_1, \dots, t_k) = u_\varepsilon^\alpha(\vartheta, t_1, \dots, t_k)\varphi(\vartheta, t_1, \dots, t_k),$$

where $\varphi \in W^{1,\infty}(\mathcal{H}; \mathbb{R}_+)$ has a compact support. Observe that φ_ε belongs to the set of admissible test functions in the sense of Definition 3.1. By (3.1), we have

$$\begin{aligned} &\int_{\mathcal{H}} u^q {u_\varepsilon}^\alpha \varphi \, d\mathcal{H} + |\alpha| \int_{\mathcal{H}} |\nabla_{\mathbb{H}} u|^p {u_\varepsilon}^{\alpha-1} \varphi \, d\mathcal{H} + \frac{1}{\alpha+1} \sum_{i=1}^k \int_{\mathcal{E}} (u_i + \varepsilon)^{\alpha+1} \varphi_i \, d\mathcal{E}_i \\ &\le \int_{\mathcal{H}} |\nabla_{\mathbb{H}} u|^{p-1} {u_\varepsilon}^\alpha |\nabla_{\mathbb{H}} \varphi| \, d\mathcal{H} + \frac{1}{\alpha+1} \sum_{i=1}^k \int_{\mathcal{H}} {u_\varepsilon}^{\alpha+1} |\varphi_{t_i}| \, d\mathcal{H}. \end{aligned} \tag{3.4}$$

Now, using Lemma 2.1, we will estimate the individual terms on the right-hand side of (3.4). For some $\varepsilon_1 > 0$, Lemma 2.1 with parameters $r = \frac{q+\alpha}{1+\alpha}$ and $r' = \frac{q+\alpha}{q-1}$ yields

$$\int_{\mathcal{H}} {u_\varepsilon}^{\alpha+1} |\varphi_{t_i}| \, d\mathcal{H} \le \varepsilon_1 \int_{\mathcal{H}} u_\varepsilon^{q+\alpha} \varphi \, d\mathcal{H} + c_{\varepsilon_1} \int_{\mathcal{H}} \Big(\frac{|\varphi_{t_i}|^r}{\varphi} \Big)^{\frac{1}{r-1}} \, d\mathcal{H},$$

from which follows

$$\begin{aligned} &\frac{1}{\alpha+1} \sum_{i=1}^k \int_{\mathcal{H}} {u_\varepsilon}^{\alpha+1} |\varphi_{t_i}| \, d\mathcal{H} \\ &\le \frac{k\varepsilon_1}{\alpha+1} \int_{\mathcal{H}} u_\varepsilon^{q+\alpha} \varphi \, d\mathcal{H} + \frac{c_{\varepsilon_1}}{\alpha+1} \sum_{i=1}^k \int_{\mathcal{H}} \Big(\frac{|\varphi_{t_i}|^r}{\varphi} \Big)^{\frac{1}{r-1}} \, d\mathcal{H}. \end{aligned} \tag{3.5}$$

For some $\varepsilon_2 > 0$, applying Lemma 2.1 with parameters p and $p' = \frac{p}{p-1}$, we obtain

$$\begin{aligned} &\int_{\mathcal{H}} |\nabla_{\mathbb{H}} u|^{p-1} {u_\varepsilon}^\alpha |\nabla_{\mathbb{H}} \varphi| \, d\mathcal{H} \\ &\le \varepsilon_2 \int_{\mathcal{H}} |\nabla_{\mathbb{H}} u|^p {u_\varepsilon}^{\alpha-1} \varphi \, d\mathcal{H} + c_{\varepsilon_2} \int_{\mathcal{H}} {u_\varepsilon}^{p+\alpha-1} |\nabla_{\mathbb{H}} \varphi|^p \varphi^{1-p} \, d\mathcal{H}. \end{aligned} \tag{3.6}$$

Again, for some $\varepsilon_3 > 0$, Lemma 2.1 with parameters $s = \frac{q+\alpha}{p+\alpha-1}$ and $s' = \frac{q+\alpha}{q-p+1}$, yields

$$\int_{\mathcal{H}} {u_\varepsilon}^{p+\alpha-1} |\nabla_{\mathbb{H}} \varphi|^p \varphi^{1-p} \, d\mathcal{H} \le \varepsilon_3 \int_{\mathcal{H}} {u_\varepsilon}^{q+\alpha} \varphi \, d\mathcal{H} + c_{\varepsilon_3} \int_{\mathcal{H}} \varphi^{1-ps'} |\nabla_{\mathbb{H}} \varphi|^{ps'} \, d\mathcal{H}. \tag{3.7}$$

