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NONLINEAR PARABOLIC EQUATIONS WITH BLOWING-UP COEFFICIENTS WITH RESPECT TO THE UNKNOWN AND WITH SOFT MEASURE DATA

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Abstract. We establish the existence of solutions for the nonlinear parabolic problem with Dirichlet homogeneous boundary conditions,

$$\frac{\partial u}{\partial t} - \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(d_i(u)\frac{\partial u}{\partial x_i}\Big) = \mu, \quad u(t=0) = u_0,$$

in a bounded domain. The coefficients $d_i(s)$ are continuous on an interval $]-\infty, m[$, there exists an index p such that $d_p(u)$ blows up at a finite value m of the unknown u, and μ is a diffuse measure.

1. Introduction

In this paper we study the existence of solutions of the problem

$$\frac{\partial u}{\partial t} - \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(d_i(u)\frac{\partial u}{\partial x_i}\Big) = \mu \quad \text{in } Q, \tag{1.1}$$

$$u(t=0) = u_0 \quad \text{in } \Omega, \tag{1.2}$$

$$u = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{1.3}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ $(N \geq 1)$, T is a positive real number, and we have set Q the cylinder $\Omega \times (0,T)$ and $\partial\Omega \times (0,T)$ its lateral surface. The coefficients $d_i(s)$ are continuous on an interval $]-\infty, m[$ of $\mathbb{R}$ (with $m > 0$) with value in $\mathbb{R}^+ \cup \{+\infty\}$, $d_i(s) \geq \alpha > 0$, and such that there exists an index p such that $\lim_{s\to m^-} d_p(s) = +\infty$, and where $u_0 \in L^1(\Omega)$, $u_0 \leq m$ a.e. in Ω and μ is a measure on Q with bounded total variation.

When problem (1.1)-(1.3) is studied, the *a priori* estimates on the above problem do not lead in general to the existence of a weak solution (i.e. in the distributional sense), there are mainly two type of difficulties in studying problem (1.1)-(1.3). One consists to define in a proper way the term $d_p(u)\frac{\partial u}{\partial x_p}$ on the subset $\{(x,t) \in Q : u(x,t) = m\}$ of Q on which $d_p(u) = +\infty$. As an example, one can not set in general $d_p(u)\frac{\partial u}{\partial x_p} = 0$ on $\{(x,t) \in Q : u(x,t) = m\}$ to obtain the equation in the sense of distributions.

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The second difficulty is represented here by the presence of an L^1 initial datum and a measure as right-hand side term in (1.1). The measure μ is just assumed to be bounded total variation over Q that do not charge the sets of zero p-capacity (see Section 2 for the definition), the so called *diffuse measures* or *soft measures*, and we will use the symbol $\mathcal{M}_0(Q)$ to denote them.

To overcome this difficulty we use the framework of renormalized solutions. This notion was introduced by Lions and DiPerna [14] for the study of Boltzmann equation. This notion was then adapted to elliptic version of (1.1)-(1.3) in Boccardo, Diaz, Giachetti, Murat [12], Lions and F. Murat [22], and Murat [22, 23]. At the same the equivalent notion of entropy solutions was developed independently by Bénilan and al. [1] for the study of nonlinear elliptic problems.

The existence of a renormalized solution of (1.1)-(1.3) was proved in [2] in the stationary case where $\mu \in L^2(\Omega)$. In the stationary and evolution cases of $u_t - div(A(x,t,u)\nabla u) = f$ in Q, where the matrix $A(x,t,s)$ blows up (uniformly with respect to (x,t)) as $s \to m^-$ and where $f \in L^1(Q)$, the existence of renormalized solution was proved by Blanchard, Guibé and Redwane in [3].

The existence and uniqueness of renormalized solution of (1.1)-(1.3) was proved in [4] in the case where $\sum_{i=1}^N \frac{\partial}{\partial x_i}\big(d_i(u)\frac{\partial u}{\partial x_i}\big)$ is replaced by the p-Laplacian operator $\Delta_p u$, $u_0 \in L^1(\Omega)$ and u_t is replaced by $b(u)_t$ and for every measure μ which does not charge the sets of null parabolic p-capacity.

Note that, the existence result in [4] is strongly based on a decomposition theorem given in [15] for diffuse measure (i.e. $\mu \in \mathcal{M}_0(Q)$), this decomposition of μ can not be easily used for problem (1.1)-(1.3). Indeed (for $p = 2$), for every $\mu \in \mathcal{M}_0(Q)$ there exist $f \in L^1(Q)$, $g \in L^2(0,T; H_0^1(\Omega))$ and $F \in L^2(0,T; H^{-1}(\Omega))$ such that

$$\mu = f + F + g_t \quad \text{in } D'(Q), \tag{1.4}$$

note that the decomposition of μ is not uniquely determined. Therefore, equation (1.1) is equivalent to

$$\frac{\partial v}{\partial t} - \sum_{i=1}^N \frac{\partial}{\partial x_i}\Big(d_i(v+g)\frac{\partial v}{\partial x_i}\Big) = f + F \quad \text{in } Q,$$

where $v = u - g$. Since $g \notin L^\infty(Q)$ in general and $\lim_{s\to m^-} d_p(s) = +\infty$, then the term $d_p(v+g)$ can not be easily handled. To overcome this difficulty, we use in this paper the following approximation property for the measure μ (see Theorem 2.2). Indeed, every $\mu \in \mathcal{M}_0(Q)$ can be strongly approximated by measures which admit decomposition (1.4) with $g \in L^\infty(Q)$ (see [17, Theorem 1.1]).

