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MULTIPLICITY OF SOLUTIONS TO A NONLOCAL CHOQUARD EQUATION INVOLVING FRACTIONAL MAGNETIC OPERATORS AND CRITICAL EXPONENT

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Abstract. In this article, we study the multiplicity of solutions to a nonlocal fractional Choquard equation involving an external magnetic potential and critical exponent, namely,

$$\begin{aligned}&(a+b[u]_{s,A}^2)(-\Delta)_A^s u+V(x)u\\&=\int_{\mathbb{R}^N}\frac{|u(y)|^{2^*_{\mu,s}}}{|x-y|^{\mu}}dy|u|^{2^*_{\mu,s}-2}u+\lambda h(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N,\end{aligned}$$

$$[u]_{s,A}=\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)^{1/2}$$

where $a\geq 0, b>0$, $0<s<\min\{1,N/4\}$, $4s\leq\mu<N$, $V:\mathbb{R}^N\to\mathbb{R}$ is a sign-changing scalar potential, $A:\mathbb{R}^N\to\mathbb{R}^N$ is the magnetic potential, $(-\Delta)_A^s$ is the fractional magnetic operator, $\lambda>0$ is a parameter, $2^*_{\mu,s}=\frac{2N-\mu}{N-2s}$ is the critical exponent in the sense of the Hardy-Littlewood-Sobolev inequality and $2<p<2^*_s$. Under suitable assumptions on a,b and λ, we obtain multiplicity of nontrivial solutions by using variational methods. In particular, we obtain the existence of infinitely many nontrivial solutions for the degenerate Kirchhoff case, that is, $a=0$, $b>0$.

1. Introduction and statement of main results

In this article we consider the multiplicity of solutions to the Choquard-Kirchhoff type problem

$$\begin{aligned}&(a+b\|u\|_{s,A}^2)(-\Delta)_A^s u+V(x)u\\&=\int_{\mathbb{R}^N}|u|^{2^*_{\mu,s}}\mathcal{K}_\mu(x-y)dy|u|^{2^*_{\mu,s}-2}u+\lambda h(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N\end{aligned} \tag{1.1}$$

where $a\geq 0, b>0$, $s\in(0,1)$, $N>\mu\geq 4s$, $2^*_{\mu,s}=\frac{2N-\mu}{N-2s}$, $2^*_s=\frac{2N}{N-2s}$, $V:\mathbb{R}^N\to\mathbb{R}$ is the scalar potential, $\mathcal{K}_\mu(x)=|x|^{-\mu}$, $A:\mathbb{R}^N\to\mathbb{R}^N$ is the magnetic potential, $h:\mathbb{R}^N\to\mathbb{R}_0^+$, $\lambda>0$ and $(-\Delta)_A^s$ is the fractional magnetic operator which, up to normalization, defined as

$$(-\Delta)_A^s u(x)=2\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)}\frac{u(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)}{|x-y|^{N+2s}}\,dy,\quad \forall x\in\mathbb{R}^N,$$

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along any $\varphi \in C_0^\infty(\mathbb{R}^N,\mathbb{C})$, see [10] and the references therein for further details on this kinds of operators. Here $B_\varepsilon(x)$ denotes the ball in $\mathbb{R}^N$ with radius $\varepsilon > 0$ centered at $x \in \mathbb{R}^N$. As showed in [44], up to correcting the operator with factor $(1-s)$ it follows that $(-\Delta)^s_A u$ converges to $-(\nabla u - \mathrm{i}A)^2 u$ in the limit $s \uparrow 1$, where

$$-(\nabla u - \mathrm{i}A)^2 u = -\Delta u + 2\mathrm{i}A(x)\cdot\nabla u + |A(x)|^2 u + \mathrm{i}u \operatorname{div} A(x).$$

Thus, up to normalization, we may think the nonlocal case as an approximation of the local case. In recent years, the following magnetic Schrödinger equations like

$$-(\nabla u - \mathrm{i}A)^2 u + V(x)u = f(x,u)$$

have been extensively studied; see [4, 12, 1, 43]. We also collect some recent results on the fractional magnetic operators; see [29, 6, 38, 39] and the references cited there.

Clearly, the operator $(-\Delta)^s_A$ is consistent with the definition of fractional Laplacian $(-\Delta)^s$ if $A \equiv 0$. For more details on the fractional Laplacian, we refer to [13]. The fractional Laplacian operator $(-\Delta)^s$ can be seen as the infinitesimal generators of Lévy stable diffusion processes (see [1]). This type of operators arises in a quite natural way in many different applications, such as, continuum mechanics, phase transition phenomena, population dynamics and game theory, as they are the typical outcome of stochastically stabilization of Lévy processes, see [3, 8, 28, 19, 7]. In the context of fractional quantum mechanics, non-linear fractional Schrödinger equation has been proposed by Laskin [20, 21] as a result of expanding the Feynman path integral, from the Brownian-like to the Lévy-like quantum mechanical paths. The literature on non-local operators and on their applications is very interesting and quite large, we refer the interested readers to see [15, 14, 30, 35, 36, 5, 41] and the references therein.

