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EXISTENCE OF SOLUTIONS TO DIFFERENTIAL INCLUSIONS WITH PRIMAL LOWER NICE FUNCTIONS

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Abstract. We prove the existence of absolutely continuous solutions to the differential inclusion
$$\dot{x}(t) \in F(x(t)) + h(t, x(t)),$$
where F is an upper semi-continuous set-valued function with compact values such that $F(x(t)) \subset \partial f(x(t))$ on $[0, T]$, where f is a primal lower nice function, and h a single valued Carathéodory perturbation.

1. Introduction

In this article we study the first-order differential inclusion
$$\begin{gathered} \dot{x}(t) \in F(x(t)) + h(t, x(t)) \quad \text{a. e. } t \geq 0 \\ x(0) = x_0 \end{gathered} \tag{1.1}$$
where F is an upper semi-continuous set-valued function with nonconvex values and h is a Carathéodory function. It is well known that the problem admits a solution when F has convex values (see [2]). When the values of F fail to be convex many results have been established in the case when $F(x) \subset \partial f(x)$, for some proper convex lower semi-continuous function f, see for instance [5]. For some extensions of these results, we refer to [1, 6, 8, 10]. The case where the convexity of f is dropped has been studied in [3] by supposing f is (Clarke) regular. This class of function is of great importance in nonsmooth analysis and optimization as it generalizes several classes of functions such as convex proper lower semi-continuous functions and uniformly regular functions. A more general problem has been studied in the infinite dimensional setting in [12], in which the author proved that for locally Lipschitz functions, the class of convex functions, the class lower-C^2 functions and the class of uniformly regular functions are strictly contained within the class of regular functions. Another class of functions which is of great interest in variational analysis and optimization is the so called primal lower nice function (pln for short). This class of functions covers all convex functions, qualified convexly composite functions, and benefits from remarkable features such the strong connection of pln functions and their Moreau envelopes, in addition to the coincidence of their proximal and Clarke subdifferential. The notion of pln functions was introduced by

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Poliquin [9] where a subdifferential characterization of these functions was given in the finite dimensional setting. Analogously to the convex case, it was proved that pln functions are completely determined by their subgradients, more precisely, if two lower semi-continuous functions are pln at some point x of their domain and have the same proximal subgradient on a neighborhood of x, then the difference between these functions is constant near x. Some local regularity properties of the Moreau envelopes and the related proximal mappings of prox regular functions and pln functions in Hilbert space have been established, see [4] and [7]. In this paper, we aim at showing that existence of solution of (1.1) holds in the context of pln lower semi-continuous functions f. The paper is organized as follows: In section 2 we recall some preliminary facts that we need in the sequel and in section 3 we prove our existence result; first in $\mathbb{R}^n$ and then in infinite dimensional Hilbert space.

2. Preliminaries

Throughout this article, H stands for a real Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, $\overline{B}(x,r)$ is the closed ball centered at x with radius r.

Definition 2.1. Let X and Y be topological spaces and F a set-valued mapping defined on X with values in the space $\mathcal{P}(Y)$ of all nonempty subsets of Y. We will say that $F : X \to \mathcal{P}(Y)$ is upper semi-continuous (usc) at $\bar{x} \in X$ if for every neighborhood U of $F(\bar{x})$ there exists a neighborhood V of $\bar{x}$ such that $F(x) \subset U$ for all $x \in V$.

Definition 2.2. Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be an extended real valued lower semi-continuous function and let $\overline{x} \in \operatorname{dom} f$, that is $f(\overline{x}) < +\infty$. The proximal subdifferential of f at $\overline{x}$ is the set $\partial_p f(\overline{x})$ of all elements $v \in H$ for which there exist $r > 0$ and $\varepsilon > 0$ such that

$$\langle v, x-\overline{x}\rangle \le f(x) - f(\overline{x}) + \varepsilon\|x-\overline{x}\|^2 \quad \text{for all } x \in B(\overline{x}, r).$$

The Fréchet subdifferential $\partial_F f(\overline{x})$ of f at $\overline{x}$ is defined by $v \in \partial_F f(\overline{x})$ provided that for each $\varepsilon > 0$, there exists some $\eta > 0$ such that for all $x \in B(\overline{x}, \eta)$,

$$\langle v, x-\overline{x}\rangle \le f(x) - f(\overline{x}) + \varepsilon\|x-\overline{x}\|.$$

