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A GENERAL PRODUCT MEASURABILITY THEOREM WITH APPLICATIONS TO VARIATIONAL INEQUALITIES

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Abstract. This work establishes the existence of measurable weak solutions to evolution problems with randomness by proving and applying a novel theorem on product measurability of limits of sequences of functions. The measurability theorem is used to show that many important existence theorems within the abstract theory of evolution inclusions or equations have straightforward generalizations to settings that include random processes or coefficients. Moreover, the convex set where the solutions are sought is not fixed but may depend on the random variables. The importance of adding randomness lies in the fact that real world processes invariably involve randomness and variability. Thus, this work expands substantially the range of applications of models with variational inequalities and differential set-inclusions.

1. Introduction

This article concerns the existence of $\mathcal{P}$-measurable solutions to evolution problems in which some of the input data, such as the operators, forcing functions or some of the system coefficients, is random. Such problems, often in the form of evolutionary variational inequalities, abound in many fields of mathematics, such as optimization and optimal control, and in applications such as in contact mechanics, populations dynamics, and many more. The interest in random inputs in variational inequalities arises from the uncertainty in the identification of the system inputs or parameters, possibly due to the process of construction or assembling of the system, and to the imprecise knowledge of the acting forces or environmental processes that may have stochastic behavior.

We first establish a very general abstract theorem on the product measurability of certain limits of sequences of functions. Then, we apply the result to variational inequalities with monotone, hemicontinuous, bounded and coercive operators that may or may not be strongly monotone, and establish the product measurability of the solutions of the variational inequalities. We note that the abstract result has many other, very diverse, applications.

The general setting is as follows. We let $(\Omega, \mathcal{F})$ be a measurable space with sample space Ω, and a σ-algebra $\mathcal{F}$. It is assumed that an evolution problem, in the form of an inequality or set-inclusion, corresponds to each $\omega \in \Omega$, with a

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solution denoted by $u(\cdot,\omega)$. When this evolution problem has a unique solution, it is usually possible to show the existence of an $\mathcal{F}$-measurable solution, i.e., a solution such that $(t,\omega) \to u(t,\omega)$ is $\mathcal{P} \equiv \mathcal{B}([0,T]) \times \mathcal{F}$ measurable, see [6]. By $\mathcal{P}$ we mean the smallest σ algebra which contains the measurable rectangles $A \times B$ where $A \in \mathcal{B}([0,T]), B \in \mathcal{F}$. The case without uniqueness is much more involved and is the focus of this work as we establish a way to deal with it. No conditions need to be made on the measurable space and in general, no specific measure is specified, although we have in mind a probability space.

Our result is very general and we illustrate it by obtaining weak solutions to a whole class of variational inequalities (see, e.g., [4, 11, 12]). Our measurability result, stated in Theorem 2.1, is a substantial improvement of the result we established in [5] and [10]. This version allows many more applications. In particular, we apply Theorem 2.1 to a very important class of evolution problems and establish in Theorem 3.1 and Theorem 3.3 the existence and measurability of the solutions. We refer to Theorem 2.1 as the *measurable selection theorem* because it establishes the existence of a $\mathcal{P}$-measurable representative in a set of limits of $\mathcal{P}$-measurable functions.

It is seen from the example that our measurable selection theorem is a powerful tool and we foresee that it will be used in many problems where randomness is important, uniqueness is unknown, and measurability of the solutions is essential. Such cases abound in contact mechanics, especially when friction is present, see, e.g., [7, 9, 10] and the many references in [12, 15]. We note that our results may be applied to the results in [13, 14], where inequality nonconvex problems were studied.

The rest of this article is as follows. The basic measurable selection theorem, Theorem 2.1, is established in Section 2. The existence of $\mathcal{P}$-measurable solutions to a variational inequality with a monotone, hemicontinuous and bounded operator is shown in Theorem 3.1 in Section 3. Then, in Theorem 3.3, we allow the convex set to be a measurable set-valued map of the random variables. This is a completely novel and somewhat surprising result never before considered as far as we know, and our measurable selection result makes it fairly easy to obtain. More general operators such as maximal monotone operators can be considered also, but we do not present this here because the necessary existence for an approximate problem is not readily available.

2. Measurable selection theorem

We establish in this section our main theorem. It asserts that under very weak conditions, in particular, the boundedness condition (2.1), there exists a measurable selection of the set of limit functions of a sequence of functions in a reflexive and separable Banach space.

We use the usual notations for Sobolev function spaces. In particular, $\mathcal{V} \equiv L^p([0,T],V)$ denotes the space of p-integrable functions defined on $[0,T]$ with values in the space V, $C([0,T])$ is the space of continuous functions on $[0,T]$ with the maximum norm, and we use the abbreviation $\|\cdot\| = \|\cdot\|_{C([0,T])}$. The numbers $p > 1$ and $p' > 1$ are conjugate, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. Also, $\langle \cdot,\cdot\rangle_V = \langle \cdot,\cdot\rangle_{V',V}$ represents the duality pairing between V and V'.

In this section $(\Omega,\mathcal{F})$ is a measurable space. We have in mind a probability space but this is not necessary, since no reference is made to any particular measure.

Also, no topological conditions are necessary on Ω. This is in contrast to the result of [2] in which a complete probability measure was assumed and Ω was assumed to be a completely regular topological space. The theorem here is a significant improvement over the one we established in [5, 10]. Moreover, the steps in the proof differ. However, for the sake of brevity, we refer to these two papers for some of the details.

