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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR ELLIPTIC EQUATIONS WITH SINGULAR GROWTH

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Abstract. In this article, we consider an elliptic problem with singular and critical growth. We prove the existence and multiplicity of solutions for the resonant and nonresonant cases.

1. Introduction

In this article we study the existence of nontrivial solutions to the semilinear elliptic problem

$$\begin{gathered} -\Delta u - \mu\frac{u}{|x|^2} = \lambda f(x)u + |u|^{2^*-2}u \quad \text{in } \Omega \setminus \{0\}, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with $0 \in \Omega$, λ and μ are positive parameters such that $0 \leq \mu < \bar{\mu} = (\frac{N-2}{2})^2$, $\bar{\mu}$ is the best constant in the Hardy inequality, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent and f is a positive singular function which will be specified later.

The study of this type of problems is motivated by its various applications. For example, it has been introduced as a model for nonlinear Schrödinger equations with a singular potential of the form:

$$-i\hbar\frac{\partial\psi}{\partial t} - \frac{\hbar^2}{2}\Delta\psi + V(x)\psi = |\psi|^{p-1}\psi, \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}^+,$$

where i is the imaginary unit and $\hbar$ denotes the Plank constant. This equation describes Bose-Einstein condensates [11, 12] and the propagation of light in some nonlinear optical materials [13].

Equation (1.1) is doubly critical due to the presence of the critical exponent and the Hardy potential. If $\lambda \leq 0$ and Ω is starshaped, using Pohozaev identity [15] one sees that (1.1) has no nontrivial solution. When $f \equiv 1$ the problem (1.1) has been widely investigated, see [3, 6, 7, 9] and the references therein.

In Jannelli [9], for $f \equiv 1$, the following was proved:

(1) If $0 \leq \mu \leq \bar{\mu} - 1$, then (1.1) has at least one solution $u \in H_0^1(\Omega)$ for all $0 < \lambda < \lambda_1^\mu$ where λ_1^μ is the first eigenvalue of the operator $(-\Delta - \frac{\mu}{|x|^2})$ in $H_0^1(\Omega)$.

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(2) If $\bar{\mu}-1<\mu<\bar{\mu}$, then (1.1) has at least one solution $u\in H_0^1(\Omega)$ for all $\mu^*<\lambda<\lambda_1^{\mu}$ where

$$\mu^*=\min_{\varphi\in H_0^1(\Omega)}\frac{\int_\Omega \frac{|\nabla\varphi(x)|^2}{|x|^{2\gamma}}dx}{\int_\Omega \frac{|\varphi(x)|^2}{|x|^{2\gamma}}dx},$$

and $\gamma=\sqrt{\bar{\mu}}+\sqrt{\bar{\mu}-\mu}$.

Ferrero and Gazzola [7] showed the existence of solutions for $\lambda\geq\lambda_1^{\mu}$; Cao and Han [3] extended the results in [7]. When $f\not\equiv 1$, is a positive measurable function, Nasri [14] extended the results of Jannelli [9] allowing f to be singular. Borrowing ideas from [3] and [7], we give existence and multiplicity results when f is a singular function. Resonant and non resonant cases are considered.

This article is organized as follows: in Section 2 we collect preliminary results and state our main results, in Section 3 we present variational properties of (1.1), and in Section 4 we complete the proofs of the main results.

2. Preliminaries and statement of main results

Throughout this article we denote by $C, C_1, C_2,\dots$ generic positive constants; B_R is the ball centered at 0 with radius R; H^{-1} is the topological dual of $H_0^1(\Omega)$; $L^p(\Omega)$ for $1\leq p\leq+\infty$, denotes the Lebesgue space with $|\cdot|_p$, its usual norm.

For all $0\leq\mu<\bar{\mu}$, we endow the Hilbert space $H_0^1(\Omega):=H_\mu(\Omega)$ with the scalar product

$$\langle u,\,v\rangle_\mu=\int_\Omega\Big(\nabla u\nabla v-\mu\frac{uv}{|x|^2}Big)dx,\quad \forall u,v\in H_\mu(\Omega),$$

and define

$$\|u\|_\mu:=\Big(\int_\Omega\big(|\nabla u|^2-\mu\frac{u^2}{|x|^2}\big)dx\Big)^{1/2},\quad \forall u\in H_\mu(\Omega).$$

By Hardy's inequality [8], this norm is equivalent to the usual norm in $H_0^1(\Omega)$. Let

$$\mathcal{F}_2=\big\{f:\Omega\to\mathbb{R}^+:\lim_{|x|\to 0}|x|^2f(x)=0 \text{ with } f\in L^\infty_{\rm loc}(\Omega\setminus\{0\})\big\}.$$

Next we state several properties to be used later in this paper.

