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CAUCHY PROBLEM FOR THE SIXTH-ORDER DAMPED MULTIDIMENSIONAL BOUSSINESQ EQUATION

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ABSTRACT. In this article, we consider the Cauchy problem for sixth-order damped Boussinesq equation in $\mathbb{R}^n$. The well-posedness of global solutions and blow-up of solutions are obtained. The asymptotic behavior of the solution is established by the multiplier method.

1. INTRODUCTION

It is well-known that the generalized Boussinesq equation, in $\mathbb{R}$,

$$u_{tt} + u_{xxxx} - u_{xx} = (f(u))_{xx}, \tag{1.1}$$

is a very important and famous nonlinear evolution equation suggested for describing the motion of water with small amplitude and long wave. There have been many results on the local and global well-posedness of problem (1.1) in [9, 10, 11, 13]. In [1], the authors studied a damped Boussinesq equation

$$u_{tt} - ku_{txx} - u_{xx} - u_{xxtt} = (f(u))_{xx}. \tag{1.2}$$

Wang and Chen [22] considered the Cauchy problem for the generalized double dispersion equation

$$u_{tt} - ku_{txx} + u_{xxxx} - u_{xx} - u_{xxtt} = (f(u))_{xx}, \tag{1.3}$$

whose well-posedness of the local and global solutions and the blow-up of the solutions were established in $\mathbb{R}$. Polat [16, 17] generalized the results obtained in [22] and proved the existence of local and global, blow-up, and asymptotic behavior of solutions for the Cauchy problem of (1.3) in $\mathbb{R}^n$.

Schneider and Eugene [18] considered another class of Boussinesq equation which characterizes the water wave problem with surface tension as follows

$$u_{tt} = u_{xx} + u_{xxtt} + \mu u_{xxxx} - u_{xxxxtt} + (u^2)_{xx}, \tag{1.4}$$

which can also be formally derived from the 2D water wave problem. For a degenerate case, they proved that the long wave limit can be described approximately by two decoupled Kawahara-equations. Wang and Mu [24, 25] studied the well-posedness of the local and global solutions, the blow-up of solutions and nonlinear scattering for small amplitude solutions to the Cauchy problem of (1.4). Piskin

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and Polat [15] considered the Cauchy problem of the multidimensional Boussinesq equation

$$u_{tt} = \Delta u + \Delta u_{tt} + \mu\Delta^2 u - \Delta^2 u_{tt} + \Delta f(u) + k\Delta u_t. \tag{1.5}$$

The existence, both locally and globally in time, the global nonexistence, and the asymptotic behavior of solutions for the Cauchy problem of equation (1.5) are established in n-dimensional space.

Wang and Esfahani [20, 21] considered the Cauchy problem associated with the sixth-order Boussinesq equation with cubic nonlinearity

$$u_{tt} = u_{xx} + \beta u_{xxxx} + u_{xxxxxx} + (u^2)_{xx}, \tag{1.6}$$

where $\beta = \pm 1$, Equation (1.6) arises as mathematical models for describing the bi-directional propagation of small amplitude and long capillary-gravity waves on the surface of shallow water for bond number (surface tension parameter) less than but very close to $\frac{1}{3}$ [2]. Equation (1.6) has been also used as the model of nonlinear lattice dynamics in elastic crystals [14]. In this article, we investigate the Cauchy problem of the sixth-order damped multidimensional Boussinesq equation

$$u_{tt} - \Delta u_{tt} - \Delta u + \Delta^2 u - \Delta^3 u - r\Delta u_t = \Delta f(u), \quad (x,t) \in \mathbb{R}^n \times (0,+\infty), \tag{1.7}$$

$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \quad x \in \mathbb{R}^n, \tag{1.8}$$

where $u(x,t)$ denotes the unknown function, $f(s)$ is the given nonlinear function, r is a constant, the subscript t indicates the partial derivation with respect to t, and Δ denotes the Laplace operator in $\mathbb{R}^n$.

Recently, the authors [27] proved the existence and asymptotic behavior of global solutions of (1.7) for all space dimensions $n \geq 1$ provided that the initial value is suitably small. In [26], the authors obtained the global existence and asymptotic decay of solutions to the problem (1.7). For the initial boundary value problem of (1.7) with $f(u) = u^2$, Zhang [28] and Lai [5, 6] established the well-posedness of strong solution and constructed the solution in the form of series in the small parameter present in the initial conditions. The long-time asymptotics was also obtained in the explicit form.

The main purpose of this paper is to study the well-posedness of the global solution and the asymptotic behavior of the global solution for the Cauchy problem (1.7)-(1.8) in $\mathbb{R}^n$. Due to the sixth-order term Δ^3, it seems difficult to construct the operator $\partial_t^2 - \Delta$ which is similar to that in [22, 16] to solve the problem (1.7)-(1.8). To overcome this difficulty, we transformed (1.7) in another way and established the corresponding estimate.

Throughout this article, we use L_p to denote the space of L^p-function on $\mathbb{R}^n$ with the norm $\|f\|_p = \|f\|_{L^p}$. H^s denotes the Sobolev space on $\mathbb{R}^n$ with norm $\|f\|_{H^s} = \|(I-\Delta)^{s/2} f\|_2$, where $1 \leq p \leq \infty, s \in \mathbb{R}$.

To prove the global well-posedness, we use the contraction mapping principle to the local-posedness of the problem (1.7)-(1.8).

Theorem 1.1. *Assume that* $s > \frac{n}{2}, \phi \in H^s, \psi \in H^{s-2}$ *and* $f(s) \in C^{[s]+1}(R)$*, then problem* (1.7)-(1.8) *admits a unique local solution* $u(x,t)$ *defined on a maximal time interval* $[0,T_0)$ *with* $u(x,t) \in C([0,T_0), H^s) \cap C^1([0,T_0), H^{s-2})$*. Moreover, if*

$$\sup_{t\in[0,T_0)} (\|u(t)\|_{H^s} + \|u_t(t)\|_{H^{s-2}}) < \infty, \tag{1.9}$$

then $T_0 = \infty$.

Now we arrive at the existence and uniqueness of global solutions for (1.7)-(1.8).

Theorem 1.2. *Assume that* $1 \le n \le 4$, $s \ge \frac{n+1}{2}$, $f(u) \in C^{[s]+1}(R)$, $F(u) = \int_0^u f(s)ds$ *or* $f'(u)$ *is bounded below, i.e. there is a constant* A_0 *such that* $f'(u) \ge A_0$ *for any* $u \in \mathbb{R}$, $|f'(u)| \le A|u|^\rho + B$, $0 < \rho \le \infty$ *for* $2 \le n \le 4, (-\Delta)^{-1/2}\psi \in L^2, \phi \in H^{s+1}$ *and* $\psi \in H^{s-1}, F(\phi) \in L^1$. *Then problem* (1.7)-(1.8) *admits a global solution* $u(x,t) \in C([0,\infty), H^s) \cap C^1([0,\infty), H^{s-2})$ *and* $(-\Delta)^{-1/2}u_t \in L^2$.

