Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 320, pp. 1–25.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

STRUCTURAL STABILITY OF RIEMANN SOLUTIONS FOR STRICTLY HYPERBOLIC SYSTEMS WITH THREE PIECEWISE CONSTANT STATES

XUEFENG WEI, CHUN SHEN

Abstract. This article concerns the wave interaction problem for a strictly hyperbolic system of conservation laws whose Riemann solutions involve delta shock waves. To cover all situations, the global solutions are constructed when the initial data are taken as three piecewise constant states. It is shown that the Riemann solutions are stable with respect to a specific small perturbation of the Riemann initial data. In addition, some interesting nonlinear phenomena are captured during the process of constructing the solutions, such as the generation and decomposition of delta shock waves.

1. Introduction

In this article, we are concerned with the hyperbolic system of conservation laws

$$\begin{aligned} u_t + (u^2)_x &= 0,\\ v_t + \Big((2u+1)v\Big)_x &= 0, \end{aligned} \tag{1.1}$$

which was used to study the behavior of a magnetohydrodynamics model (MHD) [25, 26]. System (1.1) is strictly hyperbolic whose eigenvalues are $\lambda_1 = 2u$ and $\lambda_2 = 2u+1$. Furthermore, the first characteristic field for λ_1 is genuinely nonlinear and the second one for λ_2 is linearly degenerate.

The Riemann problem is the particular Cauchy problem with the two piecewise initial data

$$(u,v)(x,0) = \begin{cases} (u_-, v_-), & x<0,\\ (u_+, v_+), & x>0, \end{cases} \tag{1.2}$$

where all the $u_\pm$ and $v_\pm$ are given constants. The Riemann problem (1.1) and (1.2) was studied by Tan [25] through the self-similar vanishing viscosity method. It was discovered in [25] that if $u_+ < u_- - 1$, then the solution of the Riemann problem (1.1) and (1.2) cannot be constructed by a combination of shock waves, rarefaction waves and contact discontinuities. In this situation, the delta shock wave should be introduced into the soluton, which is the form of the standard Dirac delta function supported on a shock wave [1, 2, 11, 12, 13, 23, 28].

2010 *Mathematics Subject Classification.* 35L65, 35L67, 76N15.
Key words and phrases. Delta shock wave; wave interaction; Riemann problem; strict hyperbolicity.

Submitted September 22, 2016. Published December 14, 2016.

With the Riemann solutions of (1.1) and (1.2) in hand, it is natural to expect the study of the so-called double Riemann problem because the Riemann problem (1.1) and (1.2) cannot describe the dynamic pictures in all the situations for (1.1). In this article, we study the Cauchy problem for (1.1) with three piecewise initial data:

$$(u, v)(x, 0) = \begin{cases} (u_-, v_-), & -\infty < x < -\varepsilon, \\ (u_m, v_m), & -\varepsilon < x < \varepsilon, \\ (u_+, v_+), & \varepsilon < x < +\infty, \end{cases} \tag{1.3}$$

where $\varepsilon > 0$ is arbitrarily small. This is the so-called perturbed Riemann problem or the double Riemann problem. For the reason that the initial data (1.3) may be regarded as a small perturbation of the corresponding Riemann initial data (1.2) with the perturbed parameter ε. In fact, we will encounter an interesting problem that if the limits $\varepsilon \to 0$ of solutions $(u_\varepsilon, v_\varepsilon)(x, t)$ are identical with the ones of the Riemann problem (1.1) and (1.2) or not, in which $(u_\varepsilon, v_\varepsilon)(x, t)$ refer to the solutions of the particular Cauchy problem (1.1) and (1.3) associated with ε accordingly.

In fact, the three piecewise initial data (1.3) have been widely used to study the wave interaction problem for some hyperbolic systems, such as the the pressureless Euler system [20, 29], the Euler system for Chaplygin gas [7, 18], a non-strictly hyperbolic system [22, 30] and various types of chromatography systems [6, 19, 24]. It is noticed that all the systems studied above belong to the so-called Temple class [27], namely the shock curves coincide with the rarefaction curves in the phase plane, such that wave interactions have relatively more simplified structures and then the global solutions may be constructed completely for these systems with the initial data (1.3). However, it is remarkable that the system (1.1) does not belong to the Temple class, such that the solutions of the perturbed Riemann problem (1.1) and (1.3) have more complicated and interesting structures. Fortunately, we discover that the propagation speeds of elementary waves for the Riemann problem (1.1) and (1.2) can be expressed concisely by the state variable u, including shock wave, rarefaction wave, contact discontinuity and delta shock wave. Thus, the global solutions of the perturbed Riemann problem (1.1) and (1.3) can be constructed in explicit forms.

The main purpose of this paper is to investigate various possible wave interactions including delta shock waves for system (1.1). Thus, we take the three piecewise initial data (1.3) instead of the Riemann initial data (1.2) such that the solutions beyond the interactions are constructed. Furthermore, it is shown that the solutions of the perturbed Riemann problem (1.1) and (1.3) converge to the corresponding ones of the Riemann problem (1.1) and (1.2) as $\varepsilon \to 0$ by dealing with this problem case by case, which shows the stability of Riemann solutions with respect to the small perturbation (1.3) of the Riemann initial data (1.2). In addition, some interesting nonlinear phenomena can be captured during the process of constructing the solutions to the perturbed Riemann problem (1.1) and (1.3). At first, we discover that a delta shock wave may be generated by the interaction between two shock waves. Secondly, it can be observed that a delta shock wave may be decomposed into a shock wave and a delta contact discontinuity during the process when it penetrates a rarefaction wave. Finally, it can be shown that infinitely many contact discontinuities may be continuously produced which have the same propagation speed during the process when a shock wave penetrates a rarefaction wave.

It should be pointed out that the following strictly hyperbolic system of conservation laws

$$\begin{aligned} u_t + \Big(\frac{u^2}{2}\Big)_x &= 0,\\ v_t + \Big((u-1)v\Big)_x &= 0, \end{aligned} \tag{1.4}$$

was introduced by Hayes and LeFloch [8]. This system has the similar property with system (1.1). It should be stressed that the interactions between the delta shock wave with the other elementary waves have been well investigated for (1.4) by Nedeljkov and Oberguggenberger [17]. The method of split delta function [14, 15, 16] was in [17] to study the strength of delta shock wave precisely.

In this article, the wave interaction problem is also considered when the delta shock wave does not appear at the initial moment for the perturbed Riemann problem (1.1) and (1.3), which was not addressed in [17]. In fact, the interactions between the delta shock wave with the other elementary waves for (1.1) have similar structures with those for (1.4). In this paper, we only use the generalized Rankine-Hugoniot conditions to calculate the strength of delta shock wave for simplicity. In addition, the stability of solutions to the Riemann problem (1.1) and (1.2) can also be analyzed when the delta shock waves are involved in the solutions to the perturbed Riemann problem (1.1) and (1.3).

This article is organized in the following way. In Section 2, some preliminaries are given, which include the Riemann solutions of (1.1) and (1.2) and the generalized Rankine-Hugoniot relations of delta shock wave. Furthermore, it is proven rigorously that the delta shock wave solution indeed satisfies the system (1.1) in the sense of distributions. In Section 3, we consider the perturbed Riemann problem (1.1) and (1.3) when the delta shock wave does not appear at the initial time. The wave interaction problems are studied in detail and then the global solutions are constructed completely. In Section 4, we consider the perturbed Riemann problem (1.1) and (1.3) when the delta shock wave is involved at the initial time. The interactions between the delta shock wave with the other elementary waves are investigated carefully, including shock wave, rarefaction wave and contact discontinuity. At the end, discussions are carried out and the conclusions are drawn in Section 5.

2. The Riemann problem

In this section, we are devoted to the Riemann problem (1.1) and (1.2), which was investigated in [25] by using the self-similar vanishing viscosity method. The eigenvalues of system (1.1) are $\lambda_1 = 2u$ and $\lambda_2 = 2u+1$, thus (1.1) is strictly hyperbolic for the reason that $\lambda_1 < \lambda_2$ holds for any u. Furthermore, the corresponding right eigenvectors are $r_1 = (-1, 2v)^T$ and $r_2 = (0, 1)^T$, respectively. Thus, we have $\nabla\lambda_1 \cdot r_1 = -2$ and $\nabla\lambda_2 \cdot r_2 = 0$, in which the symbol ∇ expresses the gradient with respect to (u, v). We know that the first characteristic field for λ_1 is genuinely nonlinear and the second one for λ_2 is always linearly degenerate. Therefore, the waves of the first family are either rarefaction waves (denoted by R) or shock waves (denoted by S) which are decided by the initial data and while the waves of the second family are always contact discontinuities (denoted by J).

We first consider the elementary wave for (1.1). For a given left state (u_-, v_-), the 1-rarefaction wave curve in the (u, v) phase plane can be expressed as $R(u_-, v_-)$:

$$\begin{gathered}\xi = \lambda_1 = 2u,\\ v \cdot e^{2u} = v_- \cdot e^{2u_-},\\ u > u_-, \quad 0 < v < v_-.\end{gathered} \tag{2.1}$$

On the other hand, the 1-shock wave curve in the (u, v) phase plane can also be expressed as $S(u_-, v_-)$:

$$\begin{gathered}\sigma = u_- + u,\\ \frac{v}{v_-} = \frac{u_- - u + 1}{u - u_- + 1},\\ u_- - 1 < u < u_-, \quad v > v_-.\end{gathered} \tag{2.2}$$

In addition, the 2-contact discontinuity curve in the (u, v) phase plane should satisfy $u = u_-$ and the corresponding propagation speed is $\tau = 2u_- + 1 = 2u + 1$.

Then, we construct the Riemann solutions of (1.1) and (1.2) for different cases. For the case $u_- < u_+$, the Riemann solution of (1.1) and (1.2) is a rarefaction wave followed by a contact discontinuity, which can be expressed as

$$(u, v)(x, t) = \begin{cases} (u_-, v_-), & x < 2u_- t,\\ \left(\frac{x}{2t}, v_- \exp(2u_- - \frac{x}{t})\right), & 2u_- t < x < 2u_+ t,\\ (u_+, v_- \exp(2u_- - 2u_+)), & 2u_+ t < x < (2u_+ + 1)t,\\ (u_+, v_+), & x > (2u_+ + 1)t. \end{cases} \tag{2.3}$$

For the case $u_- - 1 < u_+ < u_-$, the Riemann solution of (1.1) and (1.2) contains a shock wave plus a contact discontinuity, which is given by

$$(u, v)(x, t) = \begin{cases} (u_-, v_-), & x < (u_- + u_+)t,\\ (u_+, v_*), & (u_- + u_+)t < x < (2u_+ + 1)t,\\ (u_+, v_+), & x > (2u_+ + 1)t, \end{cases} \tag{2.4}$$

where

$$v_* = v_- \cdot \frac{u_- - u_+ + 1}{u_+ - u_- + 1}. \tag{2.5}$$

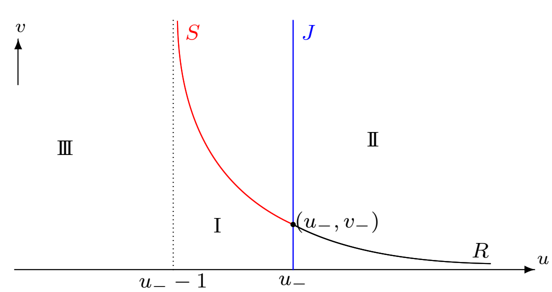


FIGURE 1. The (u, v) phase plane for system (1.1) for a given left state (u_-, v_-).