Lemma 2.1 ([5]). *Let $0\leq\mu<\bar{\mu}$, $\lambda\in\mathbb{R}$, $f\in\mathcal{F}_2$. The eigenvalue problem*

$$\begin{gathered}-\Delta e-\mu\frac{e}{|x|^2}=\lambda f(x)e\quad \text{in } \Omega\\ e=0\quad \text{on } \partial\Omega,\end{gathered}\tag{2.1}$$

admits non-trivial weak solutions in $H_0^1(\Omega)$, corresponding to

$$\lambda\in\sigma_\mu(f):=\big(\lambda_k^\mu(f)\big)_{k=1}^\infty$$

where

$$0<\lambda_1^\mu(f)\leq\lambda_2^\mu(f)\leq\dots\to+\infty.$$

if Ω is $C^{1,1}$, then all weak solutions of (2.1) *are in $H_0^1(\Omega)\cap W^{2,r}(\Omega)$ for all $1<r<\frac{2N}{N+2}$.*

Lemma 2.2 ([5]). *If $f\in\mathcal{F}_2$, then the embedding $H_0^1(\Omega)\hookrightarrow L^2(\Omega,\,fdx)$ is compact.*

For $0\leq\beta<2$, we set

$$\mathcal{F}_{2,\,\beta}=\big\{f\in\mathcal{F}_2:\exists\, 0\leq\beta<2 \text{ such that } 0<\lim_{|x|\to 0}|x|^\beta f(x)<\infty\big\}.$$

Lemma 2.3 ([5]). *Let* $2^*_\beta := \frac{2(N-\beta)}{N-2}$, *if* $f \in \mathcal{F}_{2,\,\beta}$, *then the embedding* $H^1_0(\Omega) \hookrightarrow L^q(\Omega,\, f dx)$ *is:*

- (i) *compact for all* $2 \le q < 2^*_\beta$.
- (ii) *continuous for all* $2 \le q \le 2^*_\beta$.

Definition 2.4. Let $I \in C^1(H^1_0(\Omega), \mathbb{R})$, $c \in \mathbb{R}$. We say that I satisfies the Palais-Smale condition at the level c, for short $(\mathrm{P.S})_c$, if every sequence (u_n) in $H^1_0(\Omega)$ such that

$$I(u_n) \to c \text{ in } \mathbb{R} \quad \text{and} \quad I'(u_n) \to 0 \text{ in } H^{-1}(\Omega) \quad \text{as } n \to +\infty,$$

has a convergent subsequence.

Our main results, are the following three theorems.

Theorem 2.5. *Suppose that* $f \in \mathcal{F}_{2,\,\beta}$, $\mu \in [0, \bar{\mu} - (\frac{2-\beta}{2})^2]$ *and* $\lambda \notin \sigma_\mu(f)$.

- (i) *If* $N = 3$ *and* $1 \le \beta < 2$, *then the problem* (1.1) *has at least one solution.*
- (ii) *If* $N \ge 4$ *and* $0 \le \beta < 2$, *then the problem* (1.1) *has at least one solution.*

Theorem 2.6. *Suppose that* $f \in \mathcal{F}_{2,\beta}$, $\mu \in (\bar{\mu} - (\frac{2-\beta}{2})^2,\, \bar{\mu})$ *and there exists* $\lambda^\mu_k(f) \in \sigma_\mu(f)$ *such that* $\lambda \in (\lambda_+, \lambda^\mu_k(f))$ *with* $\lambda_+ = \lambda^\mu_k(f) - S_\mu(\int_\Omega |x|^{-\beta N/2} dx)^{-2/N}$. *Assume one of the following conditions hold:*

- (i) $N = 3$ *and* $7/5 < \beta < 2$,
- (ii) $N = 4$ *and* $2/3 < \beta < 2$,
- (iii) $N \ge 5$ *and* $0 \le \beta < 2$.

Then problem (1.1) *admits* v_k *pairs of nontrivial solutions where* v_k *denotes the multiplicity of* $\lambda^\mu_k(f)$.

Theorem 2.7. *Suppose that* $f \in \mathcal{F}_{2,\,\beta}$, $\mu \in [0,\, \bar{\mu} - (\frac{N+2}{N})^2(\frac{2-\beta}{2})^2[$. *Assume one of the following conditions holds:*

- (i) $N = 3$*and* $7/5 < \beta < 2$,
- (ii) $N = 4$ *and* $2/3 < \beta < 2$,
- (iii) $N \ge 5$ *and* $0 \le \beta < 2$.

Then for all $\lambda > 0$, *the problem* (1.1) *admits at least one solution.*

We prove our results using critical point theory. However the energy functional associated to (1.1) does not satisfy (P.S) because of the lack of compactness of the embedding $H^1_0(\Omega) \hookrightarrow L^{2^*}(\Omega)$ and $H^1_0(\Omega) \hookrightarrow L^2(\Omega,\, |x|^{-2}dx)$, standard arguments are not applicable. We follow Brezis-Nirenberg's arguments in [2] to verify that the energy functional to (1.1) satisfies $(\mathrm{P.S})_c$ condition on a suitable compactness range. Then, by employing the technics introduced in [3, 7] we obtain some results on Brezis-Nirenberg type problems for an elliptic equation involving critical growth and singular coefficients.

3. Variationnal characterization

The nontrivial solutions to (1.1) are the non zero critical points of the energy functional

$$J_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\mu}{2}\int_\Omega \frac{|u|^2}{|x|^2} dx - \frac{\lambda}{2}\int_\Omega f|u|^2 dx - \frac{1}{2^*}\int_\Omega |u|^{2^*} dx. \tag{3.1}$$

Let

$$S_\mu := \inf_{u\in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla u|^2 - \mu\frac{u^2}{|x|^2})dx}{(\int_{\mathbb{R}^N} |u|^{2^*} dx)^{2/2^*}}.$$

From [16], we know that S_μ is achieved by the family of functions

$$u^*_\varepsilon(x) = \frac{C_\varepsilon}{\left(\varepsilon^2|x|^{\gamma'/\sqrt{\bar\mu}} + |x|^{\gamma/\sqrt{\bar\mu}}\right)^{\sqrt{\bar\mu}}}$$

with

$$C_\varepsilon = (\frac{4\varepsilon N(\bar\mu-\mu)}{N-2})^{\sqrt{\bar\mu}/2}, \quad \gamma = \sqrt{\bar\mu} + \sqrt{\bar\mu-\mu}, \quad \gamma' = \sqrt{\bar\mu} - \sqrt{\bar\mu-\mu}.$$

Lemma 3.1. *Assume that* $f \in \mathcal{F}_{2,\beta}$ *and* $\mu < \bar\mu$, *then* J_λ *satisfies the* $(\mathrm{PS})_c$ *condition for all* $c < \frac{1}{N}S_\mu^{N/2}$.