Combining (3.6) with (3.7), we obtain

$$\begin{aligned} &\int_{\mathcal{H}} |\nabla_{\mathbb{H}} u|^{p-1} {u_\varepsilon}^\alpha |\nabla_{\mathbb{H}} \varphi| \, d\mathcal{H} \\ &\le \varepsilon_2 \int_{\mathcal{H}} |\nabla_{\mathbb{H}} u|^p {u_\varepsilon}^{\alpha-1} \varphi \, d\mathcal{H} + c_{\varepsilon_2} \varepsilon_3 \int_{\mathcal{H}} {u_\varepsilon}^{q+\alpha} \varphi \, d\mathcal{H} \\ &\quad + c_{\varepsilon_2} c_{\varepsilon_3} \int_{\mathcal{H}} \varphi^{1-ps'} |\nabla_{\mathbb{H}} \varphi|^{ps'} \, d\mathcal{H}. \end{aligned} \tag{3.8}$$

Furthermore, substituting estimates (3.5) and (3.8) into (3.4), we obtain

$$
\begin{aligned}
&\int_{\mathcal{H}} u^q {u_\varepsilon}^{\alpha}\varphi\, d\mathcal{H} + (|\alpha|-\varepsilon_2)\int_{\mathcal{H}} |\nabla_{\mathbb{H}} u|^p {u_\varepsilon}^{\alpha-1}\varphi\, d\mathcal{H} + \frac{1}{\alpha+1}\sum_{i=1}^{k}\int_{\mathcal{E}} (u_i+\varepsilon)^{\alpha+1}\varphi_i\, d\mathcal{E}_i \\
&\le \big(\frac{k\varepsilon_1}{\alpha+1} + c_{\varepsilon_2}\varepsilon_3\big)\int_{\mathcal{H}} u_\varepsilon^{q+\alpha}\varphi\, d\mathcal{H} + \frac{c_{\varepsilon_1}}{\alpha+1}\sum_{i=1}^{k}\int_{\mathcal{H}} \big(\frac{|\varphi_{t_i}|^r}{\varphi}\big)^{\frac{1}{r-1}}\, d\mathcal{H} \\
&\quad + c_{\varepsilon_2}c_{\varepsilon_3}\int_{\mathcal{H}} \varphi^{1-ps'}|\nabla_{\mathbb{H}}\varphi|^{ps'}\, d\mathcal{H}.
\end{aligned}
$$

Passing to the limit inferior as $\varepsilon \to 0$ in the above inequality, and applying the Fatou and Lebesgue theorems, we obtain

$$
\begin{aligned}
&\big(1-\frac{k\varepsilon_1}{\alpha+1} - c_{\varepsilon_2}\varepsilon_3\big)\int_{\mathcal{H}} u^{q+\alpha}\varphi\, d\mathcal{H} + (|\alpha|-\varepsilon_2)\int_{\mathcal{H}} |\nabla_{\mathbb{H}} u|^p u^{\alpha-1}\varphi\, d\mathcal{H} \\
&+ \frac{1}{\alpha+1}\sum_{i=1}^{k}\int_{\mathcal{E}} {u_i}^{\alpha+1}\varphi_i\, d\mathcal{E}_i \\
&\le \frac{c_{\varepsilon_1}}{\alpha+1}\sum_{i=1}^{k}\int_{\mathcal{H}} \Big(\frac{|\varphi_{t_i}|^r}{\varphi}\Big)^{\frac{1}{r-1}}\, d\mathcal{H} + c_{\varepsilon_2}c_{\varepsilon_3}\int_{\mathcal{H}} \varphi^{1-ps'}|\nabla_{\mathbb{H}}\varphi|^{ps'}\, d\mathcal{H}.
\end{aligned}
$$

For $\varepsilon_1, \varepsilon_2, \varepsilon_3$ sufficiently small, we obtain

$$
\begin{aligned}
&\int_{\mathcal{H}} u^{q+\alpha}\varphi\, d\mathcal{H} + \int_{\mathcal{H}} |\nabla_{\mathbb{H}} u|^p u^{\alpha-1}\varphi\, d\mathcal{H} + \sum_{i=1}^{k}\int_{\mathcal{E}} {u_i}^{\alpha+1}\varphi_i\, d\mathcal{E}_i \\
&\le C\Big(\sum_{i=1}^{k}\int_{\mathcal{H}} \Big(\frac{|\varphi_{t_i}|^r}{\varphi}\Big)^{\frac{1}{r-1}}\, d\mathcal{H} + \int_{\mathcal{H}} \varphi^{1-ps'}|\nabla_{\mathbb{H}}\varphi|^{ps'}\, d\mathcal{H}\Big),
\end{aligned}
$$

for some constant $C > 0$, which is the desired result. □

Now we are ready to prove our main result given by Theorem 3.2.