Let us turn our attention on the case $u_+ \leq u_- - 1$. Then the nonclassical situation appears where the Riemann problem (1.1) and (1.2) cannot be solved by a combination of shock waves, rarefaction waves and contact discontinuities. To solve the Riemann problem (1.1) and (1.2) when $u_+ \leq u_- - 1$, it was shown in [25] that a solution containing a weighted δ-measure supported on a curve should be adopted. In this paper, let us use the exact definition of delta shock wave solution which was introduced by Danilov and Shelkovich [3, 4, 5] and improved by Kalisch and Mitrovic [9, 10].

Definition 2.1. Using the the two-dimensional weighted δ-measure $\beta(s)\delta_\Gamma$, which is supported on a smooth curve $\Gamma = \{(x(s), t(s)) : a < s < b\}$, we define the measure-valued solutions by

$$\langle \beta(\mathrm{s})\delta_\Gamma, \psi(\mathrm{x}, \mathrm{t})\rangle = \int_{\mathrm{a}}^{\mathrm{b}} \beta(\mathrm{s})\psi(\mathrm{x}(\mathrm{s}), \mathrm{t}(\mathrm{s}))\mathrm{ds}, \tag{2.6}$$

for all $\psi(x,t) \in C_0^\infty(R \times R_+)$.

Let $\Gamma = \{\gamma_i | i \in I\}$ be a set of curves in the closed upper half-plane $\{(x,t)|(x,t) \in (-\infty, \infty) \times [0, \infty)\}$, in which γ_i expresses a Lipschitz continuous curve and I is a finite index set. Let I_0 be the subset of I involving all the indices of curves starting from the $x-$axis and then let $\Gamma_0 = \{x_j^0 | j \in I_0\}$ be the set of initial points of all the curves γ_j when $j \in I_0$. In what follows, one may define the solutions in the distributional sense to the Cauchy problem for the system (1.1) with the delta measure initial data.

Definition 2.2. Let (u, v) be a pair of distributions, in which v has the form

$$v(x,t) = \hat{v}(x,t) + \beta(x,t)\delta(\Gamma) = \hat{v}(x,t) + \sum_{i \in I} \alpha_i(x,t)\delta(\gamma_i), \tag{2.7}$$

and $u, \hat{v} \in L^\infty(R \times R_+)$. Let us consider the initial data of type

$$(u,v)(x,0) = \Big(u_0(x), \hat{v}_0(x) + \sum_{j \in I_0} \alpha_j(x_j^0, 0)\delta(x - x_j^0)\Big), \tag{2.8}$$

in which $u_0, \hat{v}_0 \in L^\infty(R)$, then the above pair of distributions (u, v) are called as a generalized delta shock wave solution of the Cauchy problem (1.1) and (2.8) if the following integral equalities hold for any $\psi \in C_c^\infty(R \times R_+)$:

$$\int_{R_+} \int_R \big(u\psi_t + u^2\psi_x\big)\, dx\, dt + \int_R u_0(x)\psi(x,0)dx = 0, \tag{2.9}$$

and

$$\begin{aligned} &\int_{R_+} \int_R (\hat{v}\psi_t + (2u+1)\hat{v}\psi_x)\, dx\, dt + \sum_{i \in I} \int_{\gamma_i} \alpha_i(x,t) \frac{\partial \psi(x,t)}{\partial l} \\ &+ \int_R \hat{v}_0(x)\psi(x,0)dx + \sum_{k \in I_0} \alpha_k(x_k^0, 0)\psi(x_k^0, 0) = 0, \end{aligned} \tag{2.10}$$

where $\int_{\gamma_i}$ is the line integral along γ_i and $\frac{\partial \psi}{\partial l}$ is the tangential derivative of ψ.

In view of the above definitions, we use the following theorem to describe the delta shock wave solution to the Riemann problem (1.1) and (1.2) when $u_+ \leq u_- - 1$.

Theorem 2.3. *For the case $u_+ \le u_- - 1$, the Riemann solution of* (1.1) *and* (1.2) *is piecewise smooth in the form*

$$(u,v)(x,t) = \begin{cases} (u_-, v_-), & x < \sigma_\delta t, \\ (u_\delta, \beta(t)\delta(x-\sigma_\delta t)), & x = \sigma_\delta t, \\ (u_+, v_+), & x > \sigma_\delta t, \end{cases} \tag{2.11}$$

where

$$\begin{gathered} \sigma_\delta = u_- + u_+, \quad u_\delta = \frac{1}{2}(u_- + u_+ - 1), \\ \beta(t) = \Big((u_- - u_+)(v_- + v_+) - (v_+ - v_-)\Big)t. \end{gathered} \tag{2.12}$$

The measure-valued solution (2.11) *should satisfy the generalized Rankine-Hugoniot conditions*

$$\begin{gathered} \frac{dx}{dt} = \sigma_\delta, \\ \frac{d\beta(t)}{dt} = \sigma_\delta[v] - [(2u+1)v], \\ \sigma_\delta[u] = [u^2], \end{gathered} \tag{2.13}$$

and the over-compressive entropy condition

$$2u_+ + 1 < \sigma_\delta < 2u_-, \tag{2.14}$$

where $[u] = u(x(t)+0,t) - u(x(t)-0,t)$ denotes the jump of u across the discontinuity $x = x(t)$, etc.

Proof. Let us check that the measure-valued solution (2.11) with (2.12) should satisfy the system (1.1) in the sense of distributions. In other words, we need to check that (2.11) and (2.12) should satisfy

$$\begin{gathered} \int_0^\infty \int_{-\infty}^\infty \big(u\psi_t + u^2\psi_x\big)\, dx\, dt = 0, \\ \int_0^\infty \int_{-\infty}^\infty (v\psi_t + (2u+1)v\psi_x)\, dx\, dt = 0, \end{gathered} \tag{2.15}$$

for any $\psi \in C_c^\infty(R \times R_+)$. Without loss of generality, let us assume that $\sigma_\delta > 0$ for the reason that $\sigma_\delta \le 0$ can be dealt with similarly and the difference only lies in that the different integral regions are decomposed in the upper-half physical plane $(x,t) \in (R \times R_+)$.

Let us check the first equation in (2.15). By the third equation in (2.13), we have

$$\begin{aligned} &\int_0^\infty \int_{-\infty}^\infty \big(u\psi_t + u^2\psi_x\big)\, dx\, dt \\ &= \int_0^\infty \int_{-\infty}^{\sigma_\delta t} \big(u_-\psi_t + u_-^2\psi_x\big)\, dx\, dt + \int_0^\infty \int_{\sigma_\delta t}^\infty \big(u_+\psi_t + u_+^2\psi_x\big)\, dx\, dt \\ &= \int_0^\infty \int_{-\infty}^0 u_-\psi_t\, dx\, dt + \int_0^\infty \int_0^{\sigma_\delta t} u_-\psi_t\, dx\, dt + \int_0^\infty \int_{-\infty}^{\sigma_\delta t} u_-^2\psi_x\, dx\, dt \\ &\quad + \int_0^\infty \int_{\sigma_\delta t}^\infty \big(u_+\psi_t + u_+^2\psi_x\big)\, dx\, dt \end{aligned}$$

$$
\begin{aligned}
&= \int_0^{\infty}\int_{\frac{x}{\sigma_\delta}}^{\infty} u_-\psi_t\,dt\,dx + \int_0^{\infty}\int_{-\infty}^{\sigma_\delta t} u_-^2\psi_x\,dx\,dt + \int_0^{\infty}\int_0^{\frac{x}{\sigma_\delta}} u_+\psi_t\,dt\,dx \\
&\quad + \int_0^{\infty}\int_{\sigma_\delta t}^{\infty} u_+^2\psi_x\,dx\,dt \\
&= -\int_0^{\infty} u_-\psi(x,\frac{x}{\sigma_\delta})dx + \int_0^{\infty} u_-^2\psi(\sigma_\delta t,t)dt + \int_0^{\infty} u_+\psi(x,\frac{x}{\sigma_\delta})dx \\
&\quad - \int_0^{\infty} u_+^2\psi(\sigma_\delta t,t)dt \\
&= \int_0^{\infty}\Big(\sigma_\delta(u_+-u_-) - (u_+^2-u_-^2)\Big)\psi(\sigma_\delta t,t)dt = 0,
\end{aligned}
$$

in which we have used the fact that $\psi(x,t)$ is compactly support in the region $R\times R_+$.

On the other hand, taking into account the second equation in (2.13) and the relation formula $\sigma_\delta = 2u_\delta + 1$ from (2.12), we also have

$$
\begin{aligned}
&\int_0^{\infty}\int_{-\infty}^{\infty}\left(v\psi_t + (2u+1)v\psi_x\right)\,dx\,dt \\
&= \int_0^{\infty}\int_{-\infty}^{\sigma_\delta t}\left(v_-\psi_t + (2u_-+1)v_-\psi_x\right)\,dx\,dt \\
&\quad + \int_0^{\infty}\int_{\sigma_\delta t}^{\infty}\left(v_+\psi_t + (2u_++1)v_+\psi_x\right)\,dx\,dt \\
&\quad + \int_0^{\infty}\beta(t)(\psi_t(\sigma_\delta t,t) + (2u_\delta+1)\psi_x(\sigma_\delta t,t))dt \\
&= \int_0^{\infty}\int_0^{\sigma_\delta t} v_-\psi_t\,dx\,dt + \int_0^{\infty}\int_{-\infty}^{\sigma_\delta t}(2u_-+1)v_-\psi_x\,dx\,dt \\
&\quad + \int_0^{\infty}\int_{\sigma_\delta t}^{\infty} v_+\psi_t\,dx\,dt + \int_0^{\infty}\int_{\sigma_\delta t}^{\infty}(2u_++1)v_+\psi_x\,dx\,dt \\
&\quad + \int_0^{\infty}\beta(t)(\psi_t(\sigma_\delta t,t) + \sigma_\delta\psi_x(\sigma_\delta t,t))dt \\
&= \int_0^{\infty}\int_{\frac{x}{\sigma_\delta}}^{\infty} v_-\psi_t\,dt\,dx + \int_0^{\infty}\int_{-\infty}^{\sigma_\delta t}(2u_-+1)v_-\psi_x\,dx\,dt \\
&\quad + \int_0^{\infty}\int_0^{\frac{x}{\sigma_\delta}} v_+\psi_t\,dt\,dx + \int_0^{\infty}\int_{\sigma_\delta t}^{\infty}(2u_++1)v_+\psi_x\,dx\,dt \\
&\quad + \int_0^{\infty}\beta(t)d\psi(\sigma_\delta t,t) \\
&= \int_0^{\infty}(v_+-v_-)\psi(x,\frac{x}{\sigma_\delta})dx + \int_0^{\infty}((2u_-+1)v_- - (2u_++1)v_+)\psi(\sigma_\delta t,t)dt \\
&\quad + \int_0^{\infty}\beta(t)d\psi(\sigma_\delta t,t) \\
&= \int_0^{\infty}\Big(\sigma_\delta(v_+-v_-) + (2u_-+1)v_- - (2u_++1)v_+ - \beta'(t)\Big)\psi(\sigma_\delta t,t)dt = 0.
\end{aligned}
$$