The proof of the above lemma is the same that in [6]. Fix $k \in \mathbb{N}$ and let

$$H^- = \operatorname{span}\{e_1, e_2, \dots, e_k\}, \quad H^+ = (H^-)^\perp.$$

Take always $m \in \mathbb{N}$ large enough, so $B_{1/m} \subset \Omega$ and consider the function $\xi_m : \Omega \to \mathbb{R}$ defined by

$$\xi_m(x) := \begin{cases} 0 & \text{if } x \in B_{1/m}(0),\\ m|x|-1 & \text{if } x \in A_m = B_{2/m}(0)\setminus B_{1/m}(0),\\ 1 & \text{if } x \in B_{2/m}(0).\end{cases}$$

Then, as in [7], define the approximate eigenfunctions $e_i^m := \xi_m e_i$ for all $i \in \mathbb{N}$ and the space $H_m^- := \operatorname{span}\{e_i^m,\ i = 1,\dots,k\}$. For all $\varepsilon > 0$, consider the shifted functions

$$u_m^\varepsilon(x) = \begin{cases} u^*_\varepsilon(x) - u^*_\varepsilon(\frac{1}{m}) & \text{if } x \in B_{1/m}(0)\setminus\{0\},\\ 0 & \text{if } x \in \Omega\setminus B_{1/m}(0).\end{cases}$$

Lemma 3.2. *For* $f \in \mathcal{F}_{2,\beta}$, $\mu < \bar\mu$ *and* $i \neq j (i,j = 1,2,\dots,k)$, *we have:*

(i) $\|e_i^m - e_i\|_\mu \to 0$ *as* $m \to \infty$,

$$\|e_k^m\|_\mu \le \lambda_k^\mu(f) + Cm^{-2\sqrt{\bar\mu-\mu}}, \tag{3.2}$$

$$|\langle e_i^m, e_j^m\rangle_\mu| \le Cm^{-2\sqrt{\bar\mu-\mu}}, \tag{3.3}$$

$$\|e_k^m\|_{L^2_{(\Omega,f)}} \le \lambda_k^\mu(f) + Cm^{-2+\beta\sqrt{\bar\mu-\mu}}, \tag{3.4}$$

(ii) *For* $\Lambda = \{u \in H_m^- : \|u\|_{L^2(\Omega,f)} = 1\}$, *we have*

$$\max_{u\in\Lambda} \|u\|_\mu \le \lambda_k^\mu(f) + Cm^{-2\sqrt{\bar\mu-\mu}}.$$

The proof of the above lemma is essentially given in [3] with minor modifications.

Lemma 3.3. *Let* $0 \le \beta < 2$ *and* $f \in \mathcal{F}_{2,\beta}$. *For* m *large enough and* ε *small enough, we have*

$$\int_\Omega \Big(|\nabla u_m^\varepsilon| - \mu\frac{(u_m^\varepsilon)^2}{|x|^2}\Big)dx \le S_\mu^{N/2} + C\varepsilon^{N-2}m^{2\sqrt{\bar\mu-\mu}}, \tag{3.5}$$

$$\int_\Omega (u_m^\varepsilon)^{2^*} dx \ge S_\mu^{N/2} - C\varepsilon^N m^{2N\sqrt{\bar\mu-\mu}/(N-2)}. \tag{3.6}$$

$$
\begin{aligned}
&\int_\Omega f(u_m^\varepsilon)^2 dx \\
&\geq \begin{cases} C_1 \varepsilon^{\frac{\sqrt{\bar\mu}}{2\sqrt{\bar\mu-\mu}}(2-\beta)} - C\varepsilon^{2\sqrt{\bar\mu}} m^{-2+\beta+2\sqrt{\bar\mu-\mu}} & \textit{if } \mu < \bar\mu - (\frac{2-\beta}{2})^2. \\ C_2 \varepsilon^{(N-2)/2} |\ln\varepsilon| - C\varepsilon^{N-2} & \textit{if } \mu = \bar\mu - (\frac{2-\beta}{2})^2. \end{cases}
\end{aligned} \tag{3.7}
$$

Proof. For the proof of (3.5) and (3.6) we argue as in [7]. We prove only (3.7). Since $f \in \mathcal{F}_{2,\,\beta}$, we have

$$
\int_\Omega f(u_\varepsilon^*)^2 dx \geq \begin{cases} C_1 \varepsilon^{\sqrt{\bar\mu}(2-\beta)/2\sqrt{\bar\mu-\mu}} & \text{if } \mu < \bar\mu - (\frac{2-\beta}{2})^2 \\ C_2 \varepsilon^{(N-2)/2} |\ln\varepsilon| & \text{if } \mu = \bar\mu - (\frac{2-\beta}{2})^2 \end{cases}
$$

