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OSCILLATION OF SOLUTION TO SECOND-ORDER HALF-LINEAR DELAY DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. This article concerns the oscillation of solutions to second-order half-linear dynamic equations with a variable delay. By using integral averaging techniques and generalized Riccati transformations, new oscillation criteria are obtained. Our results extend Kamenev-type, Philos-type and Li-type oscillation criteria. Several examples are given to illustrate our results.

1. Introduction

Consider the second-order half-linear dynamic equation with a variable delay

$$\big(r(t)\big(x^{\Delta}(t)\big)^{\gamma}\big)^{\Delta} + q(t)f(x(\tau(t))) = 0, \quad t \geq t_0, \tag{1.1}$$

where the independent variable is in a time scale $\mathbb{T}$. Since we are interested in the oscillatory behavior of solutions near infinity, we assume that $\sup \mathbb{T} = \infty$. Recall that a solution to (1.1) is a nontrivial real function $x(t)$ such that $x(t) \in C^1_{rd}[b,\infty)$, and $r(t)\big(x^{\Delta}(t)\big)^{\gamma} \in C^1_{rd}[b,\infty)$ and satisfying (1.1) on $[c,\infty)$, where $c > b$ is chosen so that $\tau(t) \geq b$ for $t \geq c$, and C_{rd} is the space of real-valued right-dense continuous functions (see [4]). Throughout this paper, we shall restrict attention to those solutions of (1.1) which exist on some half line $[c,\infty)$ and satisfy $\sup\{|x(t)| : t > d\} > 0$ for any $d > c$. For simplicity of notation in the lemmas, theorems, and examples that follow, we use $[t_0,\infty) := [t_0,\infty)_{\mathbb{R}} \cap \mathbb{T}$ and $(x(\sigma(t)))^{\gamma} = (x^{\sigma}(t))^{\gamma} = (x^{\gamma}(t))^{\sigma}$.

The oscillation theory of difference and functional differential equations has been developed extensively during the past several years. We refer the reader to [2, 6, 10, 9, 13, 14, 16, 17, 18, 21, 23, 24, 25, 27] as well as the references cited therein. Recently, there has been an increasing interest in studying the oscillation of dynamic equations on time scales [1, 3, 5, 7, 8, 11, 12, 15, 19, 20, 22, 26, 28, 29, 30]. The oscillation problem for (1.1) and its various particular cases has been studied extensively. An important tool in the study of oscillatory behavior of solutions is the integral averaging technique which goes back as far as the classical results of Wintner [25] and Hartman [10] giving a sufficient condition for oscillation of the

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linear differential equation of the form

$$x''(t) + q(t)x(t) = 0, \quad t \in \mathbb{R}. \tag{1.2}$$

Another technique to study the oscillation problem involves the Riccati transformation

$$\omega(t) = r(t)\frac{x'(t)}{x(t)}, \quad t \in \mathbb{R}, \tag{1.3}$$

which is used to reduce the higher order equations to the first order Riccati equation (or inequality) (see [13, 16]). The result of Wintner [25] was improved by Kamenev [13] in 1978, and one of the main results is as follows.

Theorem 1.1 (Kamenev-type oscillation criteria). *Equation* (1.2) *is oscillatory, if*

$$\limsup_{t\to\infty} \frac{1}{t^n}\int_{t_0}^{t} (t-s)^n q(s)ds = \infty, \quad \text{for some } n > 1. \tag{1.4}$$

Theorem 1.1 has been extended by several authors. In 1989, Philos [16] obtained new results on oscillation by replacing the kernel function $(t-s)^n$ by a general class of functions $H(t,s)$. The following is the main result by Philos.

Theorem 1.2 (Philos-type oscillation criteria). *Let $D_0 = \{(t,s) : t > s \geq t_0,\ t,\ s \in \mathbb{R}\}$ and $D = \{(t,s) : t \geq s \geq t_0,\ t,\ s \in \mathbb{R}\}$. Suppose that there exist functions $H \in C(D,\mathbb{R})$ and $h \in C(D_0,\mathbb{R})$ which satisfy the following three conditions:*

(i) *$H(t,t) = 0$ for $t \geq t_0$, $H(t,s) > 0$ for all $(t,s) \in D_0$;*
(ii) *$H_s(t,s) \leq 0$ for all $(t,s) \in D_0$;*
(iii) *$-H_s(t,s) = h(t,s)\sqrt{H(t,s)}$, for all $(t,s) \in D_0$.*

If

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\int_{t_0}^{t} \big[H(t,s)q(s) - \frac{1}{4}h^2(t,s)\big]ds = \infty, \tag{1.5}$$

then (1.2) *is oscillatory.*

Theorems 1.1 and 1.2 cannot be applied to the Euler differential equation

$$x''(t) + \frac{\lambda}{t^2}x(t) = 0, \quad t \in \mathbb{R}, \tag{1.6}$$

where $\lambda > 0$ is a constant. In fact, (1.6) is oscillatory if $\lambda > 1/4$ and nonoscillatory if $\lambda \leq 1/4$. In 1995, Li [14] considered the linear differential equation

$$(r(t)x'(t))' + q(t)x(t) = 0, \quad t \in \mathbb{R}. \tag{1.7}$$

To improve the oscillation criteria of Philos [16] and Yan [27], Li [14] used the generalized Riccati transformation

$$\omega(t) = \Phi(t)r(t)\Big(\frac{x'(t)}{x(t)} + \phi(t)\Big), \quad t \in \mathbb{R}. \tag{1.8}$$

This has also been used to study other types of equations in [24]. Here it is assumed that $\Phi(t) > 0$ and $\phi(t)$ are differentiable functions. Using these ideas one is able to obtain some new sufficient conditions for oscillation which can be applied to equations which cannot be treated by the results using the Riccati transformation (1.3). As is pointed out in Li [14], by using the generalized Riccati transformation (1.8), Kamenev-type oscillation criteria can be applied to the Euler differential equation (1.6). The following is the main result by Li [14].

Theorem 1.3 (Li-type oscillation criteria). *Let $D_0 = \{(t,s) : t > s \geq t_0,\ t,\ s \in \mathbb{R}\}$ and $D = \{(t,s) : t \geq s \geq t_0,\ t,\ s \in \mathbb{R}\}$. Let $H \in C(D,\mathbb{R})$ satisfy the following two conditions:*

- (i) *$H(t,t) = 0$ for $t \geq t_0$, $H(t,s) > 0$ for all $(t,s) \in D_0$;*
- (ii) *$H_s(t,s) \leq 0$ for all $(t,s) \in D_0$.*

Suppose that $h \in C(D_0,\mathbb{R})$ is a continuous function with

$$-H_s(t,s) = h(t,s)\sqrt{H(t,s)} \quad \text{for all } (t,s) \in D_0.$$

Assume that there exists a function $g \in C^1[t_0,\infty)$ such that

$$\int_{t_0}^{t} a(s)r(s)h^2(t,s)ds < \infty \quad \text{for all } t \geq t_0, \tag{1.9}$$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \Big[H(t,s)\psi(s) - \frac{1}{4}a(s)r(s)h^2(t,s)\Big]ds = \infty, \tag{1.10}$$

where $a(s) = \exp\{-2\int^s g(\xi)d\xi\}$ and $\psi(s) = a(s)\{q(s) + r(s)g^2(s) - [r(s)g(s)]'\}$. Then (1.7) is oscillatory.