It is easy to check that the measure-valued solution (2.11) with (2.12) can be derived from the generalized Rankine-Hugoniot conditions (2.13) by a simple calculation. In order to ensure uniqueness, the $\delta-$entropy condition $\lambda_1(u_r) < \lambda_2(u_r) < \sigma < \lambda_1(u_l) < \lambda_2(u_l)$ should be satisfied, which leads to the over-compressive entropy condition (2.14). In other words, all the characteristics on both sides of the $\delta-$shock wave curve are incoming. Thus, it can be concluded from the above calculations that (2.11) with (2.12) is indeed the piecewise smooth solution of the Riemann problem (1.1) and (1.2) in the sense of distributions when $u_+ \leq u_- - 1$. □

Remark 2.4. One can see that the Riemann solutions of (1.1) and (1.2) can be constructed by a combination of shock waves, rarefaction waves, contact discontinuities and delta shock waves. More precisely, there are exactly three configurations of the Riemann solutions of (1.1) and (1.2) according to the relation between u_- and u_+ as follows: $R+J$ when $u_+ > u_-$, $S+J$ when $u_- - 1 < u_+ < u_-$ and δS when $u_+ \leq u_- - 1$.

3. Interactions of classical waves

To study the perturbed Riemann problem (1.1) and (1.3) is in essence to study the wave interaction problem for the system (1.1). It is remarkable that the Riemann solution of (1.1) and (1.2) may contain the delta shock wave or not. In order to cover all the cases completely, our discussion should be divided into two parts according to the appearance of delta shock wave or not at the initial time. In this section, we are mainly concerned with the wave interaction problem which does not involve the delta shock wave at the initial time. Then, we have four possibilities according to the different combinations of the classical waves from $(-\varepsilon, 0)$ and $(\varepsilon, 0)$ as follows: (1) $S+J$ and $S+J$; (2) $R+J$ and $S+J$; (3) $S+J$ and $R+J$; (4) $R+J$ and $R+J$.

Case 3.1: $S+J$ **and** $S+J$**.** First of all, let us consider the situation that both a shock wave followed by a contact discontinuity emit from the initial points $(-\varepsilon, 0)$ and $(\varepsilon, 0)$ respectively (see Figure 2). The occurrence of this case depends on the conditions $u_- - 1 < u_m < u_-$ and $u_+ < u_m < u_+ + 1$, from which we can easily get $u_+ < u_-$. The propagation speed of the first contact discontinuity J_1 is $\tau_1 = 2u_m + 1$ and that of the second shock wave S_2 is $\sigma_2 = u_m + u_+$. Thus, it is easy to see that J_1 overtakes S_2 in finite time. The intersection (x_1, t_1) is determined by

$$\begin{aligned} x_1 + \varepsilon &= (2u_m + 1)t_1, \\ x_1 - \varepsilon &= (u_m + u_+)t_1, \end{aligned} \tag{3.1}$$

which implies

$$(x_1, t_1) = \Big(\frac{\varepsilon(3u_m + u_+ + 1)}{u_m - u_+ + 1}, \frac{2\varepsilon}{u_m - u_+ + 1}\Big). \tag{3.2}$$

It can be shown that a new local Riemann problem for the system (1.1) will be formulated at the intersection (x_1, t_1) with the initial data

$$(u, v)(x, t_1) = \begin{cases} (u_1, v_1), & x < x_1, \\ (u_2, v_2), & x > x_1, \end{cases} \tag{3.3}$$

where

$$(u_1, v_1) = \Big(u_m, -v_- \cdot \frac{u_m - u_- - 1}{u_m - u_- + 1}\Big), \tag{3.4}$$

$$(u_2, v_2) = \Big(u_+, -v_m \cdot \frac{u_+ - u_m - 1}{u_+ - u_m + 1}\Big). \tag{3.5}$$

To solve the new Riemann problem (1.1) and (3.3), one can see that a new shock wave followed by a new contact discontinuity will be generated after the interaction between J_1 and S_2. Let us denote them by S_3 and J_3 respectively. One can see that the propagation speeds of S_3 and J_3 are the same as those of S_2 and J_2 respectively for the reason that $u_1 = u_m$ and $u_2 = u_+$.

Then, S_3 and S_1 intersect at the point (x_2, t_2), which can be calculated by

$$\begin{aligned} x_2 + \varepsilon &= (u_- + u_m)t_2, \\ x_2 - \varepsilon &= (u_m + u_+)t_2. \end{aligned} \tag{3.6}$$

An easy calculation leads to

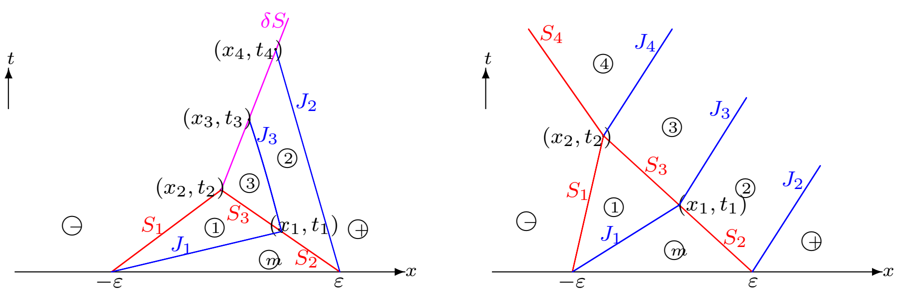
$$(x_2, t_2) = \Big(\frac{\varepsilon(2u_m + u_+ + u_-)}{u_- - u_+}, \frac{2\varepsilon}{u_- - u_+}\Big). \tag{3.7}$$

As before, a new local Riemann problem for system (1.1) will also be formulated at the intersection (x_2, t_2) with the initial data

$$(u, v)(x, t_2) = \begin{cases} (u_-, v_-), & x < x_2, \\ (u_3, v_3), & x > x_2, \end{cases} \tag{3.8}$$

in which

$$(u_3, v_3) = \Big(u_+, -v_1 \cdot \frac{u_+ - u_m - 1}{u_+ - u_m + 1}\Big). \tag{3.9}$$



(a) $u_- - 2 < u_+ < u_- - 1$ (b) $u_- - 1 < u_+ < u_-$

FIGURE 2. Interactions between $S + J$ and $S + J$ for two different situations where both $u_- - 1 < u_m < u_-$ and $u_+ < u_m < u_+ + 1$ should be satisfied.

It is easy to get $u_- - 2 < u_+ < u_-$ from $u_- - 1 < u_m < u_-$ and $u_+ < u_m < u_+ + 1$. In what follows, we can obtain two different situations according to the values u_+ and $u_- - 1$, which may be described by using the lemma below.

Lemma 3.1. *For the local Riemann problem* (1.1) *and* (3.8)*, if* $u_- - 2 < u_+ < u_- - 1$*, then a new delta shock wave will be generated; otherwise, if* $u_- - 1 < u_+ < u_-$*, then there is also a new shock wave followed by a new contact discontinuity. Finally, the global solutions of the perturbed Riemann problem* (1.1) *and* (3.1) *can be illustrated in Figure 2, in which both* $u_- - 1 < u_m < u_-$ *and* $u_+ < u_m < u_+ + 1$ *should be satisfied.*

Proof. If $u_- - 2 < u_+ < u_- - 1$, then a delta shock wave δS is generated after the coalescence of S_1 and S_3 at the point (x_2, t_2) (see Figure 2(a)), whose propagation speed and strength can be calculated by the generalized Rankine-Hugoniot conditions (2.13). Thus, one can easily get that the propagation speed of δS is $\sigma_\delta = u_- + u_+$. Consequently, the delta shock wave δS propagates with an invariant speed $u_- + u_+$ for the reason that $u_3 = u_2 = u_+$. On the other hand, the strength changes at the different growth rates due to the differences among v_3, v_2 and v_+.

In what follows, the delta shock wave δS intersects J_3 and consequently J_2 in finite time. The intersection (x_3, t_3) of δS and J_3 can be calculated by

$$\begin{gathered} x_3 - x_2 = (u_- + u_+)(t_3 - t_2), \\ x_3 - x_1 = (2u_+ + 1)(t_3 - t_1), \end{gathered} \tag{3.10}$$

which implies

$$\begin{aligned} x_3 = & \frac{\varepsilon(u_+ + u_- - 2u_m)(u_+ + u_-)}{(u_- - u_+)(u_- - u_+ - 1)} + \frac{\varepsilon(3u_m - 3u_+ - 1)(u_+ + u_-)}{(u_m - u_+ + 1)(u_- - u_+ - 1)} \\ & + \frac{\varepsilon(2u_m - u_+ - u_-)}{u_- - u_+}, \\ t_3 = & \frac{\varepsilon(u_- + u_+ - 2u_m)}{(u_- - u_+)(u_- - u_+ - 1)} + \frac{\varepsilon(3u_m - 3u_+ - 1)}{(u_m - u_+ + 1)(u_- - u_+ - 1)}. \end{aligned} \tag{3.11}$$

After δS passes through (x_3, t_3), it still propagates with the invariant speed $u_- + u_+$ and overtakes J_2 at the intersection (x_4, t_4), which is determined by

$$\begin{gathered} x_4 - x_2 = (u_- + u_+)(t_4 - t_2), \\ x_4 - \varepsilon = (2u_+ + 1)t_4, \end{gathered} \tag{3.12}$$

such that we have

$$(x_4, t_4) = \Big(\frac{2\varepsilon(2u_+ + 1)(u_- - u_m)}{(u_- - u_+)(u_- - u_+ - 1)} + \varepsilon, \frac{2\varepsilon(u_- - u_m)}{(u_- - u_+)(u_- - u_+ - 1)} \Big). \tag{3.13}$$

After the time t_4, the delta shock wave passes through J_2 and moves forwards with the same speed as before.