In Lemma 3.1 below we have the energy equality $E(t) = \|(-\Delta)^{-1/2}\psi\|_2^2 + \|\psi\|_2^2 + \|\phi\|_2^2 + \|\nabla\phi\|_2^2 + \|\Delta\phi\|_2^2 + 2\int_{\mathbb{R}^n} F(u)dx$. Then we can obtain the blow-up results by the concavity method.

Theorem 1.3. *Assume that* $r \ge 0, f(u) \in C(R)$, $\phi \in H^2, \psi \in L^2, (-\Delta)^{-1/2}\phi$, $(-\Delta)^{-1/2}\psi \in L^2, F(u) = \int_0^u f(s)ds$, $F(\phi) \in L^1$, *and there exists a constant* $\alpha > 0$ *such that*

$$f(u)u \le (\alpha + r + 2)F(u) + \frac{\alpha}{2}u^2, \quad \forall u \in \mathbb{R}. \tag{1.10}$$

Then the solution $u(x,t)$ *of* (1.7)-(1.8) *will blow up in finite time if one of the following conditions hold:*

(i) $E(0) = \|(-\Delta)^{-1/2}\psi\|_2^2 + \|\psi\|_2^2 + \|\phi\|_2^2 + \|\nabla\phi\|_2^2 + \|\Delta\phi\|_2^2 + 2\int_{\mathbb{R}^n} F(\phi)dx < 0$,
(ii) $E(0) = 0$ *and* $\big((-\Delta)^{-1/2}\phi, (-\Delta)^{-1/2}\psi\big) + (\phi,\psi) > 0$,
(iii) $E(0) > 0$ *and*

$$\big((-\Delta)^{-1/2}\phi, (-\Delta)^{-1/2}\psi\big) + (\phi,\psi) > \sqrt{2\frac{4+2r+2\alpha}{\alpha+2}E(0)(\|(-\Delta)^{-1/2}\phi\|_2^2 + \|\phi\|_2^2)}.$$

Theorem 1.4. *Let* $r > 0$ *and assume that*

$$0 \le F(u) \le f(u)u, \quad \forall u \in \mathbb{R}, \quad F(u) = \int_0^u f(s)ds.$$

Then for the global solution of problem (1.7)-(1.8), *there exist positive constants* C *and* θ *such that*

$$E(t) \le CE(0)e^{-\theta t}, \quad 0 \le t \le \infty, \tag{1.11}$$

where

$$E(t) = \frac{1}{2}(\|(-\Delta)^{-1/2}u_t\|_2^2 + \|u_t\|_2^2 + \|u\|_2^2 + \|\nabla u\|_2^2 + \|\Delta u\|_2^2) + \int_{\mathbb{R}^n} F(u)dx.$$

The article is organized as follows. In the next section, we prove Theorem 1.1 which is related to the local well-posedness for a general nonlinearity. In Section 3, we prove Theorem 1.2. The proof of the nonexistence of a global solution is given in Section 4. In the last section, the asymptotic behavior of the global solution is discussed.

2. Existence and uniqueness of the local solution

In this section, we prove the existence and the uniqueness of the local solution for (1.7)-(1.8) by contraction mapping principle. To do so, we construct the solution of the problem as a fixed point of the solution operator associated with related family of Cauchy problem for linear equation. For this purpose, we rewrite (1.7) as follows:

$$u_{tt} + \Delta^2 u = \Gamma[f(u) + ru_t + u]. \tag{2.1}$$

where $\Gamma = (I-\Delta)^{-1}\Delta$. Using the Fourier transform, it is easy to obtain

$$\Gamma f = \Delta(G * f) = G * f - f,$$

where $G(x) = \frac{1}{2}e^{-|x|}$, and $u * v$ denotes the convolution of u and v.

We start with the linear equation.

$$u_{tt} + \Delta^2 u = q(x,t), \quad x \in \mathbb{R}^n,\ t > 0, \tag{2.2}$$

with the initial value condition (1.8). To prove Theorem 1.1, we need the following lemmas.

Lemma 2.1 ([19]). *If $s > k + n/2$, where k is a nonnegative integer, then*

$$H^s(\mathbb{R}^n) \subset C^k(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n),$$

where the inclusion is continuous. In fact,

$$\sum_{|\alpha| \le k} \|\partial^\alpha u\|_{L^\infty} \le C_s \|u\|_{H^s},$$

where C_s is independent of u.

Lemma 2.2 ([3]). *Let $q \in [1,n]$ and $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$, then for any $u \in H_1^q(\mathbb{R}^n)$,*

$$\|u\|_p \le C(n,q)\|\nabla u\|_q,$$

where $C(n,q)$ is a constant dependent on n and q.

Lemma 2.3 ([23]). *Assume that $f(u) \in C^k(R), f(0) = 0, u \in H^s \cap L^\infty$ and $k = [s]+1$, where $s \ge 0$. Then*

$$\|f(u)\|_{H^s} \le K_1(W)\|u\|_{H^s},$$

if $\|u\|_\infty \le W$, where $K_1(W)$ is a constant dependent on W.

Lemma 2.4 ([23]). *Assume that $f(u) \in C^k(R), u, v \in H^s \cap L^\infty$ and $k = [s]+1$, where $s \ge 0$. Then*

$$\|f(u) - f(v)\|_{H^s} \le K_2(W)\|u - v\|_{H^s},$$

if $\|u\|_\infty \le W, \|v\|_\infty \le W$, where $K_2(W)$ is a constant dependent on W.

Lemma 2.5 ([4]). *If $1 \le p \le \infty, u(x,t) \in L^p(\mathbb{R}^n)$ for a.e. t and the function $t \mapsto \|u(\cdot,t)\|_p$ is in $L^1(I)$, where $I \subset [0,\infty)$ is an interval, then*

$$\| \int_I u(\cdot,t)\|_p \le \int_I \|u(\cdot,t)\|_p dt.$$

Lemma 2.6. *Let $s \in \mathbb{R}, \phi \in H^s, \psi \in H^{s-2}$ and $q \in L^1([0,T]; H^{s-2})$. Then for every $T > 0$, there is a unique solution $u \in C([0,T], H^s) \cap C^1([0,T], H^{s-2})$ of Cauchy problem (2.2) and (1.8). Moreover, u satisfies*

$$\|u(t)\|_{H^s} + \|u_t(t)\|_{H^{s-2}} \le C(1+T)(\|\phi\|_{H^s} + \|\psi\|_{H^{s-2}} + \int_0^t \|q(\tau)\|_{H^{s-2}} d\tau), \tag{2.3}$$

for $0 \le t \le T$, where C dependeds only on s.