The Clarke subdifferential of a lower semi-continuous function f at $\overline{x}$ is the set

$$\partial_C f(\overline{x}) := \{v \in H : f^{\uparrow}(\overline{x}; y) \ge \langle v, y\rangle,\ \forall y \in H\}.$$

where $f^{\uparrow}(\overline{x}; y)$ is the generalized Rockafellar derivative given by

$$f^{\uparrow}(\overline{x}; y) = \limsup_{\substack{t\to 0^+,\\ x\to^f \overline{x}}} \inf_{y'\to y} t^{-1}[f(x+ty') - f(x)]$$

where $x \to^f \overline{x}$ means $x \to \overline{x}$ and $f(x) \to f(\overline{x})$. If f is locally Lipschitz, the generalized Rockafellar derivative $f^{\uparrow}(\overline{x}; y)$ coincides with the Clarke directional derivative $f^0(\overline{x}, y)$ defined by

$$f^0(\overline{x}, y) = \lim_{x\to\overline{x}} \sup_{t\downarrow 0} \frac{f(x+ty) - V(x)}{t}$$

If $x \notin \operatorname{dom} f$, $\partial_C f(x) := \emptyset$. When f is convex and lower semi-continuous, one has

$$\partial_p f = \partial_C f = \partial_F f = \partial f.$$

The operator ∂f denotes the subdifferential in the sense of convex analysis.

Now, let us recall the definition of Primal Lower Nice functions [7].

Definition 2.3. Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a proper function and consider $x_0 \in \operatorname{dom} f$. The function f is said to be primal lower nice (pln for short) at x_0, if there exist positive real numbers s_0, c_0, Q_0 such that for all $x \in \overline{B}(x_0, s_0)$, for all $q \geq Q_0$ and all $v \in \partial_p f(x)$ with $\|v\| \leq c_0 q$, one has

$$f(y) \geq f(x) + \langle v, y - x\rangle - \frac{q}{2}\|y - x\|^2 \quad \text{for each } y \in \overline{B}(x_0, s_0).$$

Remark 2.4. (1) Each extended real valued convex function is primal lower nice at each point of its domain.

(2) Clearly, if f is pln at u_0 with constants s_0, c_0, Q_0, one has

$$\langle v_1 - v_2, x_1 - x_2\rangle \geq -q\|x_1 - x_2\|^2$$

for any $v_i \in \partial_p f(x_i)$ with $\|v_i\| \leq c_0 q$ whenever $q \geq Q_0$ and $x_i \in \overline{B}(u_0, s_0)$, $i = 1, 2$.

(3) If f is pln at $u_0 \in \operatorname{dom} f$ then for all x in a neighborhood of u_0, the proximal subdifferential of f at x coincides with the Fréchet subdifferential and the Clarke subdifferential of f at x, i.e, $\partial_p f(x) = \partial_F f(x) = \partial_C f(x)$. In this case, we simply denote by $\partial f(x)$ the common subdifferential, and by $\partial^0 f(x)$ its element of minimal norm for $x \in \operatorname{dom} f$.

The graph of the (proximal) subdifferential of a pln function enjoys the useful closure property.

Proposition 2.5. *Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc function which is pln at $u_0 \in \operatorname{dom} f$ with constants s_0, c_0, $Q_0 > 0$, and let T_0, T, v_0, η_0 be positive real numbers such that $T > T_0$ and $v_0 + \eta_0 = s_0$. Let $v(.) \in L^2([T_0, T], H)$ and $u(\cdot)$ be a mapping from $[T_0, T]$ into H. Let $(u_n(\cdot))_n$ be a sequence of mappings from $[T_0, T]$ into H and $(v_n(\cdot))_n$ be a sequence in $L^2([T_0, T], H)$. Assume that:*

1. *$\{u_n(t), n \in \mathbb{N}\} \subset \overline{B}(u_0, \eta_0) \cap \operatorname{dom} f$ for almost every $t \in [T_0, T]$,*
2. *$(u_n)_n$ converges almost everywhere to some mapping u with $u(t) \in \operatorname{dom} f$ for almost every $t \in [T_0, T]$,*
3. *v_n converges to v with respect to the weak topology of $L^2([T_0, T], H)$,*
4. *for each $n \geq 1$, $v_n(t) \in \partial f(u_n(t))$ for almost every $t \in [T_0, T]$.*

Then, for almost all $t \in [T_0, T]$, $v(t) \in \partial f(u(t))$.