A large number of papers was then devoted to the study the existence of renormalized solution of parabolic problems with rough data under various assumptions and in different contexts: for a review on classical results, see [5, 6, 8, 9, 18, 19, 20, 24, 25, 26, 30, 32].

We organize this article as follows. In Section 2 we give some preliminaries and, in particular, we provide the definition of parabolic capacity and some basic properties of diffuse measures. Section 3 is devoted to specifying the assumptions on d_i, u_0 and μ. We also give the definition of a renormalized solution of (1.1)-(1.3). In Section 4 we establish (Theorem 4.1) the existence of such a solution. In Section 5 (Appendix) we prove Theorem 2.3 that will be a key point in the existence result.

2. Preliminaries on Parabolic Capacity and Diffuse Measures

We recall the notion of parabolic p-capacity (with $p = 2$) associated to our problem (for further details see [29, 15]). Let $Q = \Omega \times (0,T)$ for any fixed $T > 0$, and let us recall that

$$W = \left\{u \in L^2(0,T;H_0^1(\Omega)) : u_t \in L^2(0,T;H^{-1}(\Omega))\right\},$$

endowed with its natural norm $\|\cdot\|_{L^2(0,T;H_0^1(\Omega))} + \|\cdot\|_{L^2(0,T;H^{-1}(\Omega))}$, remark that W is continuously embedded in $C([0,T];L^2(\Omega))$ and $C_c^\infty([0,T]\times\Omega)$ is dense in W. Let $U \subseteq Q$ is an open set, we define the parabolic 2-capacity of U as

$$\operatorname{cap}_2(U) = \inf\{\|u\|_W : u \in W,\, u \geq \chi_U \text{ a.e. in } Q\},$$

where as usual we set $\inf\{\emptyset\} = +\infty$. Then for any Borel set $B \subseteq Q$ we define

$$\operatorname{cap}_2(B) = inf\{\operatorname{cap}_2(U) : U \text{ is open subset of Q}, B \subseteq U\}.$$

We denote by $\mathcal{M}_b(Q)$ the set of all Radon measures with bounded variation on Q, while, as we already mentioned, $\mathcal{M}_0(Q)$ denotes the set of all measures with bounded variation over Q that do not charge the sets of zero 2-capacity, that is if $\mu \in \mathcal{M}_0(Q)$, then $\mu(E) = 0$, for all $E \subseteq Q$ such that $\operatorname{cap}_2(E) = 0$.

In [15] the authors proved the following decomposition theorem.

Theorem 2.1. *Let μ be a bounded measure on Q. If $\mu \in \mathcal{M}_0(Q)$ then there exists (f,F,g) such that $f \in L^1(Q)$, $F \in L^2(0,T;H^{-1}(\Omega))$, $g \in L^2(0,T;H_0^1(\Omega))$ and*

$$\int_Q \phi\,d\mu = \int_Q f\phi\,dx\,dt + \int_0^T \langle F,\phi\rangle\,dt - \int_0^T \langle \phi_t, g\rangle\,dt \quad \phi \in C_c^\infty([0,T]\times\Omega).$$

Such a triplet (f,F,g) will be called a decomposition of μ.

Note that the decomposition of μ is not uniquely determined. In [17] the authors proved the following approximation of diffuse measures theorem.

Theorem 2.2. *Let $\mu \in \mathcal{M}_0(Q)$, then, for every $\varepsilon > 0$ there exists $\nu \in \mathcal{M}_0(Q)$ such that*

$$\|\mu - \nu\|_{\mathcal{M}(Q)} \leq \varepsilon \quad \text{and} \quad \nu = w_t - \Delta w \text{ in } \mathcal{D}'(Q),$$

where $w \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q)$.

The following Theorem will be a key point in the existence result given in the next section. The proof follows the arguments in [27, Theorem 1.2].

Theorem 2.3. *Let $d_i \in \mathcal{C}^0(\mathbb{R}) \cap L^\infty(\mathbb{R})$ for every $i \in \{1,\dots,N\}$, $\mu \in \mathcal{M}_0(Q) \cap L^2(0,T;H^{-1}(\Omega))$ and $u_0 \in L^2(\Omega)$, let $u \in W$ be the (unique) weak solution of*

$$\begin{gathered}
\frac{\partial ua}{\partial t} - \sum_{i=1}^N \frac{\partial}{\partial x_i}\big(d_i(u)\frac{\partial u}{\partial x_i}\big) = \mu \quad \text{in } Q,\\
u = 0 \quad \text{on } (0,T)\times\partial\Omega,\\
u(t=0) = u_0 \quad \text{in } \Omega.
\end{gathered} \tag{2.1}$$

Then

$$\operatorname{cap}_2\{|u| > K\} \leq \frac{C}{\sqrt{K}} \quad \forall K \geq 1,$$

where $C > 0$ is a constant depending on $\|\mu\|_{\mathcal{M}(Q)}$, $\|u_0\|_{L^2(\Omega)}$.

The proof of the above theorem is postponed to the Appendix in Section 5.