Proof. The argument about the existence and uniqueness of the solution of the Cauchy problem for the linear problem (2.2) and (1.8) is similar to that in [19], we omit it. The solution of the linear equation is given in Fourier space by

$$\hat{u}(\xi,t)=\cos(t|\xi|^2)\hat{\phi}(\xi)+\frac{\sin(t|\xi|^2)}{|\xi|^2}|\hat{\psi}|^2+\int_0^t\frac{\sin((t-\tau)|\xi|^2)}{|\xi|^2}\hat{q}(\xi,\tau)d\tau,$$

where $\hat{}$ denotes Fourier transform with respect to x. Since

$$\|(1+|\xi|^2)^{s/2}\cos(t|\xi|^2)\hat{\phi}(\xi)\|\le\|(1+|\xi|^2)^{s/2}\hat{\phi}(\xi)\|=\|\phi\|_{H^s}$$

and

$$\begin{aligned}
&\|(1+|\xi|^2)^{s/2}\frac{\sin(t|\xi|^2)}{|\xi|^2}\hat{\psi}(\xi)\|^2\\
&=\int_{|\xi|<1}(1+|\xi|^2)^s\frac{\sin^2(t|\xi|^2)}{|\xi|^4}|\hat{\psi}(\xi)|^2d\xi+\int_{|\xi|\ge1}(1+|\xi|^2)^s\frac{\sin^2(t|\xi|^2)}{|\xi|^4}|\hat{\psi}(\xi)|^2d\xi\\
&\le t^2\int_{|\xi|<1}(1+|\xi|^2)^s|\hat{\psi}(\xi)|^2d\xi+\int_{|\xi|\ge1}(1+|\xi|^2)^s\frac{1}{|\xi|^4}|\hat{\psi}(\xi)|^2d\xi\\
&\le 4t^2\int_{|\xi|<1}(1+|\xi|^2)^{s-2}|\hat{\psi}(\xi)|^2d\xi+4\int_{|\xi|\ge1}(1+|\xi|^2)^s\frac{1}{|\xi|^4}|\hat{\psi}(\xi)|^2d\xi\\
&\le 4(1+t^2)\int_{\mathbb{R}^n}(1+|\xi|^2)^{s-2}|\hat{\psi}(\xi)|^2d\xi\\
&=4(1+t^2)\|\psi\|^2_{H^{s-2}},
\end{aligned}$$

we obtain

$$\|u(t)\|_{H^s}\le\|\phi\|_{H^s}+2(1+t)\|\psi\|_{H^{s-2}}+2(1+t)\int_0^t\|q(\tau)\|_{H^{s-2}}d\tau,$$

$$\|u_t(t)\|_{H^{s-2}}\le\|\phi\|_{H^s}+\|\psi\|_{H^{s-2}}+\int_0^t\|q(\tau)\|_{H^{s-2}}d\tau.$$

Therefore (2.3) holds. This completes the proof. □

Lemma 2.7. *The operator L is bounded on H^s for all $s\ge0$ and*

$$\|\Gamma u\|_{H^s}\le C\|u\|_{H^s},\forall u\in H^s.$$

Proof. For $u\in H^s, s\ge0$, we have

$$\|\Gamma u\|^2_{H^s}=\int_{\mathbb{R}^n}(1+|\xi|^2)^s\frac{|\xi|^4}{(1+|\xi|^2)^2}|u(\hat{|\xi|})|^2d\xi\le C\|u\|^2_{H^s}.$$

□

Proof of Theorem 1.1. We will prove the theorem in four steps.

Step 1. Define the function space

$$X(T)=C([0,T],H^s)\cap C^1([0,T],H^{s-2}),$$

which is equipped with the norm

$$\|u\|_{X(T)}=\max_{0\le t\le T}(\|u\|_{H^s}+\|u_t\|_{H^{s-2}}),\quad\forall u\in X(T).$$

It is easy to see that $X(T)$ is a Banach space. For $s>n/2$ and any initial values $\phi\in H^s,\psi\in H^{s-2}$, let $M=\|\phi\|_{H^s}+\|\psi\|_{H^{s-2}}$. Take the set

$$Y(M,T)=\{u\in X(T):\|u\|_{X(T)}\le 2CM\}.$$

Note that $Y(M,T)$ is a nonempty bounded closed convex subset of $X(T)$ for any fixed $M>0$ and $T>0$.

From Lemma 2.1, $u\in C([0,T],L^\infty)$ and $\|u\|_{L^\infty}\le C_s\|u\|_{H^s}$, if $u\in X(T)$. For $v\in Y(M,T)$, we consider the linear equation

$$u_{tt}+\Delta^2 u=\Gamma[f(v)+rv_t+v] \tag{2.4}$$

and we let S denote the map which carried v into the unique solution of (2.4) and (1.8). Our goal is to show that S has a unique fixed point in $Y(M,T)$ for appropriately chosen T. To this end, we shall employ the contraction mapping principle and Lemma 2.6.

Step 2. We shall prove that S maps $Y(M,T)$ into itself for T small enough. Let $v\in Y(M,T)$ be given. Define $q(x,t)$ by

$$q(x,t)=\Gamma[f(v)+rv_t+v].$$

Using lemmas 2.3 and 2.7, it follows easily that

$$\|q(t)\|_{H^{s-2}}\le C\|f(v)\|_{H^{s-2}}+|r|\|v_t\|_{H^{s-2}}+\|v\|_{H^{s-2}}\le C_M\|v\|_{H^s}+|r|\|v_t\|_{H^{s-2}},$$

where C_M is a constant dependent on M and s. From the above inequality we conclude that $q(x,t)\in C^1([0,T],H^{s-2})$. From Lemma 2.6, the solution $u=Sv$ of problem (2.2) and (1.8) belongs to $C([0,T],H^s)\cap C^1([0,T],H^{s-2})$ and

$$\begin{aligned}\|u(t)\|_{H^s}+\|u_t(t)\|_{H^{s-2}}&\le C(1+T)(\|\phi\|_{H^s}+\|\psi\|_{H^{s-2}}+\int_0^t\|q(\tau)\|_{H^{s-2}}d\tau)\\&\le CM+C[1+2C((C_M)+|r|)(1+T)]MT.\end{aligned}$$

By choosing T small enough, we have

$$[1+2C((C_M)+|r|)(1+T)]T\le 1, \tag{2.5}$$

then we obtain

$$\|Sv\|_{X(T)}\le 2CM. \tag{2.6}$$

Thus, if condition (2.6) holds, then S maps $Y(M,T)$ into $Y(M,T)$.