We use the following notation: C represents a generic constant that may depend on the data but is independent of n. The dependence of a constant on ω, t or k is denoted explicitly, e.g., if it depends only on ω or k, we write $C(\omega)$ or $C(k)$, while $C(\omega, t, k)$ depends on all three. The following is the main result in this work.

Theorem 2.1 (The Measurable Selection Theorem)**.** *Let V be a reflexive separable Banach space with dual V' and let p, p' be conjugate numbers. Assume that $\{u_n(t,\omega)\}_{n=1}^\infty$ is a sequence of $\mathcal{P} \equiv \mathcal{B}([0,T]) \times \mathcal{F}$-measurable functions with values in V such that for each $\omega \in \Omega$, except for a null set $N \subset \Omega$, and for each n, it satisfies*

$$\|u_n(\cdot,\omega)\|_{\mathcal{V}} \le C(\omega). \tag{2.1}$$

Then, there exists a $\mathcal{P}$-measurable function $u(t,\omega)$ such that for each $\omega \notin N$,

$$u(\cdot,\omega) \in \mathcal{V},$$

and there is a subsequence u_{n_ω} such that $u_{n_\omega}(\cdot,\omega) \to u(\cdot,\omega)$ weakly in $\mathcal{V}$ as $n_\omega \to \infty$.

We note that the bound in (2.1) need not be uniform on Ω, and u is a weak limit of a subsequence that depends on ω.

We prove the theorem in steps, presented as lemmas. First, we need the space $X = \prod_{k=1}^\infty C([0,T])$ with the product topology. Then, X is a metric space with the metric

$$d(\mathbf{f},\mathbf{g}) \equiv \sum_{k=1}^\infty 2^{-k} \frac{\|f_k - g_k\|}{1 + \|f_k - g_k\|},$$

where $\mathbf{f} = (f_1, f_2, \dots), \mathbf{g} = (g_1, g_2, \dots) \in X$ and the norm is the maximum norm on $C([0,T])$. With this metric, X is complete and separable.

The first step follows.

Lemma 2.2. *Let $\{\mathbf{f}_n\}_{n=1}^\infty$ be a sequence in X and suppose that for each component k, the sequence $\{f_{nk}\}_{n=0}^\infty$ is bounded in $C^{0,(1/p')}([0,T])$ by $C(k)$. Then, there exists a subsequence $\{\mathbf{f}_{n_j}\}$ that converges as $n_j \to \infty$ to an element $\mathbf{f} \in X$. Thus, $\{\mathbf{f}_n\}$ is pre-compact in X.*

Proof. The result follows from Tychonoff's theorem and the compactness of the embedding of $C^{0,(1/p')}$ in $C([0,1])$. $\square$

In the next step we construct a special sequence $\{\mathbf{f}_n\}_{n=1}^\infty$ in X based on the sequence $\{u_n(t,\omega)\}_{n=1}^\infty$. For $m \in \mathbb{N}$ and $\phi \in V'$, let $l_m(t) \equiv \max(0, t - (1/m))$ and define $\psi_{m,\phi} : \mathcal{V} \to C([0,T])$ by

$$(\psi_{m,\phi} u(\cdot))(t) \equiv \int_0^T \langle m\phi \mathcal{X}_{[l_m(t),t]}(s), u(s)\rangle_V \, ds = m \int_{l_m(t)}^t \langle \phi, u(s)\rangle_V \, ds.$$

Here, $\mathcal{X}_{[l_m(t),t]}(\cdot)$ is the characteristic function of the interval $[l_m(t), t]$.

Let $\mathcal{D} = \{\phi_r\}_{r=1}^\infty$ denote a countable subset of V'. Later, we will describe it more carefully. For now, all that is important is that it be countable. Then, the pairs (m, ϕ), for $\phi \in \mathcal{D}$ and $m \in \mathbb{N}$ form a countable set. Let $\{(m_k, \phi_{r_k})\}_{k=1}^{\infty}$ denote an enumeration of the pairs $(m, \phi) \in \mathbb{N} \times \mathcal{D}$. To simplify the notation, we let

$$(f_k(u(\cdot)))(t) \equiv (\psi_{m_k,\phi_{r_k}} u(\cdot))(t) = m_k \int_{l_{m_k}(t)}^{t} \langle \phi_{r_k}, u(s)\rangle_V ds.$$

We note that for each fixed $\phi \in \mathcal{D}$ there exists a subsequence such that $m_k \to \infty$ and $\phi_{r_k} = \phi$. Thus there are infinitely many indices k such that $\phi_{r_k} = \phi$.

We now choose the functions $u_n(\cdot, \omega)$ of Theorem 2.1 as $u(\cdot)$ and so we have constructed a sequence $\{\mathbf{f}_n\}_{n=1}^{\infty}$ in X, which depends on ω. We use this sequence to define the following set-valued map $\Gamma(\omega)$.