and

$$
\begin{aligned}
\int_\Omega f\big(u_m^\varepsilon\big)^2 dx &= \int_\Omega f(u_\varepsilon^*(x) - u_\varepsilon^*(\frac{1}{m}))^2 dx \\
&\geq \int_\Omega f(u_\varepsilon^*)^2 dx - 2\int_\Omega f \frac{u_\varepsilon^* C_\varepsilon}{\big(\varepsilon^2(\frac{1}{m})^{\gamma'/\sqrt{\bar\mu}} + (\frac{1}{m})^{\gamma/\sqrt{\bar\mu}}\big)^{\sqrt{\bar\mu}}} dx \\
&\geq \int_\Omega f(u_\varepsilon^*)^2 dx \\
&\quad - C\int_\Omega f \frac{\varepsilon^{2\sqrt{\bar\mu}}}{\big(\varepsilon^2|x|^{\gamma'/\sqrt{\bar\mu}} + |x|^{\gamma/\sqrt{\bar\mu}}\big)^{\sqrt{\bar\mu}}\big(\varepsilon^2(\frac{1}{m})^{\gamma'/\sqrt{\bar\mu}} + (\frac{1}{m})^{\gamma/\sqrt{\bar\mu}}\big)^{\sqrt{\bar\mu}}} dx.
\end{aligned}
$$

We have

$$
\begin{aligned}
\frac{\varepsilon^{2\sqrt{\bar\mu}}}{(\varepsilon^2(\frac{1}{m})^{\gamma'/\sqrt{\bar\mu}} + (\frac{1}{m})^{\gamma/\sqrt{\bar\mu}})^{\sqrt{\bar\mu}}} &= \frac{\varepsilon^{2\sqrt{\bar\mu}}}{\varepsilon^{2\sqrt{\bar\mu}}(\frac{1}{m})^{\gamma'}(1+\varepsilon^{-2}(\frac{1}{m})^{2\sqrt{\bar\mu-\mu}/\sqrt{\bar\mu}})^{\sqrt{\bar\mu}}} \\
&\leq \varepsilon^{2\sqrt{\bar\mu}} m^{\gamma},
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{B_{1/m}} f \frac{dx}{(\varepsilon^2|x|^{\gamma'/\sqrt{\bar\mu}} + |x|^{\gamma/\sqrt{\bar\mu}})^{\sqrt{\bar\mu}}} \\
&\leq C\int_0^{1/m} \frac{r^{N-1-\beta} dr}{(\varepsilon^2 r^{\gamma'\sqrt{\bar\mu}} + r^{\gamma/\sqrt{\bar\mu}})^{\sqrt{\bar\mu}}} \\
&\leq C\varepsilon^{\frac{(N-\beta-\gamma')\sqrt{\bar\mu}}{\sqrt{\bar\mu-\mu}} - 2\sqrt{\bar\mu}} \int_0^{1/m\varepsilon^{\frac{\sqrt{\bar\mu}}{\sqrt{\bar\mu-\mu}}}} \frac{\tau^{N-1-\beta} d\tau}{\tau^{\gamma'}\tau^{\frac{2\sqrt{\bar\mu-\mu}}{\sqrt{\bar\mu}}\sqrt{\bar\mu}}} \\
&\leq Cm^{-\gamma'-2+\beta}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\int_\Omega f(u_m^\varepsilon)^2 dx &\geq \int_\Omega f(u_\varepsilon^*)^2 dx - C\varepsilon^{2\sqrt{\bar\mu}} m^{\gamma} m^{-\gamma'-2+\beta} \\
&\geq \int_\Omega f(u_\varepsilon^*)^2 dx - C\varepsilon^{2\sqrt{\bar\mu}} m^{-2+\beta+2\sqrt{\bar\mu-\mu}}.
\end{aligned}
$$

For $\mu \leq \bar\mu - (\frac{2-\beta}{2})^2$ we find the result. $\square$

Now, we prove that the functional J_λ has Linking geometry.

Proposition 3.4. *Suppose that $f \in \mathcal{F}_{2,\beta}$ and there exists $k \in \mathbb{N}^*$ such that $\lambda_k^\mu(f) \leq \lambda < \lambda_{k+1}^\mu(f)$. Then:*

(i) *There exist* $\rho, \alpha > 0$ *such that* $J_\lambda \mid_{\partial B_\rho \cap H^+} \geq \alpha$,

(ii) *There exists* $R > \rho$ *such that* $J_\lambda \mid_{\partial Q_m^\varepsilon} \leq p(m)$ *with* $p(m) \to 0$ *as* $m \to +\infty$ *where* $Q_m^\varepsilon = (\overline{B_R} \cap H_m^-) \oplus \{r.u_m^\varepsilon : 0 < r < R\}$.

Proof. For $u \in H^+$, we have

$$\int_\Omega \Big(|\nabla u|^2 - \frac{\mu}{|x|^2}u^2\Big)dx \geq \lambda_{k+1}^\mu(f)\int_\Omega fu^2dx, \tag{3.8}$$

using (3.8), Hardy and Sobolev inequalities, we obtain

$$\begin{aligned} J_\lambda(u) &\geq \frac{1}{2}\Big(1 - \frac{\lambda}{\lambda_{k+1}^\mu(f)}\Big)\int_\Omega \Big(|\nabla u|^2 - \frac{\mu}{|x|^2}u^2\Big)dx - \frac{1}{2^*}\int_\Omega |u|^{2^*}dx \\ &\geq \frac{1}{2}\Big(1 - \frac{\lambda}{\lambda_{k+1}^\mu(f)}\Big)\big(1 - \frac{\mu}{\bar\mu}\big)|\nabla u|_2^2 - C|u|_{2^*}^{2^*}. \end{aligned}$$