Definition 2.4. A sequence of measures (μ_n) in Q is equidiffuse if for every $\eta > 0$ there exists $\delta > 0$ such that

$$\mathrm{cap}_2(E) < \delta \Longrightarrow |\mu_n|(E) < \eta \quad \forall\, n \geq 1.$$

The following result is proved in [27]:

Lemma 2.5. *Let ρ_n be a sequence of mollifiers on Q. If $\mu \in \mathcal{M}_0(Q)$, then the sequence $(\rho_n * \mu_n)$ is equidiffuse.*

Here is some notation we will use throughout the paper. For any nonnegative real number K we denote by $T_K(r) = \min(K, \max(r, -K))$ the truncation function at level K. For every $r \in \mathbb{R}$, let

$$\overline{T_K}(z) = \int_0^z T_K(s)\,ds$$

We consider the following smooth approximation of $T_K(s)$: for $m > 0$, $\eta \in]0,1[$ and $\sigma \in]0,1[$, we define $S^m_{K,\sigma} : \mathbb{R} \to \mathbb{R}$ by

$$S^{m,\eta}_{K,\sigma}(s) \; = \begin{cases} 1 & \text{if } -K+\eta \leq s \leq m-2\sigma, \\ 0 & \text{if } s \leq -K \text{ and } s \geq m-\sigma, \\ \text{affine} & \text{otherwise,} \end{cases} \tag{2.2}$$

and let us denote $T^{m,\eta}_{K,\sigma}(z) = \int_0^z S^{m,\eta}_{K,\sigma}(s)\,ds$ and

$$T^m_K(s) = \begin{cases} s & \text{if } -K \leq s \leq m, \\ -K & \text{if } s \leq -K, \\ m & \text{if } s \geq m. \end{cases}$$

By $\langle \cdot, \cdot \rangle$ we mean the duality between suitable spaces in which function are involved. In particular we will consider both the duality between $H^1_0(\Omega)$ and $H^{-1}(\Omega)$ and the duality between $H^1_0(\Omega) \cap L^\infty(\Omega)$ and $H^{-1}(\Omega) + L^1(\Omega)$.

3. Main assumptions and definition of renormalized solution

Throughout the paper, we assume that the following assumptions hold: Ω is a bounded open set on $\mathbb{R}^N$ $(N \geq 2)$, $T > 0$ is given and we set $Q = \Omega \times (0,T)$.

$$d_i \in C^0(]-\infty, m[, \mathbb{R}^+ \cup \{+\infty\}) \quad \text{with } d_i(s) < +\infty \;\forall s < m,\; \forall i \in \{1,\dots,N\}; \tag{3.1}$$

$$\exists \alpha > 0 \text{ such that } d_i(s) \geq \alpha \;\forall i \in \{1,\dots,N\},\; \forall s \in]-\infty, m[; \tag{3.2}$$

$$\exists p \in \{1,\dots,N\} \text{ such that } \lim_{s \to m^-} d_p(s) = +\infty \text{ and } \int_0^m d_p(s)\,ds < +\infty; \tag{3.3}$$

$$\mu \in \mathcal{M}_0(Q); \tag{3.4}$$

$$u_0 \in L^1(\Omega) \text{ such that } u_0 \leq m \text{ a.e. in } \Omega. \tag{3.5}$$

The definition of a renormalized solution for Problem (1.1)-(1.3) is as follows.

Definition 3.1. *Let $\mu \in \mathcal{M}_0(Q)$. A function $u \in L^1(Q)$ is a renormalized solution of Problem* (1.1)-(1.3) *if*

$$u \leq m \text{ a.e. in } Q, \quad T_K(u) \in L^2(0,T; H^1_0(\Omega)) \quad \forall K > 0; \tag{3.6}$$

$$d_i(u) \frac{\partial T^m_K(u)}{\partial x_i} \chi_{\{u<m\}} \in L^2(Q) \quad \forall K > 0,\; \forall i \in \{1,\dots,N\}, \tag{3.7}$$

if there exists a sequence of nonnegative measures $(\Lambda_K) \in \mathcal{M}(Q)$ *and a nonnegative measure* $\Gamma \in \mathcal{M}(Q)$ *such that*

$$\lim_{K\to+\infty} \|\Lambda_K\|_{\mathcal{M}(Q)} = 0, \tag{3.8}$$

$$\int_Q \varphi \, d\Gamma = 0 \quad \forall \varphi \in \mathcal{C}_0^1([0,T[), \tag{3.9}$$

and if, for every $K > 0$,

$$\frac{\partial T_K^m(u)}{\partial t} - \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(d_i(u)\frac{\partial T_K^m(u)}{\partial x_i}\chi_{\{u<m\}}\Big) = \mu + \Lambda_K + \Gamma \quad \textit{in } \mathcal{D}'(Q). \tag{3.10}$$

Remark 3.2. (1) Note that, in view of (3.6), (3.7) and (3.8) all terms in (3.10) are well defined.
(2) The study of (1.1)-(1.3) under the assumption $\int_0^m d_p(s)\,ds = +\infty$ is easier (see [28] for the elliptic case), because one can then show there exists at least a renormalized solution such that $u < m$ a.e. in Q.
(3) Let us point out that, in (3.9) the function $\varphi \in \mathcal{C}_0^1([0,T[)$ which does not depend on the variable x, we are not able to prove (3.9) with any function $\varphi \in L^2(0,T;H^1(\Omega)) \cap L^\infty(Q)$ such that $\nabla\varphi = 0$ a.e. in $\{(x,t)\,;u(x,t) = m\}$ because of a lack of regularity on u with respect to t in the parabolic case.