It follows from estimate (2.1) that for fixed k and $\omega \notin N$ the functions $\{t \to f_k(u_n(\cdot, \omega))(t)\}_{n=1}^{\infty}$ are uniformly bounded and equicontinuous since they are bounded in $C^{0,1/p'}([0, T])$. Then, by Lemma 2.2, for each n the set $\{\mathcal{X}_{N^C}(\omega)\mathbf{f}(u_j(\cdot, \omega))\}_{j=n}^{\infty}$ is pre-compact in $X = \prod_k C([0, T])$. We define for each $n \in \mathbb{N}$ a set-valued map $\Gamma^n : \Omega \to X$ by

$$\Gamma^n(\omega) \equiv \overline{\cup_{j\geq n}\{\mathcal{X}_{N^C}(\omega)\mathbf{f}(u_j(\cdot, \omega))\}},$$

where the closure is taken in X. Then $\Gamma^n(\omega)$ is compact in X. From the definition, a function $\mathbf{g}$ is in $\Gamma^n(\omega)$ if and only if $d(\mathbf{g}, \mathcal{X}_{N^C}(\omega)\mathbf{f}(w_l)) \to 0$ as $l \to \infty$, where each w_l is one of the $u_j(\cdot, \omega)$ for $j \geq n$.

The proof of the next two lemmas can be found [5] or [10].

Lemma 2.3. *The mapping $\omega \to \Gamma^n(\omega)$ is an $\mathcal{F}$-measurable set-valued map with values in X.*

Definition 2.4. *Let $\Gamma(\omega) \equiv \cap_{n=1}^{\infty}\Gamma^n(\omega)$.*

Lemma 2.5. *Γ is a nonempty $\mathcal{F}$-measurable set-valued mapping with values in compact subsets of X and there exists an $\mathcal{F}$-measurable selection $\gamma(\omega) \in \Gamma(\omega)$ such that $\gamma(t, \omega) \equiv (\gamma(\omega))(t)$ is $\mathcal{P}$-measurable and $(\prod_{k=1}^{\infty}\mathbb{R})$-valued function.*

It follows from the definition of $\Gamma(\omega)$ that, for each $\omega \notin N$, there exists a subsequence $u_{n(\omega)}(\cdot, \omega)$ of $u_n(\cdot, \omega)$ such that for each component k,

$$\gamma_k(t, \omega) = \lim_{n(\omega)\to\infty} \mathcal{X}_{N^C}(\omega)(f_k(u_{n(\omega)}(\cdot, \omega)))(t) \text{ uniformly in} t,$$

or

$$\gamma_k(t, \omega) = \lim_{n(\omega)\to\infty} m_k \int_{l_{m_k}(t)}^{t} \langle \phi_{r_k}, \mathcal{X}_{N^C}(\omega)u_{n(\omega)}(s, \omega)\rangle_V ds. \tag{2.2}$$

We now set $\gamma(t, \omega) \equiv 0$ for $\omega \in N$ and then (2.2) holds for all ω. We note that it is not clear whether $\mathcal{X}_{N^C}(\omega)(f_k(u_{n(\omega)}(\cdot, \omega)))(t)$ is $\mathcal{P}$-measurable, however, all that is needed is that the limit $\gamma(t, \omega)$ is $\mathcal{P}$-measurable. We have all the ingredients for the proof.

Proof of Theorem 2.1. By assumption, there exists a further subsequence, still denoted by $n(\omega)$, such that, the weak limit

$$\lim_{n(\omega)\to\infty} \mathcal{X}_{N^C}(\omega)u_{n(\omega)}(\cdot, \omega) = v(\cdot, \omega)$$

exists in $\mathcal{V}$. Then,

$$m_k \int_{l_{m_k}(t)}^{t} \langle \phi_{r_k}, v(s, \omega)\rangle_V ds$$

$$
\begin{aligned}
&= \lim_{n(\omega)\to\infty} m_k \int_{l_{m_k}(t)}^{t} \langle \phi_{r_k}, \mathcal{X}_{N^C}(\omega) u_{n(\omega)}(s,\omega)\rangle_V \,ds \\
&= \gamma_k(t,\omega)
\end{aligned}
$$

is product measurable. Letting $\phi \in \mathcal{D}$ be given, there exists a subsequence, denoted by k, such that $m_k \to \infty$ and $\phi_{r_k} = \phi$. Recall (m_k, ϕ_{r_k}) denoted an enumeration of the pairs $(m,\phi) \in \mathbb{N} \times \mathcal{D}$. For a given $\phi \in \mathcal{D}$ denote this sequence by m_ϕ. Thus, we have measurability of

$$
(t,\omega) \to m_\phi \int_{l_{m_\phi}(t)}^{t} \langle \phi, v(s,\omega)\rangle_V \,ds,
$$

for each $\phi \in \mathcal{D}$.

Now we describe the countable set $\mathcal{D}$. Iterate the following. Let $\phi_1 \neq 0$. Let $\mathcal{F}$ denote linearly independent subsets of V' which contain ϕ_1 such that the elements are further apart than $1/5$. Let $\mathcal{C}$ denote a maximal chain. Thus $\cup\mathcal{C}$ is also in $\mathcal{F}$. If $W := \overline{\operatorname{span} \cup\mathcal{C}}$ fails to be all of V', then there would exist $\psi \notin W$ such that the distance of ψ to the closed subspace W is at least $1/5$. Now $\mathcal{C}, \cup\{\mathcal{C} \cup \{\psi\}\}$ would violate maximality of $\mathcal{C}$. Hence $W = V'$. Now it follows that $\mathcal{C}$ must be countable since otherwise, V' would fail to be separable. Then $\mathcal{D}$ is defined as $\cup\mathcal{C}$. Let M be the set of rational linear combinations of elements of $\mathcal{D}$, hence M is dense in V'. Note that linear combinations of the ϕ_i are uniquely determined because none is a linear combination of the others. Now, we define a linear mapping on M which makes sense for (t,ω) on a certain set.