In 1996, Rogovchenko [17] proved that Theorem 1.3 holds without assumption (1.9). For the case involving a delay, we use the modified Riccati transformation

$$\omega(t) = \Phi(t)r(t)\Big(\frac{x'(t)}{x(\tau(t))} + \phi(t)\Big), \quad t \in \mathbb{R}. \tag{1.11}$$

In 2008, by using the generalized Riccati transformation

$$\omega(t) = \Phi(t)r(t)\Big(\Big(\frac{x^\Delta(t)}{x(t)}\Big)^\gamma + \phi(t)\Big), \quad t \in \mathbb{T}, \tag{1.12}$$

with $\phi(t) = 0$, Hassan [11] considered the second-order half-linear dynamic equation without delay

$$\Big(r(t)\big(x^\Delta(t)\big)^\gamma\Big)^\Delta + q(t)x^\gamma(t) = 0, \quad t \in \mathbb{T}, \tag{1.13}$$

where γ is the quotient of odd positive integers, $r(t)$ and $q(t)$ are positive rd-continuous functions on $\mathbb{T}$.

In the following, we will consider the second-order nonlinear dynamic equation with a variable delay on time scales. We will employ the generalized Riccati transformation

$$\omega(t) = \Phi(t)r(t)\Big(\Big(\frac{x^\Delta(t)}{x(\tau(t))}\Big)^\gamma + \phi(t)\Big), \quad t \in \mathbb{T}. \tag{1.14}$$

Our goal here is to establish oscillation criteria for (1.1) under very mild conditions. That is, we do not assume that any of the following conditions:

- $\gamma \geq 1$, see e.g. [8, 26];
- $r^\Delta(t) \geq 0$, see e.g. [28];
- $\int_{t_0}^{\infty} \tau(t)q(t)\Delta t = \infty$, see e.g. [7];
- $\tilde{\mathbb{T}} := \tau(\mathbb{T}) \subset \mathbb{T}$, and $\sigma \circ \tau = \tau \circ \sigma$, see e.g. [5, 26].

Rather we assume that

- (H1) $\gamma > 0$ is a quotient of odd positive integers;
- (H2) $\tau \in C_{rd}(\mathbb{T},\mathbb{R})$ is strictly increasing, $\tau(t) \leq t$ and $\lim_{t\to\infty} \tau(t) = \infty$;
- (H3) $q \in C_{rd}(\mathbb{T},\mathbb{R})$ is nonnegative for $t \geq t_0$ and not identically zero on any half-line of the form $[t_*,\infty)$;

(H4) $f \in C(\mathbb{R},\mathbb{R})$ satisfies $f(x)/x^\gamma \geq L$ for some positive constant L and all $x \neq 0$;

(H5) $r \in C_{rd}(\mathbb{T},\mathbb{R}^+)$ satisfies $\int_{t_0}^\infty \left(\frac{1}{r(s)}\right)^{1/\gamma}\Delta s = \infty$; or

(H5') $r \in C_{rd}(\mathbb{T},\mathbb{R}^+)$ satisfies $\int_{t_0}^\infty \left(\frac{1}{r(s)}\right)^{1/\gamma}\Delta s < \infty$.

In other words, by careful observation and calculation, we will show that we can obtain similar results without introducing the term $\phi(t)$ in (1.11) or (1.14). Our results extend, improve and unify a number of other existing results and handle some cases which are not covered by known criteria.

In the next section, we shall give several important lemmas, which will be used to prove our main results. In Section 3, we shall establish several new oscillation criteria for (1.1). Finally, in Section 4, by means of several examples, we illustrate our results.

2. Preliminary results on time scales

The following lemmas will be needed in the proofs of our results. Lemma 2.1 can be found in [5, 8]. Lemma 2.2 can be found in [4, Theorem 1.14]. Lemma 2.3 is similar to Zhang and Wang [28, Lemma 2.3].

Lemma 2.1. *Assume condition* (H1) *holds and* $x^\gamma(t) \in C^1_{rd}([b,\infty)_\mathbb{T},\mathbb{R})$*. Then*

$$(x^\gamma(t))^\Delta \geq \begin{cases} \gamma\left(x^\sigma(t)\right)^{\gamma-1}(t)x^\Delta(t), & 0<\gamma\leq 1,\\ \gamma\left(x(t)\right)^{\gamma-1}(t)x^\Delta(t), & \gamma \geq 1.\end{cases}$$

Lemma 2.2 (Mean Value Theorem)**.** *Let f be a continuous function on $[a,b]$ that is differentiable on $[a,b)$. Then there exist η, $\xi \in [a,b)$ such that*

$$f^\Delta(\xi) \leq \frac{f(b)-f(a)}{b-a} \leq f^\Delta(\eta).$$

Lemma 2.3. *Let $\psi(u) = a_0u - b_0(u-c_0)^{(\gamma+1)/\gamma}$ where $\gamma > 0$ is a quotient of odd positive integers, a_0 and $c_0 \in \mathbb{R}$, and $b_0 > 0$. Then $\psi(u)$ attains its maximum value at $u^* = c_0 + \left(\frac{a_0\gamma}{b_0(\gamma+1)}\right)^\gamma$, and*

$$\max_{u\in\mathbb{R}} \psi(u) = \psi(u^*) = a_0c_0 + \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}}\frac{a_0^{\gamma+1}}{b_0^\gamma}.$$

The proof of the above lemma is simple, it can be obtained directly through a change of variables from Zhang and Wang [28, Lemma 2.3]. We omit it.

3. Main results

Theorem 3.1. *Assume that conditions* (H1)–(H5) *hold. Also assume that there exists a function $\Phi \in C^1_{rd}(\mathbb{T},\mathbb{R}^+)$ such that*

$$\limsup_{t\to\infty}\Big\{L\Phi(t)\int_t^\infty q(s)\Delta s + \int_{t_0}^t \Big[L\Phi(s)q(s) - \frac{r^*(s)\big(\Phi_+^\Delta(s)\big)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\big(\tau^\Delta(s)\Phi(s)\big)^\gamma}\Big]\Delta s\Big\} = \infty, \tag{3.1}$$

where $\Phi_+^\Delta(s) = \max\{\Phi^\Delta(s),0\}$ and $r^*(s) = \max\{r(\xi) | \tau(s) \leq \xi < \tau^\sigma(s)\}$. Then (1.1) *is oscillatory.*

Proof. Suppose to the contrary that (1.1) has a nonoscillatory solution on $[t_0, \infty)$. Without loss of generality, we may assume that $x(t) > 0$ for $t \geq t_0$. From condition (H1), we shall only consider this case, since the substitution $z(t) = -x(t)$ transforms (1.1) into an equation of the same form. Then there exists $t_1 \geq t_0$ such that $x(\tau(t)) > 0$ when $t \geq t_1$. In view of (1.1), conditions (H3) and (H4), we immediately get

$$\big(r(t)\big(x^{\Delta}(t)\big)^{\gamma}\big)^{\Delta} = -q(t)f(x(\tau(t))) \leq -Lq(t)\,(x(\tau(t)))^{\gamma} \leq 0 \quad \text{for } t \geq t_1. \tag{3.2}$$