Thus, the strength of delta shock wave δS can be calculated respectively by

$$\beta(t) = \Big((u_- - u_+)(v_- + v_3) - (v_3 - v_-) \Big)(t - t_2), \quad \text{for } t_2 \le t \le t_3, \tag{3.14}$$

$$\beta(t) = \beta(t_3) + \Big((u_- - u_+)(v_- + v_2) - (v_2 - v_-) \Big)(t - t_3), \quad \text{for } t_3 < t \le t_4, \tag{3.15}$$

$$\beta(t) = \beta(t_4) + \Big((u_- - u_+)(v_- + v_+) - (v_+ - v_-) \Big)(t - t_4), \quad \text{for } t > t_4. \tag{3.16}$$

Otherwise, if $u_- - 1 < u_+ < u_-$, then a new shock wave (denotes by S_4) followed by a new contact discontinuity (denotes by J_4) appears after the time t_2 (see Figure 2(b)). The propagation speed of S_4 is $\sigma_4 = u_- + u_+$, which leads to $\sigma_3 < \sigma_4 < \sigma_1$ for $u_+ < u_m < u_-$. The propagation speed of J_4 is still $2u_+ + 1$, which is equal to those of J_3 and J_2. It is clear that the intermediate state between S_4 and J_4 is given by

$$(u_4, v_4) = \Big(u_+, -v_- \cdot \frac{u_+ - u_- - 1}{u_+ - u_- + 1} \Big). \tag{3.17}$$

Thus, the global solutions of the perturbed Riemann problem (1.1) and (1.3) can be illustrated by Figure 2(a) for $u_- - 2 < u_+ < u_- - 1$ and Figure 2(b) for $u_- - 1 < u_+ < u_-$, respectively. The proof is complete. $\square$

Case 3.2: $R + J$ **and** $S + J$**.** In this case, we investigate the interaction between a rarefaction wave followed by a contact discontinuity starting from $(-\varepsilon, 0)$ and a shock wave followed by a contact discontinuity starting from $(\varepsilon, 0)$ (see Figure 3). This case happens if and only if $u_- < u_m$ and $u_+ < u_m < u_+ + 1$, such that we have $u_- - 1 < u_+$. The propagation speed of the contact discontinuity J_1 is $\tau_1 = 2u_m + 1$ and that of the shock wave S_1 is $\sigma_1 = u_m + u_+$. Thus, it is easy to see that J_1 and S_1 intersect in finite time. The intersection (x_1, t_1) is also given by (3.2) which is the same as that of Case3.1.

Analogously, the new local Riemann problem will also be formulated at (x_1, t_1), which also gives rise to a new shock wave S_2 and a new contact discontinuity J_3 after the time t_1. For the propagation speed of the shock wave S_2 is $\sigma_2 = u_m + u_+$ and that of the wave front in the rarefaction wave R is $2u_m$, S_2 and R will meet whose first intersection (x_2, t_2) is determined by

$$\begin{gathered} x_2 + \varepsilon = 2u_m t_2, \\ x_2 - \varepsilon = (u_m + u_+) t_2, \end{gathered} \tag{3.18}$$

which means that

$$(x_2, t_2) = \Big(\frac{\varepsilon(3u_m + u_+)}{u_m - u_+}, \frac{2\varepsilon}{u_m - u_+} \Big). \tag{3.19}$$

The shock wave begins to enter the rarefaction wave fan after the interaction of S_2 and R happens. At the same time, the new shock wave S_3 and contact discontinuity J_4 are generated and propagate forwards. Here we use $\Gamma : x = x(t)$ to express the curve of S_3 who has the changing state variables (u_4, v_4) on the left-hand side and (u_5, v_5) on the right-hand side. It follows from (2.1) that

$$\begin{gathered} v_4 \cdot e^{2u_4} = v_- \cdot e^{2u_-}, \\ x + \varepsilon = 2u_4 t, \end{gathered} \tag{3.20}$$

which enables us to have

$$(u_4, v_4) = \Big(\frac{x + \varepsilon}{2t}, v_- \exp\Big(2u_- - \frac{x + \varepsilon}{t}\Big) \Big). \tag{3.21}$$

On the other hand, we always have $u_5 = u_3 = u_+$ for all the characteristic lines in the rarefaction wave become the contact discontinuities when across the shock wave S_3. In addition, if the matched state (u_5, v_5) can be connected to the corresponding one (u_4, v_4) by a shock wave, then they should satisfy

$$\frac{v_5}{v_4} = \frac{u_4 - u_5 + 1}{u_5 - u_4 + 1}. \tag{3.22}$$

Through a tedious and detailed calculation, one obtains

$$(u_5, v_5) = \Big(u_+, v_- \cdot \frac{x + \varepsilon - 2t(u_+ - 1)}{2t(u_+ + 1) - x - \varepsilon} \cdot \exp\Big(2u_- - \frac{x + \varepsilon}{t}\Big) \Big). \tag{3.23}$$

Thus, the curve of S_3 is determined by

$$\sigma_3(t) = \frac{dx}{dt} = u_+ + \frac{x + \varepsilon}{2t}, \tag{3.24}$$

in which the initial condition $x(t_2) = x_2$ is given by (3.19), whose unique solution may be written as

$$x(t) = 2u_+t + 2\sqrt{2\varepsilon t(u_m - u_+)} - \varepsilon, \qquad t \geq t_2. \tag{3.25}$$

It follows from (3.24) and (3.25) that

$$\frac{d^2x}{dt^2} = -\frac{1}{2}\sqrt{\frac{2\varepsilon(u_m - u_+)}{t^3}} < 0, \tag{3.26}$$

which means that S_3 begins to decelerate and is not a straight line any more after the time t_2.

Taking into account the comparison of values between u_- and u_+, let us also use the following lemma to explain that if S_3 is able to cancel the rarefaction wave R completely or not.

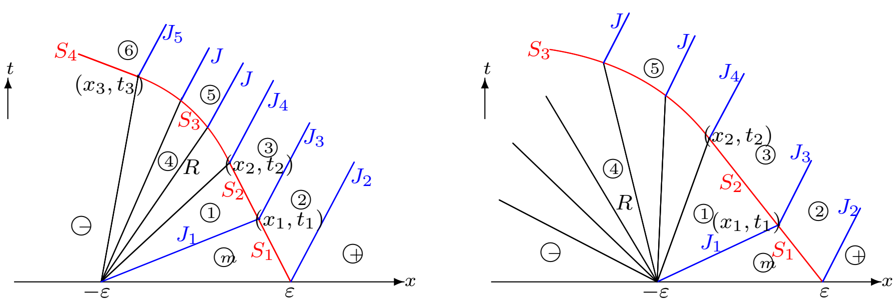


FIGURE 3. Interaction between $R + J$ and $S + J$ for two situations where both $u_- < u_m$ and $u_+ < u_m < u_+ + 1$ should be satisfied.

Lemma 3.2. *If $u_- - 1 < u_+ < u_-$, then the shock wave S_3 has the ability to cancel the whole rarefaction wave R completely in finite time. Otherwise, if $u_+ > u_-$, then the shock wave S_3 has no ability to penetrate the rarefaction wave R completely in finite time and finally takes the line $x + \varepsilon = 2u_+t$ as its asymptote.*

Proof. If $u_- - 1 < u_+ < u_-$, then the shock wave S_3 is continuous to penetrate the rarefaction wave R and is able to cancel the whole rarefaction wave R completely in finite time (see Figure 3(a)). During the process of penetration, the local Riemann problem will be formulated on every point of the shock curve S_3 and new contact discontinuities will be continuously produced along with the shock curve S_3. This is due to the fact that the shock curve S_3 has the varying left state (u_4, v_4) supported on each characteristic line in the rarefaction wave fan R which should be connected with the matched right state (u_5, v_5) exactly by a shock wave followed by a contact discontinuity. It is clear to see that all the contact discontinuities have the same propagation speed $2u_+ + 1$, thus they are parallel to each other. Finally, S_3 will meet the wave back in the rarefaction wave R at the intersection (x_3, t_3), which can be calculated by

$$\begin{gathered} x_3 + \varepsilon = 2u_-t_3, \\ x_3 = 2u_+t_3 + 2\sqrt{2\varepsilon t_3(u_m - u_+)} - \varepsilon. \end{gathered} \tag{3.27}$$

Thus, we have

$$(x_3, t_3) = \Big(\frac{4\varepsilon u_-(u_m - u_+)}{(u_- - u_+)^2} - \varepsilon, \frac{2\varepsilon(u_m - u_+)}{(u_- - u_+)^2}\Big). \tag{3.28}$$

After the time t_3, the shock wave is denoted with S_4 whose propagation speed is $\sigma_4 = u_- + u_+$. The state (u_6, v_6) between S_4 and J_5 has the same representation as that in (3.17). Consequently, a new shock wave S_4 followed by a contact discontinuity J_5 will be generated after the time t_3, in which the propagation speeds of S_4 and J_5 are $\sigma_4 = u_- + u_+$ and $\tau_5 = 2u_+ + 1$ respectively.

Otherwise, if $u_+ > u_-$, then the shock wave S_3 is continuous to penetrate the rarefaction wave R but unable to cancel the whole R completely in finite time (see Figure 3(b)). During the process of penetration, it can be derived from (3.24) and (3.25) that

$$\sigma_3(t) = \frac{dx}{dt} = 2u_+ + \sqrt{\frac{2\varepsilon(u_m - u_+)}{t}}. \tag{3.29}$$

Thus, it is shown that $\sigma_3(t) \to 2u_+$ as $t \to \infty$ for given $\varepsilon > 0$. Thus, when $u_+ > u_-$, the shock wave S_3 has the characteristic line $x(t) = 2u_+t - \varepsilon$ in the rarefaction wave fan R as its asymptote in the end. $\square$

Case 3.3: $S + J$ **and** $R + J$**.** Let us consider the situation that the shock wave S_1 plus the contact discontinuity J_1 emanates from $(-\varepsilon, o)$ and the rarefaction wave R_1 plus another contact discontinuity J_2 emits from $(\varepsilon, 0)$ (see Fig.4). The occurrence of this case depends on the conditions $u_- - 1 < u_m < u_-$ and $u_m < u_+$, which implies that $u_- - 1 < u_+$. The propagation speed of J_1 is $\tau_1 = 2u_m + 1$ and that of the wave back in the rarefaction wave R_1 is $2u_m$. Thus, it is easy to see that J_1 and R_1 meet at a time. The intersection (x_1, t_1) is determined by

$$\begin{aligned} x_1 + \varepsilon &= (2u_m + 1)t_1, \\ x_1 - \varepsilon &= 2u_m t_1, \end{aligned} \tag{3.30}$$

which implies

$$(x_1, t_1) = \Big(\varepsilon(4u_m + 1), 2\varepsilon\Big). \tag{3.31}$$

The contact discontinuity J_1 begins to enter the rarefaction wave fan after the interaction between J_1 and R_1 occurs. It can be derived directly from (2.1) that the state variable (u_2, v_2) in R_1 can be determined by

$$\begin{aligned} v_m \cdot e^{2u_m} &= v_2 \cdot e^{2u_2}, \\ x - \varepsilon &= 2u_2 t, \end{aligned} \tag{3.32}$$

which implies

$$(u_2, v_2) = \Big(\frac{x - \varepsilon}{2t}, v_m \exp\Big(2u_m - \frac{x - \varepsilon}{t}\Big)\Big). \tag{3.33}$$

It is remarkable that the values of u on the both sides of the contact discontinuity should be equal. In other words, the rarefaction wave cannot change its direction when across J_1. Thus, the state variable (u_3, v_3) in R_2 can also be determined by

$$\begin{aligned} v_1 \cdot e^{2u_1} &= v_3 \cdot e^{2u_3}, \\ x - \varepsilon &= 2u_3 t, \end{aligned} \tag{3.34}$$

such that

$$(u_3, v_3) = \Big(\frac{x - \varepsilon}{2t}, v_1 \exp\Big(2u_1 - \frac{x - \varepsilon}{t}\Big)\Big), \tag{3.35}$$

in which (u_1, v_1) is given by (3.4). Therefore, the contact discontinuity J_1 crosses the rarefaction wave R_1 with a varying propagation speed, which is determined by

$$\begin{aligned} \frac{dx}{dt} &= 2u_2 + 1, \\ x - \varepsilon &= 2u_2 t, \end{aligned} \tag{3.36}$$

together with the initial condition $x_1 = x(t_1)$ given by (3.31), which enables us to obtain

$$x(t) = t \ln \frac{t}{2\varepsilon} + 2u_m t + \varepsilon. \tag{3.37}$$

It follows from (3.36) and (3.37) that

$$\frac{d^2 x}{dt^2} = \frac{1}{t} > 0, \tag{3.38}$$

which means that J_3 begins to accelerate and is not a straight line any more after the interaction between J_1 and R_1.