Definition 2.6. Let E be the set of points (t,ω) such that the following limit exists for each $\phi \in \mathcal{D}$

$$
\Lambda(t,\omega)\phi := \lim_{m_\phi\to\infty} m_\phi \int_{l_{m_\phi}(t)}^{t} \langle \phi, v(s,\omega)\rangle ds.
$$

Extend this mapping linearly. That is, for $\psi \in M, \psi := \sum_i a_i \phi_i$,

$$
\Lambda(t,\omega)\psi := \sum_i a_i \Lambda(t,\omega)\phi_i = \sum_i a_i \Big(\lim_{m_{\phi_i}\to\infty} m_{\phi_i} \int_{l_{m_{\phi_i}}(t)}^{t} \langle \phi_i, v(s,\omega)\rangle ds \Big).
$$

Thus $(t,\omega) \to \Lambda(t,\omega)\psi$ is product measurable, being the sum of limits of product measurable functions. Let G denote those (t,ω) in E such that there exists a constant $C(t,\omega)$ such that for all $\psi \in M$,

$$
|\Lambda(t,\omega)\psi| \leq C(t,\omega)\|\psi\|.
$$

We note that E is a product measurable set because it was shown above that $(t,\omega) \to m_\phi \int_{l_{m_\phi}(t)}^{t} \langle \phi, v(s,\omega)\rangle ds$ is product measurable and in general, the set of points where a sequence of measurable functions converges is a measurable set.

Lemma 2.7. *G is product measurable.*

Proof. This follows from the formula

$$
E \cap G^C = \cap_n \cup_{\psi\in M} \{(t,\omega) : |\Lambda(t,\omega)\psi| > n\|\psi\|\}
$$

which is clearly product measurable because $(t,\omega) \to \Lambda(t,\omega)\psi$ is. Thus, since E is measurable, it follows that $E \cap G = G$ is too. $\square$

For $(t,\omega) \in G, \Lambda(t,\omega)$ has a unique extension to all of V, the dual space of V', still denoted as $\Lambda(t,\omega)$. By the Riesz representation theorem, for $(t,\omega) \in G$,there exists $u(t,\omega) \in V$,

$$\Lambda(t,\omega)\psi = \langle \psi, u(t,\omega)\rangle_V.$$

Thus, $(t,\omega) \to \mathcal{X}_G(t,\omega)u(t,\omega)$ is product measurable by the Pettis theorem. It has been shown that G is measurable and we let $u = 0$ off G. Next, we will show using the fundamental theorem of calculus that for each ω the set $\{t : (t,\omega) \in G\}$ is all but a set of Lebesgue measure zero.

We now fix ω. By the fundamental theorem of calculus,

$$\lim_{m\to\infty} m \int_{l_m(t)}^t v(s,\omega)ds = v(t,\omega) \quad \text{in } V,$$

for a.e. t, say for all $t \notin N(\omega) \subseteq [0,T]$. Of course we do not know that $\omega \to v(t,\omega)$ is measurable. However, the existence of this limit for $t \notin N(\omega)$ implies that for every $\phi \in V'$,

$$\lim_{m\to\infty} \big| m \int_{l_m(t)}^t \langle \phi, v(s,\omega)\rangle ds \big| \le C(t,\omega)\|\phi\|,$$

for some $C(t,\omega)$. Here m does not depend on ϕ. Thus, in particular, this holds for a subsequence and so for each $t \notin N(\omega), (t,\omega) \in G$ because for each $\phi \in \mathcal{D}$,

$$\lim_{m_\phi\to\infty} m_\phi \int_{l_{m_\phi}(t)}^t \langle \phi, v(s,\omega)\rangle ds \text{ exists and satisfies the above inequality.}$$

Hence, for all $\psi \in M$,

$$\Lambda(t,\omega)\psi = \langle \psi, u(t,\omega)\rangle_V,$$

where u is product measurable.

Also, for $t \notin N(\omega)$ and $\phi \in \mathcal{D}$,

$$\langle \phi, u(t,\omega)\rangle_V = \Lambda(t,\omega)\phi \equiv \lim_{m_\phi\to\infty} m_\phi \int_{l_{m_\phi}(t)}^t \langle \phi, v(s,\omega)\rangle ds = \langle \phi, v(t,\omega)\rangle_V,$$

therefore, for all $\phi \in M$,

$$\langle \phi, u(t,\omega)\rangle_V = \langle \phi, v(t,\omega)\rangle_V,$$

and hence $u(t,\omega) = v(t,\omega)$. Thus, for each ω, the product measurable function u satisfies $u(t,\omega) = v(t,\omega)$ for a.e. t. Hence $u(\cdot,\omega) = v(\cdot,\omega)$ in $\mathcal{V}$. This completes the proof of Theorem 1. □

We show next how condition (2.1) can be verified by taking expectation when we deal with a probability space. Thus, we let $(\Omega,\mathcal{F},P)$ be a probability space, where P is the probability function.