Equation (1.1) is a nonlocal elliptic type equation and covers in particular for $s = 1, 2^*_{\mu,s} = 2, A \equiv 0$ the Choquard-Pekar equation, which appears as a model in quantum theory of a polaron at rest, see [37]. The time-dependent form of (1.1) also describes the self-gravitational collapse of a quantum mechanical wave function, in which context it is called Hartree equation or the Newton-Schrodinger eqution [31]. In recent years, the Choquard and related equations have been studied by many authors, see [23, 24, 27, 32] and the references therein. Very recently, D'Avenia, Siciliano and Squassina studied the existence, regularity and asymptotic of the solutions for the following fractional Choquard equation

$$(-\Delta)^s u + \omega u = (\mathcal{K}_\alpha * |u|^p)|u|^{p-2}u, \quad u \in H^s(\mathbb{R}^N), \tag{1.2}$$

where $s \in (0,1)$, $\omega > 0$, $N \geq 3$, $1 + \frac{\alpha}{N} < p < \frac{N+\alpha}{N-2s}$, $\alpha \in (0,N)$ and $\mathcal{K}_\alpha(x) = |x|^{\alpha-N}$. The existence of groundstates for fractional Choquard equations with general nonlinearities was obtained by Shen, Gao and Yang [42] using variational methods. In [40], Pucci, Xiang and Zhang extended equation (1.2) to the fractional p-Laplacian and obtained several existence results by using variational methods.

Fiscella and Valdinoci [17] proposed a stationary Kirchhoff variational model, in bounded regular domains of $\mathbb{R}^N$, which takes into account the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string, and obtained the existence and multiplicity of solutions for fractional Kirchhoff problems by using variational method and the concentration-compactness principle. Nyamoradi [34] studied a class of fractional Kirchhoff type equation in a bounded domain Ω and obtained three solutions by using three critical point theorem. For more details about stationary Kirchhoff problems involving the fractional Laplacian, we refer the interested readers to [10, 16, 45, 46, 47].

Inspired by the above cited papers, we consider the critical case of (1.2) and prove multiplicity results depending on λ, a b and N. In particular, when $N > \mu = 4s, a = 0, b > 0, V \equiv 0$, we obtain infinitely many solutions for (1.1) by applying critical point theory. Since equation (1.1) contains a critical nonlinearity, it is difficult to get the global (PS) condition. To overcome this difficulty, we borrow some tricks from articles [33, 25].

Definition 1.1. We say that $u \in D_A^s(\mathbb{R}^N, \mathbb{C})$ is a weak solution of (1.1), if

$$\begin{aligned}
&(a + b\|u\|_{s,A}^2) \\
&\times \Re \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y))\overline{(\varphi(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}\varphi(y))}}{|x-y|^{N+2s}}\,dx\,dy \\
&+ \Re \int_{\mathbb{R}^N} V(x) u\overline{\varphi} dx \\
&= \Re \int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}-2} u\overline{\varphi} dx + \lambda \Re \int_{\mathbb{R}^N} h(x)|u|^{p-2}u\overline{\varphi} dx,
\end{aligned}$$

for any $\varphi \in D_A^s(\mathbb{R}^N, \mathbb{C})$.

The best constant of Hardy-Littlehood-Sobolev inequality is

$$S_{H,L} := \inf_{u\in D_A^s(\mathbb{R}^N,\mathbb{C})\setminus\{0\}} \frac{[u]_{s,A}^2}{\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy\right)^{\frac{1}{2^*_{\mu,s}}}}. \tag{1.3}$$

Theorem 1.2. *Assume that* $s \in (0,1)$, $N > \mu \geq 4s$, $V \in L^{\frac{N}{2s}}(\mathbb{R}^N)$, $2 < p < 2_s^*$, $h \geq 0$, $h \not\equiv 0, h \in L^{\frac{2_s^*}{2_s^*-p}}(\mathbb{R}^N)$ *and* $A \in C(\mathbb{R}^N, \mathbb{R}^N)$. *If* $\mu = 4s$, $a \geq 0$ *and* $b > S_{H,L}^{-2^*_{\mu,s}}$ *or* $\mu > 4s$, $a > 0$, $b > 0$ *and*

$$a > (2 - 2^*_{\mu,s})\Big(\frac{b}{2^*_{\mu,s}-1}\Big)^{-\frac{2^*_{\mu,s}-1}{2-2^*_{\mu,s}}} S_{H,L}^{-\frac{2^*_{\mu,s}}{2-2^*_{\mu,s}}}, \tag{1.4}$$

then there exists $\lambda^* > 0$ *such that* (1.1) *admits at least two nontrivial solutions in* $D_A^s(\mathbb{R}^N, \mathbb{C})$ *for all* $\lambda > \lambda^*$.