4. Existence of solutions

This section is devoted to establish the following existence theorem.

Theorem 4.1. *Under assumptions* (3.1)-(3.7) *there exists at least a renormalized solution* u *of Problem* (1.1)-(1.3).

Proof. The proof is divided into 4 steps. In Step 1, we introduce an approximate problem. Step 2 is devoted to establish a few *a priori* estimates. At last, Step 3 and Step 4 are devoted to prove that u satisfies (3.7), (3.8), (3.9) and (3.10) of Definition 3.1.

Step 1. Let us introduce the following regularization of the data: for $n \geq 1$ fixed

$$d_i^n(s) = d_i(T_{m-\frac{1}{n}}(s^+) - T_n(s^-)) \quad \forall s \in \mathbb{R},\ \forall i \in \{1,\dots,N\}, \tag{4.1}$$

$$u_{0n} \in C_c^\infty(\Omega) : u_{0n} \to u_0 \text{ strongly in } L^1(\Omega) \text{ as } n \text{ tends to } +\infty, \tag{4.2}$$

we consider a sequence of mollifiers (ρ_n), and we define the convolution $\rho_n * \mu$ for every $(t,x) \in Q$ by

$$\mu^n(t,x) = \rho_n * \mu(t,x) = \int_Q \rho_n(t-s, x-y)d\mu(s,y). \tag{4.3}$$

Let us now consider the regularized problem

$$\frac{\partial u_n}{\partial t} - \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(d_i^n(u_n)\frac{\partial u_n}{\partial x_i}\Big) = \mu^n \quad \text{in } Q, \tag{4.4}$$

$$u_n(t=0) = u_{0n} \quad \text{in } \Omega, \tag{4.5}$$

$$u_n = 0 \quad \text{on } \partial\Omega \times (0,T). \tag{4.6}$$

As a consequence, proving existence of a weak solution $u_n \in L^2(0,T;H_0^1(\Omega))$ of (4.4)-(4.6) is an easy task (see e.g. [21]).

Step 2. Using $T_K(u_n)$ as a test function in (4.4) leads to

$$\int_\Omega \overline{T_K}(u^n)\,dx+\sum_{i=1}^N\int_Q d_i^n(u^n)\Big|\frac{\partial T_K(u^n)}{\partial x_i}\Big|^2\,dx\,dt\le K(\|\mu_n\|_{L^1(Q)}+\|u_0\|_{L^1(\Omega)}) \tag{4.7}$$

for almost every t in $(0,T)$, and where $\overline{T_K}(r)=\int_0^r T_K(s)\,ds$. The properties $\overline{T_K}\big(\overline{T_K}\ge 0,\overline{T_K}(s)\ge |s|-1\ \forall s\in\mathbb{R}\big)$, and since $\|\mu^n\|_{L^1(Q)}$ and $\|u_{0n}\|_{L^1(\Omega)}$ are bounded, we deduce from (4.7) that

$$u^n \text{ is bounded in } L^\infty(0,T;L^1(\Omega)), \tag{4.8}$$

$$T_K(u^n) \text{ is bounded in } L^2(0,T;H_0^1(\Omega)), \tag{4.9}$$

$$d_i^n(u^n)^{1/2}\frac{\partial T_K(u^n)}{\partial x_i} \text{ is bounded in } L^2(Q) \tag{4.10}$$

independently of n for any $K\ge 0$ and any $i\in\{1,2,\dots,N\}$.

In view of (3.1)-(3.3), we have that for any $K\ge 0$,

$$\Big|\int_0^{u^n} d_i^n(s)\chi_{\{-K\le s\le m\}}\,dx\Big|\le\int_{-K}^m d_i(s)\,ds\equiv C_K<+\infty,$$

then we can use $\int_0^{u^n} d_i^n(s)\chi_{\{-K\le s\le m\}}\,ds$ in $L^2(0,T;H_0^1(\Omega))\cap L^\infty(Q_T)$ as a test function in (4.4) obtaining

$$\begin{aligned}&\int_\Omega\int_0^{u^n}\int_0^z d_i^n(s)\chi_{\{-K\le s\le m\}}\,ds\,dz\,dx+\int_Q(d_i^n(u^n))^2\Big|\frac{\partial T_K^m(u^n)}{\partial x_i}\Big|^2\,dx\,dt\\&\le(\|\mu_n\|_{L^1(Q_T)}+\|u_0\|_{L^1(\Omega)})\max_i\int_{-K}^m d_i(s)\,ds\end{aligned} \tag{4.11}$$

for all $i\in\{1,2,\dots,N\}$. Since $\int_\Omega\int_0^{u^n}\int_0^z d_i^n(s)\,ds\,dz\,dx$ is positive and $\|\mu^n\|_{L^1(Q)}$ and $\|u_{0n}\|_{L^1(\Omega)}$ are bounded, from (4.11) we deduce that

$$d^n(u^n)\nabla T_K^m(u^n) \text{ is bounded in } (L^2(Q))^N. \tag{4.12}$$

For any $S\in W^{2,\infty}(\mathbb{R})$ such that S' has a compact support ($\operatorname{supp}(S')\subset[-K,m]$), we have

$$S(u^n) \text{ is bounded in } L^2(0,T;H_0^1(\Omega)), \tag{4.13}$$

$$\frac{\partial S(u^n)}{\partial t} \text{ is bounded in } L^1(Q)+L^2(0,T;H^{-1}(\Omega)), \tag{4.14}$$