Proposition 2.8. *Let $\{u_n(\cdot,\omega)\}_{n=1}^\infty$ be a sequence of functions in $L^p(\Omega,\mathcal{V})$ such that*

$$\sup_n \int_\Omega \|u_n\|_{\mathcal{V}}^p dP = C < \infty. \tag{2.3}$$

Then, there is a set N of measure zero and a subsequence $\{u_{n,n}(\cdot,\omega)\}_{n=1}^\infty$ such that for all $\omega \notin N$,

$$\sup_n \|u_{n,n}(\cdot,\omega)\|_{\mathcal{V}} \le C(\omega) < \infty.$$

Proof. First, we know that there is a set $\hat{N}$ of measure zero such that for $\omega \notin \hat{N}$, $\|u_n(\cdot,\omega)\|_{\mathcal{V}}^p < \infty$ for all n. Indeed, we just take the union of the exceptional sets, one for each n. We have,

$$\begin{aligned}
&P\Big(\{\omega : \liminf_{n\to\infty} \|u_n(\cdot,\omega)\|_{\mathcal{V}}^p \geq M_1\}\Big)\\
&\leq \frac{1}{M_1}\int_\Omega \liminf_{n\to\infty} \|u_n(\cdot,\omega)\|_{\mathcal{V}}^p dP\\
&\leq \frac{1}{M_1}\liminf_{n\to\infty}\int_\Omega \|u_n(\cdot,\omega)\|_{\mathcal{V}}^p dP \leq \frac{C}{M_1}.
\end{aligned}$$

Therefore, by choosing a sufficiently large M_1, we obtain that the set

$$B_1 \equiv \{\omega : \liminf_{n\to\infty} \|u_n(\cdot,\omega)\|_{\mathcal{V}}^p \geq M_1\}$$

has measure less than 1/2. Letting $G_1 \equiv \Omega \setminus B_1$, it follows that $P(G_1) > 1/2$ and for $\omega \in G_1$,

$$\liminf_{n\to\infty} \|u_n(\cdot,\omega)\|_{\mathcal{V}}^p < M_1.$$

It follows that there is a subsequence $\{u_{1,n}(\cdot,\omega)\}_{n=1}^\infty$ of $\{u_n(\cdot,\omega)\}_{n=1}^\infty$ such that for all n and $\omega \in G_1$,

$$\|u_{1,n}(\cdot,\omega)\|_{\mathcal{V}}^p < M_1.$$

Next, the same reasoning leads to

$$P\Big(\big\{\omega \in B_1 : \liminf_{n\to\infty} \|u_{1,n}(\cdot,\omega)\|_{\mathcal{V}}^p \geq M_2\big\}\Big) \leq \frac{C}{M_2},$$

and so for a sufficiently large $M_2 > M_1$,

$$B_2 \equiv \big\{\omega \in B_1 : \liminf_{n\to\infty} \|u_{1,n}(\cdot,\omega)\|_{\mathcal{V}}^p \geq M_2\big\}$$

has measure less than 1/4. Let G_2 be such that $B_2 \cup G_2 = B_1$. Then, for $\omega \in G_2$ it follows that $\liminf_{n\to\infty} \|u_{1,n}(\cdot,\omega)\|_{\mathcal{V}}^p < M_2$. Thus, there is a further subsequence $\{u_{2,n}(\cdot,\omega)\}_{n=1}^\infty$ of $\{u_{1,n}(\cdot,\omega)\}_{n=1}^\infty$ such that for $\omega \in G_2$,

$$\|u_{2,n}(\cdot,\omega)\|_{\mathcal{V}}^p < M_2.$$

Continuing this way, we find a sequence of a subsequence, the subsequence $\{u_{i,n}\}_{n=1}^\infty$ being a subsequence of the $\{u_{(i-1),n}\}_{n=1}^\infty$ such that for all n,

$$\|u_{i,n}(\cdot,\omega)\|_{\mathcal{V}}^p < M_i \quad \text{if } \omega \in \cup_{j=1}^i G_i,$$

where $\Omega \setminus \cup_{j=1}^i G_i \equiv B_i$ satisfying $P(B_i) < 2^{-i}$, $B_{i+1} \subseteq B_i$. Letting $N \equiv \cap_i B_i \cup \hat{N}$, it follows that $P(N) = 0$. Now, we consider the diagonal sequence $\{u_{n,n}\}_{n=1}^\infty$. If $\omega \notin N$, then it is in some G_i and so for all n sufficiently large, say $n \geq k$, $\|u_{n,n}(\cdot,\omega)\|_{\mathcal{V}}^p \leq M_i$. Therefore, for that ω, it follows that for all n,

$$\|u_{n,n}(\cdot,\omega)\|_{\mathcal{V}}^p \leq M_i + \max(\|u_{m,m}(\cdot,\omega)\|_{\mathcal{V}}^p, m < k) \equiv C(\omega) < \infty.$$

□

This result leads to the following important corollary.

Corollary 2.9. *Let V be a reflexive separable Banach space with dual V' and p, p' be conjugate numbers and let $\{u_n(t,\omega)\}_{n=1}^\infty$ be a sequence of $\mathcal{P}$-measurable functions with values in V that satisfies the estimate*

$$\sup_n \int_\Omega \|u_n\|_{\mathcal{V}}^p dP = C < \infty.$$

Then, there exists a $\mathcal{P}$-measurable function $u(t,\omega)$ such that for each ω not in a null set N,

$$u(\cdot,\omega) \in \mathcal{V},$$

and there is a subsequence u_{n_ω} such that $u_{n_\omega}(\cdot,\omega) \to u(\cdot,\omega)$ weakly in $\mathcal{V}$, as $n_\omega \to \infty$.

Proof. It follows from the proposition that there is a set of measure zero N and a subsequence, still denoted with subscript n, such that for $\omega \notin N$,

$$\sup_n \|u_n(\cdot,\omega)\|_{\mathcal{V}} \le C(\omega) < \infty.$$

Then, we apply Theorem 2.1 to $u_n \mathcal{X}_{[0,T]\times N}$ and obtain the desired conclusion. $\square$

The following proposition is not surprising, being a consequence of the above measurable selection theorem.