Hence, we can choose $|\nabla u|_2 = \rho$ sufficiently small and $\alpha > 0$ such that

$$J_\lambda \mid_{\partial B_\rho \cap H^+} \geq \alpha.$$

For any $u \in H_m^-$, from the estimates of Lemma 3.2 we obtain

$$\begin{aligned} J_\lambda(u) &\leq C_1 m^{-2\sqrt{\bar\mu-\mu}}\int_\Omega fu^2dx - \frac{1}{2^*}\int_\Omega u^{2^*}dx \\ &\leq C_2 m^{-2\sqrt{\bar\mu-\mu}}|u|_{2^*}^2 - \frac{1}{2^*}|u|_{2^*}^{2^*} \\ &\leq C_3 m^{-N\sqrt{\bar\mu-\mu}}. \end{aligned} \tag{3.9}$$

Consequently,

$$\lim_{m\to\infty}\max_{u\in H_m^-} J_\lambda(u) = 0.$$

On the other hand,

$$J_\lambda(ru_m^\varepsilon) \leq \frac{r^2}{2}\|u_m^\varepsilon\|_\mu - \frac{r^{2^*}}{2^*}|u_m^\varepsilon|_{2^*}^{2^*};$$

then $J_\lambda(ru_m^\varepsilon)$ becomes negative if $r = R$ and R large enough. Therefore

$$J_\lambda(u) \leq Cm^{-N\sqrt{\bar\mu-\mu}} \quad \text{for all } u \in H_m^- \cup \{H_m^- \oplus Ru_m^\varepsilon\}.$$

Since $\max_{0\leq r\leq R} J_\lambda(ru_m^\varepsilon) < +\infty$ as $v \in H_m^- \oplus \mathbb{R}^+u_m^\varepsilon$, we may write $v = u + ru_m^\varepsilon$ with $|\operatorname{supp}(u_m^\varepsilon) \cap \operatorname{supp}(u)| = 0$, then for large R,

$$J_\lambda \mid_{\partial Q_m^\varepsilon} \leq 0.$$

□

4. PROOF OF THEOREM 2.5

Lemma 4.1. *Suppose that* $f \in \mathcal{F}_{2,\beta}$ *and* $\mu \in [0, \bar\mu - (\frac{2-\beta}{2})^2]$. *Then*

$$J_\lambda(t_\varepsilon u_m^\varepsilon) < \frac{1}{N}S_\mu^{N/2} \quad \textit{for } \varepsilon \textit{ small enough.}$$

Proof. Assume by contradiction that for all $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that

$$J_\lambda(t_\varepsilon u_m^\varepsilon) \geq \frac{1}{N}S_\mu^{N/2}, \tag{4.1}$$

then we affirm that there exists a subsequence of (t_ε) such that $t_\varepsilon \to t_0$. If not suppose that $t_\varepsilon \to +\infty$, then $J_\lambda(t_\varepsilon u_m^\varepsilon) \to -\infty$ when $\varepsilon \to 0$, which contradicts (4.1), thus (t_ε) is bounded and there exists $t_0 \geq 0$ such that $t_\varepsilon \to t_0$. If $t_0 = 0$, using the continuity of the embedding, we obtain that $\int_\Omega f u_m^\varepsilon dx$ and $|u_m^\varepsilon|_{2^*}$ are bounded, the same for $\|u_m^\varepsilon\|_\mu$. We have

$$\frac{t_\varepsilon^2}{2}\Big[\int_\Omega |\nabla u_m^\varepsilon|^2 dx - \frac{\mu}{2}\int_\Omega \frac{(u_m^\varepsilon)^2}{|x|^2}dx - \frac{\lambda t_\varepsilon^2}{2}\int_\Omega f(u_m^\varepsilon)^2 dx\Big] - \frac{t_\varepsilon^{2^*}}{2^*}\int_\Omega (u_m^\varepsilon)^{2^*}dx = o(1),$$

which is in contradiction with (4.1). So $t_\varepsilon \to t_0 > 0$. Using (3.5) and (3.6) and letting $\varepsilon \to 0$, it follows that

$$\begin{gathered}\frac{1}{2}\|t_\varepsilon u_m^\varepsilon\|_\mu^2 \le \frac{1}{2}S_\mu^{N/2} + \frac{t_\varepsilon^2-1}{2}S_\mu^{N/2} + C\varepsilon^{N-2}m^{2\sqrt{\bar\mu-\mu}},\\ -\frac{1}{2^*}|t_\varepsilon u_m^\varepsilon|_{2^*}^{2^*} \le -\frac{1}{2^*}S_\mu^{N/2} - \frac{1}{2^*}(t_\varepsilon^{2^*}-1)S_\mu^{N/2} + C\varepsilon^N m^{2N\sqrt{\bar\mu-\mu}/(N-2)}.\end{gathered}$$

By adding these two equations, we obtain

$$\frac{1}{2}\|t_\varepsilon u_m^\varepsilon\|_\mu^2 - \frac{1}{2^*}|t_\varepsilon u_m^\varepsilon|_{2^*}^{2^*} \le \frac{1}{N}S_\mu^{N/2} + \frac{1}{2}\Big(t_\varepsilon^2 - 1 - \frac{N-2}{N}\big(t_\varepsilon^{2^*}-1\big)\Big)S_\mu^{N/2} + C\varepsilon^{N-2}.$$