Step 3. We shall also claim that for T small enough, S is a strictly contractive map. Let $T>0$ and $v,\bar v\in Y(M,T)$ be given. Set $u=Sv,\bar u=S\bar v,U=u-\bar u,V=v-\bar v$ and note that U satisfies

$$U_{tt}+\Delta^2U=Q(x,t),(x,t)\in\mathbb{R}^n\times(0,+\infty), \tag{2.7}$$

$$U(x,0)=U_t(x,0)=0, \tag{2.8}$$

where $Q(x,t)$ is defined by

$$Q(x,t)=\Gamma[f(v)-f(\bar v)]+r\Gamma[V_t]+\Gamma[V]. \tag{2.9}$$

Observed that S has the smoothness required to apply Lemma 2.6 to problem (2.7) and (2.8). By Lemmas 2.4, 2.6 and 2.7, from (2.9) we obtain

$$\begin{aligned}&\|U(t)\|_{H^s}+\|U_t(t)\|_{H^{s-2}}\\&\le C(1+T)\int_0^t[\|f(v(\tau))-f(\bar v(\tau))\|_{H^{s-2}}+|r|\|V_t\|_{H^{s-2}}+\|V\|_{H^{s-2}}]d\tau\\&\le C(1+T)[C_M\max_{0\le t\le T}\|V(t)\|_{H^s}+|r|\max_{0\le t\le T}\|V_t(t)\|_{H^{s-2}}]T.\end{aligned}$$

Hence, we obtain

$$\|U(t)\|_{X(T)}\le C(1+T)[C_M+|r|+C]T\|V(t)\|_{X(T)}.$$

By choosing T so small that (2.5) holds and

$$(1+T)[C_M+|r|+C] < 1/C, \tag{2.10}$$

then

$$\|Sv-S\bar{v}\|_{X(T)} < \|v-\bar{v}\|_{X(T)}.$$

This shows that $S : Y(M,T) \to Y(M,T)$ is strictly contractive.

Step 4. From the contraction mapping principle, it follows that for appropriately chosen $T>0$, S has a unique fixed point $u(x,t) \in Y(M,T)$, which is a strong solution of problem (1.7)-(1.8). Similarly to [25], we can prove uniqueness and local Lipschitz dependence with respect to the initial data in the space $Y(M,T)$. Using uniqueness we can extend the result in the space $C([0,T],H^s)\cap C^1([0,T],H^{s-2})$ by a standard technique. □

3. Existence and uniqueness of a global solution

In this section, we prove the existence and the uniqueness of the global solution for problem (1.7)-(1.8). For this purpose, we are going to make a priori estimates of the local solutions for problem (1.7)-(1.8).

Lemma 3.1. *Suppose that $f(u)\in C(R), F(u)=\int_0^u f(s)ds, \phi\in H^2, (-\Delta)^{\frac{1}{2}}\psi\in L^2, \psi\in L^2$, and $F(\phi)\in L^1$. Then for the solution $u(x,t)$ of the problem* (1.7)-(1.8)*, it follows that*

$$\begin{aligned}E(t)&=\|(-\Delta)^{-1/2}u_t\|_2^2+\|u_t\|_2^2+\|u\|_2^2+\|\nabla u\|_2^2+\|\Delta u\|_2^2\\&\quad+2r\int_0^t\|u_\tau\|_2^2d\tau+2\int_{\mathbb{R}^n}F(u)dx=E(0).\end{aligned}\tag{3.1}$$

Here and in the sequel $(-\Delta)^{-\alpha}u(x)=\mathcal{F}^{-1}[|x|^{-2\alpha}\mathcal{F}u(x)]$, $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote Fourier transformation and inverse Fourier transformation in $\mathbb{R}^n$ respectively.

Proof. Multiplying both sides of (1.7) by $(-\Delta)^{-1}u_t$, integrating the product over $\mathbb{R}^n$ and integrating by parts, we obtain

$$\begin{aligned}&(u_{tt}-\Delta u-\Delta u_{tt}+\Delta^2u-\Delta^3u-r\Delta u_t-\Delta f(u),(-\Delta)^{-1}u_t)\\&=((-\Delta)^{-1}u_{tt}+u+u_{tt}-\Delta u+\Delta^2u+ru_t+f(u),u_t)\\&=((-\Delta)^{-1/2}u_{tt},(-\Delta)^{-1/2}u_t)+(u,u_t)+(u_{tt},u_t)+(\Delta^2u,u_t)+(\Delta u,u_t)\\&\quad+r(u_t,u_t)+(f(u),u_t)=0.\end{aligned}$$

So,

$$\begin{aligned}&\frac{d}{dt}[\|(-\Delta)^{-1/2}u_t\|_2^2+\|u_t\|_2^2+\|u\|_2^2+\|\Delta u\|_2^2+\|\nabla u\|_2^2\\&+2r\int_0^t\|u_\tau\|_2^2d\tau+2\int_{\mathbb{R}^n}F(u)dx]=0.\end{aligned}$$

The lemma is proved. □

Lemma 3.2. *Suppose that the assumptions of Lemma 3.1 hold and $F(u)\ge 0$ or $f'(u)$ is bounded below, i.e there is a constant A_0 such that $f'(u)\ge A_0$ for any $u\in\mathbb{R}$, then the solution $u(x,t)$ of problem* (1.7)-(1.8) *has the estimate*

$$E_1(t)=\|(-\Delta)^{-1/2}u_t\|_2^2+\|u_t\|_2^2+\|u\|_2^2+\|\nabla u\|_2^2+\|\Delta u\|_2^2\le M_1(T), \tag{3.2}$$

for all $t \in [0, T]$. *Here and in the sequel* $M_i(T)(i = 1, 2, \dots)$ *are constants dependent on* T.