Proposition 2.10. *Let $f(\cdot,\omega) \in \mathcal{V}'$. If $\omega \to f(\cdot,\omega)$ is measurable into $\mathcal{V}'$, then for each ω, there exists a representative $\hat{f}(\cdot,\omega) \in \mathcal{V}', \hat{f}(\cdot,\omega) = f(\cdot,\omega)$ in $\mathcal{V}'$ such that $(t,\omega) \to \hat{f}(t,\omega)$ is product measurable. If $f(\cdot,\omega) \in \mathcal{V}'$ and $(t,\omega) \to f(t,\omega)$ is product measurable, then $\omega \to f(\cdot,\omega)$ is measurable into $\mathcal{V}'$. The same conclusions hold when $\mathcal{V}'$ is replaced with $\mathcal{V}$.*

Proof. Since f is measurable into $\mathcal{V}'$, there exist simple functions f_n such that

$$\lim_{n\to\infty} \|f_n(\omega) - f(\omega)\|_{\mathcal{V}'} = 0,\ \|f_n(\omega)\| \le 2\|f(\omega)\|_{\mathcal{V}'} \equiv C(\omega).$$

Now, one of these simple functions is of the form

$$\sum_{i=1}^{M} c_i \mathcal{X}_{E_i}(\omega),$$

where $c_i \in \mathcal{V}'$. Therefore, there is no loss of generality in assuming that $c_i(t) = \sum_{j=1}^{N} d_j^i \mathcal{X}_{F_j}(t)$, where $d_j^i \in V'$. Hence, we can assume each f_n is product measurable into $\mathcal{B}(V') \times \mathcal{F}$. Then, by Theorem 2.1, there exists $\hat{f}(\cdot,\omega) \in \mathcal{V}'$ such that $\hat{f}$ is product measurable and a subsequence $f_{n(\omega)}$ converging weakly in $\mathcal{V}'$ to $\hat{f}(\cdot,\omega)$ for each ω. Thus $f_{n(\omega)}(\omega) \to f(\omega)$ strongly in $\mathcal{V}'$ and $f_{n(\omega)}(\omega) \to \hat{f}(\omega)$ weakly in $\mathcal{V}'$. Therefore, $\hat{f}(\omega) = f(\omega)$ in $\mathcal{V}'$ and so it can be assumed that if f is measurable into $\mathcal{V}'$, then for each ω, it has a representative $\hat{f}(\omega)$ such that $(t,\omega) \to \hat{f}(t,\omega)$ is product measurable.

If f is product measurable into V' and each $f(\cdot,\omega) \in \mathcal{V}'$, does it follow that f is measurable into $\mathcal{V}'$? By measurability,

$$f(t,\omega) = \lim_{n\to\infty} \sum_{i=1}^{m_n} c_i^n \mathcal{X}_{E_i^n}(t,\omega) = \lim_{n\to\infty} f_n(t,\omega),$$

where E_i^n is product measurable and we can assume $\|f_n(t,\omega)\|_{V'} \le 2\|f(t,\omega)\|$. Then by product measurability, $\omega \to f_n(\cdot,\omega)$ is measurable into $\mathcal{V}'$ because if $g \in \mathcal{V}$,

$$\omega \to \langle f_n(\cdot,\omega), g\rangle$$

is of the form

$$\omega \to \sum_{i=1}^{m_n} \int_0^T \langle c_i^n \mathcal{X}_{E_i^n}(t,\omega), g(t)\rangle dt$$

which is
$$\omega \to \sum_{i=1}^{m_n} \int_0^T \langle c_i^n, g(t)\rangle \mathcal{X}_{E_i^n}(t,\omega)dt,$$
and this is $\mathcal{F}$-measurable since E_i^n is product measurable. Thus, it is measurable into $\mathcal{V}'$ as desired and
$$\langle f(\cdot,\omega), g\rangle = \lim_{n\to\infty} \langle f_n(\cdot,\omega), g\rangle,\ \omega \to \langle f_n(\cdot,\omega), g\rangle \text{ is } \mathcal{F}\text{-measurable.}$$
Obviously, the conclusion is the same for these two conditions if $\mathcal{V}'$ is replaced with $\mathcal{V}$, since by the Pettis Theorem, $\omega \to \langle f(\cdot,\omega), g\rangle$ is measurable into $\mathcal{V}'$. □

3. Measurability of weak solutions of variational inequalities

In this section, we provide an important application of the measurable selection theorem, Theorem 2.1, to weak solutions of a broad class of variational inequalities.

In what follows $(\Omega,\mathcal{F})$ is a measurable space; $p > 1$; $V \subseteq H = H' \subseteq V'$ and each space is a separable Hilbert space that is dense in the following one; and $\mathcal{H} \equiv L^2([0,T],H)$. Then, $\mathcal{V} \subseteq \mathcal{H} = \mathcal{H}' \subseteq \mathcal{V}'$. As above, we denote by $\langle\cdot,\cdot\rangle_{\mathcal{V}} = \langle\cdot,\cdot\rangle_{\mathcal{V}',\mathcal{V}}$ the duality pairing between $\mathcal{V}$ and $\mathcal{V}'$, and $\mathcal{P}$ denotes the product σ-algebra which is the smallest σ algebra that contains the sets of the form $B \times F$ where B is Borel in $[0,T]$ and $F \in \mathcal{F}$.