For the proof of the above proposition, we refer the reader to [7].

Proposition 2.6 ([11])**.** *Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous and pln function at $\overline{x} \in \operatorname{dom} f$ with constants ϵ, c, T and f is bounded from above on $B(\overline{x}, \epsilon)$. Assume also that ∂f is included in the Clarke subdifferential of f. Then the function f is Lipschitz continuous and DC (difference of convex functions) near $\overline{x}$. In fact, if $\epsilon > 0$ is such that f is also bounded from below on $B(\overline{x}, \epsilon)$, then for any $\alpha \in]0, 1[$ the function f is Lipschitz continuous and DC on $B(\overline{x}, \alpha\epsilon)$*

For more details about pln functions we refer to [4] and [9].

3. Main results

Let us recall first the existence result for the subdifferential operator of a pln function obtained in [7].

Theorem 3.1. *Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc function. Consider $T_0 \in [0, +\infty[$ and let $x_0 \in \operatorname{dom} f$ be such that f is pln at x_0 with constants s_0, c_0, Q_0. Then, there exist a real number $T \in]T_0, +\infty[$ and a unique absolutely continuous mapping $x : [T_0, T] \to B(x_0, s_0)$ which is a solution of the problem*

$$\begin{gathered} 0 \in \dot{x}(t) + \partial f(x(t)) \quad \text{for a.e } t \in [T_0, T], \\ x(T_0) = x_0. \end{gathered} \tag{3.1}$$

Further, in the particular case when $x_0 \in dom\ \partial f$, the solution $x(\cdot)$ above is actually Lipschitz continuous and it also satisfies:

1. *$f \circ x(\cdot)$ is Lipschitz continuous on $[T_0, T]$.*
2. *For almost every $t \in [T_0, T]$, the derivative $(f \circ x)'(t)$ exists and*
$$(f \circ x)'(t) = -\|\dot{x}(t)\|^2,$$
and for any $T_0 \le s \le t \le T$:
$$f(x(t)) - f(x(s)) = -\int_s^t \|\dot{x}(\tau)\|^2 d\tau.$$

Now we are able to establish existence result for the problem (1.1) in the context of finite dimensional space $\mathbb{R}^n$.

Theorem 3.2. *Under the following assumptions:*

- (H1) *$\Omega \subset \mathbb{R}^n$ is an open set and $F : \Omega \to \mathbb{R}^n$ is an upper semi-continuous compact valued multifunction.*
- (H2) *$f : \Omega \to \mathbb{R} \cup \{+\infty\}$ is a lower semi-continuous function, pln at $x_0 \in \operatorname{dom} \partial f$ with constants s_0, c_0, Q_0 such that*
$$F(x) \subset \partial f(x) \quad \forall x \in \Omega.$$
- (H3) *$h : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function, i. e. for every $x \in \mathbb{R}^n$, $h(\cdot, x)$ is measurable, $h(t, \cdot)$ is continuous, and there exists $m(\cdot) \in L^2(\mathbb{R}_+^*)$ such that*
$$\|h(t, x)\| \le (1 + \|x\|)m(t), \quad \forall x \in \mathbb{R}^n, \ \text{a.e. } t \in \mathbb{R}.$$

Then, there exist $T > 0$, and an absolutely continuous solution $x : [0, T] \to \mathbb{R}^n$ of the differential inclusion

$$\begin{gathered} \dot{x}(t) \in F(x(t)) + h(t, x(t)) \quad \text{a.e. } t \in [0, T], \\ x(0) = x_0. \end{gathered}$$