Proof. If $F(u) \geq 0$, then from energy identity (3.1), we obtain

$$E_1(t) \leq E(0) + 2|r| \int_0^t \|u_\tau\|_2^2 d\tau.$$

It follows from Gronwall's inequality and the above inequality that

$$E_1(t) \leq E(0)e^{2|r|T}. \tag{3.3}$$

If $f'(u)$ is bounded below. Let $f_0(u) = f(u) - r_0 u$, where $k_0 = min\{A_0, 0\} (\leq 0)$, then $f_0(0) = 0, f_0'(u) = f'(u) - r_0 \geq 0$ and $f_0(u)$ is a monotonically increasing function. Then $F_0(u) = \int_0^u f_0(s)ds \geq 0$ and $F(u) = \int_0^u f(s)ds = \int_0^u (f_0(s) + r_0 s)ds = F_0(u) + \frac{r_0}{2}u^2$. From (3.1), we have

$$\begin{aligned}
&E_1(t) + 2\int_{\mathbb{R}^n} F_0(u)dx \\
&= E(0) - 2r\int_0^t \|u_\tau\|_2^2 d\tau - r_0\|u\|_2^2 \\
&= E(0) - 2r\int_0^t \|u_\tau\|_2^2 d\tau - r_0\|u_0\|_2^2 + \int_0^t (r_0^2\|u\|_2^2 + \|u_\tau\|_2^2)d\tau \\
&\leq E(0) - r_0\|u_0\|_2^2 + (2|r| + 1 + r_0^2)\int_0^t (\|u\|_2^2 + \|u_\tau\|_2^2)d\tau.
\end{aligned}$$

It follows from Gronwall's inequality and the above inequality that

$$E_1(t) \leq (E(0) - r_0\|u_0\|_2^2)\exp[(2|r| + 1 + r_0^2)T]. \tag{3.4}$$

We get (3.2) from inequalities (3.3) and (3.4). The lemma is proved. ☐

Lemma 3.3. *Under the conditions of Lemma 3.2, assume that* $1 \leq n \leq 4, f(u) \in C^2(R)$ *and* $|f'(u)| \leq A|u|^\rho + B, 0 < \rho < \infty$ *for* $2 \leq n \leq 4, \phi \in H^3$ *and* $\psi \in H^1$, *then the solution* $u(x, t)$ *of problem* (1.7)-(1.8) *has the estimation*

$$E_2(t) = \|u_t\|_2^2 + \|\nabla u\|_2^2 + \|\nabla u_t\|_2^2 + \|\Delta u\|_2^2 + \|\nabla^3 u\|_2^2 \leq M_2(T), \quad \forall t \in [0, T]. \tag{3.5}$$

Proof. Multiplying (1.7) by u_t and integrating the product over $\mathbb{R}^n$, we obtain

$$\frac{d}{dt}E_2(t) + 2r\|\nabla u_t\|_2^2 + 2(\nabla f(u), \nabla u_t) = 0. \tag{3.6}$$

When $n = 1$, we conclude from Lemma 2.1 and 3.2 that $u \in L^\infty$. Therefore, from (3.6), Hölder inequality, Cauchy inequality, Lemma 2.3 and (3.2), we obtain

$$\begin{aligned}
\frac{d}{dt}E_2(t) &\leq 2|r|\|\nabla u_t\|_2^2 + 2|(\nabla f(u), \nabla u_t)| \\
&\leq 2|r|\|\nabla u_t\|_2^2 + 2\|\nabla f(u)\|_2\|\nabla u_t\|_2 \\
&\leq 2|r|\|\nabla u_t\|_2^2 + 2K_1(W)(\|u\|_\infty)(\|u\|_2 + \|\nabla u\|_2)\|\nabla u_t\|_2 \\
&\leq C_1(M_1(t))(\|\nabla u\|_2^2 + \|\nabla u_t\|_2^2),
\end{aligned} \tag{3.7}$$

where and in the sequel $C_i(M_j(t))(i = 1, 2, \dots, j = 1, 2, \dots)$ are constants depending on $M_j(t)$. Integrating (3.7) with respect to t and using the Gronwall's inequality, we obtain (3.5).

In the case $2 \le n \le 4$, from Hölder inequality, Lemma 2.2, Cauchy inequality and (3.2), we have

$$\begin{aligned}\int_{\mathbb{R}^n} \nabla f(u)\nabla u_t dx &\le A\|u^\rho\|_\infty \|\nabla u\|_2^2 \|\nabla u_t\|_2 + B\|\nabla u\|_2\|\nabla u_t\|_2 \\ &\le \frac{A}{2}(C_2\|\Delta u\|_2^2\|\nabla u\|_2^2 + \|\nabla u_t\|_2^2) + \frac{B}{2}(\|\nabla u\|_2^2 + \|\nabla u_t\|_2^2) \\ &\le \frac{A}{2}(C_2(M_1(t))\|\Delta u\|_2^2 + \|\nabla u_t\|_2^2) + \frac{B}{2}(M_1(t) + \|\nabla u_t\|_2^2).\end{aligned}$$

Substitute the above inequality in (3.6) to obtain

$$\begin{aligned}\frac{d}{dt}E_2(t) &\le 2|r|\|\nabla u_t\|_2^2 + 2|(\nabla f(u), \nabla u_t)| \\ &\le BM_1(t) + C_3M_1(t)(\|\Delta u\|_2^2 + \|\nabla u_t\|_2^2).\end{aligned} \tag{3.8}$$

Integrating (3.8) with respect to t and using the Gronwall's inequality, we obtain (3.5). The lemma is proved. □

Lemma 3.4. *Under the conditions of Lemma 3.3, assume that* $s \ge 2, f(u) \in C^{[s]}(R), \phi \in H^{s+1}, \psi \in H^{s-1}$, *then the solution* $u(x,t)$ *of problem* (1.7)-(1.8) *has the estimate*

$$\begin{aligned}E_3(t) &= \|\nabla^{s-2}u_t\|_2^2 + \|\nabla^{s-1}u\|_2^2 + \|\nabla^{s-1}u_t\|_2^2 + \|\nabla^s u\|_2^2 + \|\nabla^{s+1}u\|_2^2 \\ &\le M_3(T), \quad \forall t \in [0,T].\end{aligned} \tag{3.9}$$

Proof. Multiplying (1.7) by $\Delta^{s-2}u_t$ and integrating the product over $\mathbb{R}^n$, we obtain

$$\frac{d}{dt}E_3(t) + 2r\|\nabla^{s-1}u_t\|_2^2 + 2(\nabla^{s-1}f(u), \nabla^{s-1}u_t) = 0. \tag{3.10}$$

From Lemmas 2.2 and 3.3, we know that $u \in L^\infty$. From Hö lder inequality, Cauchy inequality, Lemma 2.3 and (3.2) we obtain

$$\begin{aligned}\frac{d}{dt}E_3(t) &\le 2|r|\|\nabla^{s-1}u_t\|_2^2 + 2|(\nabla^{s-1}f(u), \nabla^{s-1}u_t)| \\ &\le 2|r|\|\nabla^{s-1}u_t\|_2^2 + 2K_1(W)(\|u\|_\infty)(\|u\|_2 + \|\nabla^{s-1}u\|_2)\|\nabla^{s-1}u_t\|_2 \\ &\le C_4(M_1(t))(\|\nabla^{s-1}u\|_2^2 + \|\nabla^{s-1}u_t\|_2^2).\end{aligned}$$

