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BIFURCATION FOR ELLIPTIC FORTH-ORDER PROBLEMS WITH QUASILINEAR SOURCE TERM

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Abstract. We study the bifurcations of the semilinear elliptic forth-order problem with Navier boundary conditions

$$\Delta^2 u - \operatorname{div}(c(x)\nabla u) = \lambda f(u) \quad \text{in } \Omega,$$
$$\Delta u = u = 0 \quad \text{on } \partial\Omega.$$

Where $\Omega \subset \mathbb{R}^n$, $n \geq 2$ is a smooth bounded domain, f is a positive, increasing and convex source term and $c(x)$ is a smooth positive function on $\overline{\Omega}$ such that the L^∞-norm of its gradient is small enough. We prove the existence, uniqueness and stability of positive solutions. We also show the existence of critical value λ^* and the uniqueness of its extremal solutions.

1. Introduction and statement of main results

In the literature the term 'bifurcation' is used in a general way to indicate stability changes, structural changes in a system etc.. The foundations of the theory has been laid by Poincaré who studied branching of solutions in many problem in celestial mechanics and bifurcation, i.e. splitting into two parts, of rotating fluid masses when the rotational velocity reached a certain value.

The non-linearity of a phenomenon can have several origins. It often results from the geometry. Wish show very interesting characteristics, namely the existence of multiple solutions, the presence of bifurcations, the passage of a solution to another through loss of stability.

The Bifurcations are one of the most interesting events and surprising of nonlinear systems. We say that a system has a bifurcation if an infinitesimal variation of its parameters causes a sudden change of regime.

The main interest of non-linear physics lies in its ability to explain the evolution of the problems: a phenomenon usually depends on a number of parameters, called control parameters that control the evolution of the system. By variation of the parameters and the result of non-linearities, the system may undergo transitions. In math, they are bifurcations.

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Various authors have studied the existence of weak solutions for the bifurcation problem

$$\begin{gathered} -\Delta u = \lambda f(u) \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^n$, $n \geq 2$.

Mironescu and Rădulescu have proved in [15] that there exists $0 < \lambda^* < \infty$, a critical value of the parameter λ, such that (1.1) has a minimal, positive, classical solution u_λ for $0 < \lambda < \lambda^*$ and does not have a weak solution for $\lambda > \lambda^*$. Abid et al generalized in [1] the same result for the Bi-laplace operator. Now, let

$$a := \lim_{t\to\infty} \frac{f(t)}{t}.$$

The value a was be crucial in the study of (E_{λ^*}) and of the behavior of u_λ when λ approaches λ^*.

Also in dimension 4, Wei in [18], have studied the behavior of solutions to the following non-linear eigenvalue problem (1.1). More precisely, when $f(u) = e^u$, we can see that (1.1) is issued from the geometry by prescribing the so-called $\mathcal{Q}$-curvature. For more details, see [3] and [4].

Our main interest here is the study of a bifurcation problem for $\lambda > 0$,

$$\begin{gathered} \Delta^2 u - \operatorname{div}(c(x)\nabla u) = \lambda f(u) \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ \Delta u = u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

Where Ω be a smooth bounded domain in $\mathbb{R}^n$ $(n \geq 2)$, $c(x)$ is a smooth positive function on $\overline{\Omega}$ and f is a positive, increasing and convex smooth function on $(0, +\infty)$, which verifies

$$\lim_{t\to\infty} \frac{f(t)}{t} = a \in (0, \infty).$$

In this paper, we show how the critical problem behaves when he is considered with the Navier boundary condition, we have to use the maximum principle which assured with smallest condition: There exists $\epsilon = \epsilon(n, \Omega)$ such that

$$\|\nabla c\|_\infty \ll \epsilon$$

For more detail, see [8].

Throughout this article, we denote by $\|\cdot\|_2$, the $L^2(\Omega)$-norm, whereas we denote by $\|\cdot\|$, the $H^2(\Omega) \cap H_0^1(\Omega)$-norm given by

$$\|u\|^2 = \int_\Omega |\Delta u|^2.$$

We say that $u \in H^2(\Omega) \cap H_0^1(\Omega)$ is a weak solution of (1.2), if $f(u) \in L^1(\Omega)$ and

$$\int_\Omega \Delta u \cdot \Delta\varphi + \int_\Omega c(x)\nabla u \cdot \nabla\varphi = \lambda \int_\Omega f(u)\varphi, \quad \forall \varphi \in C^2(\overline{\Omega}) \cap H^2(\Omega) \cap H_0^1(\Omega).$$

Such solutions are usually known as weak energy solutions. For short, we will refer to them simply as solutions wish is assuredly by the next lemma.