Proof. Since Ω is open, there exists $r > 0$ such that the compact set $K = \overline{B}(x_0, r)$ is contained in Ω. Moreover, by (H_1) and [2, Proposition 1.1.3], $F(K) = \underset{x \in K}{\cup} F(x)$ is compact, hence there exists M such that

$$\sup\{\|u\| : u \in F(x), x \in K\} \le M. \tag{3.2}$$

Since f is pln at x_0, there exist positive real numbers s_0, c_0, Q_0 such that for all $x \in \overline{B}(x_0, s_0)$, $q \ge Q_0$ and $u \in \partial_p f(x)$, with $\|u\| \le c_0 q$ one has

$$f(y) \ge f(x) + \langle u, y - x \rangle - \frac{q}{2}\|y - x\|^2$$

for each $y \in \overline{B}(x_0, s_0)$. Let us choose $T' > 0$ such that

$$\int_0^{T'} (M + \alpha m(t)) dt < \frac{r}{2},$$

where $\alpha = 1+\|x_0\|+\frac{r}{2}$. Set $r_0 = \min(\frac{r}{2}, s_0)$, take $T = \min(\frac{r_0}{M}, T')$ and let $I = [0, T]$. For each integer $n \geq 1$ and for $0 \leq i \leq n-1$, we set $t_n^i = \frac{iT}{n}$; $I_n^i = [t_n^i, t_n^{i+1}[$, and for every $t \in I_n^i$, we define

$$x_n(t) = x_n^i + (t - t_n^i)u_n^i + \int_{i\frac{T}{n}}^t h(s, x_n^i)ds, \tag{3.3}$$

where $x_n(0) = x_n^0 = x_0$ and

$$x_n(t_n^i) = x_n^i = x_n^{i-1} + \frac{T}{n}u_n^{i-1} \quad \text{for every } i \in \{1, 2, \dots, n\}, \tag{3.4}$$

$$u_n^i \in F(x_n^i), \quad \text{for every } i \in \{0, 1, 2, \dots, n\}. \tag{3.5}$$

(x_n) is well defined on $[0, T]$. Clearly one has for every $i \in \{1, 2, \dots, n\}$,

$$\begin{aligned}
x_n^i - x_n^0 &= \frac{T}{n}(u_n^0 + u_n^1 + \dots + u_n^{i-1}) \\
&\leq \frac{T}{n}(\|u_n^0\| + \|u_n^1\| + \|u_n^2\| + \dots + \|u_n^{i-1}\| \\
&\leq \frac{iTM}{n} \leq \frac{r}{2},
\end{aligned}$$

proving that

$$x_n(t_n^i) = x_n^i \in \overline{B}(x_0, \frac{r}{2}). \tag{3.6}$$

By (3.2) and (3.3) for all $t \in [t_n^i, t_n^{i+1}[$, we obtain

$$\begin{aligned}
\|x_n(t) - x_n(t_n^i)\| &\leq \int_{t_n^i}^t (M + (1 + \|x_0\| + \frac{r}{2})m(\tau))d\tau \\
&\leq \int_{t_n^i}^t (M + \alpha m(\tau))d\tau < \frac{r}{2}.
\end{aligned} \tag{3.7}$$

From (3.6) and (3.7) one can deduce that

$$\begin{aligned}
\|x_n(t) - x_0\| &\leq \|x_n(t) - x_n(t_n^i)\| + \|x_n(t_n^i) - x_0\| \\
&\leq \frac{r}{2} + \frac{r}{2} = r,
\end{aligned}$$

and so

$$x_n(t) \in \overline{B}(x_0, r), \quad \text{for each } t \in [0, T]. \tag{3.8}$$

By (3.3) we have

$$\dot{x}_n(t) = u_n^i + h(t, x_n(t)) \quad \forall t \in]t_n^i, t_n^{i+1}[, \tag{3.9}$$

the last equality with (3.2) ensures that

$$\|\dot{x}_n(t)\| \leq M + \alpha m(t) \quad \text{for a.e } t \in [0, T], \tag{3.10}$$

then

$$\int_0^T \|\dot{x}_n(t)\|^2 dt \leq \int_0^T (M + \alpha m(t))^2 dt,$$

and so the sequence $(\dot{x}_n)_n$ is bounded in $L^2([0, T], \mathbb{R}^n)$. Further, for all t, $s \in [0, T]$, $0 \leq s < t \leq T$, one has

$$\|x_n(t) - x_n(s)\| = \|\int_s^t \dot{x}_n(\tau)d\tau\| \leq \int_s^t (M + \alpha m(\tau))d\tau, \tag{3.11}$$

hence by [2, Theorem 0.3.4], there exist a subsequence, still denoted by $(x_n)_n$ and an absolute continuous function, $x : [0,T] \to \mathbb{R}^n$, such that $(x_n)_n$ converges uniformly on $C_{\mathbb{R}^n}([0,T])$ to x and $(\dot{x}_n)_n$ converges weakly in $L^2([0,T];\mathbb{R}^n)$ to $\dot{x}$. Let us define step functions from $[0,T]$ to $[0,T]$ by