Furthermore, it is shown that the contact discontinuity J_1 has the ability to penetrate the entire rarefaction wave R_1 fully in finite time and the terminal point (x_2, t_2) can be calculated by

$$\begin{aligned} x_2 &= t_2 \ln \frac{t_2}{2\varepsilon} + 2u_m t_2 + \varepsilon, \\ x_2 - \varepsilon &= 2u_+ t_2, \end{aligned} \tag{3.39}$$

which implies

$$(x_2, t_2) = \Big(4\varepsilon u_+ \exp(2u_+ - 2u_m) + \varepsilon, 2\varepsilon \exp(2u_+ - 2u_m) \Big). \tag{3.40}$$

After the time t_2, the contact discontinuity is denoted with J_3. The state between J_2 and J_3 is given by

$$(u_4, v_4) = (u_+, v_m \exp(2u_m - 2u_+)). \tag{3.41}$$

The contact discontinuity J_3 is parallel to J_2 for the reason that $u_4 = u_+$. Similarly, the state (u_5, v_5) between R_2 and J_3 can be calculated by

$$(u_5, v_5) = (u_+, v_1 \exp(2u_1 - 2u_+)). \tag{3.42}$$

Now, let us consider the interaction between S_1 and R_2. The propagation speed of S_1 is $\sigma_1 = u_- + u_m$ and that of the wave back in the rarefaction wave R_2 is still equal to $2u_m$. Thus, it is easy to see that S_1 catches up with R_2 in finite time and the interaction (x_3, t_3) can be calculated by

$$\begin{aligned} x_3 - \varepsilon &= 2u_m t_3, \\ x_3 + \varepsilon &= (u_- + u_m) t_3, \end{aligned} \tag{3.43}$$

which implies

$$(x_3, t_3) = \Big(\frac{\varepsilon(3u_m + u_-)}{u_- - u_m}, \frac{2\varepsilon}{u_- - u_m} \Big). \tag{3.44}$$

After the time t_3, the shock wave enters the region of the rarefaction wave fan R_2 and is denoted with S_2 during the process of penetration. It is noticed that the state on the right-hand side of the shock wave S_2 is (u_3, v_3), in which u_3 varies from u_m to u_+ for $u_1 = u_m$ and $u_5 = u_+$. To study the problem that the shock wave S_2 penetrates the rarefaction wave R_2 is essential to study infinitely many local Riemann problems. There is still a shock wave followed by a contact discontinuity for a local Riemann problem provided that $u_- - 1 < u_3 < u_-$. Thus, there are infinitely many contact discontinuities generated during the process of penetration. As before, the value u_3 does not change when these contact discontinuities pass through the rarefaction wave R_2. Therefore, the curve of the shock wave S_2 can be determined by

$$\begin{gathered}\frac{dx}{dt} = u_- + u_3,\\ x - \varepsilon = 2u_3 t,\end{gathered} \tag{3.45}$$

and the initial condition $x_3 = x(t_3)$ is given by (3.44), which has a unique solution

$$x(t) = 2u_- t - 2\sqrt{2\varepsilon t(u_- - u_m)} + \varepsilon. \tag{3.46}$$

It follows from (3.45) and (3.46) that

$$\frac{d^2x}{dt^2} = \frac{1}{2}\sqrt{\frac{2\varepsilon(u_- - u_m)}{t^3}} > 0, \tag{3.47}$$

which means that S_2 also accelerates during the process of penetration.

As in Lemma 3.2, according to the values u_+ and $u_- - 1$, let us also use the following lemma to explain that if S_2 is able to cancel the rarefaction wave R_2 completely or not.

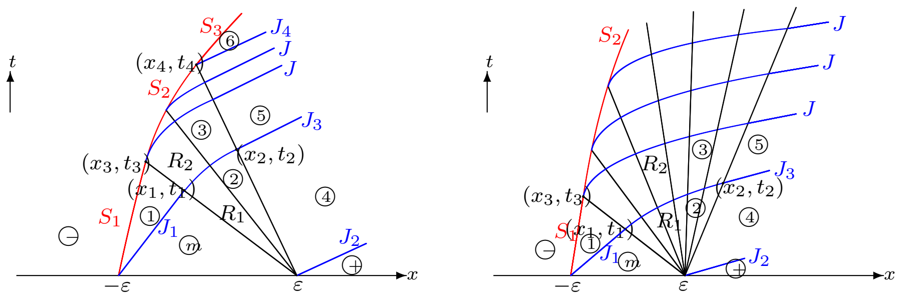


FIGURE 4. Interactions between $S + J$ and $R + J$ for two different situations when both $u_- - 1 < u_m < u_-$ and $u_m < u_+$ should be satisfied.

Lemma 3.3. *If $u_- - 1 < u_+ < u_-$, then the shock wave S_2 has the ability to cancel the whole rarefaction wave R_2 completely in finite time and a new shock wave followed by a new contact discontinuity will be generated at last. Otherwise, if $u_+ > u_-$, then the shock wave S_2 is unable to pass through the whole R_2 completely and finally takes the line $x - \varepsilon = 2u_- t$ as its asymptote.*

Proof. If $u_- - 1 < u_+ < u_-$, then the shock wave S_2 is able to penetrate the whole R_2 fully at (x_4, t_4) (see Figure 4(a)), which can be calculated by

$$\begin{gathered}x_4 - \varepsilon = 2u_+ t_4,\\ x_4 = 2u_- t_4 - 2\sqrt{2\varepsilon t_4(u_- - u_m)} + \varepsilon,\end{gathered} \tag{3.48}$$

namely

$$(x_4, t_4) = \Big(\frac{4\varepsilon u_+(u_- - u_m)}{(u_- - u_+)^2} + \varepsilon, \frac{2\varepsilon(u_- - u_m)}{(u_- - u_+)^2}\Big). \tag{3.49}$$

After the time t_4, we denote the shock wave by S_3 whose propagation speed is $\sigma_3 = u_- + u_+$. Obviously, a new contact discontinuity J_4 will be produced with the speed $\tau_4 = 2u_+ + 1$. In addition, the state (u_6, v_6) between S_3 and J_4 is given by

$$(u_6, v_6) = \Big(u_+, -v_- \cdot \frac{u_+ - u_- - 1}{u_+ - u_- + 1}\Big). \tag{3.50}$$

Otherwise, if $u_+ > u_-$, then the shock wave S_2 is unable to cancel the whole R_2 completely in finite time and ultimately has $x(t) = 2u_- t + \varepsilon$ as its asymptote (see Figure 4(b)). $\square$

Case 3.4: $R+J$ and $R+J$. In this case, we are concerned with the situation that both a rarefaction wave followed by a contact discontinuity start from the initial points $(-\varepsilon, 0)$ and $(\varepsilon, 0)$ respectively. The occurrence of this case depends on the condition $u_- < u_m < u_+$. For this case, we need only to consider the situation that a contact discontinuity penetrates a rarefaction wave, which can be dealt with similarly to that for Case 3.3. The details are omitted here.

4. Interactions of delta shock waves with classical waves

It is known that the delta shock wave occurs in the Riemann solution of (1.1) and (1.2) for some specific Riemann initial data. It is interesting to investigate the interactions between the delta shock wave with the other elementary waves, including shock wave, rarefaction wave and contact discontinuity. As before, we continue to study the perturbed Riemann problem (1.1) and (1.3) but we need to require that at least one delta shock wave generates at $(-\varepsilon, 0)$ or $(\varepsilon, 0)$. More precisely, we need to consider five possibilities when the delta shock wave is involved at the initial time, according to the different combinations from $(-\varepsilon, 0)$ and $(\varepsilon, 0)$ as follows: (1) δS and $S + J$; (2) $S + J$ and δS; (3) δS and δS; (4) δS and $R + J$; (5) $R + J$ and δS.

In addition, it should be emphasized that the interactions between the delta shock wave with the other elementary waves for the system (1.4) have been considered in [17] by using the method of split delta function. In order for completeness and self-contained, we utilize a somewhat different technique from that in [17] to study the perturbed Riemann problem (1.1) and (1.3) when the delta shock wave is involved at the initial moment. In addition, the systems (1.1) and (1.4) are derived from very different physical models and used to describe different physical phenomena. Let us see [8, 25] for the detailed comparison and contrast between the two systems.

Case 4.1: δS and $S + J$. For this case, we draw our attention on the interaction between a delta shock wave starting from $(-\varepsilon, 0)$ and a shock wave followed by a contact discontinuity starting from $(\varepsilon, 0)$ (see Figure 5(a)). This requires that both $u_m \le u_- - 1$ and $u_m - 1 < u_+ < u_m$ should be satisfied. The propagation speed of the delta shock wave δS_1 is $\sigma_{\delta_1} = u_- + u_m$ and that of the shock wave S is $\sigma = u_m + u_+$. Thus, δS_1 overtakes S in finite time whose intersection is given by

$$(x_1, t_1) = \Big(\frac{\varepsilon(u_- + 2u_m + u_+)}{u_- - u_+}, \frac{2\varepsilon}{u_- - u_+}\Big). \tag{4.1}$$

In addition, the strength of delta shock wave before the time t_1 can be calculated by

$$\beta(t) = \Big((u_- - u_m)(v_- + v_m) - (v_m - v_-)\Big)t \quad \text{for } 0 \le t \le t_1. \tag{4.2}$$

Then, the generalized Riemann problem for the system (1.1) with the delta type initial data is formulated at (x_1, t_1), in which the initial data are expressed as

$$u|_{t=t_1} = \begin{cases} u_-, & x < x_1, \\ u_1, & x > x_1, \end{cases} \qquad v|_{t=t_1} = \beta(t_1)\delta_{(x_1,t_1)} + \begin{cases} v_-, & x < x_1 \\ v_1, & x > x_1, \end{cases} \tag{4.3}$$

where $(u_1, v_1) = (u_+, -v_m \cdot \frac{u_+ - u_m - 1}{u_+ - u_m + 1})$ can be calculated by the same formula as in (3.5).