Theorem 1.3. *Assume that* $s \in (0,1)$, $N > \mu = 4s$, $a = 0$, $b > S_{H,L}^{-2^*_{\mu,s}}$, $V \equiv 0$, $2 < p < 2_s^*$, $h \geq 0$, $h \not\equiv 0$, $h \in L^{\frac{2_s^*}{2_s^*-p}}(\mathbb{R}^N)$ *and* $A \in C(\mathbb{R}^N, \mathbb{R}^N)$. *Then* (1.1) *has infinitely many pairs of solutions in* $D_A^s(\mathbb{R}^N, \mathbb{C})$ *for all* $\lambda > 0$. *Moreover, any nontrivial solution* $u \in D_A^s(\mathbb{R}^N, \mathbb{C}) \setminus \{0\}$ *satisfies*

$$[u]_{s,A} \leq \left[\frac{\lambda\|h\|_{L^{\frac{2_s^*}{2_s^*-p}}(\mathbb{R}^N)}}{S^{p/2}(b - S_{H,L}^{-2^*_{\mu,s}})}\right]^{\frac{1}{4-p}},$$

where S is the best constant of the embedding $D_A^s(\mathbb{R}^N, \mathbb{C}) \hookrightarrow L^{2_s^*}(\mathbb{R}^N, \mathbb{C})$ *defined by*

$$S := \inf_{u\in D_A^s(\mathbb{R}^N,\mathbb{C})\setminus\{0\}} \frac{[u]_{s,A}^2}{\|u\|_{L^{2_s^*}(\mathbb{R}^N,\mathbb{C})}^2}. \tag{1.5}$$

Remark 1.4. We say that equation (1.1) is non-degenerate if $a > 0, b \geq 0$; and degenerate if $a = 0, b > 0$. To the best of our knowledge, this article is the first to deal with the multiplicity of solutions for fractional Choquard-Kirchhoff type equations with external magnetic operator and critical exponent.

This article is organized as follows. In Section 2, we recall some necessary definitions and properties of spaces $D^s(\mathbb{R}^N)$ and $D^s_A(\mathbb{R}^N, \mathbb{C})$. In Section 3, the multiplicity of solutions of (1.1) is obtained by using variational methods.

2. Preliminaries

In this section, we first give some basic results of fractional Sobolev spaces that will be used later. Let $N > 1$, $0 < s < 1$ be real number satisfying $2s < N$ and the fractional critical exponent 2^*_s be defined as $2^*_s = \frac{2N}{N-2s}$. The fractional Sobolev space $D^s(\mathbb{R}^N)$ is defined as the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$[u]_s = \Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \Big)^{1/2},$$

The embedding $D^s(\mathbb{R}^N) \hookrightarrow L^{2^*_s}(\mathbb{R}^N)$ is continuous by [13, Theorem 6.7]. Suppose that $A : \mathbb{R}^N \to \mathbb{R}^N$ is a continuous function. Consider the magnetic Gagliardo semi-norm defined by

$$[u]_{s,A} := \Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \Big)^{1/2},$$

and define $D^s_A(\mathbb{R}^N, \mathbb{C})$ as the closure of $C_0^\infty(\mathbb{R}^N, \mathbb{C})$ with respect to $[\cdot]_{s,A}$.

Lemma 2.1. *For each $u \in D^s_A(\mathbb{R}^N, \mathbb{C})$ it holds $|u| \in D^s(\mathbb{R}^N)$. More precisely,*

$$\big[|u|\big]_s \leq [u]_{s,A}, \quad \textit{for all } u \in D^s_A(\mathbb{R}^N, \mathbb{C}).$$

Proof. The proof follows by using the pointwise diamagnetic inequality

$$\big||u(x)| - |u(y)|\big| \leq \big|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u(y)\big|,$$

for a.e. $x, y \in \mathbb{R}^N$, see [10, Lemma 3.1, Remark 3.2]. □

Finally, we introduce the well-known Hardy-Littlewood-Sobolev inequality, see [22].

Lemma 2.2. *Assume $1 < r, t < \infty$ and $0 < \mu < N$ with $\frac{1}{r} + \frac{1}{t} + \frac{\mu}{N} = 2$. If $u \in L^r(\mathbb{R}^N)$ and $v \in L^t(\mathbb{R}^N)$, then there exists $C(N, \alpha, r, t) > 0$ such that*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)||v(y)|}{|x-y|^\mu} \, dx \, dy \leq C(N, \mu, r, t) \|u\|_{L^r(\mathbb{R}^N)} \|v\|_{L^t(\mathbb{R}^N)}.$$

3. Proof of Theorem 1.2

The functional associated with (1.1) is defined as

$$\begin{aligned} \mathcal{I}(u) = & \frac{a}{2}[u]^2_{s,A} + \frac{b}{4}[u]^4_{s,A} + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|u|^2 dx \\ & - \frac{1}{22_{\mu,s}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_{\mu,s}} |u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu} \, dx \, dy - \frac{\lambda}{p} \int_{\mathbb{R}^N} h(x)|u|^p dx. \end{aligned}$$

for all $u \in D^s_A(\mathbb{R}^N, \mathbb{C})$.