Proof of Theorem 3.2. Suppose that u is a nontrivial global weak solution to (1.1). Let us consider the test function

$$
\begin{aligned}
\varphi_R(\vartheta, t) &= \varphi_R(x, y, \tau, t) \\
&= \phi^\omega\Big(\frac{t_1^{2\theta_1} + \dots + t_k^{2\theta_1} + |x|^{4\theta_2} + |y|^{4\theta_2} + \tau^{2\theta_2}}{R^{4\theta_2}}\Big), \quad R > 0,\ \omega \gg 1,
\end{aligned}
$$

where $\phi \in C_0^\infty(\mathbb{R}^+)$ is a decreasing function satisfying

$$
\phi(z) = \begin{cases} 1 & \text{if } 0 \le z \le 1 \\ 0 & \text{if } z \ge 2, \end{cases}
$$

and θ_j, $j = 1, 2$, are positive parameters, whose exact values will be specified later. Let

$$
\rho = \frac{t_1^{2\theta_1} + \dots + t_k^{2\theta_1} + |x|^{4\theta_2} + |y|^{4\theta_2} + \tau^{2\theta_2}}{R^{4\theta_2}}.
$$

Clearly φ_R has support in

$$
\Omega_R = \{(\vartheta, t) \in \mathcal{H} :\ 0 \le \rho \le 2\},
$$

while $(\varphi_R)_{t_i}$ $(i=1,2,\dots,k)$ and $\nabla_{\mathbb{H}}\varphi_R$ have support in

$$\Theta_R=\{(\vartheta,t)\in\mathcal{H}:\ 1\le\rho\le 2\}.$$

A simple computation yields

$$\partial_{t_i}\varphi_R(t\vartheta)=2\theta_1\omega t_i^{2\theta_1-1}R^{-4\theta_2}\phi^{\omega-1}(\rho)\phi'(\rho),\quad i=1,2,\dots,k$$

while

$$\begin{aligned}\nabla_{\mathbb{H}}\varphi_R(t,\vartheta)|^2&=16\theta_2^2\omega^2R^{-8\theta_2}(\phi'(\rho))^2\phi^{2\omega-2}(\rho)\Big((|x|^2+|y|^2)\tau^{4\theta_2-2}\\&\quad+(|x|^{8\theta_2-2}+|y|^{8\theta_2-2})+2\tau^{2\theta_2-1}\sum_{i=1}^N x_iy_i(|x|^{4\theta_2-2}-|y|^{4\theta_2-2})\Big).\end{aligned}$$

Then, for all $(t,\vartheta)\in\Omega_R$ and $i=1,2,\dots,k$, we have

$$R|\nabla_{\mathbb{H}}\varphi_R|+R^{2\theta_2/\theta_1}|\partial_{t_i}\varphi_R|\le C|\phi'(\rho)|\phi^{\omega-1}(\rho). \tag{3.9}$$

For simplicity, in the sequel, we will write φ instead of φ_R. Let us consider now the change of variables

$$(t_1,\dots,t_k,x,y,\tau)=(t,\vartheta)\mapsto(\widetilde{t_1},\dots,\widetilde{t_k},\widetilde{x},\widetilde{y},\widetilde{\tau})=(\widetilde{t},\widetilde{\vartheta}),$$

where

$$\widetilde{t}=R^{-2\theta_2/\theta_1}t,\quad \widetilde{x}=R^{-1}x,\quad \widetilde{y}=R^{-1}y,\quad \widetilde{\tau}=R^{-2}\tau,\quad d\widetilde{\mathcal{H}}=d\widetilde{t}d\widetilde{\vartheta}.$$

In the same way, let

$$\begin{gathered}\widetilde{\rho}=\widetilde{t_1}^{2\theta_1}+\dots+\widetilde{t_k}^{2\theta_1}+|\widetilde{x}|^{4\theta_2}+|\widetilde{y}|^{4\theta_2}+\widetilde{\tau}^{2\theta_2},\\ \widetilde{\Omega}=\{(\widetilde{x},\widetilde{y},\widetilde{\tau},\widetilde{t})\in\mathcal{H}:0\le\widetilde{\rho}\le 2\},\\ \widetilde{\Theta}=\{(\widetilde{x},\widetilde{y},\widetilde{\tau},\widetilde{t})\in\mathcal{H}:1\le\widetilde{\rho}\le 2\}.\end{gathered}$$