So $r(t)(x^{\Delta}(t))^{\gamma}$ is eventually of one sign. We assert that $r(t)(x^{\Delta}(t))^{\gamma} > 0$ for $t \geq t_1$. To see this, we suppose not. Then there exists a $t_2 \geq t_1$ such that $r(t_2)(x^{\Delta}(t_2))^{\gamma} = \alpha \leq 0$ and $r(t)(x^{\Delta}(t))^{\gamma} \leq \alpha$ for all $t \geq t_2$. If $\alpha = 0$ and $r(t)(x^{\Delta}(t))^{\gamma} = 0$ for all $t \geq t_2$, from condition (H3) and (3.2), we have $f(x(t)) \equiv 0$, which contradicts the fact that $f(x) > 0$ for $x > 0$. Therefore it follows that $\alpha < 0$. From condition (H5) we have

$$x(t) \leq x(t_2) + \alpha^{1/\gamma} \int_{t_2}^{t} \Big(\frac{1}{r(s)}\Big)^{1/\gamma} \Delta s \to -\infty \quad \text{as} \quad t \to \infty,$$

which contradicts the fact that $x(t) > 0$. Thus we have

$$x(t) > 0, \quad x^{\Delta}(t) > 0 \quad \text{and} \quad \big(r(t)(x^{\Delta}(t))^{\gamma}\big)^{\Delta} \leq 0 \quad \text{for } t \geq t_1. \tag{3.3}$$

Define

$$\omega(t) = \Phi(t)r(t)\Big(\frac{x^{\Delta}(t)}{x(\tau(t))}\Big)^{\gamma} \quad \text{for } t \geq T \geq t_1. \tag{3.4}$$

From (1.1), (3.2) and (3.3), we see that

$$r(t)(x^{\Delta}(t))^{\gamma} \geq L\int_t^{\infty} q(s)x^{\gamma}(\tau(s))\Delta s \geq Lx^{\gamma}(\tau(t))\int_t^{\infty} q(s)\Delta s.$$

It follows that

$$\omega(t) = \Phi(t)r(t)\Big(\frac{x^{\Delta}(t)}{x(\tau(t))}\Big)^{\gamma} \geq L\Phi(t)\int_t^{\infty} q(s)\Delta s > 0 \quad \text{for} \quad t \geq T. \tag{3.5}$$

Now, by the product rule and the quotient rule, from (3.2), (3.4) and (3.5), we obtain

$$\begin{aligned}
\omega^{\Delta} &= [r(x^{\Delta})^{\gamma}]^{\Delta}\frac{\Phi}{(x\circ\tau)^{\gamma}} + [r(x^{\Delta})^{\gamma}]^{\sigma}\Big[\frac{\Phi}{(x\circ\tau)^{\gamma}}\Big]^{\Delta} \\
&= \Phi\frac{[r(x^{\Delta})^{\gamma}]^{\Delta}}{(x\circ\tau)^{\gamma}} + [r(x^{\Delta})^{\gamma}]^{\sigma}\Big[\frac{\Phi^{\Delta}}{(x\circ\tau^{\sigma})^{\gamma}} - \frac{\Phi[(x\circ\tau)^{\gamma}]^{\Delta}}{(x\circ\tau)^{\gamma}(x\circ\tau^{\sigma})^{\gamma}}\Big] \\
&\leq -L\Phi q + \frac{\Phi^{\Delta}}{\Phi^{\sigma}}\omega^{\sigma} - \Phi\frac{[r(x^{\Delta})^{\gamma}]^{\sigma}[(x\circ\tau)^{\gamma}]^{\Delta}}{(x\circ\tau)^{\gamma}(x\circ\tau^{\sigma})^{\gamma}} \\
&\leq -L\Phi q + \frac{\Phi_{+}^{\Delta}}{\Phi^{\sigma}}\omega^{\sigma} - \Phi\frac{[r(x^{\Delta})^{\gamma}]^{\sigma}[(x\circ\tau)^{\gamma}]^{\Delta}}{(x\circ\tau)^{\gamma}(x\circ\tau^{\sigma})^{\gamma}}.
\end{aligned} \tag{3.6}$$

By Lemma 2.1, we obtain

$$[(x\circ\tau)^{\gamma}]^{\Delta} \geq \begin{cases} \gamma\,(x\circ\tau^{\sigma})^{\gamma-1}\,(x\circ\tau)^{\Delta}, & 0 < \gamma \leq 1, \\ \gamma(x\circ\tau)^{\gamma-1}(x\circ\tau)^{\Delta}, & \gamma \geq 1. \end{cases} \tag{3.7}$$

Fix $t \in \mathbb{T}^\kappa$. If $\sigma(t) > t$, by Lemma 2.2, we obtain

$$\begin{aligned}(x \circ \tau)^\Delta(t) &= \frac{(x \circ \tau^\sigma)(t) - (x \circ \tau)(t)}{\sigma(t) - t} \\ &= \frac{(x \circ \tau^\sigma)(t) - (x \circ \tau)(t)}{\tau^\sigma(t) - \tau(t)} \tau^\Delta(t) \\ &\geq x^\Delta(\xi)\tau^\Delta(t),\end{aligned} \tag{3.8}$$

where $\xi \in [\tau(t),\ \tau^\sigma(t))$. If $\sigma(t) = t$, by condition (H2), we obtain $\sigma(\tau(t)) = \tau(\sigma(t)) = \tau(t)$ and

$$(x \circ \tau)^\Delta(t) = x'(\tau(t))\tau'(t), \tag{3.9}$$

Using (3.7), (3.8) and (3.9) in (3.6), we have

$$\begin{aligned}\omega^\Delta &\leq -L\Phi q + \frac{\Phi_+^\Delta}{\Phi^\sigma}\omega^\sigma - \begin{cases} \gamma\Phi \frac{[r(x^\Delta)^\gamma]^\sigma (x\circ\tau^\sigma)^{\gamma-1} x^\Delta(\xi)\tau^\Delta}{(x\circ\tau)^\gamma (x\circ\tau^\sigma)^\gamma}, & 0 < \gamma \leq 1 \\ \gamma\Phi \frac{\left[r(x^\Delta)^\gamma\right]^\sigma (x\circ\tau)^{\gamma-1} x^\Delta(\xi)\tau^\Delta}{(x\circ\tau)^\gamma (x\circ\tau^\sigma)^\gamma}, & \gamma \geq 1 \end{cases} \\ &= -L\Phi q + \frac{\Phi_+^\Delta}{\Phi^\sigma}\omega^\sigma - \begin{cases} \gamma\Phi\tau^\Delta \frac{[r(x^\Delta)^\gamma]^\sigma}{(x\circ\tau^\sigma)^{\gamma+1}} \frac{(x\circ\tau^\sigma)^\gamma}{(x\circ\tau)^\gamma} x^\Delta(\xi), & 0 < \gamma \leq 1, \\ \gamma\Phi\tau^\Delta \frac{[r(x^\Delta)^\gamma]^\sigma}{(x\circ\tau^\sigma)^{\gamma+1}} \frac{x\circ\tau^\sigma}{x\circ\tau} x^\Delta(\xi), & \gamma \geq 1. \end{cases}\end{aligned} \tag{3.10}$$