For $u_1 = u_+ < u_m \leq u_- - 1$, a new delta shock wave δS_2 will be generated after the interaction between δS_1 and S. The curve of δS_2 is determined by the equation

$$\sigma_{\delta_2} = \frac{dx}{dt} = u_- + u_1 = u_- + u_+ \tag{4.4}$$

and the initial condition $x(t_1) = x_1$ given by (4.1), which enables us to get the unique exact solution

$$x(t) = (u_- + u_+)\Big(t - \frac{2\varepsilon}{u_- - u_+}\Big) + \frac{\varepsilon(u_- + 2u_m + u_+)}{u_- - u_+} \quad \text{for } t \geq t_1. \tag{4.5}$$

The propagation speed of δS_2 is $\sigma_{\delta_2} = u_- + u_+$ and that of the contact discontinuity J is $\tau = 2u_+ + 1$. Thus, it is easy to see that δS_2 intersects J at a time. The intersection (x_2, t_2) is determined by

$$\begin{gathered} x_2 - x_1 = (u_- + u_+)(t_2 - t_1), \\ x_2 - \varepsilon = (2u_+ + 1)t_2, \end{gathered} \tag{4.6}$$

which means

$$(x_2, t_2) = \Big(\frac{2\varepsilon(2u_+ + 1)(u_- - u_m)}{(u_- - u_+)(u_- - u_+ - 1)} + \varepsilon, \frac{2\varepsilon(u_- - u_m)}{(u_- - u_+)(u_- - u_+ - 1)}\Big). \tag{4.7}$$

In addition, the strength of delta shock wave between t_1 and t_2 can also be calculated by

$$\beta(t) = \beta(t_1) + \Big((u_- - u_+)(v_- + v_1) - (v_1 - v_-)\Big)(t - t_1) \quad \text{for } t_1 < t \leq t_2. \tag{4.8}$$

After the time t_2, the delta shock wave passes through J with the same propagation speed as before, but is at the different growth rates for the strength of delta shock wave. This is due to the fact that the propagation speed of delta shock wave is calculated by $\sigma_\delta = u_l + u_r$ which is controlled completely by the state variable u, where u_l stands for the state on the left-hand side of delta shock curve and u_r expresses the state on the right-hand side of delta shock curve. Taking into account $u_1 = u_+$, we always have the states $u_l = u_-$ and $u_r = u_+$ on two sides of delta shock curve such that the propagation speed of delta shock wave keeps $u_- + u_+$ invariant after the time t_1. On the other hand, the strength of delta shock wave depends on both the state variables u and v. Thus, the growth rate of delta shock wave also changes when across J for the reason that the state variable v changes from v_1 to v_+. After the time t_2, the strength of delta shock wave can be calculated by

$$\beta(t) = \beta(t_2) + \Big((u_- - u_+)(v_- + v_+) - (v_+ - v_-)\Big)(t - t_2) \quad \text{for } t > t_2. \tag{4.9}$$

Case 4.2: $S + J$ and δS. In this case, we are concerned with the situation that a shock wave followed by a contact discontinuity starts from $(-\varepsilon, 0)$ and a delta shock wave emits from $(\varepsilon, 0)$. This case arises when both $u_- - 1 < u_m < u_-$ and $u_+ \le u_m - 1$ occur. This situation can be dealt with similarly to that for Case 4.1. Let us draw Figure 5(b) to illustrate this situation for comparison and then give the detailed explanations for Figure 5(b) below. The propagation speeds of J and δS_1 are given by $\tau = 2u_m + 1$ and $\sigma_{\delta_1} = u_m + u_+$ respectively, such that they will meet at the intersection given by

$$(x_1, t_1) = \Big(\frac{\varepsilon(3u_m + 1 + u_+)}{u_m + 1 - u_+}, \frac{2\varepsilon}{u_m + 1 - u_+}\Big). \tag{4.10}$$

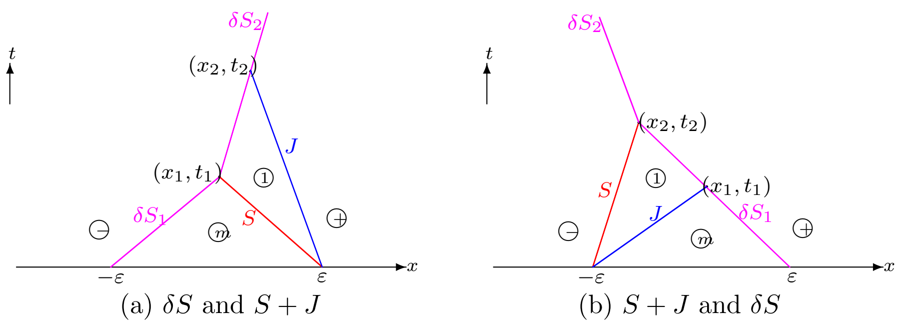


FIGURE 5. Interaction between δS and $S + J$ when $u_m \le u_- - 1$ and $u_m - 1 < u_+ < u_m$ on the left, and interaction between $S + J$ and δS when $u_- - 1 < u_m < u_-$ and $u_+ \le u_m - 1$ on the right.
.

The state (u_1, v_1) between S and J can be calculated by the same formula as in (3.4). For $u_1 = u_m$, the delta shock wave δS_1 cannot change its direction when across J. Thus, the intersection of S and δS_1 can be calculated by

$$\begin{aligned} x_2 + \varepsilon &= (u_- + u_m)t_2, \\ x_2 - \varepsilon &= (u_m + u_+)t_2, \end{aligned} \tag{4.11}$$

in which $u_- + u_m$ is the propagation speed of S, such that we have

$$(x_2, t_2) = \Big(\frac{\varepsilon(u_- + 2u_m + u_+)}{u_- - u_+}, \frac{2\varepsilon}{u_- - u_+}\Big). \tag{4.12}$$

Consequently, the generalized Riemann problem for the system (1.1) with the delta type initial data is also formulated at (x_2, t_2), in which the initial data are expressed as

$$u|_{t=t_2} = \begin{cases} u_-, & x < x_2, \\ u_+, & x > x_2, \end{cases} \qquad v|_{t=t_2} = \beta(t_2)\delta_{(x_2,t_2)} + \begin{cases} v_-, & x < x_2 \\ v_+, & x > x_2. \end{cases} \tag{4.13}$$

We can obtain $u_+ < u_- - 1$ from $u_m < u_-$ and $u_+ \le u_m - 1$, thus the interaction between S and δS_1 generates a new delta shock wave denoted with δS_2 in Figure 5(b).

In addition, the strength of delta shock wave can be calculated respectively by

$$\beta(t) = \Big((u_m - u_+)(v_m + v_+) - (v_+ - v_m)\Big)t \quad \text{for } 0 \le t \le t_1, \tag{4.14}$$

$$\beta(t)=\beta(t_1)+\Big((u_m-u_+)(v_1+v_+)-(v_+-v_1)\Big)(t-t_1)\quad \text{for } t_1<t\le t_2, \tag{4.15}$$

$$\beta(t)=\beta(t_2)+\Big((u_--u_+)(v_-+v_+)-(v_+-v_-)\Big)(t-t_2)\quad \text{for } t>t_2, \tag{4.16}$$

in which $v_1=-v_-\cdot\frac{u_m-u_--1}{u_m-u_-+1}$ is given by (3.4).

Case 4.3: δS **and** δS**.** In this case, we consider the interaction of two delta shock waves starting from $(-\varepsilon,0)$ and $(\varepsilon,0)$ respectively. This case happens if and only if $u_m\le u_--1$ and $u_+\le u_m-1$. Let us use δS_1 and δS_2 to denote the delta shock waves originating from the initial points $(-\varepsilon,0)$ and $(\varepsilon,0)$ respectively. The propagation speed of δS_1 is u_-+u_m and that of δS_2 is u_m+u_+, and thus they will meet in finite time and the intersection of δS_1 and δS_2 can also be calculated by the formula (4.12). Before the time $t_1=\frac{2\varepsilon}{u_--u_+}$, the strengths of δS_1 and δS_2 can be calculated respectively by

$$\beta_1(t)=\Big((u_--u_m)(v_-+v_m)-(v_m-v_-)\Big)t, \tag{4.17}$$

$$\beta_2(t)=\Big((u_m-u_+)(v_m+v_+)-(v_+-v_m)\Big)t. \tag{4.18}$$

At the point (x_1,t_1), the delta-type initial data can also be formulated as

$$u|_{t=t_1}=\begin{cases}u_-, & x<x_1,\\ u_+, & x>x_1,\end{cases}\qquad v|_{t=t_1}=\beta(t_1)\delta_{(x_1,t_1)}+\begin{cases}v_-, & x<x_1\\ v_+, & x>x_1,\end{cases} \tag{4.19}$$

in which the strength $\beta(t_1)=\beta_1(t_1)+\beta_2(t_1)$ is the sum of the strengths of δS_1 and δS_2 at the point (x_1,t_1) and thus can be calculated by

$$\beta(t_1)=\frac{2\varepsilon}{u_--u_+}\cdot\Big((u_--u_m)(v_-+v_m)+(u_m-u_+)(v_m+v_+)-(v_+-v_-)\Big). \tag{4.20}$$

It can be obtained that $u_+\le u_--2$, thus the wave interaction has a relatively simpler structure for this case, namely two delta shock waves coalesce into one delta shock wave when they meet. Let us use δS_3 to denote the new delta shock wave whose strength can be calculated by

$$\beta(t)=\beta(t_1)+\Big((u_--u_+)(v_-+v_+)-(v_+-v_-)\Big)(t-t_1)\quad \text{for } t>t_1, \tag{4.21}$$

in which $t_1=\frac{2\varepsilon}{u_--u_+}$ and $\beta(t_1)$ is given by (4.20).