$$\theta_n(t) = t_n^i \quad \text{for all } t \in [t_n^i, t_n^{i+1}[,\ \theta_n(T) = T,$$

then by (3.3), (3.5), and (3.9),

$$\dot{x}_n(t) - h(t, x_n(t)) \in F(x_n(\theta_n(t)) \subset \partial f(x_n(\theta_n(t)), \tag{3.12}$$

and by (3.6), $x_n(\theta_n(t)) \in \overline{B}(x_0, r)$, for any $t \in [0,T]$. We have $|\theta_n(t) - t| \le \frac{T}{n}$ for each $t \in [0,T]$, then $\theta_n(t) \to t$ uniformly on $[0,T]$, further, by the uniform convergence of (x_n) and (θ_n), we conclude that $x_n(\theta_n(t)) \to x(t)$. By (H_2) and Proposition 2.5, we obtain

$$\dot{x}(t) - h(t, x(t)) \in \partial f(x). \tag{3.13}$$

By Theorem 3.1, the maps $t \to x(t)$ and $t \to f(x(t))$ are Lipschitzian, so by [3, Proposition 3.4]

$$\frac{d}{dt} f(\dot{x}(t)) = \langle \dot{x}(t), \dot{x}(t) - h(t, x(t)) \rangle$$

for a.e $t \in [0,T]$, hence

$$f(x(T)) - f(x(0)) = \int_0^T \|\dot{x}(\tau)\|^2 d\tau - \int_0^T \langle \dot{x}(\tau), h(\tau, x(\tau)) \rangle d\tau. \tag{3.14}$$

On the other hand, since

$$\dot{x}_n(t) - h(t, x_n(t_n^i) \in F(x_n(t_n^i)) \subset \partial f(x_n(t_n^i)),\ \forall t \in]t_n^i, t_n^{i+1}[,$$

and using the fact that f is pln on x_0, we obtain for all $q \ge \max(Q_0, \frac{M}{c_0})$

$$\begin{aligned}
& f(x_n(t_n^{i+1}) - f(x_n(t_n^i)) \\
& \ge \langle \dot{x}_n(t) - h(t, x_n(t_n^i)), x_n(t_n^{i+1}) - (x_n(t_n^i) \rangle - \frac{q}{2} \|x_n(t_n^{i+1}) - x_n(t_n^i)\|^2 \\
& \ge \langle \dot{x}_n(t) - h(t, x_n(t_n^i)), \int_{t_n^i}^{t_n^{i+1}} \dot{x}_n(t) dt \rangle - \frac{q}{2} \|x_n(t_n^{i+1}) - x_n(t_n^i)\|^2 \\
& \ge \int_{t_n^i}^{t_n^{i+1}} \|\dot{x}(t)\|^2 dt - \int_{t_n^i}^{t_n^{i+1}} \langle h(t, x_n(t_n^i)), \dot{x}_n(t) \rangle dt - \frac{q}{2} \|x_n(t_n^{i+1}) - x_n(t_n^i)\|^2 \\
& \ge \int_{t_n^i}^{t_n^{i+1}} \|\dot{x}(t)\|^2 dt - \int_{t_n^i}^{t_n^{i+1}} \langle h(t, x_n(t_n^i)), \dot{x}_n(t) \rangle dt - \frac{q}{2} (\frac{T^2}{n^2} \|u_n^i\|^2) \\
& \ge \int_{t_n^i}^{t_n^{i+1}} \|\dot{x}(t)\|^2 dt - \int_{t_n^i}^{t_n^{i+1}} \langle h(t, x_n(t_n^i)), \dot{x}_n(t) \rangle dt - \frac{q}{2} (\frac{T^2}{n^2} M^2).
\end{aligned}$$

By adding we obtain

$$f(x_n(T)) - f(x_0) \ge \int_0^T \|\dot{x}_n(t)\|^2 dt - \int_0^T \langle h(t, x_n(t_n^i)), \dot{x}_n(t) \rangle dt - \frac{qT^2M^2}{2n^2}. \tag{3.15}$$