From (3.3) and condition (H2), it is easy to see that $(x \circ \tau^\sigma)(t) \geq (x \circ \tau)(t)$. Therefore, for $\gamma > 0$, from (3.10), we obtain

$$\omega^\Delta \leq -L\Phi q + \frac{\Phi_+^\Delta}{\Phi^\sigma}\omega^\sigma - \gamma\Phi\tau^\Delta \frac{[r(x^\Delta)^\gamma]^\sigma}{(x \circ \tau^\sigma)^{\gamma+1}} x^\Delta(\xi), \tag{3.11}$$

where $\xi \in [\tau(t),\ \tau^\sigma(t))$. From (3.3) and condition (H2), we have

$$r(\xi)\left(x^\Delta(\xi)\right)^\gamma \geq r(\tau^\sigma(t))\left(x^\Delta(\tau^\sigma(t))\right)^\gamma \geq r(\sigma(t))\left(x^\Delta(\sigma(t))\right)^\gamma.$$

Therefore,

$$x^\Delta(\xi) \geq \left(r(\sigma(t))\left(x^\Delta(\sigma(t))\right)^\gamma\right)^{1/\gamma} (r^*(t))^{-1/\gamma}, \tag{3.12}$$

where $r^*(t) = \max\{r(\xi) | \tau(t) \leq \xi < \tau^\sigma(t)\}$. Using (3.12) in (3.11), we have

$$\begin{aligned}\omega^\Delta &\leq -L\Phi q + \frac{\Phi_+^\Delta}{\Phi^\sigma}\omega^\sigma - \gamma\Phi\tau^\Delta \Big[\frac{\left[r(x^\Delta)^\gamma\right]^{\frac{\gamma+1}{\gamma}}}{(x \circ \tau)^{\gamma+1}}\Big]^\sigma (r^*)^{-1/\gamma} \\ &= -L\Phi q + \frac{\Phi_+^\Delta}{\Phi^\sigma}\omega^\sigma - \gamma\Phi\tau^\Delta \Big[\frac{\omega^\sigma}{\Phi^\sigma}\Big]^{1+\frac{1}{\gamma}} (r^*)^{-1/\gamma} \\ &= -L\Phi q + \frac{\Phi_+^\Delta}{\Phi^\sigma}\omega^\sigma - \gamma\Phi\tau^\Delta (r^*)^{-1/\gamma} (\Phi^\sigma)^{-\frac{\gamma+1}{\gamma}} \left(\omega^\sigma\right)^{\frac{\gamma+1}{\gamma}}.\end{aligned} \tag{3.13}$$

Let

$$a_0 = \frac{\Phi_+^\Delta}{\Phi^\sigma}, \quad b_0 = \gamma\Phi\tau^\Delta (r^*)^{-1/\gamma} (\Phi^\sigma)^{-\frac{\gamma+1}{\gamma}}, \quad c_0 = 0. \tag{3.14}$$

From Lemma 2.3, (3.13) and (3.14), we have

$$\omega^\Delta \leq -L\Phi q + \frac{r^*(\Phi_+^\Delta)^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\Phi\tau^\Delta)^\gamma} \quad \text{for } t \geq T.$$

Integrating the above inequality from T to t, we obtain

$$\omega(t) \leq \omega(T) - \int_T^t \Big[L\Phi(s)q(s) - \frac{r^*(s)\left(\Phi_+^\Delta(s)\right)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\left(\tau^\Delta(s)\Phi(s)\right)^\gamma}\Big]\Delta s.$$

From (3.5), we have

$$\Phi(t)\int_t^\infty q(s)\Delta s+\int_T^t\Big[L\Phi(s)q(s)-\frac{r^*(s)\left(\Phi_+^\Delta(s)\right)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\left(\tau^\Delta(s)\Phi(s)\right)^\gamma}\Big]\Delta s\le\omega(T).$$

Taking the lim sup on both sides of the above inequality as $t\to\infty$, we obtain a contradiction to condition (3.1). This completes the proof of Theorem 3.1. $\square$

Let $\Phi(t)=t$, then $\Phi^\Delta(t)=1$ and Theorem 3.1 yields the following result.

Corollary 3.2. *Suppose that conditions* (H1)–(H5) *hold. If*

$$\limsup_{t\to\infty}\Big\{Lt\int_t^\infty q(s)\Delta s+\int_{t_0}^t\big[Lsq(s)-\frac{r^*(s)}{(\gamma+1)^{\gamma+1}\left(\tau^\Delta(s)\right)^\gamma s^\gamma}\big]\Delta s\Big\}=\infty,$$

where $r^*(s)=\max\{r(\xi)|\tau(s)\le\xi<\tau^\sigma(s)\}$, *then* (1.1) *is oscillatory.*

Remark 3.3. Theorem 3.1 is new, since we have the term $\Phi(t)\int_t^\infty q(s)\Delta s$ in (3.1). It should be noted that the term $\Phi(t)\int_t^\infty q(s)\Delta s$ in (3.1) is important, and Theorem 3.1 can be applied to different equations which cannot be covered by the results established in [1, 3, 5, 7, 8, 10, 11, 13, 14, 15, 16, 17, 19, 20, 22, 23, 24, 25, 26, 28, 29]. We shall illustrate the importance of this term in Example 4.1.

Theorem 3.4. *Suppose that conditions* (H1)–(H5) *hold. Let* $D_0=\{(t,s):t>s\ge t_0,\ t,\ s\in\mathbb{T}\}$ *and* $D=\{(t,s):t\ge s\ge t_0,\ t,\ s\in\mathbb{T}\}$*. Moreover, suppose that there exist functions* $H\in C(D,\mathbb{R})$, $h\in C(D_0,\mathbb{R})$, *and* $\Phi(t)\in C^1_{rd}(\mathbb{T},\mathbb{R}^+)$, *such that the following three conditions hold:*

- (i) $H(t,t)=0$ *for all* $t\ge t_0$, $H(t,s)>0$ *for all* $(t,s)\in D_0$;
- (ii) H *has a continuous and non-positive partial derivative on* D_0 *with respect to the second variable;*
- (iii) $-[H(t,s)\Phi(s)]^{\Delta_s}=h(t,s)[H(t,s)\Phi(s)]^{\frac{\gamma}{\gamma+1}}$, *for all* $(t,s)\in D_0$.

If

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t\Big[LH(t,s)\Phi(s)q(s)-\frac{r^*(s)h^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}(\tau^\Delta(s))^\gamma}\Big]\Delta s=\infty,\quad(3.15)$$

where $r^*(s)=\max\{r(\xi)|\tau(s)\le\xi<\tau^\sigma(s)\}$, *then* (1.1) *is oscillatory.*

Proof. Suppose to the contrary that $x(t)$ is a nonoscillatory solution of (1.1). Define $\omega(t)$ as in (1.14), where $\phi\in C^1_{rd}(\mathbb{T},\mathbb{R})$. We see that

$$\omega(t)=\Phi(t)r(t)\Big(\Big(\frac{x^\Delta(t)}{x(\tau(t))}\Big)^\gamma+\phi(t)\Big)\ge\Phi(t)r(t)\phi(t)\quad\text{for}\quad t\ge T\ge t_1.\quad(3.16)$$