Case 4.4: δS **and** $R+J$**.** In this case, let us investigate the interaction between a delta shock wave δS_1 emanating from $(-\varepsilon,0)$ and a rarefaction wave R_1 followed by a contact discontinuity J_1 starting from $(\varepsilon,0)$ (see Fig.6). This is possible to happen if and only if both $u_m\le u_--1$ and $u_m<u_+$ hold. The propagation speed of the delta shock wave δS_1 is $\sigma_{\delta_1}=u_-+u_m$ and the wave back in the rarefaction wave propagates with the speed $2u_m$. Thus, δS_1 and R intersect at the point

$$(x_1,t_1)=\Big(\frac{\varepsilon(3u_m+u_-)}{u_--u_m},\frac{2\varepsilon}{u_--u_m}\Big), \tag{4.22}$$

and the strength of delta shock wave at (x_1,t_1) is given by

$$\beta(t_1)=\Big((u_--u_m)(v_-+v_m)-(v_m-v_-)\Big)t_1. \tag{4.23}$$

After the time t_1, the delta shock wave enters the rarefaction wave fan R_1 where it is denoted with δS_2, whose expression can be calculated by

$$\sigma_{\delta_2}(t) = \frac{dx}{dt} = u_- + \frac{x-\varepsilon}{2t}, \tag{4.24}$$

where the initial condition $x(t_1) = x_1$ is given by (4.22), which has a unique solution

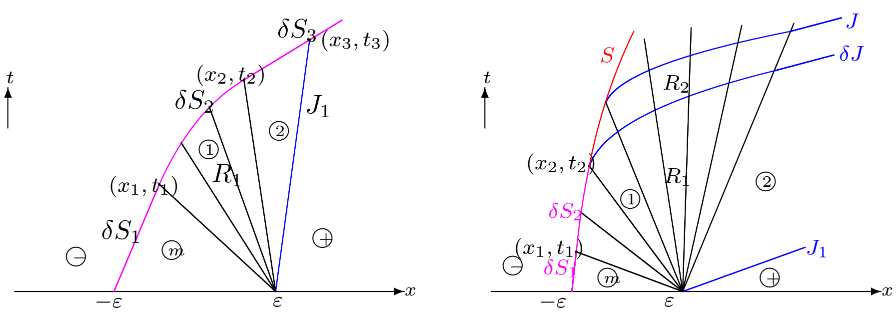
$$x(t) = 2u_-t - 2\sqrt{2\varepsilon t(u_- - u_m)} + \varepsilon, \quad \text{for } t \geq t_1. \tag{4.25}$$

Thus, we have $\frac{d^2x}{dt^2} > 0$, namely δS_2 begins to accelerate when it enters the rarefaction wave fan R_1. As before, it is easy to get that the state (u_1, v_1) in R_1 and the state (u_2, v_2) between R_1 and J_1 are given respectively by

$$(u_1, v_1) = \Big(\frac{x-\varepsilon}{2t}, v_m \exp(2u_m - \frac{x-\varepsilon}{t})\Big), \tag{4.26}$$

$$(u_2, v_2) = \Big(u_+, v_m \exp(2u_m - 2u_+)\Big). \tag{4.27}$$

In view of the different relations among the values u_+, u_- and $u_- - 1$, the discussions should be divided into three different cases which can be fully depicted in the following lemma.



(a) $u_+ < u_- - 1$ (b) $u_+ > u_-$

FIGURE 6. Interactions between δS and $R + J$ for two different situations where both $u_m \leq u_- - 1$ and $u_m < u_+$ should be satisfied.

Lemma 4.1. *If $u_+ < u_- - 1$, then the delta shock wave is able to cancel the whole rarefaction wave R_1 completely in finite time. Otherwise, if $u_- - 1 < u_+$, then the delta shock wave is unable to cancel the whole R_1 completely in finite time and is divided into a shock wave and a delta contact discontinuity when it passes through the characteristic line with the state satisfying $u_1 = u_- - 1$.*

Proof. If $u_+ < u_- - 1$, then δS_2 is able to cancel the whole R_1 at (x_2, t_2) which can be calculated by

$$\begin{gathered} x_2 - \varepsilon = 2u_+t_2, \\ x_2 = 2u_-t_3 - 2\sqrt{2\varepsilon t_2(u_- - u_m)} + \varepsilon, \end{gathered} \tag{4.28}$$

namely,

$$(x_2, t_2) = \Big(\frac{4\varepsilon u_+(u_- - u_m)}{(u_- - u_+)^2} + \varepsilon, \frac{2\varepsilon(u_- - u_m)}{(u_- - u_+)^2}\Big). \tag{4.29}$$

The strength of δS_2 during the process of penetration can be calculated by

$$\beta(t) = \beta(t_1) + \Big((u_- - u_1)(v_- + v_1) - (v_1 - v_-)\Big)(t - t_1), \quad \text{for } t_1 < t \leq t_2, \tag{4.30}$$

in which t_1, $\beta(t_1)$ and (u_1, v_1) are given by (4.22), (4.23) and (4.26) respectively. After the time t_2, the situation is similar to that for Case 4.1. In other words, the delta shock wave δS_3 propagates with the invariant speed $u_- + u_+$, only the strength of δS_3 adds up at the different rates when it passes through J_1.

Otherwise, if $u_- - 1 < u_+$, then the delta shock wave is unable to cross the entire R_1 in finite time and should be decomposed into a shock wave S and a delta contact discontinuity δJ when it passes through the characteristic line with $u_1 = u_- - 1$ in the rarefaction wave fan R_1. This is due to the fact that the inequality $u - 1 < u_+$ cannot always hold for the varying states (u, v) supported on the characteristic lines in the rarefaction wave fan R_1. More precisely, the critical point for the decomposition of δS_2 into S and δJ can be calculated by

$$\begin{gathered} x_2 - \varepsilon = 2(u_- - 1)t_2, \\ x_2 = 2u_- t_2 - 2\sqrt{2\varepsilon t_2(u_- - u_m)} + \varepsilon, \end{gathered} \tag{4.31}$$

which enables us to have

$$(x_2, t_2) = \Big(4\varepsilon(u_- - u_m)(u_- - 1) + \varepsilon, 2\varepsilon(u_- - u_m)\Big). \tag{4.32}$$

The curve of the delta contact discontinuity δJ can be calculated as

$$\tau_\delta(t) = \frac{dx}{dt} = \frac{x - \varepsilon}{t} + 1, \tag{4.33}$$

in which the initial condition $x(t_2) = x_2$ is given by (4.32). Analogously, the expression of δJ can be given in the form

$$x(t) = t\Big(\ln t - \ln(2\varepsilon(u_- - u_m)) + 2u_- - 2\Big) + \varepsilon, \quad t \geq t_2. \tag{4.34}$$

As in Case 3.3, the delta contact discontinuity δJ is able to penetrate the whole R_1 in finite time and finally propagates forwards with the invariant speed $2u_+ + 1$.

Let us turn our attention on the shock wave S. Actually, the shock wave S is the continuation of the delta shock wave δS_2. Thus, the curve of the shock wave S can also be expressed by (4.25). Consequently, the shock wave S continues to penetrate the rarefaction wave R_2 and the situation is similar to that for Case 3.3. For the reason that the propagation speed of S depends entirely on the state variable u on both sides of the shock curve, in which the left-hand state variable u is always u_- and the right-hand state variable u supported on each characteristic line in the rarefaction wave fan remains unchanged when the characteristic line passes through δJ and J. That is to say, we need also consider two subcases $u_- - 1 < u_+ < u_-$ and $u_+ > u_-$, which can be illustrated by Lemma 3.3. □

Case 4.5: $R + J$ and δS. In the end, we study the situation when the rarefaction wave R followed by the contact discontinuity J_1 starts from $(-\varepsilon, 0)$ and the delta shock wave δS_1 starts from $(\varepsilon, 0)$ (see Fig.7). This case arises when both $u_- < u_m$ and $u_+ \leq u_m - 1$ happen. The propagation speeds of J_1 and δS_1 are $\tau_1 = 2u_m + 1$ and $\sigma_{\delta_1} = u_m + u_+$, respectively. Thus, δS_1 and J_1 meet in finite time whose intersection is

$$(x_1, t_1) = \Big(\frac{\varepsilon(3u_m + u_+ + 1)}{u_m - u_+ + 1}, \frac{2\varepsilon}{u_m - u_+ + 1}\Big). \tag{4.35}$$

The strength of δS_1 at (x_1, t_1) is

$$\beta(t_1) = \Big((u_m - u_+)(v_m + v_+) - (v_+ - v_m)\Big)t_1. \tag{4.36}$$

After the time t_1, the delta shock wave δS_1 passes through J_1 with the same speed as before and consequently enters the rarefaction wave R from the point (x_2, t_2) which can be obtained as

$$(x_2, t_2) = \Big(\frac{\varepsilon(3u_m + u_+)}{u_m - u_+}, \frac{2\varepsilon}{u_m - u_+}\Big). \tag{4.37}$$

The strength of δS_1 at (x_2, t_2) can be calculated as

$$\beta(t_2) = \beta(t_1) + \Big((u_1 - u_+)(v_1 + v_+) - (v_+ - v_1)\Big)(t_2 - t_1), \tag{4.38}$$

in which

$$(u_1, v_1) = \Big(u_m, v_- \exp(2u_- - 2u_m)\Big). \tag{4.39}$$

After the time t_2, the delta shock wave enters the rarefaction wave fan R where it is denoted with δS_2. As before, the curve of δS_2 is determined by

$$\sigma_{\delta_2}(t) = \frac{dx}{dt} = u_+ + \frac{x + \varepsilon}{2t}, \tag{4.40}$$

in which the initial condition $x(t_2) = x_2$ is given by (4.37). It is easy to obtain a unique solution

$$x(t) = 2u_+t + 2\sqrt{2\varepsilon t(u_m - u_+)} - \varepsilon, \qquad t \geq t_2. \tag{4.41}$$

It follows from (4.40) and (4.41) that $\frac{d^2x}{dt^2} < 0$, which means that δS_2 begins to decelerate and is not a straight line any more.

The state (u_2, v_2) in the rarefaction wave R can be obtained as

$$(u_2, v_2) = \Big(\frac{x + \varepsilon}{2t}, v_m \exp\Big(2u_- - \frac{x + \varepsilon}{t}\Big)\Big). \tag{4.42}$$

As in Lemma 4.1, one can see that if $u_- > u_+ + 1$, then the delta shock wave is able to cancel the whole rarefaction wave R completely in finite time and consequently propagates with the invariant speed $u_- + u_+$. Otherwise, if $u_- < u_+ + 1$, then the delta shock wave cannot cross the whole rarefaction wave R completely in finite time and may be also divided into a shock wave and a delta contact discontinuity when it passes through the characteristic line with the state satisfying $u_2 = u_+ + 1$ in the rarefaction wave fan R.