independently of n. In fact, as a consequence of (4.9), by Stampacchia's Theorem, we obtain (4.13). To show that (4.14) holds true, we multiply the equation (4.4) by $S'(u^n)$ to obtain

$$\begin{aligned}\frac{\partial S(u^n)}{\partial t}&=\sum_{i=1}^N\frac{\partial}{\partial x_i}\Big(d_i^n(u_n)\frac{\partial S(u_n)}{\partial x_i}\Big)\\&\quad-\sum_{i=1}^N d_i^n(u_n)\Big|\frac{\partial u_n}{\partial x_i}\Big|^2 S''(u^n)+\mu^n S'(u^n)\quad\text{in } \mathcal{D}'(Q),\end{aligned} \tag{4.15}$$

as a consequence of (4.3), (4.10), (4.12), we obtain (4.14).

Arguing again as in [5, 6, 7, 9] estimates (4.13) and (4.14) imply that, for a subsequence still indexed by n,

$$u^n \to u \quad \text{almost every where in } Q, \tag{4.16}$$

$$T_K(u^n) \rightharpoonup T_K(u) \quad \text{weakly in } L^2(0,T;H_0^1(\Omega)), \tag{4.17}$$

$$(d^n(u^n))^{1/2}\nabla T_K(u^n) \rightharpoonup X_K \quad \text{weakly in } (L^2(Q))^N, \tag{4.18}$$

$$d^n(u^n)\nabla T_K^m(u^n) \rightharpoonup Y_K \quad \text{weakly in } (L^2(Q))^N, \tag{4.19}$$

as n tends to $+\infty$, for any $K > 0$.

Using the admissible test function $T_{2m}^+(u^n) - T_m^+(u^n)$ in (4.4) and the Poincaré inequality, leads to

$$d_p(m-\frac{1}{n})\int_Q \Big|T_{2m}^+(u^n) - T_m^+(u^n)\Big|^2\,dx\,dt \le m(\|\mu_n\|_{L^1(Q)} + \|u_{0n}\|_{L^1(\Omega)}). \tag{4.20}$$

In view of (3.3), (4.2) and (4.16) (since $d_p(m-\frac{1}{n}) \to +\infty$ as n tends $+\infty$) passing to the limit in (4.20) as n tends to $+\infty$, we deduce that $T_{2m}^+(u) - T_m^+(u) = 0$ a.e. in Q, hence

$$u \le m \quad \text{a.e. in } Q. \tag{4.21}$$

Now, in view of (4.18), (4.19) and (4.21) we deduce

$$X_K = d(u)^{1/2}\nabla T_K(u) \text{ and } Y_K = d(u)\nabla T_K^m(u) \quad \text{a.e. in } \{(x,t)\in Q : u(x,t) < m\}, \tag{4.22}$$

for any $K \ge 0$.

For fixed $K \ge 1, \eta \in]0,1[$ and $\sigma \in]0,1[$, we define the functions, $h_{K,\eta}$ and Z_σ by

$$h_{K,\eta}(s) = \begin{cases} 0 & \text{if } -K \le s \\ -1 & \text{if } s \le -K-\eta \\ \text{affine} & \text{otherwise,} \end{cases} \quad Z_\sigma(s) = \begin{cases} 0 & \text{if } s \le m-2\sigma \\ 1 & \text{if } s \ge m-\sigma \\ \text{affine} & \text{otherwise.} \end{cases} \tag{4.23}$$

We remark that $\max(\|h_{K,\eta}\|_{L^\infty(\mathbb{R})}, \|Z_\sigma\|_{L^\infty(\mathbb{R})}) = 1$ for any $K \ge 1$ any $0 < \eta < 1$ and any $0 < \sigma < 1$. Using the admissible test functions $h_{K,\eta}(u^n)$ and $Z_\sigma(u^n)$ in (4.4) leads to

$$\begin{aligned} &\int_\Omega \overline{h_{K,\eta}}(u^n(T))\,dx + \sum_{i=1}^N \int_Q d_i^n(u_n)\frac{\partial u_n}{\partial x_i}\frac{\partial h_{K,\eta}(u_n)}{\partial x_i}\,dx\,dt \\ &= \int_Q h_{K,\eta}(u_n)\mu_n\,dx\,dt + \int_\Omega \overline{h_{K,\eta}}(u_{0n})\,dx, \end{aligned} \tag{4.24}$$

and

$$\begin{aligned} &\int_\Omega \overline{Z_\sigma}(u^n(T))\,dx + \sum_{i=1}^N \int_Q d_i^n(u_n)\frac{\partial u_n}{\partial x_i}\frac{\partial Z_{K,\sigma}(u_n)}{\partial x_i}\,dx\,dt \\ &= \int_Q Z_{K,\sigma}(u_n)\mu_n\,dx\,dt + \int_\Omega \overline{Z_{K,\sigma}}(u_{0n})\,dx, \end{aligned} \tag{4.25}$$

where

$$\overline{h_{K,\eta}}(r) = \int_0^r h_{K,\eta}(s)\,ds \ge 0, \quad \overline{Z_\sigma}(r) = \int_0^r Z_\sigma(s)\,ds \ge 0.$$