From $V \in L^{\frac{N}{2s}}(\mathbb{R}^N)$ and $h \in L^{\frac{2_s^*}{2_s^*-p}}(\mathbb{R}^N)$, the Hardy-Littlehood-Sobolev inequality and the fractional Sobolev inequality, one can show that $\mathcal{I}$ is well-defined, of class C^1 and

$$\begin{aligned}
&\langle \mathcal{I}'(u), v\rangle \\
&= (a + b[u]_{s,A}^2) \\
&\times \Re \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[u(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}u(y)]\overline{[v(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}v(y)]}}{|x-y|^{N+2s}}\,dx\,dy \\
&\quad + \Re\int_{\mathbb{R}^N} V u\bar{v}dx - \Re\int_{\mathbb{R}^N} (\mathcal{K} * |u|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}-2}u\bar{v}dx - \lambda\Re\int_{\mathbb{R}^N} h|u|^{p-2}u\bar{v}dx,
\end{aligned}$$

for all $u, v \in D_A^s(\mathbb{R}^N, \mathbb{C})$. Hence a critical point of $\mathcal{I}$ is a (weak) solution of (1.1).

Definition 3.1. For any $c \in \mathbb{R}$, $\{u_n\}$ is called a $(PS)_c$ sequence of $\mathcal{I}$ in $D_A^s(\mathbb{R}^N, \mathbb{C})$, if $\mathcal{I}(u_n) \to c$ and $\mathcal{I}'(u_n) \to 0$ as $n \to \infty$. We say that $\mathcal{I}$ satisfies $(PS)_c$ condition if any $(PS)_c$ sequence of $\mathcal{I}$ admits a convergent subsequence in $D_A^s(\mathbb{R}^N, \mathbb{C})$.

Now we give a key lemma for proving the main results.

Lemma 3.2. *Under the conditions of Theorem 1.2, functional $\mathcal{I}$ satisfies the $(PS)_c$ conditions in $D_A^s(\mathbb{R}^N, \mathbb{C})$ for all $\lambda > 0$.*

Proof. Suppose that $\{u_n\} \subset D_A^s(\mathbb{R}^N, \mathbb{C})$ is a $(PS)_c$ sequence of functional $\mathcal{I}$, i.e.

$$\mathcal{I}(u_n) \to c, \quad \mathcal{I}'(u_n) \to 0$$

as $n \to \infty$.

By Hölder's inequality, (1.3) and (1.5), we deuce

$$\begin{aligned}
\mathcal{I}(u) &\geq \frac{a}{2}[u]_{s,A}^2 + \frac{b}{4}[u]_{s,A}^4 - \frac{1}{2}S^{-1}\|V\|_{L^{\frac{N}{2s}}(\mathbb{R}^N)}[u]_{s,A}^2 &(3.1)\\
&\quad - \frac{1}{22^*_{\mu,s}} S_{H,L}^{-2^*_{\mu,s}}[u]_{s,A}^{22^*_{\mu,s}} - \frac{1}{p}S^{-\frac{p}{2}}\lambda\|h\|_{L^{\frac{2_s^*}{2_s^*-p}}(\mathbb{R}^N)}[u]_{s,A}^p, &(3.2)
\end{aligned}$$

for all $u \in D_A^s(\mathbb{R}^N, \mathbb{C})$. When $\mu = 4s$, since $\frac{2}{2^*_{\mu,s}} S_{H,L}^{-4} < b$, $2^*_{\mu,s} = 2$ and $2 < p < 2_s^* < 4$ by $N > 4s$, it follows that $\mathcal{I}$ is coercive and bounded from below on $D_A^s(\mathbb{R}^N, \mathbb{C})$. When $N > \mu \geq 4s$, since $a > 0, b > 0$, $2^*_{\mu,s} < 2$ and $2_s^* < 4$, it follows that $\mathcal{I}$ is coercive and bounded from below on $D_A^s(\mathbb{R}^N, \mathbb{C})$. Hence, $\{u_n\}$ is bounded in $D_A^s(\mathbb{R}^N, \mathbb{C})$. Then there exists $u \in D_A^s(\mathbb{R}^N, \mathbb{C})$ such that, up to a subsequence, it follows that

$$\begin{gathered}
u_n \rightharpoonup u \quad \text{in } D_A^s(\mathbb{R}^N, \mathbb{C}) \text{ and in } L^{2_s^*}(\mathbb{R}^N, \mathbb{C}),\\
u_n \to u \quad \text{a.e. in } \mathbb{R}^N \text{ and in } L^p_{\text{loc}}(\mathbb{R}^N),\ 1 \leq p < 2_s^*,\\
|u_n|^{2_s^*-2}u_n \rightharpoonup |u|^{2_s^*-2}u \quad \text{weakly in } L^{\frac{2_s^*}{2_s^*-1}}(\mathbb{R}^N, \mathbb{C}),
\end{gathered} \tag{3.3}$$

as $n \to \infty$. We first show that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} V(x)|u_n|^2dx = \int_{\mathbb{R}^N} V(x)|u|^2dx. \tag{3.4}$$