In a manner similar to the proof of Theorem 3.1, we can prove the inequality

$$\omega^\Delta\le-L\Phi q+\Phi(r\phi)^\Delta+\frac{\Phi^\Delta}{\Phi^\sigma}\omega^\sigma-\gamma\Phi\tau^\Delta(r^*)^{-1/\gamma}(\Phi^\sigma)^{-\frac{\gamma+1}{\gamma}}\left(\omega^\sigma-(r\phi\Phi)^\sigma\right)^{\frac{\gamma+1}{\gamma}}.\quad(3.17)$$

Multiplying (3.17) (with t replaced by s) by $H(t,s)$, integrating with respect to s from T to t for $t\ge T\ge t_2$, using the following integration by parts formula (see [4]),

$$\int_a^b f(t)g^\Delta(t)\Delta t=[f(t)g(t)]_a^b-\int_a^b f^\Delta(t)g^\sigma(t)\Delta t,\quad(3.18)$$

and rearranging the terms, by condition (i) and (iii) we find that

$$
\begin{aligned}
&\int_T^t H(t,s)\Phi\left(Lq-(r\phi)^{\Delta}\right)\Delta s\\
&\le -\int_T^t H(t,s)\omega^{\Delta}(s)\Delta s+\int_T^t H(t,s)\frac{\Phi^{\Delta}}{\Phi^{\sigma}}\omega^{\sigma}\Delta s\\
&\quad -\int_T^t \Big[\frac{H(t,s)\gamma\Phi\tau^{\Delta}}{(r^*)^{1/\gamma}(\Phi^{\sigma})^{\frac{\gamma+1}{\gamma}}}\left(\omega^{\sigma}-(r\phi\Phi)^{\sigma}\right)^{\frac{\gamma+1}{\gamma}}\Big]\Delta s\\
&= -H(t,s)\omega(s)\mid_T^t+\int_T^t \Big[\big[H^{\Delta_s}(t,s)+H(t,s)\frac{\Phi^{\Delta}}{\Phi^{\sigma}}\big]\omega^{\sigma}\\
&\quad -\frac{H(t,s)\gamma\Phi\tau^{\Delta}}{(r^*)^{1/\gamma}(\Phi^{\sigma})^{\frac{\gamma+1}{\gamma}}}\left(\omega^{\sigma}-(r\phi\Phi)^{\sigma}\right)^{\frac{\gamma+1}{\gamma}}\Big]\Delta s\\
&= H(t,T)\omega(T)+\int_T^t \Big[-\frac{h(t,s)}{\Phi^{\sigma}(s)}(H(t,s)\Phi)^{\frac{\gamma}{\gamma+1}}\omega^{\sigma}\\
&\quad -\frac{H(t,s)\gamma\Phi\tau^{\Delta}}{(r^*)^{1/\gamma}(\Phi^{\sigma})^{\frac{\gamma+1}{\gamma}}}\left(\omega^{\sigma}-(r\phi\Phi)^{\sigma}\right)^{\frac{\gamma+1}{\gamma}}\Big]\Delta s.
\end{aligned}
\tag{3.19}
$$

Fix $t\ge T$, and set

$$
a_0=-\frac{h(t,s)}{\Phi^{\sigma}}(H(t,s)\Phi(s))^{\frac{\gamma}{\gamma+1}},\quad b_0=\frac{H(t,s)\gamma\Phi\tau^{\Delta}}{(r^*)^{1/\gamma}(\Phi^{\sigma})^{\frac{\gamma+1}{\gamma}}},\quad c_0=(r\phi\Phi)^{\sigma}. \tag{3.20}
$$

Then, by Lemma 2.3, (3.19) and (3.20), we have

$$
\begin{aligned}
&\int_T^t H(t,s)\Phi(s)\left(Lq(s)-(r(s)\phi(s))^{\Delta}\right)\Delta s\\
&\le H(t,T)\omega(T)+\int_T^t \Big[-(r(s)\phi(s))^{\sigma}h(t,s)(H(t,s)\Phi(s))^{\frac{\gamma}{\gamma+1}}\\
&\quad +\frac{r^*(s)h^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}(\tau^{\Delta})^{\gamma}}\Big]\Delta s.
\end{aligned}
\tag{3.21}
$$

From (3.21) and condition (iii) we obtain

$$
\begin{aligned}
&H(t,T)\omega(T)\\
&\ge \int_T^t \Big[H(t,s)\Phi\left(Lq-(r\phi)^{\Delta}\right)+(r\phi)^{\sigma}h(t,s)(H(t,s)\Phi)^{\frac{\gamma}{\gamma+1}}-\frac{r^*(s)h^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}(\tau^{\Delta})^{\gamma}}\Big]\Delta s\\
&= \int_T^t \Big[H(t,s)\Phi\left(Lq(s)-(r\phi)^{\Delta}\right)-(r\phi)^{\sigma}(H(t,s)\Phi)^{\Delta}-\frac{r^*(s)h^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}(\tau^{\Delta})^{\gamma}}\Big]\Delta s\\
&= \int_T^t \Big[LH(t,s)\Phi q-(H(t,s)\Phi r\phi)^{\Delta}-\frac{r^*(s)h^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}(\tau^{\Delta})^{\gamma}}\Big]\Delta s\\
&= H(t,T)\Phi(T)r(T)\phi(T)+\int_T^t \Big[LH(t,s)\Phi q-\frac{r^*(s)h^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}(\tau^{\Delta})^{\gamma}}\Big]\Delta s.
\end{aligned}
$$

From (3.16) and condition (ii) it is easy to see that

$$
\int_T^t \Big[LH(t,s)\Phi(s)q(s)-\frac{r^*(s)h^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}(\tau^{\Delta})^{\gamma}}\Big]\Delta s\le H(t,T)[\omega(T)-\Phi(T)r(T)\phi(T)]
$$

$$\leq H(t,t_0)[\omega(T)-\Phi(T)r(T)\phi(T)].$$

It follows that for $t \geq t_0$,

$$\begin{aligned}
&\int_{t_0}^{t}\Big[LH(t,s)\Phi(s)q(s)-\frac{r^*(s)h^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}(\tau^{\Delta})^{\gamma}}\Big]\Delta s\\
&=\int_{t_0}^{T}\Big[LH(t,s)\Phi(s)q(s)-\frac{r^*(s)h^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}(\tau^{\Delta})^{\gamma}}\Big]\Delta s\\
&\quad+\int_{T}^{t}\Big[LH(t,s)\Phi(s)q(s)-\frac{r^*(s)h^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}(\tau^{\Delta})^{\gamma}}\Big]\Delta s\\
&\leq H(t,t_0)\int_{t_0}^{T}L\Phi(s)q(s)\Delta s+H(t,t_0)[\omega(T)-\Phi(T)r(T)\phi(T)].
\end{aligned}$$

That is,

$$\begin{aligned}
&\frac{1}{H(t,t_0)}\int_{t_0}^{t}\Big[LH(t,s)\Phi(s)q(s)-\frac{r^*(s)h^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}(\tau^{\Delta})^{\gamma}}\Big]\Delta s\\
&\leq\int_{t_0}^{T}L\Phi(s)q(s)\Delta s+\omega(T)-\Phi(T)r(T)\phi(T)<\infty,
\end{aligned}$$

which is a contradiction to (3.15). This completes the proof. □

From the proof of Theorem 3.4, it is easy to see that the term $\phi(t)$ appearing in (1.14) is not important, and we can obtain the same result without $\phi(t)$. Suppose $\phi(t)=0$. Then replacing inequality (3.17) by (3.13), we can prove the Theorem 3.5, which improves Theorem 3.4 when $h(t,s)$ is oscillatory or $h(t,s)\geq 0$.