As $\frac{qT^2M^2}{2n^2} \to 0$, the convergence of $(x_n(\cdot))$ in $L^2([0,T],\mathbb{R}^n)$-norm and of $(\dot{x}_n)$ in the weak topology of $L^2([0,T],\mathbb{R}^n)$ implies that

$$\lim_{n \to +\infty} \int_0^T \langle h(t, x_n(t_n^i)), \dot{x}_n(t) \rangle dt = \int_0^T \langle h(t, x(t)), \dot{x}(t) \rangle dt.$$

Taking the limit superior in (3.15) and using the continuity of f, we obtain

$$f(x(T)) - f(x_0) \geq \limsup_{n\to+\infty} \int_0^T \|\dot{x}_n(t)\|^2 dt - \int_0^T \langle h(t,x(t)), \dot{x}(t)\rangle dt, \tag{3.16}$$

By (3.14) we obtain

$$\|\dot{x}(t)\|^2_{L^2} \geq \limsup_{n\to+\infty} \|\dot{x}_n(t)\|^2_{L^2}$$

By the weak lower semi-continuity of the norm, one has

$$\|\dot{x}(t)\|^2_{L^2} \leq \inf_{n\to+\infty} \|\dot{x}_n(t)\|^2_{L^2}.$$

So that

$$\|\dot{x}(t)\|^2_{L^2} = \lim_{n\to+\infty} \|\dot{x}_n(t)\|^2_{L^2}.$$

By (H1), the graph of F is closed and since $(x_n(t), \dot{x}_n(t) - h_n(t,x_n(t)))$ converges to $(x(t), \dot{x}(t) - h(t,x(t)))$ on a complement of a null set, we conclude from (3.12) that

$$\dot{x}(t) - h(t,x(t)) \in F(x(t)), \text{ a.e on } [0,T].$$

□

Next, we prove an existence result for the problem (1.1) in the case of infinite dimensional Hilbert space. For this purpose we use the same techniques given by Yarou [12].

Theorem 3.3. *Under the following assumptions:*

- (H1) *$\Omega \subset H$ is an open set in H and $F : \Omega \rightharpoonup H$ is an upper semi-continuous compact valued multifunction.*
- (H2') *$f : \Omega \to \mathbb{R}\cup\{+\infty\}$ is a lower semi-continuous function, pln at $x_0 \in \operatorname{dom}\partial f$ with constants s_0, c_0, Q_0 such that*
  $$F(x) \subset \partial f(x) \quad \forall x \in \Omega, \quad F(x) \subset (1 + \|x\|)K \quad (K \text{ a compact set}).$$
  and f is bounded from above on $\overline{B}(x_0, s_0)$,
- (H3') *$h : \mathbb{R}^+ \times H \to H$ is measurable in t and Lipschitz in x, and for any bounded subset B of H, there is a compact set K_1 such that $h(t,x) \in K_1$ for all $(t,x) \in \mathbb{R}^+ \times B$,*

Then there exist $T > 0$ and an absolutely continuous function $x : [0,T] \to H$ solution of the differential inclusion (1.1).

Proof. Since Ω is open, there exists $r > 0$ such that $\overline{B}(x_0, r)$ is contained in Ω. By the definition of f, for all $x \in \overline{B}(x_0, s_0)$, for all $q \geq Q_0$ and all $u \in \partial_p f(x)$, with $\|u\| \leq c_0 q$, one has

$$f(y) \geq f(x) + \langle u, y - x\rangle - \frac{q}{2}\|y - x\|^2.$$

Since f is lower semi-continuous on x_0, taking s_0 smaller if necessary, we may suppose that f is bounded from bellow on $\overline{B}(x_0, s_0)$. Let us fix $\beta \in]0,1[$, Proposition 2.6 implies that there is $L > 0$ such that $\partial_p f(x) \subset L\overline{B}$, whenever $x \in \overline{B}(x_0, \beta s_0)$. By our assumption (H3') there is a positive constant m such that $h(t,x) \in K_1 \subset mB$ for all $(t,x) \in \mathbb{R}^+ \times \overline{B}(x_0, r)$. Moreover, by (H2'), there exists a positive constant m_1 such that for any $x \in \overline{B}(x_0, r)$, $F(x) \subset (1 + \|x_0\| + r)K \subset m_1 B$. Choose T such that