Since $V \in L^{\frac{N}{2s}}(\mathbb{R}^N)$, for any $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that

$$\Big(\int_{\mathbb{R}^N\setminus B_{R_\varepsilon}(0)} |V(x)|^{\frac{N}{2s}}dx\Big)^{2s/N} < \varepsilon.$$

By Hölder's inequality, we deduce

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^N\setminus B_{R_\varepsilon}(0)} V(x)(|u_n|^2-|u|^2)dx\Big| \\
&\le \Big(\int_{\mathbb{R}^N\setminus B_{R_\varepsilon}(0)} |V(x)|^{\frac{N}{2s}}dx\Big)^{2s/N}\|u_n\|^2_{L^{2^*_s}(\mathbb{R}^N)} \\
&\quad + \Big(\int_{\mathbb{R}^N\setminus B_{R_\varepsilon}(0)} |V(x)|^{\frac{N}{2s}}dx\Big)^{2s/N}\|u\|^2_{L^{2^*_s}(\mathbb{R}^N)} \\
&\le C\Big(\int_{\mathbb{R}^N\setminus B_{R_\varepsilon}(0)} |V(x)|^{\frac{N}{2s}}dx\Big)^{2s/N} \le C\varepsilon.
\end{aligned}
\tag{3.5}
$$

On the other hand, by the boundedness of $\{u_n\}$, for any measurable non-empt subset $\Omega\subset B_{R_\varepsilon}$, we have

$$
\Big|\int_\Omega V(x)(|u_n|^2+|u|^2)dx\Big| \le C\Big(\int_\Omega |V(x)|^{\frac{N}{2s}}dx\Big)^{2s/N}.
$$

It follows from $V\in L^{\frac{N}{2s}}(\mathbb{R}^N)$ that the sequence $\{V(x)(|u_n|^2-|u|^2)\}$ is equi-integrable in $L^1(B_{R_\varepsilon}(0))$. Thus the Vitali convergence theorem implies

$$
\lim_{n\to\infty}\int_{B_{R_\varepsilon}(0)} V(x)|u_n|^2dx = \int_{B_{R_\varepsilon}(0)} V(x)|u|^2dx. \tag{3.6}
$$

Combining (3.5) with (3.6), we obtain the desired result (3.4). By using a similar discussion, we can deduce from $h\in L^{2^*_s}(\mathbb{R}^N)$ that

$$
\lim_{n\to\infty}\int_{\mathbb{R}^N} h(x)|u_n|^p dx = \int_{\mathbb{R}^N} h(x)|u|^p dx. \tag{3.7}
$$

Let $w_n=u_n-u$. Then by (3.3), we obtain

$$
\begin{gathered}
{}^2_{s,A} = [w_n]^2_{s,A} + [u]^2_{s,A} + o(1), \\
[u_n]^4_{s,A} = [w_n]^4_{s,A} + [u]^4_{s,A} + 2[u_n]^2_{s,A}[u]^2_{s,A} + o(1).
\end{gathered}
\tag{3.8}
$$

By the Brezis-Lieb type lemma (see [18]), one has

$$
\begin{aligned}
&\int_{\mathbb{R}^N}(\mathcal{K}_\mu * |w_n|^{2^*_{\mu,s}})|w_n|^{2^*_{\mu,s}}dx \\
&= \int_{\mathbb{R}^N}(\mathcal{K}_\mu * |u_n|^{2^*_{\mu,s}})|u_n|^{2^*_{\mu,s}}dx - \int_{\mathbb{R}^N}(\mathcal{K}_\mu * |u|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}}dx + o(1).
\end{aligned}
\tag{3.9}
$$

Without loss of generality, we assume that $\lim_{n\to\infty}[w_n]_{s,A}=\eta$. From $\{u_n\}$ is a $(PS)_c$ sequence and the boundedness of $\{u_n\}$, we have

$$
\begin{aligned}
\langle \mathcal{I}'(u_n),u_n\rangle &= a[u_n]^2_{s,A} + b[u_n]^4_{s,A} + \int_{\mathbb{R}^N} V(x)|u_n|^2dx \\
&\quad - \int_{\mathbb{R}^N}(\mathcal{K}_\mu * |u_n|^{2^*_{\mu,s}})|u_n|^{2^*_{\mu,s}}dx - \lambda\int_{\mathbb{R}^N} h(x)|u_n|^p dx = o(1)
\end{aligned}
\tag{3.10}
$$

and

$$
\begin{aligned}
\lim_{n\to\infty}\langle \mathcal{I}'(u_n),u\rangle &= a[u]^2_{s,A} + b[u]^4_{s,A} + b\eta^2[u]^2_{s,A} + \int_{\mathbb{R}^N} V(x)|u|^2dx \\
&\quad - \int_{\mathbb{R}^N}(\mathcal{K}_\mu * |u|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}}dx - \lambda\int_{\mathbb{R}^N} h(x)|u|^p dx = 0.
\end{aligned}
\tag{3.11}
$$