Lemma 1.1. *Since $f(t) \leq at + f(0)$, if $u \in H^2(\Omega) \cap H_0^1(\Omega)$ is a weak solution of (1.2), it is easily seen by a standard bootstrap argument that u is always a classical solution.*

For more details, see [8, Proposition 7.15]. In the rest of this article, we denote by a solution of (1.2) any weak or classical solution.

Definition 1.2. We say that a solution u_λ of (1.2) is *minimal* if $u_\lambda \leq u$ in Ω for any solution u of (1.2).

Definition 1.3. We say that $u \in H^2(\Omega) \cap H_0^1(\Omega)$ is a supersolution (resp. subsolution) of (1.2) if $f(u) \in L^1(\Omega)$ and

$$\Delta^2 u - \operatorname{div}(c(x)\nabla u) \geq \lambda f(u) \quad (\text{resp. } \leq \lambda f(u)) \quad \text{in } \mathcal{D}'(\Omega).$$

Definition 1.4. A solution u of (1.2) is stable if and only if the first eigenvalue of the linearized operator

$$v \mapsto L_\lambda(v) := \Delta^2 v - \operatorname{div}(c(x)\nabla v) - \lambda f'(u)v,$$

given by

$$\eta_1(\lambda, u) := \inf_{\varphi \in H^2(\Omega)\cap H_0^1(\Omega)-\{0\}} \frac{\int_\Omega |\Delta\varphi|^2 + \int_\Omega c(x)|\nabla\varphi|^2 - \lambda \int_\Omega f'(u)\varphi^2}{\|\varphi\|_2^2},$$

is nonnegative.

If $\eta_1(\lambda, u) < 0$, the solution u is said to be *unstable*.

Let v_1 be a positive eigenfunction (see [8, section 3.1.3]) associated with the first eigenvalue λ_1 of the operator $\Delta^2 - \operatorname{div}(c(x)\nabla)$ with Navier boundary conditions, namely

$$\begin{gathered} \Delta^2 v_1 - \operatorname{div}(c(x)\nabla v_1) = \lambda_1 v_1 \quad \text{in } \Omega, \\ \Delta v_1 = v_1 = 0 \quad \text{on } \partial\Omega, \\ \|v_1\|_2 = 1. \end{gathered} \tag{1.3}$$

Next, we let

$$\Lambda := \{\lambda > 0 : (1.2) \text{ admits a solution and } \lambda^* := \sup \Lambda \leq +\infty.$$

We also let

$$r_0 := \inf_{t>0} \frac{f(t)}{t}.$$

The two values a and r_0 that we have already defined will be important in the bifurcation phenomena. More precisely, in the frame of the critical value λ^*.

Theorem 1.5. *There exists a critical value $\lambda^* \in (0, \infty)$ such that the following properties hold:*

(i) *For any $\lambda \in (0, \lambda^*)$, problem* (1.2) *has a minimal solution u_λ, which is the unique stable solution of* (1.2).
(ii) *For any $\lambda \in (0, \lambda_1/a)$, u_λ is the unique solution of problem* (1.2).
(iii) *The mapping $\lambda \longmapsto u_\lambda$ is increasing.*
(iv) *$u^* := \lim_{\lambda\to\lambda^*} u_\lambda$ is a solution stable of the problem* (1.2) *with λ in stead of λ. In particular, $\eta_1(\lambda^*, u^*) = 0$.*

An important role in our arguments will be played by

$$l := \lim_{t\to\infty} \big(f(t) - at\big).$$

We distinguish two situations strongly depending on the sign of l.

Theorem 1.6. *Assume that $l \geq 0$. Then*

(i) *$\lambda^* = \lambda_1/a$;*

(ii) *problem* (1.2) *with* λ *in stead of* λ *has no solution;*
(iii) $\lim_{\lambda\to\lambda^*} u_\lambda = \infty$ *uniformly on compact subsets of* Ω.

Theorem 1.7. *Assume* $l < 0$. *Then the critical value* λ^* *belongs to* $(\lambda_1/a, \lambda_1/r_0)$ *and* (1.2) *with* λ *in stead of* λ *has a unique solution* u^*. *In this case,* (1.2) *has an unstable solution* v_λ *for any* $\lambda \in (\lambda_1/a, \lambda^*)$ *and the sequence* $(v_\lambda)_\lambda$ *has the following properties:*

(i) $\lim_{\lambda\to\lambda_1/a} v_\lambda = \infty$ *uniformly on compact subsets of* Ω*;*
(ii) $\lim_{\lambda\to\lambda^*} v_\lambda = u^*$ *uniformly in* Ω.

2. Proof of Theorem 1.5

The basic idea is to apply the barrier method, when the existence of the critical value λ^* is a consequence of the following auxiliary result.