Theorem 3.5. *Suppose that all conditions hold as in Theorem 3.4. Also assume that*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\Big[LH(t,s)\Phi(s)q(s)-\frac{r^*(s)(h_-(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\tau^{\Delta}(s))^{\gamma}}\Big]\Delta s=\infty,$$

where $r^*(s)=\max\{r(\xi)|\tau(s)\leq\xi<\tau^{\sigma}(s)\}$ *and* $h_-(t,s)=\max\{-h(t,s),0\}$. *Then* (1.1) *is oscillatory.*

If $\mathbb{T}=\mathbb{R}$, we have $r^*(t)=r(\tau(t))$. In a manner similar to the proof of Theorems 3.1 and 3.4, we can prove the following results for (1.13).

Theorem 3.6. *Suppose that* $\mathbb{T}=\mathbb{R}$ *and* $r(t)>0$ *hold. Also, assume that there exists a function* $\Phi(t)\in C^1(\mathbb{R},\mathbb{R}^+)$ *such that*

$$\limsup_{t\to\infty}\Big\{\Phi(t)\int_{t}^{\infty}q(s)ds+\int_{t_0}^{t}\Big[\Phi(s)q(s)-\frac{r(\tau(s))\left(\Phi_+'(s)\right)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\Phi^{\gamma}(s)}\Big]ds\Big\}=\infty,$$

where $\Phi_+(s)=\max\{\Phi(s),0\}$. *Then* (1.13) *is oscillatory.*

Theorem 3.7. *Suppose that conditions* $\mathbb{T}=\mathbb{R}$ *and* $r(t)>0$ *hold. Let* $D_0=\{(t,s): t>s\geq t_0,\, t,\, s\in\mathbb{T}\}$ *and* $D=\{(t,s): t\geq s\geq t_0,\, t,\, s\in\mathbb{T}\}$. *Moreover, suppose that there exist functions* $H\in C(D,\mathbb{R})$, $h\in C(D_0,\mathbb{R})$, *and* $\Phi(t)\in C^1(\mathbb{R},\mathbb{R}^+)$, *such that the following three conditions hold:*

(i) $H(t,t)=0$ *for all* $t\geq t_0$, $H(t,s)>0$ *for all* $(t,s)\in D_0$;

(ii) H *has a continuous and non-positive partial derivative on* D_0 *with respect to the second variable;*

(iii) $-[H(t,s)\Phi(s)]' = h(t,s)[H(t,s)\Phi(s)]^{\frac{\gamma}{\gamma+1}}$, *for all* $(t,s) \in D_0$.

If

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \Big[H(t,s)\Phi(s)q(s) - \frac{r(\tau(s)(h_-(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}}\Big] ds = \infty, \tag{3.22}$$

where $h_-(t,s) = \max\{-h(t,s), 0\}$, *then* (1.13) *is oscillatory.*

Remark 3.8. From Theorems 3.1, 3.4 and 3.5, we can present different explicit sufficient conditions for the oscillation of (1.1) by appropriate choices of $\Phi(s)$ and $H(t,s)$. For instance, we may choose $\Phi(s)$ to be $1, s$, etc.; we may choose $H(t,\, s) = (t-s)^k$, or $H(t,s) = [R(t) - R(s)]^k$, for $t \geq s \geq t_0$, where $k > 1$ is a constant, and $R(t) = \int_{t_0}^{t} 1/r(s)\Delta s$ for $t \geq t_0$.

Remark 3.9. If we take $\mathbb{T} = \mathbb{R}$, $r(t) = 1$, $f(x) = x$, $\tau(t) = t$, $\gamma = 1$, $H(t,\, s) = (t-s)^k$ and $\Phi(s) = 1$, then Theorem 3.7 reduces to Theorem 1.1. If we take $\mathbb{T} = \mathbb{R}$, $f(x) = x$, $\tau(t) = t$, $\gamma = 1$ and $\Phi(s) = 1$, then Theorem 3.7 reduces to Theorem 1.2. If we take $\mathbb{T} = \mathbb{R}$, $f(x) = x$, $\tau(t) = t$, $\gamma = 1$, then Theorem 3.7 reduces to Theorem 1.3. It is particularly interesting that we can get condition (1.5) from condition (3.22). We can see this from the following proof. Since $\mathbb{T} = \mathbb{R}$, $f(x) = x$, $\tau(t) = t$, $\gamma = 1$, and $\Phi(s) = a(s) = \exp\{-2\int^{s} g(\xi)d\xi\}$, from (3.22) and condition (iii) of Theorem 3.7, we obtain

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \Big[H(t,s)\Phi(s)q(s) - \frac{r(s)h^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}}\Big] ds \\
&= \limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \Big[H(t,s)aq - \frac{1}{4} r(s) \frac{[H_s(t,s)a(s) + H(t,s)a'(s)]^2}{H(t,s)a(s)}\Big] ds \\
&= \limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \Big[H(t,s)aq - \frac{1}{4} ra \frac{H_s^2(t,s)}{H(t,s)} - \frac{1}{2} r(s)H_s a'(s) \\
&\quad - \frac{H(t,s)r(s)(a'(s))^2}{4a(s)}\Big] ds \\
&= \limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \Big[H(t,s)aq - \frac{1}{4} ra \frac{H_s^2(t,s)}{H(t,s)} + a(s)r(s)g(s)H_s \\
&\quad - H(t,s)r(s)(g(s))^2\Big] ds \\
&= \limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \Big[H(t,s)a\{q + rg^2 - (rg)'\} - \frac{1}{4} ra \frac{H_s^2(t,s)}{H(t,s)} \\
&\quad + (H(t,s)arg)'\Big] ds \\
&= \limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \Big[H(t,s)a\{q + rg^2 - (rg)'\} - \frac{1}{4} ra \frac{H_s^2(t,s)}{H(t,s)}\Big] ds \\
&\quad - a(t_0)r(t_0)g(t_0).
\end{aligned}$$

Therefore, our results unify Li-type oscillation criteria.

Remark 3.10. Theorem 3.4 improves the corresponding results established by Yan [27], Sahiner [19], Wu et al [26], and Chen [5]. Also, when $T = N$ it improves the oscillation results in Thandapani et al [21].

As we have seen before, we can obtain different corollaries from Theorem 3.7 by choosing different $\Phi(t)$. Next, we consider the case when (H5') holds. We remark that since the crucial step in obtaining Theorem 3.1 is to show that eventually positive and eventually increasing solutions of (1.1) do not exist, then we have an analogue of Theorems 3.1 and 3.4.