Hence, using (4.2), (4.3) and dropping a nonnegative term,

$$\begin{aligned} &\sum_{i=1}^{N} \frac{1}{\eta} \int_{\{-K-\eta\le u^n\le -K\}} d_i^n(u^n)\Big|\frac{\partial u^n}{\partial x_i}\Big|^2\,dx\,dt \\ &\le \int_{\{u^n\le -K\}} |\mu^n|\,dx\,dt + \int_{\{u_{n0}\le -K\}} |u_{0n}|\,dx \le C_1, \end{aligned} \tag{4.26}$$

and

$$\begin{aligned} &\sum_{i=1}^{N} \frac{1}{\sigma} \int_{\{m-2\sigma\le u^n\le m-\sigma\}} d_i^n(u^n)\Big|\frac{\partial u^n}{\partial x_i}\Big|^2\,dx\,dt \\ &\le \int_{\{u^n\ge m-2\sigma\}} Z_\sigma(u_n)\mu^n\,dx\,dt + \int_{\{u_{n0}\ge m-2\sigma\}} |u_{0n}|\,dx \le C_2. \end{aligned} \tag{4.27}$$

Thus, there exists a bounded Radon measures λ_K^n and ν_σ such that, as η tends to zero and n tends to infinity

$$\lambda_K^{n,\eta} \equiv \sum_{i=1}^{N} \frac{1}{\eta} d_i^n(u^n)\Big|\frac{\partial u^n}{\partial x_i}\Big|^2 \chi_{\{-K-\eta\le u^n\le -K\}} \rightharpoonup \lambda_K^n \quad *\text{-weakly in } \mathcal{M}(Q), \tag{4.28}$$

and

$$\nu_\sigma^n \equiv \sum_{i=1}^{N} \frac{1}{\sigma} d_i^n(u^n)\Big|\frac{\partial u^n}{\partial x_i}\Big|^2 \chi_{\{m-2\sigma\le u^n\le m-\sigma\}} \rightharpoonup \nu_\sigma \quad *\text{-weakly in } \mathcal{M}(Q). \tag{4.29}$$

Step 3. In this step, u is shown to satisfy (3.10). For all real numbers $\eta > 0$, $\sigma > 0$ and $K > 0$, let $S_{K,\sigma}^{m,\eta}$ be the function defined by (2.2), and let us denote $T_{K,\sigma}^{m,\eta}(z) = \int_0^z S_{K,\sigma}^{m,\eta}(s)\,ds$. Since $\operatorname{supp}(S_{K,\sigma}^{m,\eta})' \subset [-K-\eta, -K] \cup [m-2\sigma, m-\sigma]$, the equation (4.15) with $S = T_{K,\sigma}^{m,\eta}$ gives

$$\begin{aligned} &\frac{\partial T_{K,\sigma}^{m,\eta}(u^n)}{\partial t} - \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(d_i^n(u_n)\frac{\partial T_{K,\sigma}^{m,\eta}(u_n)}{\partial x_i}\Big) \\ &= \mu^n + (S_{K,\sigma}^{m,\eta}(u^n) - 1)\mu^n + \frac{1}{\eta}\sum_{i=1}^{N} d_i^n(u_n)\Big|\frac{\partial u_n}{\partial x_i}\Big|^2 \chi_{\{-K-\eta<u_n<-K\}} \\ &\quad + \frac{1}{\sigma}\sum_{i=1}^{N} d_i^n(u_n)\Big|\frac{\partial u_n}{\partial x_i}\Big|^2 \chi_{\{m-2\sigma<u_n<m-\sigma\}} \end{aligned} \tag{4.30}$$

in $\mathcal{D}'(Q)$. Passing to the limit in (4.30) as η tends to zero, and using (4.17), (4.19), (4.21), (4.22), (4.28) and (4.29), we deduce

$$\begin{aligned} &\frac{\partial T_{K,\sigma}^{m}(u^n)}{\partial t} - \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(d_i^n(u_n)\frac{\partial T_{K,\sigma}^{m}(u_n)}{\partial x_i}\Big) \\ &= \mu^n - \mu^n\chi_{\{u^n<-K\}} - \mu^n Z_\sigma(u^n) + \lambda_K^n + \nu_\sigma^n \end{aligned} \tag{4.31}$$

in $\mathcal{D}'(Q)$. Now, using the properties of convolution $\mu_n = \rho_n * \mu$ and in view of (4.26), (4.27), (4.28) and (4.29), we deduce that $\Lambda_K^n \equiv -\mu^n\chi_{\{u^n<-K\}} + \lambda_K^n$ and $\Gamma_\sigma^n \equiv -\mu^n Z_\sigma(u^n) + \nu_\sigma^n$ are bounded in $L^1(Q)$. Then there exists a bounded measures Λ_K and Γ_σ such that $(-\mu^n\chi_{\{u^n<-K\}} + \lambda_K^n)_n$ converges to Λ_K and $(-\mu^n Z_\sigma(u^n) + \nu^n)_n$

converges to Γ_σ in $*-$weakly in $\mathcal{M}(Q)$. From (4.16), (4.17), (4.19), (4.21), (4.22) and (4.31) We deduce that u satisfies

$$\frac{\partial T^m_{K,\sigma}(u)}{\partial t}-\sum_{i=1}^N\frac{\partial}{\partial x_i}\Big(d_i^n(u)\frac{\partial T^m_{K,\sigma}(u)}{\partial x_i}\chi_{\{u<m\}}\Big)=\mu+\Lambda_K+\Gamma_\sigma \text{ in } \mathcal{D}'(Q). \tag{4.32}$$