Using the above change of variables and (3.9), we obtain

$$\int_{\mathcal{H}}\Big(\frac{|\varphi_{t_i}|^r}{\varphi}\Big)^{\frac{1}{r-1}}\,d\mathcal{H}\le CR^{Q+2\frac{\theta_2}{\theta_1}(k-\frac{r}{r-1})}\int_{\mathcal{H}}\phi^{\omega-\frac{r}{r-1}}|\phi'|^{\frac{r}{r-1}}\,d\widetilde{\mathcal{H}}, \tag{3.10}$$

$$\int_{\mathcal{H}}\varphi^{1-ps'}|\nabla_{\mathbb{H}}\varphi|^{ps'}\,d\mathcal{H}\le CR^{Q+2\frac{\theta_2}{\theta_1}k-ps'}\int_{\mathcal{H}}\phi^{\omega-ps'}|\phi'|^{ps'}\,d\widetilde{\mathcal{H}}. \tag{3.11}$$

Setting

$$\frac{\theta_2}{\theta_1}=\frac{ps'(r-1)}{2r},$$

we have

$$Q+2\frac{\theta_2}{\theta_1}(k-\frac{r}{r-1})=Q+2\frac{\theta_2}{\theta_1}k-ps'=Q-\frac{p(q+\alpha)}{q-p+1}+\frac{p(q-1)k}{q-p+1}. \tag{3.12}$$

Using (3.3), (3.10)–(3.12), we obtain

$$\int_{\mathcal{H}}u^{q+\alpha}\varphi\,d\mathcal{H}\le CR^{Q-\frac{p(q+\alpha)}{q-p+1}+\frac{p(q-1)k}{q-p+1}}. \tag{3.13}$$

Furthermore, noting that

$$Q-\frac{p(q+\alpha)}{q-p+1}+\frac{p(q-1)k}{q-p+1}<0$$

for

$$q<\frac{pk+Q(p-1)}{Q+(k-1)p}$$

and some $\alpha \in (\delta, 0)$ sufficiently small. Under the above condition, letting $R \to \infty$ in (3.13) and using the monotone convergence theorem, we obtain

$$\int_{\mathcal{H}} u^{q+\alpha}\, d\mathcal{H} \leq 0,$$

which contradicts our assumption about u.

Finally, the limit case

$$q = \frac{pk + Q(p-1)}{Q + (k-1)p}$$

can be treated by the same way as in [18]. □

We now consider some examples where we can apply Theorem 3.2. Applying Theorem 3.2 with $p = 2$ and $k = 1$, we obtain the following Heisenberg version of Fujita exponent [19].

Corollary 3.4. *If $1 < q \leq 1 + \frac{2}{Q}$, then the problem*

$$\begin{gathered} u_t - \Delta_{\mathbb{H}} u \geq u^q \quad \text{in } \mathcal{H}, \\ u \geq 0, \quad \text{a.e. in } \mathcal{H}, \\ u_{|_{t=0}} = u_0 \geq 0, \quad \text{in } \mathbb{H}, \end{gathered}$$

where $\mathcal{H} = \mathbb{H} \times (0, \infty)$ and $u_0 \in L^1_{\rm loc}(\mathbb{H})$, has no nontrivial global weak solution.

Next, applying Theorem 3.2 with $p = 2$ and $k = 2$, we obtain the following result, which is an extension of [11, Theorem 2.1] in the case $\alpha = 2$, $m = 1$, $s = \ell = r = 0$, to the Heisenberg group.

Corollary 3.5. *If $1 < q \leq 1 + \frac{2}{Q+2}$, then the problem*

$$\begin{gathered} u_{t_1} + u_{t_2} - \Delta_{\mathbb{H}} u \geq u^q \quad \text{in } \mathcal{H}, \\ u \geq 0, \quad \text{a.e. in } \mathcal{H}, \\ u_{|_{t_i=0}} = u_i \geq 0, \quad \text{in } \mathcal{E}, \end{gathered}$$

where $\mathcal{H} = \mathbb{H} \times (0, \infty)^2$, $\mathcal{E} = \mathbb{H} \times (0, \infty)$, and $u_i \in L^1_{\rm loc}(\mathcal{E})$, $i = 1, 2$, has no nontrivial global weak solution.

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Mohamed Jleli
Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia
E-mail address: jleli@ksu.edu.sa

Mokhtar Kirane
Laboratoire LaSIE, Pôle Sciences et Technologies, Université de La Rochelle, Avenue M. Crépeau, 17042 La Rochelle, France
E-mail address: mkirane@univ-lr.fr

Bessem Samet
Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia
E-mail address: bsamet@ksu.edu.sa