Here we have used that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}(\mathcal{K}_\mu*|u_n|^{2^*_{\mu,s}})|u_n|^{2^*_{\mu,s}-2}u_n u dx=\int_{\mathbb{R}^N}(\mathcal{K}_\mu*|u_n|^{2^*_{\mu,s}})|u_n|^{2^*_{\mu,s}}dx. \tag{3.12}$$

Indeed, by the Hardy-Littlewood-Sobolev inequality, the Riesz potential defines a linear continuous map from $L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)$ to $L^{\frac{2N}{\mu}}(\mathbb{R}^N)$. Then

$$\mathcal{K}_\mu*|u_n|^{2^*_{\mu,s}}\rightharpoonup\mathcal{K}_\mu*|u|^{2^*_{\mu,s}}\quad\text{in } L^{\frac{2N}{\mu}}(\mathbb{R}^N), \tag{3.13}$$

as $n\to\infty$. Note that for any measurable subset $U\subset\mathbb{R}^N$, we have

$$\int_U\big||u_n|^{2_{\mu,s}*-2}u_n u\big|^{\frac{2^*_s}{2^*_{\mu,s}}}dx\le\|u_n\|_{L^{2^*_s}(\mathbb{R}^N)}^{\frac{2^*_{\mu,s}-1}{2^*_{\mu,s}}}\|u\|_{L^{2^*_s}(U)}^{\frac{1}{2^*_{\mu,s}}},$$

which implies that $\{||u_n|^{2^*_{\mu,s}-2}u_n u|^{\frac{2^*_s}{2^*_{\mu,s}}}\}$ is equi-integrable in $L^1(\mathbb{R}^N)$. Observe that $|u_n|^{2^*_{\mu,s}-2}u_n u\to|u|^{2^*_{\mu,s}}$ a.e. in $\mathbb{R}^N$, then the Vitali convergence theorem yields

$$|u_n|^{2^*_{\mu,s}-2}u_n u\to|u|^{2^*_{\mu,s}}\quad\text{in } L^{\frac{2^*_s}{2^*_{\mu,s}}}(\mathbb{R}^N). \tag{3.14}$$

Combining (3.13) with (3.14) and $\frac{2^*_s}{2^*_{\mu,s}}=\frac{2N}{2N-\mu}$, we obtain the desired result (3.12).

It follows from (3.10) and (3.11) that

$$\begin{aligned}&a[u]^2_{s,A}+a[w_n]^2_{s,A}+b[u]^4_{s,A}+b[w_n]^4_{s,A}+2b[w_n]^2_{s,A}[u]^2_{s,A}\\&-\int_{\mathbb{R}^N}(\mathcal{K}_\mu*|u|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}}dx-\int_{\mathbb{R}^N}(\mathcal{K}_\mu*|w_n|^{2^*_{\mu,s}})|w_n|^{2^*_{\mu,s}}dx=o(1).\end{aligned}$$

Then

$$a[w_n]^2_{s,A}+b[w_n]^4_{s,A}+b[w_n]^2_{s,A}[u]^2_{s,A}-\int_{\mathbb{R}^N}(\mathcal{K}_\mu*|w_n|^{2^*_{\mu,s}})|w_n|^{2^*_{\mu,s}}dx=o(1).$$

From the definition of $S_{H,L}$, we obtain

$$\int_{\mathbb{R}^N}(\mathcal{K}_\mu*|w_n|^{2^*_{\mu,s}})|w_n|^{2^*_{\mu,s}}dx\le S_{H,L}^{2^*_{\mu,s}}[w_n]_{s,A}^{22^*_{\mu,s}}.$$

Using this and letting $n\to\infty$, we arrive at the inequality

$$a\eta^2+b\eta^2[u]^2_{s,A}+b\eta^4\le S_{H,L}^{-2^*_{\mu,s}}\eta^{22^*_{\mu,s}},$$

which implies

$$a\eta^2+b\eta^4\le S_{H,L}^{-2^*_{\mu,s}}\eta^{22^*_{\mu,s}}. \tag{3.15}$$

When $\mu=4s$ and $S_{H,L}^{-2^*_{\mu,s}}<b$, it follows from (3.15) that $\eta=0$. Thus, $u_n\to u$ in $D^s_A(\mathbb{R}^N,\mathbb{C})$.