To complete this step, we use

$$\begin{aligned}\int_Q|\Gamma_\sigma|\,dx\,dt&\le\liminf_{n\to+\infty}\int_Q|\Gamma^n_\sigma|\,dx\,dt\\&=\liminf_{n\to+\infty}\int_Q|-\mu^nZ_\sigma(u^n)+\nu^n_\sigma|\,dx\,dt\\&\le 2\|\mu\|_{\mathcal{M}(Q)}+\|u_0\|_{L^1(\Omega)}\end{aligned}$$

then there exists a bounded measure Γ such that Γ_σ converges to Γ in $*-$weakly in $\mathcal{M}(Q)$. Therefore, as σ tends to zero in (4.32), it is easy to see that u satisfies (3.10).

Step 4. In this step, Λ_K and Γ are shown to satisfy (3.8) and (3.9). From (4.26) and (4.28) we deduce

$$\begin{aligned}\|\Lambda^n_K\|_{L^1(Q)}&=\|\ -\mu^n\chi_{\{u^n<-K\}}+\lambda^n_K\|_{L^1(Q)}\\&\le 2\int_{\{u^n<-K\}}|\mu^n|\,dx\,dt+\int_{\{u_{0n}<-K\}}|u_{0n}|\,dx.\end{aligned}\tag{4.33}$$

Since

$$\|\lambda_K\|_{\mathcal{M}(Q)}\le\liminf_{n\to+\infty}\|\mu^n\chi_{\{u^n<-K\}}+\lambda^n_K\|_{\mathcal{M}(Q)},$$

the sequence (μ_n) is equidiffuse, and the function u_{0n} converges to u_0 strongly in $L^1(\Omega)$, we deduce from Theorem 2.3 and (4.33) that $\|\Lambda_K\|_{\mathcal{M}(Q)}$ tends to zero as K tends to infinity, then we obtain (3.8).

On the other hand, for all $\varphi\in\mathcal{C}^1_0([0,T[)$, we can write

$$\int_Q\varphi\,d\Gamma=\lim_{\sigma\to0}\int_Q\varphi\,d\Gamma_\sigma=\lim_{\sigma\to0}\lim_{n\to+\infty}\int_Q\varphi\Gamma^n_\sigma\,dx\,dt \tag{4.34}$$

where

$$\Gamma^n_\sigma\equiv\frac{1}{\sigma}\sum_{i=1}^N d_i^n(u_n)\Big|\frac{\partial u_n}{\partial x_i}\Big|^2\chi_{\{m-2\sigma<u_n<m-\sigma\}}-Z_\sigma(u^n)\mu^n.$$

Using the admissible function $Z_\sigma(u^n)\varphi$ in (4.4), since $\varphi\in\mathcal{C}^1_0([0,T[)$, it is easy to see that

$$\begin{aligned}&\int_\Omega\overline{Z_\sigma}(u^n_0)\varphi(0)\,dx+\int_Q\overline{Z_\sigma}(u^n)\varphi_t\,dx\,dt\\&=\frac{1}{\sigma}\sum_{i=1}^N\int_{\{m-2\sigma<u_n<m-\sigma\}}d_i^n(u_n)\Big|\frac{\partial u_n}{\partial x_i}\Big|^2\varphi\,dx\,dt-\int_Q Z_\sigma(u^n)\mu^n\varphi\,dx\,dt\\&\equiv\int_Q\varphi\Gamma^n_\sigma\,dx\,dt,\end{aligned}\tag{4.35}$$

where $\overline{Z_\sigma}(r)=\int_0^r Z_\sigma(s)\,ds$. Next we pass to the limit in (4.35) as n tends to infinity, and then σ tends to zero. Since $\overline{Z_\sigma}(u^n)$ converges to $\overline{Z_\sigma}(u)$ strongly in $L^1(Q)$ and

$\overline{Z_\sigma}(u_0^n)$ converges to $\overline{Z_\sigma}(u_0)$ strongly in $L^1(\Omega)$ as n tends to infinity, we deduce

$$\begin{gathered}\lim_{n\to+\infty}\int_Q \overline{Z_\sigma}(u^n)\varphi_t\,dx = \int_Q \overline{Z_\sigma}(u)\varphi_t\,dx\\ \lim_{n\to+\infty}\int_\Omega \overline{Z_\sigma}(u_0^n)\varphi\,dx = \int_\Omega \overline{Z_\sigma}(u_0)\varphi\,dx\end{gathered}\tag{4.36}$$

Moreover, since $\overline{Z_\sigma}(r)$ converges to $(r-m)^+$ for all $r\in\mathbb{R}$ and $u\le m, u_0\le m$ almost everywhere, then it is easy to see that

$$\lim_{\sigma\to0}\lim_{n\to+\infty}\int_Q \overline{Z_\sigma}(u^n)\varphi_t\,dx = \int_Q (u-m)^+\varphi_t\,dx = 0,\tag{4.37}$$

$$\lim_{\sigma\to0}\lim_{n\to+\infty}\int_\Omega \overline{Z_\sigma}(u_0^n)\varphi\,dx = \int_\Omega (u_0-m)^+\varphi\,dx = 0.\tag{4.38}$$

Then, from (4.34), (4.35), (4.37) and (4.38) we deduce (3.9).