Integrating the above inequality with respect to t and using the Gronwall's inequality, we obtain (3.9). The lemma is proved. □

Proof of Theorem 1.2. From Theorem 1.1, we need only to show that

$$\sup_{t\in[0,T_0]} (\|u(t)\|_{H^s} + \|u_t(t)\|_{H^{s-2}}) < \infty.$$

From Lemmas 3.2–3.4, we obtain

$$\|u(t)\|_{H^s} + \|u_t(t)\|_{H^{s-2}} < M_4(T), \forall t \in [0,T),$$

where $M_4(T)$ is a constant dependent on T. Therefore, from the above inequality, problem (1.7)-(1.8) has a unique global solution $u(x,t) \in C([0,\infty), H^s) \cap C^1([0,\infty), H^{s-2})$ and $(-\Delta)^{-1/2}u_t \in L^2$. The theorem is proved. □

4. Blow-up of solutions

In this section, we give the proof of the blow-up of the solution for problem (1.7)-(1.8). For this purpose, we give the following lemma which is a generalization of Levine's result [7, 8].

Lemma 4.1. *Suppose that for $t \geq 0$, a positive, twice differential function $I(t)$ satisfies the inequality*

$$I''(t)I(t) - (1+\varepsilon)(I'(t))^2 \geq -2L_1 I(t)I'(t) - L_2(I(t))^2,$$

where $\varepsilon > 0$ and L_1, L_2 are constants. If $I(0) > 0$, $I'(0) > \gamma_2\nu^{-1}I(0)$ and $L_1+L_2 > 0$, then $I(t)$ tends to infinity as

$$t \to t_1 \leq t_2 = \frac{1}{2\sqrt{L_1^2+\nu L_2}} \ln \frac{\gamma_1 I(0) + \nu I'(0)}{\gamma_1 I(0) + \nu I'(0)},$$

where $\gamma_{1,2} = -L_1 \mp \sqrt{L_1^2 + \nu L_2}$. If $I(0) > 0$, $I'(0) > 0$ and $L_1 = L_2 = 0$, then $I(t) \to \infty$ as $t \to t_1 \leq t_2 = I(0)/\nu I'(0)$.

Proof of Theorem 1.3. Suppose $T = +\infty$, let

$$I(t) = \|(-\Delta)^{-1/2}u\|_2^2 + \|u\|_2^2 + \beta(t+\tau)^2, \tag{4.1}$$

where $\beta, \tau \geq 0$ to be defined later. Then

$$I'(t) = 2((-\Delta)^{-1/2}u_t, (-\Delta)^{-1/2}u) + 2\beta(t+\tau) + 2(u, u_t). \tag{4.2}$$

So,

$$\begin{aligned}(I'(t))^2 &\leq 4[\|(-\Delta)^{-1/2}u\|_2^2 + \|u\|_2^2 + \beta(t+\tau)^2][\|(-\Delta)^{-1/2}u_t\|_2^2 + \|u_t\|_2^2 + \beta] \\ &= 4I(t)[\|(-\Delta)^{-1/2}u_t\|_2^2 + \|u_t\|_2^2 + \beta].\end{aligned} \tag{4.3}$$

By (1.7), we obtain

$$\begin{aligned}I''(t) &= 2\|(-\Delta)^{-1/2}u_t\|_2^2 + 2((-\Delta)^{-1/2}u, (-\Delta)^{-1/2}u_{tt}) + 2\|u_t\|_2^2 + 2(u, u_{tt}) \\ &\quad + 2\beta \\ &= 2\|(-\Delta)^{-1/2}u_t\|_2^2 + 2\|u_t\|_2^2 + 2\beta + 2(u, (-\Delta)^{-1}u_{tt} + u_{tt}) \\ &= 2\|(-\Delta)^{-1/2}u_t\|_2^2 + 2\|u_t\|_2^2 + 2\beta - 2(u, u - \Delta u + \Delta^2 u + ru_t + f(u)) \\ &= 2\|(-\Delta)^{-1/2}u_t\|_2^2 + 2\|u_t\|_2^2 + 2\beta - 2\|u\|_2^2 - 2\|\nabla u\|_2^2 - 2\|\Delta u\|_2^2 \\ &\quad - 2r(u, u_t) - 2\int_{\mathbb{R}^n} uf(u)dx.\end{aligned} \tag{4.4}$$

With the aid of the Cauchy inequality we obtain

$$\begin{aligned}2r(u, u_t) &\leq r(\|u\|_2^2 + \|u_t\|_2^2) \\ &= r[E(0) - \|(-\Delta)^{-1/2}u_t\|_2^2 - \|\nabla u\|_2^2 - \|\Delta u\|_2^2 \\ &\quad - 2r\int_0^t \|u_\tau\|_2^2 d\tau - 2\int_{\mathbb{R}^n} F(u)dx].\end{aligned} \tag{4.5}$$

It follows from (4.1)-(4.5) that

$$\begin{aligned}
&I(t)I''(t) - (1+\frac{\alpha}{4})(I'(t))^2 \\
&\ge I(t)I''(t) - (4+\alpha)I(t)[\|(-\Delta)^{-1/2}u_t\|_2^2 + \|\Delta u\|_2^2 + \|u_t\|_2^2 + \beta] \\
&\ge I(t)\{2\|(-\Delta)^{-1/2}u_t\|_2^2 + 2\|u_t\|_2^2 + 2\beta - 2\|\Delta u\|_2^2 - 2\|u\|_2^2 - 2\|\nabla u\|_2^2 \\
&\quad - 2r(u,u_t) - 2\int_{\mathbb{R}^n} uf(u)dx - (4+\alpha)[\|(-\Delta)^{-1/2}u_t\|_2^2 + \|u_t\|_2^2 + \beta]\} \\
&\ge I(t)\{(r-\alpha-2)\|(-\Delta)^{-1/2}u_t\|_2^2 + (-2-\alpha)\|u_t\|_2^2 + (-4-\alpha)\beta \\
&\quad + (r-2)(\|\nabla u\|_2^2 + \|\Delta u\|_2^2) + \int_{\mathbb{R}^n}[2rF(u) - 2uf(u) - 2u^2]dx \\
&\quad + 2r^2\int_0^t \|u_\tau\|_2^2 d\tau - rE(0)\}.
\end{aligned} \tag{4.6}$$