$$0 < T < \frac{r_0}{(m_1 + m)}, \tag{3.17}$$

where $r_0 = \min(\frac{r}{2}, \beta s_0)$. Let $I = [0, T]$, for each integer $n \geq 1$ and for $0 \leq i \leq n-1$, we set $t_n^i = \frac{iT}{n}$; $I_n^i = [t_n^i, t_n^{i+1}[$, and consider the same discretization (3.3), (3.4) and (3.5). We obtain (3.6), (3.7) and (3.8). By (3.3) we have for every $t \in [0,T]\backslash\{t_n^i\}$,

$$\dot{x}_n(t) = u_n^i + h(t, x_n(t)). \tag{3.18}$$

The above equality with (3.17) ensures that

$$\|\dot{x}_n(t)\| \leq (m_1 + m) \quad \text{for a.e } t \in [0, T], \tag{3.19}$$

then the sequence $(\dot{x}_n)_n$ is bounded in $L^1([0,T], H)$. Then $(\dot{x}_n)_n$ converges weakly in $L^1([0,T]; H)$ to $\dot{x}$.

Following the same steps of proof of Theorem 3.2, we obtain: for all t, $s \in [0,T]$, $0 \leq s < t \leq T$,

$$\begin{aligned} \|x_n(t) - x_n(s)\| &= \|x_0 + \int_0^t \dot{x}_n(\tau)d\tau - x_0 - \int_0^s \dot{x}_n(\tau)d\tau\| \\ &= \| \int_s^t \dot{x}_n(\tau)d\tau\| \\ &\leq \int_s^t \|\dot{x}_n(\tau)\|d\tau \leq \int_s^t (m_1 + m)d\tau \\ &\leq (m_1 + m)|t - s|. \end{aligned}$$

So that the sequence $(x_n)_n$ is an equi-Lipschitz subset of $C_H([0,T])$, and the set $\{x_n(t) : n \in \mathbb{N}^*\}$ is relatively compact in H for every $t \in [0,T]$ since $x_n(t) \in x_0 + (K_1 + (1 + \|x_0\| + r)K)[0,T] := K_2$, hence by Ascoli Theorem, there exists a subsequence, still denoted by $(x_n)_n$ and an absolute continuous function, $x : [0,T] \to H$, such that $(x_n)_n$ converges uniformly on $C_H([0,T])$ to x. Further, the sequence $(u_n)_{n\in\mathbb{N}}$, is relatively $\sigma(L^1([0,T],H); (L^\infty([0,T],H))$-compact since we have almost everywhere

$$u_n(t) \in (1 + \|x_0\| + r)K. \quad \forall n \in \mathbb{N}^*. \tag{3.20}$$

Therefore, by extracting subsequences if necessary, we can assume that there exists $u \in L^1([0,T], H)$ such that $u_n \to u$ for $\sigma(L^1([0,T],H); (L^\infty([0,T],H))$-topology. Also, we have $h(\cdot, x_n\,(\theta_n(\cdot)) \to h(\cdot, x\,(\cdot))$ in the norm of the space $L^1([0,T],H)$. Consequently, one has for all $t \in [0,T]$

$$x(t) = x_0 + \lim_{n\to\infty} \int_0^t [h(s, x_n(\theta_n(s))) + u_n(s)]ds = x_0 + \int_0^t [h(s, x(s)) + u(s)]ds,$$

which gives the equality

$$\dot{x}(t) = h(s, x(s)) + u(s) \quad \text{for almost every } t \in [0,T].$$

Since u_n converges weakly to u, $\dot{x}_n$ converges weakly to $\dot{x}$. By construction, we have for a.e $t \in [0,T]$

$$\dot{x}_n(t) - h(s, x_n(\theta_n(t))) = u_n(t) \in F(x_n(\theta_n(t))),$$

and by (H2'),

$$u_n(t) \in F(x_n(\theta_n(t))) \subset (1 + \|x_n(\theta_n(t))\|)K \subset (1 + \|x_0\| + r)K.$$

Then u_n is in the fixed compact set $(1 + \|x_0\| + r)K$, consequently it converges strongly to u which gives the strong convergence of $\dot{x}_n$. Since the graph of F is

closed, we get
$$\dot{x}(t) \in F(x(t)) + h(t, x(t)) \quad \text{a.e. on } [0, T].$$
Therefore, the differential inclusion (1.1) admits a solution. $\square$

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