Let $\mathcal{K}$ be a closed convex subset of $\mathcal{V}$ and let, for the sake of convenience, $0 \in \mathcal{K}$. Note that if $\mathcal{K}$ is closed and convex in $\mathcal{H}$, then $\mathcal{K} \cap \mathcal{V}$ is closed and convex in $\mathcal{V}$. Let $A(\cdot,\omega) : \mathcal{V} \to \mathcal{V}'$ be monotone, hemicontinuous, bounded and coercive operator, i.e.,
$$\lim_{\|u\|\to\infty} \frac{\langle A(u,\omega), u\rangle_{\mathcal{V}}}{\|u\|_{\mathcal{V}}} = \infty. \tag{3.1}$$
We assume that $\omega \to A(u(\omega),\omega)$ is measurable into $\mathcal{V}'$ whenever $\omega \to u(\omega)$ is measurable into $\mathcal{V}$.

Let P be a penalization operator associated with $\mathcal{K}$, as discussed in [11]. Thus,
$$Pu(\omega) = F(u(\omega) - \operatorname{proj}_{\mathcal{K}} u(\omega)),$$
where F is the duality map for p satisfying $\|Fx\| = \|x\|^{p-1}$ and $\langle Fx, x\rangle = \|x\|^p$, [11]. As is well known, P is monotone and demicontinuous, since $P(u) = 0$ on $\mathcal{K}$ and is nonzero for $u \notin \mathcal{K}$.

We assume that $(t,\omega) \to f(t,\omega)$ is product measurable. Under these assumptions, for each $n \in \mathbb{N}$, there exists a solution u_n to the penalized problem,
$$\begin{gathered} u_n' + A(u_n(\omega),\omega) + nP(u_n(\cdot,\omega)) = f(\cdot,\omega) \text{ in } \mathcal{V}', \\ u_n(0,\omega) = 0, \end{gathered} \tag{3.2}$$
such that $(t,\omega) \to u_n(t,\omega)$ is product measurable. Such a solution can be obtained by using arguments similar to those in stochastic partial differential equations.

We also assume, as in [11], the existence of a regularizing sequence such that if $u \in \mathcal{K}$, then there exists $u_i \in \mathcal{K}$ such that $u_i' \in \mathcal{V}', u_i(0) = 0$ and
$$\limsup_{i\to\infty} \langle u_i', u_i - u\rangle_{\mathcal{V}} \le 0.$$

Then, using standard arguments, one can pass to an appropriate limit and obtain the first part of the following theorem.

Theorem 3.1. *Suppose $A(\cdot,\omega)$ is monotone, hemicontinuous, bounded and coercive as a map from $\mathcal{V}$ to $\mathcal{V}'$. Suppose also that when $\omega \to u(\omega)$ is measurable into $\mathcal{V}$, then $\omega \to A(u(\omega),\omega)$ is measurable into $\mathcal{V}'$. Let $\mathcal{K}$ be a closed and convex subset of $\mathcal{V}$ containing 0. Let there be a regularizing sequence $\{u_i\}$ for each $u \in \mathcal{K}$ satisfying $u_i(0) = 0, u_i' \in \mathcal{V}', u_i \in \mathcal{K}$,*

$$\limsup_{i\to\infty}\langle u_i', u_i - u\rangle \le 0.$$

Then, for each ω there exists a solution to the variational inequality

$$\langle v', u - v\rangle + \langle A(u(\cdot,\omega),\omega), u(\cdot,\omega) - v\rangle \le \langle f(\cdot,\omega), u - v\rangle$$

that is valid for all $v \in \mathcal{K}$, such that $(Bv)' \in \mathcal{V}', Bv(0) = 0$, and $(t,\omega) \to u(t,\omega)$ is $\mathcal{B}([0,T]) \times \mathcal{F}$-measurable.

Proof. It only remains to verify the assertion about measurability. This follows from Theorem 2.1, since one can obtain an estimate of the right form for the measurable functions $u_n(\cdot,\omega)$ and $u_n^*(\cdot,\omega)$. Then, the above argument shows that a subsequence has a convergent subsequence which converges to a solution. $\square$

We note that one can replace the coercivity condition (3.1) with a weaker one involving $\lambda I + A$ for large enough λ, for $p \ge 2$, see [8] for details.

Next, we describe a surprising generalization of Theorem 3.1 in which the convex closed set is not fixed, but depends on the random variable, i.e., $\mathcal{K} = \mathcal{K}(\omega)$. The proof is exactly the same as long as the penalization operator satisfies that $\omega \to P(u(\omega),\omega) = F(u(\omega) - \operatorname{proj}_{\mathcal{K}(\omega)} u(\omega))$ is measurable into $\mathcal{V}'$ whenever $\omega \to u(\omega)$ is measurable into $\mathcal{V}$.

First, we need the following result that describes the conditions on the set-valued mapping $\omega \to \mathcal{K}(\omega)$ that make it satisfy this property.