By the fact that $\max_{x\ge 0}\Big(x^2 - 1 - \frac{N-2}{N}(x^{2^*}-1)\Big) = 0$, we obtain

$$\frac{1}{2}\|t_\varepsilon^2 u_m^\varepsilon\|_\mu^2 - \frac{1}{2^*}\int_\Omega (t_\varepsilon u_m^\varepsilon)^{2^*} \le \frac{1}{N}S_\mu^{N/2} + C\varepsilon^{N-2}.$$

We will estimate $\int_\Omega f(t_\varepsilon u_m^\varepsilon)^2$ for $\mu \le \bar\mu - (\frac{2-\beta}{2})^2$. For $q = 1/2^{1/\gamma'}$, we can take ε small enough so that

$$\varepsilon^{\sqrt{\bar\mu}/\sqrt{\bar\mu-\mu}} < \frac{1}{qm}.$$

Hence there exists $C > 0$ such that

$$\varepsilon^2|x|^{\gamma'/\sqrt{\bar\mu}} + |x|^{\gamma/\sqrt{\bar\mu}} \le C|x|^{\gamma/\sqrt{\bar\mu}}, \quad \forall |x| \ge \varepsilon^{\sqrt{\bar\mu}/\gamma}.$$

and

$$\begin{aligned}\int_\Omega f(t_\varepsilon u_m^\varepsilon)^2 &\ge C\int_{\varepsilon^{\sqrt{\bar\mu}/\gamma}}^{1/qm} r^{-\beta}\Big(u_\varepsilon^*(r) - u_\varepsilon^*(\frac{1}{m})\Big)^2 r^{N-1}dr\\ &\ge C\int_{\varepsilon^{\sqrt{\bar\mu}/\gamma}}^{1/qm} r^{-\beta}(u_\varepsilon^*(r))^2 r^{N-1}dr\\ &\ge CC_\varepsilon^2\int_{\varepsilon^{\sqrt{\bar\mu}/\gamma}}^{1/qm} r^{-\beta}r^{-2\gamma}r^{N-1}dr\\ &\ge CC_\varepsilon^2\int_{\varepsilon^{\sqrt{\bar\mu}/\gamma}}^{1/qm} r^{-\beta+1-2\sqrt{\bar\mu-\mu}}dr.\end{aligned}$$

To continue we distinguish two cases:
(1) $\mu < \bar\mu - (\frac{2-\beta}{2})^2$,

$$\begin{aligned}\int_\Omega f(t_\varepsilon u_m^\varepsilon)^2 dx &\ge C\varepsilon^{2\sqrt{\bar\mu}}\varepsilon^{2(\sqrt{\bar\mu}/\gamma)(2-\beta-2\sqrt{\bar\mu-\mu})}\\ &\ge C\varepsilon^{N-2}\varepsilon^{2(\sqrt{\bar\mu}/\gamma)(2-\beta-2\sqrt{\bar\mu-\mu})}.\end{aligned}$$

(2) $\mu = \bar\mu - (\frac{2-\beta}{2})^2$

$$\begin{aligned}\int_\Omega f(t_\varepsilon u_m^\varepsilon)^2 dx &\ge CC_\varepsilon^2\int_{\varepsilon^{\frac{\sqrt{\bar\mu}}{\gamma}}}^{1/qm} r^{-\beta+1-2\sqrt{\bar\mu-\mu}}dr\\ &\ge C\varepsilon^{2\sqrt{\bar\mu}}|\ln\varepsilon^{2(\sqrt{\bar\mu}/\gamma)}|.\end{aligned}$$

Thus $J_\lambda(t_\varepsilon u_m^\varepsilon) < \frac{1}{N}S_\mu^{N/2}$ for $\mu \in [0, \bar{\mu} - (\frac{2-\beta}{2})^2]$. ☐

Proof of Theorem 2.5. The proof is based on Linking Theorem [1]. We have

$$\inf_{h\in\Gamma} \max_{u\in Q_m^\varepsilon} J_\lambda(h(u)) \le \max_{u\in Q_m^\varepsilon} J_\lambda(u). \tag{4.2}$$

It suffices to show that

$$\max_{u\in Q_m^\varepsilon} J_\lambda(u) < \frac{1}{N}S_\mu^{N/2}$$

Arguing by contradiction, suppose that

$$\max_{u\in Q_m^\varepsilon} J_\lambda(u) \ge \frac{1}{N}S_\mu^{N/2} \quad \forall m \in \mathbb{N}, \quad \forall \varepsilon > 0.$$

Since $\{v \in Q_m^\varepsilon : J_\lambda(v) \ge 0\}$ is a compact set then the upper bound in (4.2) is achieved thus, for all $\varepsilon > 0$ there exist $\omega_\varepsilon \in H_m^-$ and $t_\varepsilon \ge 0$ such that for $v_\varepsilon := \omega_\varepsilon + t_\varepsilon u_m^\varepsilon$, we have

$$J_\lambda(v_\varepsilon) := \sup_{u\in Q_m^\varepsilon} J_\lambda(u) \ge \frac{1}{N}S_\mu^{N/2},$$

i.e.,

$$\frac{1}{2}\|v_\varepsilon\|_\mu^2 - \frac{\lambda}{2}\int_\Omega f v_\varepsilon^2 dx - \frac{1}{2^*}\int_\Omega v_\varepsilon^{2^*} dx \ge \frac{1}{N}S_\mu^{N/2}, \quad \forall \varepsilon > 0. \tag{4.3}$$