From (3.1), we obtain

$$\begin{aligned}
&(r-\alpha-2)\|(-\Delta)^{-1/2}u_t\|_2^2 + (-2-\alpha)\|u_t\|_2^2 + (r-2)(\|\nabla u\|_2^2 + \|\Delta u\|_2^2) \\
&\ge (-\alpha-2)(\|(-\Delta)^{-1/2}u_t\|_2^2 + \|\nabla u\|_2^2 + \|\Delta u\|_2^2 + \|u_t\|_2^2) \\
&= (\alpha+2)(\|u\|_2^2 + 2r\int_0^t \|u_\tau\|_2^2 d\tau + 2\int_{\mathbb{R}^n} F(u)dx - E(0)).
\end{aligned}$$

Thus, from the above inequality, (1.10) and (4.6), we have

$$\begin{aligned}
&I(t)I''(t) - (1+\frac{\alpha}{4})(I'(t))^2 \\
&\ge I(t)\Big\{ -(4+\alpha)\beta - (2+\alpha+r)E(0) + \int_{\mathbb{R}^n}[2(2+\alpha+r)F(u) \\
&\quad + \alpha u^2 - 2uf(u)\big]dx + (2r(2+\alpha) + 2r^2)\int_0^t \|u_\tau\|_2^2 d\tau\Big\} \\
&\ge -\big[(4+\alpha)\beta + (2+\alpha+r)E(0)\big]I(t).
\end{aligned} \tag{4.7}$$

If $E(0) < 0$, taking $\beta = -\frac{2+\alpha+r}{4+\alpha}E(0) > 0$, then

$$I(t)I''(t) - (1+\frac{\alpha}{4})(I'(t))^2 \ge 0.$$

We may choose τ so large that $I'(t) > 0$. From Lemma 4.1 we know that $I(t)$ becomes infinite at a time T_1 at most equal to

$$T_1 = \frac{4I(0)}{\alpha I'(t)} < \infty.$$

If $E(0) = 0$, taking $\beta = 0$, from (4.7), we obtain

$$I(t)I''(t) - (1+\frac{\alpha}{4})(I'(t))^2 \ge 0.$$

Also $I'(t) > 0$ by assumption (ii), Thus, we obtain from Lemma 4.1 that $I(t)$ becomes infinite at a time T_2 at most equal to

$$T_2 = \frac{4I(0)}{\alpha I'(t)} < \infty.$$

If $E(0) > 0$, then taking $\beta = 0$, inequality (4.7) becomes

$$I(t)I''(t) - (1 + \frac{\alpha}{4})(I'(t))^2 \geq -(2 + \alpha + r)E(0)I(t). \tag{4.8}$$

Define $J(t) = (I(t))^{-\lambda}$, where $\lambda = \alpha/4$. Then

$$\begin{aligned} J'(t) &= -\lambda(I(t))^{-\lambda-1}I'(t), \\ J''(t) &= -\lambda(I(t))^{-\lambda-2}[I(t)I''(t) - (1+\lambda)(I'(t))^2] \\ &\leq \lambda(2 + r + 4\lambda)E(0)(I(t))^{-\lambda-1}, \end{aligned} \tag{4.9}$$

where inequality (4.8) is used. Assumption (iii) implies $J'(0) < 0$. Let

$$t^* = \sup\{t | J'(\tau) < 0, \tau \in (0, t)\}. \tag{4.10}$$

By the continuity of $J'(t)$, t^* is positive. Multiplying (4.9) by $2J'(t)$ yields

$$\begin{aligned} [(J'(t))^2]' &\geq -2\lambda^2(2 + r + 4\lambda)E(0)(I(t))^{-2\lambda-2}I'(t) \\ &= 2\lambda^2\frac{2 + r + 4\lambda}{2\lambda + 1}E(0)[I(t)^{-2\lambda-1}]'. \end{aligned} \tag{4.11}$$

Integrate with respect to t over $[0, t)$ to obtain

$$\begin{aligned} (J'(t))^2 &\geq 2\lambda^2\frac{2 + r + 4\lambda}{2\lambda + 1}E(0)(I(t))^{-2\lambda-1} \\ &\quad + (J'(0))^2 - 2\lambda^2\frac{2 + r + 4\lambda}{2\lambda + 1}E(0)(I(0))^{-2\lambda-1} \\ &\geq (J'(0))^2 - 2\lambda^2\frac{2 + r + 4\lambda}{2\lambda + 1}E(0)(I(0))^{-2\lambda-1}. \end{aligned}$$

From assumption (iii), we obtain

$$(J'(0))^2 - 2\lambda^2\frac{r + 2 + 4\lambda}{2\lambda + 1}E(0)(I(0))^{-2\lambda-1} > 0.$$

Hence by continuity of $J'(t)$, we have

$$J'(t) \leq -[(J'(0))^2 - 2\lambda^2\frac{2 + r + 4\lambda}{2\lambda + 1}E(0)(I(0))^{-2\lambda-1}]^{1/2} \tag{4.12}$$

for $0 \leq t < t^*$. By the definition of t^*, it follows that (4.12) holds for all $t \geq 0$. Therefore,

$$J(t) \leq J(0) - [(J'(0))^2 - 2\lambda^2\frac{2 + r + 4\lambda}{2\lambda + 1}E(0)(I(0))^{-2\lambda-1}]^{1/2}t, \quad \forall t > 0.$$

So $J(T_1) = 0$ for some T_1 and

$$0 < T_1 \leq T_2 = J(0)/[(J'(0))^2 - [\lambda^2(2 + \lambda + r)/(4\lambda + 8)]E(0)(I(0))^{-(\lambda+2)/2}]^{1/2}.$$

Thus, $I(t)$ becomes infinite at a time T_1.