Proposition 3.2. *Let $\omega \to \mathcal{K}(\omega)$ be measurable into $\mathcal{V}$. Then, $\omega \to \operatorname{proj}_{\mathcal{K}(\omega)} u(\omega)$ is also measurable into $\mathcal{V}$ whenever $\omega \to u(\omega)$ is measurable.*

Proof. It follows from standard results on measurable multi-functions, see e.g., [3], that there is a countable collection $\{w_n(\omega)\}$, $\omega \to w_n(\omega)$ being measurable and $w_n(\omega) \in \mathcal{K}(\omega)$, for each ω, such that for each ω we have $\mathcal{K}(\omega) = \overline{\cup_n w_n(\omega)}$. Let

$$d_n(\omega) \equiv \min\{\|u(\omega) - w_k(\omega)\|, k \le n\}.$$

Let $u_1(\omega) \equiv w_1(\omega)$. Set $u_2(\omega) = w_1(\omega)$ on the set

$$\{\omega : \|u(\omega) - w_1(\omega)\| < \{\|u(\omega) - w_2(\omega)\|\}\}$$

and $u_2(\omega) \equiv w_2(\omega)$ off this set. Thus, $\|u_2(\omega) - u(\omega)\| = d_2$. Let

$$\begin{gathered}
u_3(\omega) = w_1(\omega) \quad \text{on } S_1 \equiv \big\{\omega : \|u(\omega) - w_1(\omega)\| < \|u(\omega) - w_j(\omega)\|,\ j = 2,3\big\},\\
u_3(\omega) = w_2(\omega) \quad \text{on } S_1 \cap \big\{\omega : \|u(\omega) - w_1(\omega)\| < \|u(\omega) - w_j(\omega)\|,\ j = 3\big\},\\
u_3(\omega) = w_3(\omega) \text{ on the remainder of } \Omega.
\end{gathered}$$

Thus, $\|u_3(\omega) - u(\omega)\| = d_3$.

We continue in this way, at each step n obtaining $u_n(\omega)$ such that

$$\|u_n(\omega) - u(\omega)\| = d_n(\omega),$$

and $u_n(\omega) \in \mathcal{K}(\omega)$ with u_n measurable. Thus, in effect, we pick the closest of all the $w_k(\omega)$ for $k \leq n$ as the value of $u_n(\omega)$ and u_n is measurable. Then, by the density of $\{w_n(\omega)\}$ in $\mathcal{K}(\omega)$, for each ω we have that $\{u_n(\omega)\}$ is a minimizing sequence for

$$\lambda(\omega) \equiv \inf\{\|u(\omega) - z\| : z \in \mathcal{K}(\omega)\}.$$

Then, it follows that $u_n(\omega) \to \operatorname{proj}_{\mathcal{K}(\omega)} u(\omega)$ weakly in $\mathcal{V}$. Indeed, suppose the sequence fails to converge to $\operatorname{proj}_{\mathcal{K}(\omega)} u(\omega)$. Since the sequence is minimizing, it is bounded. Thus, there is a subsequence, still denoted as $u_n(\omega)$ that converges to $q(\omega) \neq \operatorname{proj}_{\mathcal{K}(\omega)} u(\omega)$. Then,

$$\lambda(\omega) = \lim_{n\to\infty} \|u(\omega) - u_n(\omega)\| \geq \|u(\omega) - q(\omega)\|,$$

because a convex and lower semicontinuous function is weakly lower semicontinuous.This implies that $q(\omega) = \operatorname{proj}_{\mathcal{K}(\omega)}(u(\omega))$ because the projection map is well defined thanks to strict convexity of the norm used, which is a contradiction. Hence, $\operatorname{proj}_{\mathcal{K}(\omega)} u(\omega) = \lim_{n\to\infty} u_n(\omega)$ and so is a measurable function. It follows that $\omega \to P(u(\omega), \omega)$ is measurable into $\mathcal{V}$. □

The following resutl is now immediate, which we state as a theorem because of its importance.

Theorem 3.3. *Suppose that: $A(\cdot,\omega)$ is monotone, hemicontinuous, bounded and coercive as a map from $\mathcal{V}$ to $\mathcal{V}'$; when $\omega \to u(\omega)$ is measurable into $\mathcal{V}$ then $\omega \to A(u(\omega),\omega)$ is measurable into $\mathcal{V}'$; $\mathcal{K}(\omega)$ is a closed and convex subset of $\mathcal{V}$ containing 0; and $\omega \to \mathcal{K}(\omega)$ is a set-valued measurable multifunction. Moreover, suppose that there is a regularizing sequence $\{u_i\}$, for each $u \in \mathcal{K}$, satisfying $u_i(0) = 0, u_i' \in \mathcal{V}', u_i \in \mathcal{K}$,*

$$\limsup_{i\to\infty} \langle u_i', u_i - u\rangle \leq 0.$$

Then, for each ω, there exists a solution $u(\omega)$ to the variational inequality

$$\langle v', u(\cdot,\omega) - v\rangle + \langle A(u(\cdot,\omega),\omega), u(\cdot,\omega) - v\rangle \leq \langle f(\cdot,\omega), u(\cdot,\omega) - v\rangle$$

valid for all $v \in \mathcal{K}(\omega)$ such that $v' \in \mathcal{V}', Bv(0) = 0$, and $(t,\omega) \to u(t,\omega)$, is $\mathcal{B}([0,T]) \times \mathcal{F}$-measurable.

Proof. The proof is identical to the proof of Theorem 3.1. We obtain a measurable solution to (3.2) in which P is replaced with $P(\cdot,\omega)$. Then, we follow exactly the same steps and finally use Theorem 2.1 to obtain the measurability of a solution to the variational inequality. □

The above result, in addition top its importance, is quite interesting. Indeed, it is not obvious that if $u(\omega) \in \mathcal{K}(\omega)$, for each ω, that $(t,\omega) \to u(t,\omega)$ has any kind of product measurability. This would not be obvious even if $\mathcal{K}$ were independent of ω. However, the theorem says that there is a measurable solution to the variational inequality, as long as $\omega \to \mathcal{K}(\omega)$ is a set-valued measurable multifunction.

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