Using the proof of Lemma 4.1, we obtain that (t_ε) admits a convergent subsequence, (ω_ε) is bounded and thus

$$t_\varepsilon \to t_0 > 0, \quad \omega_\varepsilon \to \omega_0 \in H_m^-.$$

By the Lemma 3.2 and the fact that $\lambda \in \big(\lambda_k^\mu(f), \lambda_{k+1}^\mu(f)\big)$, we obtain

$$\begin{aligned} J_\lambda(\omega_\varepsilon) &= \frac{1}{2}\|\omega_\varepsilon\|_\mu^2 - \frac{\lambda}{2}\int_\Omega f\omega_\varepsilon^2 dx - \frac{1}{2^*}\int_\Omega \omega_\varepsilon^{2^*} dx \\ &\le \frac{\lambda_k^\mu(f) + o(1)}{2}\big|\omega_\varepsilon\big|_2^2 - \frac{\lambda}{2}\big|\omega_\varepsilon\big|_2^2 \le 0 \end{aligned}$$

for m large enough. Using (4.3) and proceeding in the same way that Lemma 4.1, we obtain

$$J_\lambda\big(v_\varepsilon\big) = J_\lambda\big(\omega_\varepsilon\big) + J_\lambda\big(t_\varepsilon u_m^\varepsilon\big) \le J_\lambda\big(t_\varepsilon u_m^\varepsilon\big) < \frac{1}{N}S_\mu^{N/2}$$

which is absurd. ☐

5. PROOF OF THEOREM 2.6

Let

$$\lambda_+ = \min\{\lambda_j^\mu(f) \in \sigma : \lambda < \lambda_j^\mu(f)\},$$

denote by $M\big(\lambda_j^\mu(f)\big)$ the eigenspace correpsonding to $\lambda_j^\mu(f)$. We put

$$M^+ = \overline{\oplus_{\lambda_j^\mu(f)\ge\lambda_+} M\big(\lambda_j^\mu(f)\big)}^{H_\mu}, \quad M^- = \oplus_{\lambda_j^\mu(f)\le\lambda_+} M\big(\lambda_j^\mu(f)\big),$$

suppose that $\lambda_+ - \lambda < S_\mu(\int_\Omega |x|^{-\beta N/2}dx)^{-2/N}$.

Lemma 5.1. *We have*

$$\beta_\lambda = \sup_{u\in M^-} J_\lambda(u) \le \frac{1}{N}\big(\lambda_+ - \lambda\big)^{N/2}\int_\Omega |x|^{-\beta N/2} < \frac{1}{N}S_\mu^{N/2}.$$

Moreover, there exist $\rho_\lambda > 0$ and $\delta_\lambda \in \big(0, \beta_\lambda\big)$ such that $J_\lambda(u) \ge \delta_\lambda$ for all $u \in M^+$ with $\|u\|_\mu = \rho_\lambda$

Proof. For all $u \in M^-$ we have $\|u\|_\mu^2 \leq \lambda_+ \int_\Omega fu^2dx$. Since M^- is a finite dimension space, using Hölder inequality and knowing that

$$\max_{t\geq 0}(A\frac{t^2}{2} - B\frac{t^{2^*}}{2^*}) = \frac{1}{N}A(\frac{A}{B})^{(N-2)/2} \quad \text{for all } A, B > 0,$$

we obtain

$$\begin{aligned} J_\lambda(u) &= \frac{1}{2}\int_\Omega \left|\nabla u\right|^2 dx - \frac{\mu}{2}\int_\Omega \frac{u^2}{\left|x\right|^2}dx - \frac{\lambda}{2}\int_\Omega fu^2dx - \frac{1}{2^*}\int_\Omega |u|^{2^*}dx \\ &\leq \frac{1}{2}\big(\lambda_+ - \lambda\big)\int_\Omega fu^2dx - \frac{1}{2^*}\int_\Omega |u|^{2^*}dx \\ &\leq \frac{1}{2}\big(\lambda_+ - \lambda\big)\int_\Omega |x|^{-\beta}u^2dx - \frac{1}{2^*}\int_\Omega |u|^{2^*}dx \\ &\leq \int_\Omega \max_{t\geq 0}\Big(\frac{1}{2}\big(\lambda_+ - \lambda\big)|x|^{-\beta}t^2 - \frac{1}{2^*}t^{2^*}\Big)dx. \end{aligned}$$

Let $u \in M^+$, by the inequality $\|u\|_\mu^2 \geq \lambda_+ \int_\Omega fu^2dx$ and $\|u\|_\mu^2 \geq S_\mu\big|u\big|_{2^*}^2$, we have

$$\begin{aligned} J_\lambda(u) &\geq \frac{\lambda_+ - \lambda}{2\lambda_+}\|u\|_\mu^2 - \frac{1}{S_\mu^{2/2^*}2^*}\|u\|_\mu^{2^*} \\ &\geq \max_{t\geq 0}\Big(\frac{\lambda_+ - \lambda}{2\lambda_+}t^2 - \frac{1}{S_\mu^{2/2^*}2^*}t^{2^*}\Big) \\ &= \frac{1}{N}\Big(\frac{\lambda_+ - \lambda}{2\lambda_+}\Big)^{N/2}S_\mu^{N/2}. \end{aligned}$$

If we take

$$\rho_\lambda = \Big(\big(\frac{\lambda_+ - \lambda}{\lambda_+}\big)S_\mu^{2/2^*}\Big)^{(N-2)/4}, \quad \delta_\lambda < \frac{1}{N}\big(\frac{\lambda_+ - \lambda}{\lambda_+}\big)^{N/2}S_\mu^{N/2},$$

then we obtain $J_\lambda(u) \geq \delta_\lambda$ for all $u \in M^+ \cap \partial B_{\rho_\lambda}$. It remains to show that $\delta_\lambda < \beta_\lambda$. Indeed, since $M^+ \cap M^- = M\big(\lambda_+\big)$, we have $M^+ \cap M^- \cap B_{\rho_\lambda} \neq \emptyset$ and all $u \in M^+ \cap M^- \cap B_{\rho_\lambda}$ satisfies $\delta_\lambda < J_\lambda(u) \leq \beta_\lambda = \sup_{u\in M^-} J_\lambda(u)$.