Theorem 3.11. *Suppose that conditions* (H1)–(H4) *and* (H5') *hold. Let* $\Phi(t)$ *be defined as in Theorem 3.1 such that* (3.1) *holds. Assume further that*

$$\int_{t_0}^{\infty} q(s)\Delta s = \infty \quad \text{and} \quad \int_{t_0}^{\infty} \Big[\frac{1}{r(s)}\int_{t_0}^{s} q(u)\Delta u\Big]^{1/\gamma}\Delta s = \infty. \tag{3.23}$$

Then every solution of (1.1) *oscillates or converges to zero.*

Proof. Suppose to the contrary that (1.1) has a nonoscillatory solution on $[t_0,\infty)_{\mathbb{T}}$. Without loss of generality, we may assume that $x(t) > 0$ for $t \ge t_0$. In view of Theorem 3.1 we see that $x^{\Delta}(t)$ is eventually negative or eventually positive. If $x^{\Delta}(t)$ is eventually positive, we are then back to the proof of Theorem 3.1 and we obtain a contradiction to condition (3.1). If $x^{\Delta}(t)$ is eventually negative, then $\lim_{t\to\infty} x(t) = M \ge 0$. We claim that $M = 0$. If not, then $x(t) \ge M > 0$. From (H3), there exists $t_1 \ge t_0$ such that $f(t, x(\tau(t))) \ge q(t)\left(x(\tau(t))\right)^{\gamma} \ge q(t)M^{\gamma}$, for $t \ge t_1$. Therefore, from (1.1), we have

$$\Big(r(t)\big(x^{\Delta}(t)\big)^{\gamma}\Big)^{\Delta} = -f(t, x(\tau(t))) \le -q(t)M^{\gamma} \le 0, \quad \text{for } t \ge t_1.$$

Integrating the above inequality from t_1 to t, we obtain

$$r(t)\big(x^{\Delta}(t)\big)^{\gamma} \le r(t_1)\left(x^{\Delta}(t_1)\right)^{\gamma} - M^{\gamma}\int_{t_1}^{t} q(u)\Delta u.$$

In view of (3.23), it is possible to choose t_2 sufficiently large such that for all $t \ge t_2$,

$$r(t)\big(x^{\Delta}(t)\big)^{\gamma} \le -\frac{M^{\gamma}}{2}\int_{t_2}^{t} q(u)\Delta u. \tag{3.24}$$

Therefore

$$x^{\Delta}(t) \le -\frac{M}{2^{1/\gamma}}\Big[\frac{1}{r(t)}\int_{t_2}^{t} q(u)\Delta u\Big]^{1/\gamma}.$$

Integrating both sides of the last inequality from t_3 to t, we obtain

$$x(t) \le x(t_3) - \frac{M}{2^{1/\gamma}}\int_{t_3}^{t}\Big[\frac{1}{r(s)}\int_{t_2}^{s} q(u)\Delta u\Big]^{1/\gamma}\Delta s.$$

So it follows from (3.23) that $x(t)$ is eventually negative, a contradiction. This completes the proof. □

In a manner similar to the proof of Theorems 3.11, we can prove the following result.

Theorem 3.12. *Suppose that conditions* (H1)–(H4) *and* (H5') *hold. Let* $H \in C(D,\mathbb{R})$, $h \in C(D_0,\mathbb{R})$, *and* $\Phi(t) \in C^1_{rd}(\mathbb{T},\mathbb{R}^+)$ *be defined as in Theorem 3.4 such that* (3.15) *holds. Assume that*

$$\int_{t_0}^{\infty} q(s)\Delta s = \infty \quad \text{and} \quad \int_{t_0}^{\infty} \Big[\frac{1}{r(s)}\int_{t_0}^{s} q(u)\Delta u\Big]^{1/\gamma}\Delta s = \infty.$$

Then every solution of (1.1) *oscillates or converges to zero.*

4. Examples

Let us consider the following examples to better understand our results.

Example 4.1. Consider the second-order half-linear dynamic delay equation

$$\Big(\big(x^\Delta(t)\big)^\gamma\Big)^\Delta + \frac{\lambda}{t\sigma^\gamma(t)}\ (x(\tau(t)))^\gamma = 0, \tag{4.1}$$

where $\lambda > 0$ and $0 < \gamma \le 1$ is a quotient of odd positive integers.

Here $L = 1$ and $r^*(t) = r(t) = 1$. For arbitrary time scale $\mathbb{T}$, we take $\Phi(t) = t$. Since

$$\begin{aligned}\Big(\frac{1}{t^\gamma}\Big)^\Delta &= -\frac{(t^\gamma)^\Delta}{t^\gamma\sigma^\gamma(t)} = -\frac{1}{t^\gamma\sigma^\gamma(t)}\frac{\sigma^\gamma(t)-t^\gamma}{\sigma(t)-t} = -\frac{1}{t^\gamma\sigma^\gamma(t)}\frac{\gamma\eta^{\gamma-1}(\sigma(t)-t)}{\sigma(t)-t}\\ &\ge -\frac{\gamma t^{\gamma-1}}{t^\gamma\sigma^\gamma(t)} = -\frac{\gamma}{t\sigma^\gamma(t)}, \quad \eta\in[t,\sigma(t)].\end{aligned} \tag{4.2}$$

Therefore, from Corollary 3.2 and (4.2), we obtain

$$\begin{aligned}&\limsup_{t\to\infty}\Big\{Lt\int_t^\infty q(s)\Delta s + \int_T^t \Big[Lsq(s) - \frac{r^*(s)}{(\gamma+1)^{\gamma+1}\left(\tau^\Delta(s)\right)^\gamma s^\gamma}\Big]\Delta s\Big\}\\ &= \limsup_{t\to\infty}\Big\{t\int_t^\infty \frac{\lambda}{s\sigma^\gamma(s)}\Delta s + \int_T^t\Big[\frac{\lambda}{\sigma^\gamma(s)} - \frac{1}{(\gamma+1)^{\gamma+1}\left(\tau^\Delta(s)\right)^\gamma s^\gamma}\Big]\Delta s\Big\}\\ &\ge \limsup_{t\to\infty}\Big\{\frac{\lambda}{\gamma}t^{1-\gamma} + \int_T^t\Big[\frac{\lambda}{\sigma^\gamma(s)} - \frac{1}{(\gamma+1)^{\gamma+1}\left(\tau^\Delta(s)\right)^\gamma s^\gamma}\Big]\Delta s\Big\}.\end{aligned} \tag{4.3}$$

If $\mathbb{T} = \mathbb{R}$ and $\tau(t) = t - \tau$ for $\tau \ge 0$, then conditions (H1)–(H5) are satisfied. For $0 < \gamma < 1$ and $\lambda > \gamma/(\gamma+1)^{\gamma+1}$, we have

$$\begin{aligned}&\limsup_{t\to\infty}\Big\{\frac{\lambda}{\gamma}t^{1-\gamma} + \int_T^t\Big[\frac{\lambda}{\sigma^\gamma(s)} - \frac{1}{(\gamma+1)^{\gamma+1}\left(\tau^\Delta(s)\right)^\gamma s^\gamma}\Big]\Delta s\Big\}\\ &= \limsup_{t\to\infty}\Big\{\frac{\lambda}{\gamma}t^{1-\gamma} + \int_T^t\Big[\frac{\lambda}{s^\gamma} - \frac{1}{(\gamma+1)^{\gamma+1}s^\gamma}\Big]ds\Big\}\\ &= \limsup_{t\to\infty}\Big\{\frac{\lambda}{\gamma} + \frac{1}{1-\gamma}\Big(\lambda - \frac{1}{(\gamma+1)^{\gamma+1}}\Big)\Big\}t^{1-\gamma} - \frac{1}{1-\gamma}\Big(\lambda - \frac{1}{(\gamma+1)^{\gamma+1}}\Big)T^{1-\gamma}\\ &= \limsup_{t\to\infty}\frac{1}{\gamma(1-\gamma)}\Big(\lambda - \frac{\gamma}{(\gamma+1)^{\gamma+1}}\Big)t^{1-\gamma} - \frac{1}{1-\gamma}\Big(\lambda - \frac{\gamma}{(\gamma+1)^{\gamma+1}}\Big)T^{1-\gamma} = \infty.\end{aligned} \tag{4.4}$$

Hence (4.1) is oscillatory when $0 < \gamma < 1$ and $\lambda > \gamma/(\gamma+1)^{\gamma+1}$.