Therefore, $I(t)$ becomes infinite at a time T_1 under either assumptions. We have a contradiction with the fact that the maximal time of existence is infinite. Hence the maximal time of existence is finite. This completes the proof. □

5. Asymptotic behavior of solution

Proof of Theorem 1.4. Let $u(x,t)$ be a global solution of (1.7)-(1.8). Multiplying (1.7) by $(-\Delta)^{-1}u_t$ and integrating on $\mathbb{R}^n$ it follows that

$$\frac{d}{dt}E(t) + r\|u_t\|_2^2 = 0. \tag{5.1}$$

Multiplying (5.1) by e^{kt} we have

$$\frac{d}{dt}(e^{kt}E(t)) + re^{kt}\|u_t\|^2 = ke^{kt}E(t). \tag{5.2}$$

Integrating (5.2) over $(0,t)$, we obtain

$$\begin{aligned}
&e^{kt}E(t) + r\int_0^t e^{r\tau}\|u_\tau\|_2^2 d\tau \\
&= E(0) + k\int_0^t e^{k\tau}E(\tau)d\tau \\
&= E(0) + \frac{k}{2}\int_0^t e^{k\tau}(\|(-\Delta)^{-1/2}u_\tau\|_2^2 + \|u_\tau\|_2^2 + \|\Delta u\|_2^2 + \|\nabla u\|_2^2 + \|u\|_2^2)d\tau \\
&\quad + k\int_0^t e^{k\tau}\Big(\int_{\mathbb{R}^n} F(u)dx\Big)d\tau.
\end{aligned} \tag{5.3}$$

From $0 \le F(u) \le f(u)u$ and (1.7), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} F(u)dx \\
&\le \int_{\mathbb{R}^n} f(u)udx \\
&= -((-\Delta)^{-1}u_{tt} + u_{tt} + \Delta^2 u + u - \Delta u + ru_t, u) \\
&= -((-\Delta)^{-1}u_{tt}, u) - (u_{tt}, u) - (\Delta^2 u, u) - \|u\|_2^2 - \|\nabla u\|_2^2 - \frac{r}{2}\frac{d}{dt}\|u\|_2^2 \\
&= -\|\nabla u\|_2^2 - \|\Delta u\|_2^2 - \|u\|_2^2 - ((-\Delta)^{-1}u_{tt}, u) - (u_{tt}, u) - \frac{r}{2}\frac{d}{dt}\|u\|_2^2.
\end{aligned} \tag{5.4}$$

Hence we have

$$\begin{aligned}
&k\int_0^t e^{k\tau}\int_{\mathbb{R}^n} F(u)\,dx\,d\tau \\
&\le k\int_0^t e^{k\tau}[-\|\nabla u\|_2^2 - \|\Delta u\|_2^2 - \|u\|_2^2 - ((-\Delta)^{-1}u_{\tau\tau}, u) - (u_{\tau\tau}, u) \\
&\quad - \frac{r}{2}\frac{d}{d\tau}\|u\|_2^2]d\tau.
\end{aligned} \tag{5.5}$$

We will estimate the terms on the right-hand side of (5.5) separately. Integrating by parts and using Young's inequality, we obtain

$$
\begin{aligned}
&-\int_0^t e^{k\tau}((-\Delta)^{-1}u_{\tau\tau},u)d\tau\\
&=-\int_0^t e^{k\tau}(\frac{d}{d\tau}((-\Delta)^{-1}u_\tau,u)-\|(-\Delta)^{-1/2}u_\tau\|)d\tau\\
&=-e^{kt}((-\Delta)^{-1/2}u_t,(-\Delta)^{-1/2}u)+((-\Delta)^{-1/2}\psi,(-\Delta)^{-1/2}\phi)\\
&\quad+k\int_0^t e^{k\tau}((-\Delta)^{-1/2}u_\tau,(-\Delta)^{-1/2}u)d\tau+\int_0^t e^{k\tau}\|(-\Delta)^{-1/2}u_\tau\|_2^2d\tau\\
&\leq\frac{1}{2}e^{k\tau}(\|(-\Delta)^{-1/2}u_t\|_2^2+\|(-\Delta)^{-1/2}u\|_2^2)\\
&\quad+(\|(-\Delta)^{-1/2}\psi\|_2^2+\|(-\Delta)^{-1/2}\phi\|_2^2)\\
&\quad+\frac{k}{2}\int_0^t e^{k\tau}(\|(-\Delta)^{-1/2}u_\tau\|_2^2+\|(-\Delta)^{-1/2}u\|_2^2)d\tau\\
&\quad+\int_0^t e^{k\tau}\|(-\Delta)^{-1/2}u_\tau\|_2^2d\tau.
\end{aligned}
\tag{5.6}
$$

Similarly using integration by parts and Young's inequality, we obtain

$$
\begin{aligned}
&-\int_0^t e^{k\tau}(u_{\tau\tau},u)d\tau\\
&=-\int_0^t e^{k\tau}(\frac{d}{d\tau}(u_\tau,u)-\|u_\tau\|_2^2)d\tau\\
&=-e^{k\tau}(u_\tau,u)+(\psi,\phi)+k\int_0^t e^{k\tau}(u_\tau,u)d\tau+\int_0^t e^{k\tau}\|u_\tau\|_2^2d\tau\\
&\leq\frac{1}{2}e^{k\tau}(\|u_\tau\|_2^2+\|u\|_2^2)+\frac{1}{2}(\|\psi\|_2^2+\|\phi\|_2^2)\\
&\quad+\frac{k}{2}\int_0^t e^{k\tau}(\|u_\tau\|_2^2+\|u\|_2^2)d\tau+\int_0^t e^{k\tau}\|u_\tau\|_2^2d\tau.
\end{aligned}
\tag{5.7}
$$

For the last term, by using integration by parts, we have

$$
-\frac{r}{2}\int_0^t e^{k\tau}\frac{d}{d\tau}\|u\|_2^2d\tau=-\frac{r}{2}e^{k\tau}\|u\|_2^2+\frac{r}{2}\|\phi\|_2^2+\frac{r}{2}k\int_0^t e^{k\tau}\|u\|_2^2d\tau. \tag{5.8}
$$

Substituting (5.6)-(5.8) into (5.4) and (5.5), it follows that there exist positive constants C_0, C_1,C_2 and C_3 such that

$$
\begin{aligned}
&e^{k\tau}E(t)+r\int_0^t e^{rt}\|u_\tau\|_2^2d\tau\\
&\leq C_0E(0)+C_1ke^{kt}E(t)+C_2k^2\int_0^t e^{k\tau}E(\tau)d\tau+C_3k\int_0^t e^{k\tau}E(\tau)d\tau.
\end{aligned}
\tag{5.9}
$$

Taking k satisfying $0<k<\frac{1}{2C_1}$, then from (5.9) and $r>0$, we obtain

$$
e^{kt}E(t)\leq 2C_0E(0)+(2C_2k^2+2C_3k)\int_0^t e^{k\tau}E(\tau)d\tau,
$$

which together with the Gronwall inequality gives

$$e^{kt}E(t) \le 2C_0E(0)e^{2C_2k^2t+2C_3kt}, \quad 0 \le t < \infty,$$
$$E(t) \le 2C_0E(0)e^{-(k-2C_2k^2-2C_3k)t}, \quad 0 \le t \le \infty.$$

Again taking k satisfying $0 < k < \min\{\frac{1}{2C_1}, \frac{1-2C_3}{2C_2}\}$, we can obtain (1.11), where $\theta = k - 2C_2k^2 - 2C_3k > 0$. The proof is complete. □

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