To complete the proof it suffices to apply [4, Theorem 2.5] with $H = H_\mu$, $W = M^-$ and $V = M^+$, $\beta = \frac{1}{N}S_\mu^{N/2}$, $\delta = \delta_\lambda$, $\beta' = \beta_\lambda$, $\rho = \rho_\lambda$. □

6. Proof of Theorem 2.7

Proposition 6.1. *Let* $f \in \mathcal{F}_{2,\beta}$ *and* $\mu < \bar{\mu} - (\frac{N+2}{N})^2(\frac{2-\beta}{2})^2$. *Then for all* $\lambda > 0$, $c < \frac{1}{N}S_\mu^{\frac{N}{2}}$.

Proof. Without loss of generality, we can assume that there exists k such that $\lambda_k^\mu(f) \leq \lambda < \lambda_{k+1}^\mu(f)$.

Let $\max_{u\in Q_m^\varepsilon} J_\lambda(u) = J_\lambda(w_m + t_m^\varepsilon u_m^\varepsilon)$, where $w_m \in H_m^-$. Using the same calculation as in the second point of Proposition 3.4, we have

$$J_\lambda(w_m) \leq Cm^{-N\sqrt{\bar{\mu}-\mu}}.$$

By choosing $\varepsilon = m^{-\frac{N+2}{N-2}\sqrt{\bar{\mu}-\mu}}$,

$$\int_\Omega \big(|\nabla u_m^\varepsilon|^2 - \mu\frac{(u_m^\varepsilon)^2}{|x|^2}\big)dx \leq S_\mu^{N/2} + Cm^{-N\sqrt{\bar{\mu}-\mu}},$$

$$\int_\Omega (u_m^\varepsilon)^{2^*}dx \geq S_\mu^{N/2} - Cm^{(-N^2/(N-2))\sqrt{\bar{\mu}-\mu}},$$
$$\int_\Omega f(u_m^\varepsilon)^2 dx \geq Cm^{-(N+2)(\frac{2-\beta}{2})}.$$

and

$$\begin{aligned}
c &\leq \max_{u\in Q_m^\varepsilon} J_\lambda(u)\\
&\leq J_\lambda(w_m + t_m^\varepsilon u_m^\varepsilon)\\
&\leq J_\lambda(w_m) + J_\lambda(t_m^\varepsilon u_m^\varepsilon)\\
&\leq Cm^{-N\sqrt{\bar{\mu}-\mu}} + \frac{(t_m^\varepsilon)^2}{2}\int_\Omega(|\nabla u_m^\varepsilon|^2 - \mu\frac{(u_m^\varepsilon)^2}{|x|^2} - \lambda\int_\Omega f(u_m^\varepsilon)^2)dx\\
&\quad - \frac{(t_m^\varepsilon)^{2^*}}{2^*}\int_\Omega (u_m^\varepsilon)^{2^*}dx\\
&\leq Cm^{-N\sqrt{\bar{\mu}-\mu}} + \frac{(t_m^\varepsilon)^2}{2}(S_\mu^{\frac{N}{2}} + Cm^{-N\sqrt{\bar{\mu}-\mu}} - Cm^{-(N+2)(\frac{2-\beta}{2})})\\
&\quad - \frac{(t_m^\varepsilon)^{2^*}}{2^*}(S_\mu^{\frac{N}{2}} - Cm^{-\frac{N^2}{N-2}\sqrt{\bar{\mu}-\mu}})\\
&\leq Cm^{-N\sqrt{\bar{\mu}-\mu}} + \frac{1}{N}(S_\mu^{\frac{N}{2}} + Cm^{-N\sqrt{\bar{\mu}-\mu}} - Cm^{-(N+2)(\frac{2-\beta}{2})})\\
&\quad \times\Big(\frac{S_\mu^{\frac{N}{2}} + Cm^{-N\sqrt{\bar{\mu}-\mu}} - Cm^{-(N+2)(\frac{2-\beta}{2})}}{S_\mu^{\frac{N}{2}} - Cm^{-\frac{N^2}{N-2}\sqrt{\bar{\mu}-\mu}}}\Big)^{\frac{N-2}{2}}.
\end{aligned}$$

Note that for $\mu < \bar{\mu} - (\frac{N+2}{N})^2(\frac{2-\beta}{2})^2$, we have $(N+2)(\frac{2-\beta}{2}) < N\sqrt{\bar{\mu}-\mu} < \frac{N^2}{N-2}\sqrt{\bar{\mu}-\mu}$ and we deduce that

$$c \leq \frac{1}{N}S_\mu^{N/2} + Cm^{-N\sqrt{\bar{\mu}-\mu}} - Cm^{-(N+2)(\frac{2-\beta}{2})} < \frac{1}{N}S_\mu^{N/2}.$$

□

Proof of Theorem 2.7. From Lemma 3.1 and Proposition 6.1, J_λ satisfies the hypotheses of Linking Theorem [1], moreover $\partial B_\rho \cap H^+$ and ∂Q_m^ε are linked. Hence c is a critical value of J_λ and u is a nontrivial solution of the problem (1.1). □

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