When $\mu>4s$, it follows from (3.15) and the Young inequality that

$$\begin{aligned}a\eta^2+b\eta^4&\le\frac{1}{\frac{1}{2^*_{\mu,s}-1}}(\eta^{42^*_{\mu,s}-4})^{\frac{1}{2^*_{\mu,s}-1}}\Big[\Big(\frac{b}{2^*_{\mu,s}-1}\Big)^{2^*_{\mu,s}-1}\Big]^{\frac{1}{2^*_{\mu,s}-1}}\\&\quad+\frac{1}{\frac{1}{2-2^*_{\mu,s}}}\Big(\frac{b}{2^*_{\mu,s}-1}\Big)^{-\frac{2^*_{\mu,s}-1}{2-2^*_{\mu,s}}}S_{H,L}^{-\frac{2^*_{\mu,s}}{2-2^*_{\mu,s}}}\Big(\eta^{4-22^*_{\mu,s}}\Big)^{\frac{1}{2-2^*_{\mu,s}}}\\&=b\eta^4+(2-2^*_{\mu,s})\Big(\frac{b}{2^*_{\mu,s}-1}\Big)^{-\frac{2^*_{\mu,s}-1}{2-2^*_{\mu,s}}}S_{H,L}^{-\frac{2^*_{\mu,s}}{2-2^*_{\mu,s}}}\eta^2.\end{aligned}$$

Consequently,

$$\Big\{a-(2-2^*_{\mu,s})\Big(\frac{b}{2^*_{\mu,s}-1}\Big)^{-\frac{2^*_{\mu,s}-1}{2-2^*_{\mu,s}}}S_{H,L}^{-\frac{2^*_{\mu,s}}{2-2^*_{\mu,s}}}\Big\}\eta^2\leq 0,$$

which together with (1.4) implies that $\eta=0$. Hence $u_n\to u$ in $D^s_A(\mathbb{R}^N,\mathbb{C})$. $\square$

Remark 3.3. Clearly, when $a=0, V\equiv 0$, $\mu=4s$, $2<p<2^*_s$ and $b>S_{H,L}^{-2^*_{\mu,s}}$, the functional $\mathcal{I}$ also satisfies the $(PS)_c$ condition in $D^s_A(\mathbb{R}^N,\mathbb{C})$.

Proof of Theorem 1.2. We first show that (1.1) has a nontrivial global minimizer solution. By (3.3), we know $m:=\inf_{u\in D^s_A(\mathbb{R}^N,\mathbb{C})}\mathcal{I}(u)$ is well-defined. Now we claim that there exists $\lambda^*>0$ such that $m<0$ for all $\lambda>\lambda^*$. Actually, we can choose $\varphi_0\in D^s_A(\mathbb{R}^N,\mathbb{C})$ with $[\varphi_0]_{s,A}=1$ and $\int_{\mathbb{R}^N}h(x)|\varphi_0|^pdx>0$, then

$$\begin{aligned}\mathcal{I}(\varphi_0)&\leq\frac{a}{2}+\frac12\|V\|_{L^{\frac{N}{2s}}(\mathbb{R}^N)}S^{-1}+\frac{b}{4}-\frac{1}{22^*_{\mu,s}}\int_{\mathbb{R}^N}(\mathcal{K}_\mu*|\varphi_0|^{2^*_{\mu,s}})|\varphi_0|^{2^*_{\mu,s}}dx\\&\quad-\frac{\lambda}{p}\int_{\mathbb{R}^N}h(x)|\varphi_0|^pdx\\&\leq\frac{a}{2}+\frac12\|V\|_{L^{\frac{N}{2s}}(\mathbb{R}^N)}S^{-1}+\frac{b}{4}-\frac{\lambda}{p}\int_{\mathbb{R}^N}h(x)|\varphi_0|^pdx<0,\end{aligned}$$

for all $\lambda>\frac{p(\frac{a}{2}+\frac12\|V\|_{L^{\frac{N}{2s}}(\mathbb{R}^N)}S^{-1}+\frac{b}{4})}{\int_{\mathbb{R}^N}h(x)|\varphi_0|^pdx}$. Hence our claim holds true. Further, by Lemma 3.2 and [26, Theorem 4.4], there exists $u_1\in D^s_A(\mathbb{R}^N,\mathbb{C})$ such that $\mathcal{I}(u_1)=m$. Therefore, u_1 is a nontrivial global minimizer solution of (1.1) with $\mathcal{I}(u_1)<0$.

Now we prove that (1.1) has a mountain pass solution. Since $p\in(2,2^*_s)$, we obtain that 0 a local minimum point of $\mathcal{I}$ in $D^s_A(\mathbb{R}^N,\mathbb{C})$. Define

$$c=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\mathcal{I}(\gamma(t)),$$

where $\Gamma=\{\gamma\in C([0,1],D^s_A(\mathbb{R}^N,\mathbb{C})):\gamma(0)=0,\gamma(1)=u_1\}$. Then $c>0$. By Lemma 3.2, we know that $\mathcal{I}$ satisfies the conditions of the mountain-pass lemma (see [2, Theorem 2.1]). Then there exists $u_2\in D^s_A(\mathbb{R}^N,\mathbb{C})$ such that $\mathcal{I}(u_2)=c>0$ and $\mathcal{I}'(u_2)=0$. Thus, u_2 is a nontrivial solution of equation (1.1). $\square$

To obtain the existence of infinitely many solutions, we introduce the following theorem (see [9]).