As a conclusion from Step 1, Step 2, Step 3 and Step 4, the proof is complete. □

5. Appendix: Proof of Theorem 2.3

Sketch of the Proof. For simplicity we assume that $\mu\ge0$ and $u_0\ge0$. Using the admissible test function $T_K(u)$ in (2.1) leads to

$$\begin{aligned}&\int_\Omega \overline{T_K}(u)\,dx + \sum_{i=1}^n\int_Q \Big|d_i(u)^{1/2}\frac{\partial T_K(u)}{\partial x_i}\Big|^2\,dx\,dt\\ &\le K\big[\|\mu\|_{\mathcal{M}(Q)} + \|u_0\|_{L^1(\Omega)}\big] \equiv KM,\end{aligned}\tag{5.1}$$

for almost any t in $]0,T[$ and where $\overline{T_K}(r)=\int_0^r T_K(s)\,ds$. Since $\frac12 T_K^2(r)\le\overline{T_K}(r)\le Kr$, from (5.1) we deduce that

$$\max\big\{\|T_K(u)\|^2_{L^\infty(L^2(\Omega))} : \|\nabla T_K(u)\|^2_{L^2(Q)}\big\}\le KM,\quad \|T_K(u)\|^2_{L^2(H_0^1(\Omega))}\le K\frac{M}{\alpha}.\tag{5.2}$$

Moreover, for $i\in\{1,\dots,N\}$ let us choose $\int_0^{T_K(u)} d_i(r)\,dr \in L^2(0,T;H_0^1(\Omega))\cap L^\infty(Q)$ as test function in 2.1. Then

$$\sum_{i=1}^n\int_Q \Big|d_i(u)\frac{\partial T_K(u)}{\partial x_i}\Big|^2\,dx\,dt \le K\big[\|\mu\|_{\mathcal{M}(Q)} + \|u_0\|_{L^1(\Omega)}\big]\|d_i\|_{L^\infty(\mathbb{R})}.\tag{5.3}$$

Let $v\in W$ be the solution of

$$\begin{gathered}-\frac{\partial v}{\partial t} - \sum_{i=1}^N \frac{\partial}{\partial x_i}\big(d_i(u)\frac{\partial v}{\partial x_i}\big) = -2\sum_{i=1}^N\frac{\partial}{\partial x_i}\big(d_i(u)\frac{\partial T_K(u)}{\partial x_i}\big)\quad\text{in } Q,\\ v=0\quad\text{on } (0,T)\times\partial\Omega,\\ v(t=T) = T_K(u(t=T))\quad\text{in }\Omega.\end{gathered}\tag{5.4}$$

Using the admissible test function v in (5.4) and integrate between τ and T, and by Young's inequality we obtain

$$\begin{aligned}&\int_\Omega \frac{|v(\tau)|^2}{2}\,dx + \frac{\alpha}{2}\int_Q |\nabla v|^2\,dx\,dt\\ &\le C\sum_{i=1}^n\int_Q\Big|d_i(u)\frac{\partial T_K(u)}{\partial x_i}\Big|^2\,dx\,dt + \int_\Omega \overline{T_K}(u(t=T))\,dx\end{aligned}\tag{5.5}$$

In view of (5.2), (5.3) and (5.5), we deduce that

$$\max\left\{\|v\|^2_{L^\infty(0,T;L^2(\Omega))} : \|\nabla v\|^2_{L^2(Q)}\right\} \le CKM. \tag{5.6}$$

Moreover, by (5.4) we obtain

$$\|v_t\|_{L^2(0,T;H^{-1}(\Omega))} \le C\Big(\|v\|_{L^2(0,T;H^1_0(\Omega))} + \|T_K(u)\|_{L^2(0,T;H^1_0(\Omega))}\Big). \tag{5.7}$$

Hence, by (5.6) and (5.7) we conclude that

$$\|v\|_W \le C\sqrt{K}. \tag{5.8}$$

Since $\mu \ge 0$ and $u_0 \ge 0$, it follows that

$$\frac{\partial ua}{\partial t} - \sum_{i=1}^N \frac{\partial}{\partial x_i}(d_i(u)\frac{\partial u}{\partial x_i}) \ge 0$$

and $u \ge 0$ in Q, and by a nonlinear version of Kato's inequality for parabolic equations (see [27]), we deduce that

$$\frac{\partial T_K(u)}{\partial t} - \sum_{i=1}^N \frac{\partial}{\partial x_i}(d_i(u)\frac{\partial T_K(u)}{\partial x_i}) \ge 0,$$

hence by (5.4), we obtain

$$-\frac{\partial v}{\partial t} - \sum_{i=1}^N \frac{\partial}{\partial x_i}(d_i(u)\frac{\partial v}{\partial x_i}) \ge -\frac{\partial T_K(u)}{\partial t} - \sum_{i=1}^N \frac{\partial}{\partial x_i}(d_i(u)\frac{\partial T_K(u)}{\partial x_i}) \quad \text{in } \mathcal{D}'(Q).$$

Now using the standard comparison argument, we easily see that $v \ge T_K(u)$ a.e. in Q, hence $v \ge K$ a.e. on $\{u > K\}$, and by (5.8) we conclude that

$$cap_2\{u > K\} \le \big\|\frac{v}{K}\big\|_W \le \frac{C}{\sqrt{K}},$$

the proof is complete. □

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