Note that when $\mathbb{T} = \mathbb{R}$, $\gamma = 1$ and $\tau(t) = t$, from (4.4) we see that (4.1) is also oscillatory provided $\lambda > 1/4$, which is the sharp condition for the Euler differential equation (1.6) to be oscillatory. When $\mathbb{T} = \mathbb{R}$ and $0 < \gamma < 1$, from Example 4.1 we obtain $\lambda > \gamma/(\gamma+1)^{\gamma+1}$. However, to the best of our knowledge, the results in [1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29] yield $\lambda > 1/(\gamma+1)^{\gamma+1}$.

Next, we consider the quantum time scale $\mathbb{T} = q^{\mathbb{N}} = \{q^n : n \in \mathbb{N}\}$, where $q > 1$, $q \in \mathbb{R}$, and $\tau(t) = \frac{t}{\tau}$ for $\tau = q^m$, $m \in \mathbb{N}$ and $m < N$, then conditions $(H_1) - (H_5)$ are satisfied. Noting that

$$\Big(\frac{t}{t^\gamma}\Big)^\Delta = \frac{1}{\sigma^\gamma(t)} - \frac{t\,(t^\gamma)^\Delta}{t^\gamma\sigma^\gamma(t)} = \frac{1}{\sigma^\gamma(t)} - \frac{t}{t^\gamma\sigma^\gamma(t)}\frac{\sigma^\gamma(t)-t^\gamma}{\sigma(t)-t}$$

$$
\begin{aligned}
&= \frac{1}{\sigma^\gamma(t)} - \frac{t}{t^\gamma\sigma^\gamma(t)}\frac{\gamma\eta^{\gamma-1}(\sigma(t)-t)}{\sigma(t)-t}\\
&\geq \frac{1}{\sigma^\gamma(t)} - \frac{t\gamma t^{\gamma-1}}{t^\gamma\sigma^\gamma(t)} = \frac{1-\gamma}{\sigma^\gamma(t)}, \quad \eta\in[t,\sigma(t)],
\end{aligned}
$$

from (4.3) we obtain

$$
\begin{aligned}
&\limsup_{t\to\infty}\Big\{\frac{\lambda}{\gamma}t^{1-\gamma} + \int_T^t\Big[\frac{\lambda}{\sigma^\gamma(s)} - \frac{1}{(\gamma+1)^{\gamma+1}\left(\tau^\Delta(s)\right)^\gamma s^\gamma}\Big]\Delta s\Big\}\\
&\geq \limsup_{t\to\infty}\Big\{\frac{\lambda}{\gamma}t^{1-\gamma} - \frac{\lambda}{\gamma}T^{1-\gamma} + \int_T^t\Big[\frac{\lambda}{\sigma^\gamma(s)} - \frac{\tau^\gamma}{(\gamma+1)^{\gamma+1}s^\gamma}\Big]\Delta s\Big\}\\
&= \limsup_{t\to\infty}\Big\{\frac{\lambda}{\gamma}\int_T^t(\frac{t}{t^\gamma})^\Delta\Delta s + \int_T^t\Big[\frac{\lambda}{\sigma^\gamma(s)} - \frac{\tau^\gamma}{(\gamma+1)^{\gamma+1}s^\gamma}\Big]\Delta s\Big\}\\
&\geq \limsup_{t\to\infty}\Big\{\frac{\lambda}{\gamma}\int_T^t\frac{1-\gamma}{\sigma^\gamma(s)}\Delta s + \int_T^t\Big[\frac{\lambda}{\sigma^\gamma(s)} - \frac{\tau^\gamma}{(\gamma+1)^{\gamma+1}s^\gamma}\Big]\Delta s\Big\}\\
&= \limsup_{t\to\infty}\int_T^t\Big[\frac{\lambda}{\gamma\sigma^\gamma(s)} - \frac{\tau^\gamma}{(\gamma+1)^{\gamma+1}s^\gamma}\Big]\Delta s = \infty.
\end{aligned}
$$

Since $\sigma(s) = qs$, (4.1) is oscillatory when $\lambda > \gamma(q\tau)^\gamma/(\gamma+1)^{\gamma+1}$. However, to the best of our knowledge, the results in [1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29] give the estimate $\lambda > (q\tau)^\gamma/(\gamma+1)^{\gamma+1}$. Therefore, our results improve the corresponding results in these references, even for $\tau(t) = t$.

Example 4.2. Consider the second-order dynamic delay equation

$$
\Big(\frac{1}{t}x^\Delta(t)\Big)^\Delta + \frac{\lambda}{t^3}x(\tau(t)) = 0, \quad t\in\mathbb{T}. \tag{4.5}
$$

Note that in the case $\mathbb{T} = \mathbb{R}$, and $\tau(t) = t$, we see that $\lambda = 1$ which is the sharp condition for (4.5) to be oscillatory, and $x(t) = 1/t$ is a solution when $\lambda = 1$. Here we take $\Phi(s) = s^2$, and $H(t,s) = (t-s)^2$. From Theorem 3.4, we obtain

$$
\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\Big[H(t,s)\Phi(s)q(s) - \frac{r^*(s)(h(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\tau^\Delta(s))^\gamma}\Big]\Delta s\\
&= \limsup_{t\to\infty}t^{-2}\int_T^t\Big[(t-s)^2s^2\frac{\lambda}{s^3} - \frac{(t-2s)^2}{\tau(s)\tau^\Delta(s)}\Big]\Delta s.
\end{aligned} \tag{4.6}
$$

If $\mathbb{T} = \mathbb{R}$ and $\tau(t) = t$, then conditions (H1)–(H5) are satisfied. By (4.6) we obtain

$$
\begin{aligned}
&\limsup_{t\to\infty}t^{-2}\int_T^t\Big[(t-s)^2\frac{\lambda}{s} - (t-2s)^2\frac{1}{s}\Big]ds\\
&= \limsup_{t\to\infty}t^{-2}\int_T^t\Big[(t-s)^2\frac{\lambda}{s} - (t-s)^2\frac{1}{s} + 2s(t-s)\frac{1}{s} - s^2\frac{1}{s}\Big]ds\\
&= \limsup_{t\to\infty}t^{-2}\int_T^t\Big[\frac{\lambda-1}{s}(t-s)^2 + 2t - 3s\Big]ds\\
&= \limsup_{t\to\infty}t^{-2}\int_T^t\Big[\frac{\lambda-1}{s}t^2 + 2(2-\lambda)t + (\lambda-4)s\Big]ds = \infty.
\end{aligned} \tag{4.7}
$$

From Theorem 3.4, (4.5) is oscillatory for $\lambda > 1$. Moreover, Our results are established for arbitrary time scales.

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