Theorem 3.4 ([9, Theorem 5.2.23])**.** *Let X be a Banach space, and $J\in C^1(X,\mathbb{R})$ be an even functional satisfying the $(PS)_c$ condition. Assume $\alpha<\beta$ and either $J(0)<\alpha$ or $J(0)>\beta$. If further,*

1. *there are an m-dimensional linear subspace E and a constant $\rho>0$ such that $\sup_{E\cap\partial B_\rho(0)}J(u)\leq\beta$, where $\partial B_\rho(0)=\{u\in X:\|u\|=\rho\}$;*
2. *there is a j-dimensional linear subspace F such that $\inf_{F^\perp}J(u)>\alpha$, where $F^\perp$ is a complementary space of F;*
3. *$m>j$,*

then J has at least $m-j$ pairs of distinct critical points.

Proof of Theorem 1.3. Clearly, $\mathcal{I}$ is an even functional. By Remark 3.3, $\mathcal{I}$ satisfies the $(PS)_c$ condition. Choose $E=D^s_A(\mathbb{R}^N,\mathbb{C})$ and $F=\emptyset$, then $F^\perp=D^s_A(\mathbb{R}^N,\mathbb{C})$.

We can choose $\phi_0 \in D_A^s(\mathbb{R}^N, \mathbb{C})$ such that $[\phi_0]_{s,A} = 1$ and $\int_{\mathbb{R}^N} h(x)|\phi_0|^p dx > 0$. Then

$$\begin{aligned}\mathcal{I}(t\phi_0) &= \frac{b}{4}t^4[\phi_0]_{s,A}^4 - t^{22^*_{\mu,s}}\frac{1}{22^*_{\mu,s}}\int_{\mathbb{R}^N}(\mathcal{K} * |\phi_0|^{2^*_{\mu,s}})|\phi_0|^{2^*_{\mu,s}}dx \\ &\quad - t^p\frac{\lambda}{p}\int_{\mathbb{R}^N} h(x)|\phi_0|^p dx \\ &\le \frac{b}{4}t^4[\phi_0]_{s,A}^4 - \frac{\lambda}{p}t^p\int_{\mathbb{R}^N} h(x)|\phi_0|^p dx \\ &= \Big[\frac{b}{4}t^{4-p}[\phi_0]_{s,A}^4 - \frac{\lambda}{p}\int_{\mathbb{R}^N} h(x)|\phi_0|^p dx\Big]t^p,\end{aligned}$$

for all $t > 0$. It follows from $2 < p < 4$ that there exist $\beta < 0$ and $\rho > 0$ small enough such that $\mathcal{I}(t\phi_0) \le \beta < 0$ for all $0 < t < \rho$. Thus, we obtain

$$\sup_{E\cap\partial B_\rho(0)} \mathcal{I}(u) \le \beta < 0,$$

where $\partial B_\rho(0) := \{u \in D_A^s(\mathbb{R}^N, \mathbb{C}) : [u]_{s,A} = \rho\}$. By (3.3), we easily deduce

$$\inf_{u\in F^\perp} \mathcal{I}(u) > -\infty.$$

Therefore, $\mathcal{I}$ satisfies the conditions of Theorem 3.4. Hence $\mathcal{I}$ has infinitely many pairs distinct critical points in $D_A^s(\mathbb{R}^N, \mathbb{C})$, that is, equation (1.1) has infinitely many pairs distinct solutions. Let $u \in D_A^s(\mathbb{R}^N, \mathbb{C}) \setminus \{0\}$ be a solution of (1.1). Then

$$b[u]_{s,A}^4 = \int_{\mathbb{R}^N}(\mathcal{K}_\mu * |u|^{2^*_s})|u|^{2^*_s}dx + \lambda\int_{\mathbb{R}^N} h(x)|u|^p dx.$$

It follows that

$$b[u]_{s,A}^4 \le S_{H,L}^{-2^*_{\mu,s}}[u]_{s,A}^{22^*_{\mu,s}} + \lambda\|h\|_{L^{\frac{2^*_s}{2^*_s-p}}(\mathbb{R}^N)} S^{-\frac{p}{2}}[u]_{s,A}^p.$$

Since $b > S^{-2^*_{\mu,s}}$ and $2^*_{\mu,s} = 2$ by $\mu = 4s$, we have

$$(b - S_{H,L}^{-2^*_{\mu,s}})[u]_{s,A}^4 \le \lambda\|h\|_{L^{\frac{2^*_s}{2^*_s-p}}(\mathbb{R}^N)} S^{-\frac{p}{2}}[u]_{s,A}^p.$$

Further,

$$\Big[(b - S_{H,L}^{-2^*_{\mu,s}})[u]_{s,A}^{4-p} - \lambda\|h\|_{L^{\frac{2^*_s}{2^*_s-p}}(\mathbb{R}^N)} S^{-\frac{p}{2}}\Big][u]_{s,A}^p \le 0,$$

which implies

$$[u]_{s,A} \le \Big[\frac{\lambda\|h\|_{L^{\frac{2^*_s}{2^*_s-p}}(\mathbb{R}^N)} S^{-\frac{p}{2}}}{b - S_{H,L}^{-2^*_{\mu,s}}}\Big]^{\frac{1}{4-p}}.$$

This completes the proof. □

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