Analogously, the critical point for the decomposition of δS_2 into S and δJ can be calculated by

$$\begin{gathered} x_3 + \varepsilon = 2(u_+ + 1)t_3, \\ x_3 = 2u_+t_3 + 2\sqrt{2\varepsilon t_3(u_m - u_+)} - \varepsilon, \end{gathered} \tag{4.43}$$

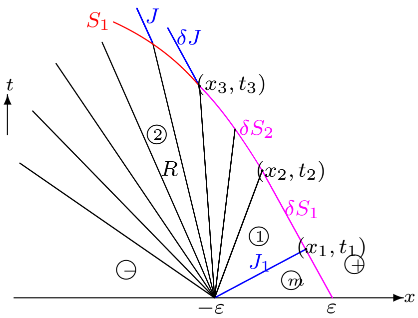
which enables us to obtain

$$(x_3, t_3) = \Big(4\varepsilon(u_m - u_+)(u_+ + 1) - \varepsilon, 2\varepsilon(u_m - u_+)\Big). \tag{4.44}$$

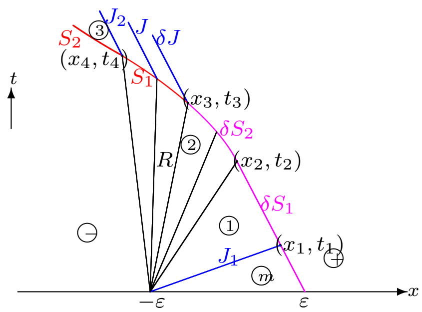
Consequently, the curve of δJ can be expressed as

$$x(t) = (2u_+ + 1)(t - 2\varepsilon(u_m - u_+)) + 4\varepsilon(u_m - u_+)(u_+ + 1) - \varepsilon. \tag{4.45}$$

On the other hand, the shock wave S is still the continuation of the delta shock wave δS_2 and can be expressed by (4.41). Consequently, the shock wave S continues to penetrate the rarefaction wave R. If $u_+ < u_- < u_+ + 1$, then the shock wave S



(a) $u_+ > u_-$



(b) $u_- - 1 < u_+ < u_-$

FIGURE 7. Interactions between $R + J$ and δS for two different situations where both $u_- < u_m$ and $u_+ \leq u_m - 1$ should be satisfied.

is able to penetrate the whole R completely in finite time. Otherwise, if $u_- < u_+$, then the shock wave S cannot penetrate R completely in finite time and finally has the line $x + \varepsilon = 2u_+t$ as its asymptote.

5. Discussions and conclusions

So far, the wave interaction problems for the system (1.1) have been investigated in detail. The global solutions of the perturbed Riemann problem (1.1) and (1.3) are constructed fully for all the situations. Now, we are in a position to consider whether the limits of the solutions of the perturbed Riemann problem (1.1) and (1.3) are the corresponding ones of the Riemann problem (1.1) and (1.2) or not as $\varepsilon \to 0$. Let us take Case 4.4 as an example to explain our problem in detail.

With $u_m \leq u_- - 1$ and $u_m < u_+$ in mind for Case 4.4, we first consider the situation that $u_+ \leq u_- - 1$. It is easy to see that all the points $(-\varepsilon, 0)$, $(\varepsilon, 0)$, (x_1, t_1), (x_2, t_2) and (x_3, t_3) tend to the origin $(0, 0)$ and coincide with each other, such that there is only a delta shock wave with the propagation speed $u_- + u_+$ in the limit situation. If $u_- - 1 < u_+ < u_-$, then the shock wave is able to cancel the whole rarefaction wave completely. It can be seen from (4.23) and (4.30) that $\beta(t_2)$ tends to zero for the reason that t_2 tends to zero as $\varepsilon \to 0$, which implies that the delta shock wave disappears and the delta contact discontinuity becomes the contact discontinuity in the limit situation. Furthermore, we can see that all the contact discontinuities coincide with each other in the limit situation for they have the same propagation speed and start from the origin $(0, 0)$. Thus, there is still $S + J$ for $u_- - 1 < u_+ < u_-$ in the limit situation. Otherwise, if $u_+ > u_-$, then the limit situation is similar and the difference only lies in that the shock wave is unable to cancel the rarefaction wave completely. Thus, there is $R + J$ for $u_+ > u_-$ in the limit situation. Gathering the above results together, we can see that the limits of the solutions of the perturbed Riemann problem (1.1) and (1.3) are identical with the corresponding ones of the Riemann problem (1.1) and (1.2) as $\varepsilon \to 0$ for Case 4.4.

The above method can also be generalized to the other cases and one can discover that the large-time asymptotic solutions of the perturbed Riemann problem (1.1) and (1.3) indeed coincide with the corresponding ones of the Riemann problem (1.1) and (1.2). That is to say, the large-time asymptotic solutions of the perturbed Riemann problem (1.1) and (1.3) is the delta shock wave for $u_+ \leq u_- - 1$, the shock wave followed by the contact discontinuity for $u_- - 1 < u_+ < u_-$ and the rarefaction wave followed by the contact discontinuity for $u_+ > u_-$. Let us call that the solutions of the Riemann problem (1.1) and (1.2) are stable with respect to the specific small perturbations (1.3) of the Riemann initial data (1.2) provided that the solutions of the perturbed Riemann problem (1.1) and (1.3) converge to the ones of the corresponding Riemann problem (1.1) and (1.2) as $\varepsilon \to 0$ in the sense of distributions in all kinds of situations. In a word, we can summarize our results in the following theorem.

Theorem 5.1. *The limits of the solutions to the perturbed Riemann problem* (1.1) *and* (1.3) *are identical with the corresponding ones to the Riemann problem* (1.1) *and* (1.2) *as* $\varepsilon \to 0$ *for all kinds of situations. Thus, the conclusion can be drawn that the solutions to the Riemann problem* (1.1) *and* (1.2) *are stable with respect to such a local small perturbation* (1.3) *of the Riemann initial data* (1.2).

Acknowledgements. The authors would like to thank the anonymous referees for their useful comments on the original manuscript. This work is partially supported by the Shandong Provincial Natural Science Foundation (ZR2014AM024) and the National Natural Science Foundation of China (11441002).

References

[1] G. Q. Chen, H.Liu; *Formation of δ-shocks and vacuum states in the vanishing pressure limit of solutions to the Euler equations for isentropic fluids*, SIAM J. Math. Anal., 34 (2003), 925-938.
[2] M. Colombeau; *A method of projection of delta waves in Godunov scheme and application to pressureless fluid dynamics*, SIAM J. Numer. Anal., 48 (2010), 1900-1919.
[3] V. G. Danilov, D. Mitrovic; *Delta shock wave formation in the case of triangular hyperbolic system of conservation laws*, J. Differential Equations, 245 (2008), 3704-3734.
[4] V. G. Danilov, V. M. Shelkovich; *Dynamics of propagation and interaction of δ-shock waves in conservation law systems*, J. Differential Equations, 221 (2005), 333-381.
[5] V. G. Danilov, V. M. Shelkovich; *Delta-shock waves type solution of hyperbolic systems of conservation laws*, Q. Appl. Math., 63 (2005), 401-427.
[6] L. Guo, L. Pan, G. Yin; *The perturbed Riemann problem and delta contact discontinuity in chromatography equations*, Nonlinear Analysis,TMA, 106(2014),110-123.
[7] L. Guo, Y. Zhang, G.Yin; *Interactions of delta shock waves for the Chaplygin gas dynamic equations with split delta functions*, J. Math. Anal. Appl., 410 (2014), 190-201.
[8] B. T. Hayes, P. G. LeFloch; *Measure solutions to a strictly hyperbolic system of conservation laws*, Nonlinearity, 9 (1996), 1547-1563.
[9] H. Kalisch, D. Mitrovic; *Singular solutions of a fully nonlinear 2×2 system of conservation laws*, Proceedings of the Edinburgh Mathematical Society, 55 (2012), 711-729.
[10] H. Kalisch, D. Mitrovic; *Singular solutions for the shallow-water equations*, IMA J. Appl.Math., 77 (2012), 340-350.
[11] B. L. Keyfitz, H. C. Kranzer; *Spaces of weighted measures for conservation laws with singular shock solutions*, J. Differential Equations, 118 (1995), 420-451.
[12] B. L. Keyfitz, C. Tsikkou; *Conservaing the wrong variables in gas dynamics: a Riemann problem with singular shocks*, Q. Appl. Math., 70 (2012), 407-436.
[13] J. Li, T. Zhang, S. Yang; *The Two-Dimensional Riemann Problem in Gas Dynamics*, Pitman Monographs and Surveys in Pure and Applied Mathematics, 98, Longman Scientific and Technical, 1998.
[14] M. Nedeljkov; *Delta and singular delta locus for one dimensional systems of conservation laws*, Math. Meth. Appl.Sci., 27 (2004), 931-955.

[15] M. Nedeljkov; *Singular shock waves in interactions*, Q. Appl. Math., 66 (2008), 281-302.

[16] M. Nedeljkov; *Shadow waves: entropies and interactions for delta and singular shocks*, Arch. Rational Mech. Anal., 197 (2010), 487-537.

[17] M. Nedeljkov, M. Oberguggenberger; *Interactions of delta shock waves in a strictly hyperbolic system of conservation laws*, J. Math. Anal. Appl., 344 (2008), 1143-1157.

[18] A.Qu, Z. Wang; *Stability of the Riemann solutions for a Chaplygin gas*, J. Math. Anal. Appl., 409 (2014), 347-361.

[19] C. Shen; *The asymptotic behaviors of solutions to the perturbed Riemann problem near the singular curve for the chromatography system*, J. Nonlinear Math. Phys., 22 (2015), 76-101.

[20] C. Shen, M. Sun; *Interactions of delta shock waves for the transport equations with split delta functions*, J. Math. Anal. Appl., 351 (2009), 747-755.

[21] C. Shen, M. Sun; *Formation of delta shocks and vacuum states in the vanishing pressure limit of Riemann solutions to the perturbed Aw-Rascle model*, J. Differential Equations, 249 (2010), 3024-3051.

[22] C. Shen, M. Sun; *Stability of the Riemann solutions for a nonstrictly hyperbolic system of conservation laws*, Nonlinear Analysis, TMA, 73 (2010), 3284-3294.

[23] W. Sheng, T. Zhang; *The Riemann problem for the transportation equations in gas dynamics*, Mem. Amer. Math. Soc., 137 (N654) (1999), AMS: Providence.

[24] M. Sun; *Interactions of delta shock waves for the chromatography equations*, Applied Mathematics Letters, 26 (2013), 631-637.

[25] D. Tan; *Riemann problem for hyperbolic systems of conservation laws with no classical wave solutions*, Q. Appl. Math., 53 (1993), 765-776.

[26] D. Tan, T. Zhang, Y. Zheng; *Delta-shock waves as limits of vanishing viscosity for hyperbolic systems of conservation laws*, J. Differential Equations, 112 (1994), 1-32.

[27] B. Temple; *Systems of conservation laws with invariant submanifolds*, Trans. Amer. Math. Soc., 280 (1983), 781-795.

[28] H. Yang; *Riemann problem for a class of coupled hyperbolic systems of conservation laws*, J. Differential Equations, 159 (1999), 447-484.

[29] H. Yang, W. Sun; *The Riemann problem with delta initial data for a class of coupled hyperbolic systems of conservation laws*, Nonlinear Analysis, TMA, 67 (2007), 3041-3049.

[30] A. Yao, W. Sheng; *Interaction of elementary waves on boundary for a hyperbolic system of conservation laws*, Math. Meth. Appl. Sci., 31 (2008), 1369-1381.

Xuefeng Wei
School of Mathematics and Statistics Science, Ludong University, Yantai, Shandong Province 264025, China
E-mail address: `50816179@qq.com`

Chun Shen (corresponding author)
School of Mathematics and Statistics Science, Ludong University, Yantai, Shandong Province 264025, China
E-mail address: `shenchun3641@sina.com`