Lemma 2.1. *Problem* (1.2) *has no solution for any* $\lambda > \lambda_1/r_0$, *but has at least one solution provided* λ *is positive and small enough.*

Proof. To show that (1.2) has a solution, we use the barrier method. To this aim, let $\overline{w} \in H^4(\Omega)$ that satisfies

$$\begin{gathered}\Delta^2\overline{w} - \operatorname{div}(c(x)\nabla\overline{w}) = 1 \quad \text{in } \Omega\\ \Delta\overline{w} = \overline{w} = 0 \quad \text{on } \partial\Omega.\end{gathered}$$

The choice of $\overline{w}$ implies that $\overline{w}$ is a super-solution of (1.2) for $\lambda \le 1/f(\|\overline{w}\|_\infty)$. Notice that for any $\lambda > 0$, the function $\underline{w} \equiv 0$ is a sub-solution of (1.2) since $f(0) > 0$.

Next, we define a sequence $w_n \in H^4(\Omega)$ by

$$\begin{gathered}\Delta^2 w_{n+1} - \operatorname{div}(c(x)\nabla w_{n+1}) = \lambda f(w_n) \quad \text{in } \Omega\\ \Delta w_{n+1} = w_{n+1} = 0 \quad \text{on } \partial\Omega.\end{gathered} \tag{2.1}$$

The maximum principle (see [8]] implies that

$$\underline{w} \le w_n \le w_{n+1} \le \overline{w} \quad \text{for all } n \in \mathbb{N},$$

so that the sequence $(w_n)_{n\ge0}$ is increasing and bounded, then it converges. It follows that problem (1.2) has a solution.

Assume now that u is a solution of (1.2) for some $\lambda > 0$. Using v_1 given in (1.3) as a test function and integrating by parts, we obtain

$$\begin{aligned}\lambda_1\int_\Omega v_1 u &= \int_\Omega (\Delta^2 v_1 - \operatorname{div}(c(x)\nabla v_1))u\\ &= \int_\Omega \Delta^2 u v_1 + \int_\Omega c(x)\nabla u\cdot\nabla v_1\\ &= \int_\Omega \Delta^2 u v_1 - \int_\Omega \operatorname{div}(c(x)\nabla u) v_1\\ &= \lambda\int_\Omega f(u) v_1\\ &\ge \lambda r_0 \int_\Omega u v_1.\end{aligned}$$

This yields

$$(\lambda_1 - \lambda r_0)\int_\Omega v_1 u \ge 0.$$

Since $v_1 > 0$ and $u > 0$, we conclude that the parameter λ should belong to $(0, \lambda_1/r_0)$. This completes our proof. $\square$

Another useful result is stated in what follows.

Lemma 2.2. *Assume that* (1.2) *has a solution for some* $\lambda \in (0, \lambda^*)$. *Then there exists a minimal solution denoted by* u_λ. *Moreover, for any* $\lambda' \in (0, \lambda)$, *problem* (1.2) *with* λ' *instead of* λ *has a solution.*

Proof. Fix $\lambda \in (0, \lambda^*)$ and let u be a solution of (1.2). As above, we use the barrier method to obtain a minimal solution of (1.2). The basic idea is to prove by induction that the sequence $(w_n)_{n\geq 0}$ defined in (2.1) is increasing and bounded by u, so it converges to some solution u_λ. Since u_λ is independent of the choice of u, then it is a minimal solution.

Now, if u is a solution of (1.2), then u is a super-solution for the problem (1.2) with λ' instead of λ for any λ' in $(0, \lambda)$ and 0 can be used always as a sub-solution. These complete the proof. $\square$

Remark 2.3. Thanks to lemmas 2.1 and 2.2, the set Λ is an interval bounded and not empty.

Proof of (i) of Theorem 1.5. First, we claim that u_λ is stable. Indeed, arguing by contradiction, i.e. the first eigenvalue $\eta_1(\lambda, u_\lambda)$ is negative. Then, there exists an eigenfunction $\psi \in H^4(\Omega)$ such that

$$\begin{gathered}
\Delta^2\psi - \operatorname{div}(c(x)\nabla\psi) - \lambda f'(u_\lambda)\psi = \eta_1\psi \quad \text{in } \Omega \\
\psi > 0 \quad \text{in } \Omega \\
\Delta\psi = \psi = 0 \quad \text{on } \partial\Omega.
\end{gathered}$$

Consider $u^\varepsilon := u_\lambda - \varepsilon\psi$. Hence, by linearity, we have

$$\begin{aligned}
&\Delta^2 u^\varepsilon - \operatorname{div}(c(x)\nabla u^\varepsilon) - \lambda f(u^\varepsilon) \\
&= \lambda f(u_\lambda) - \varepsilon(\Delta^2\psi - \operatorname{div}(c(x)\nabla\psi)) - \lambda f(u_\lambda - \varepsilon\psi) \\
&= \lambda f(u_\lambda) - \varepsilon(\lambda f'(u_\lambda)\psi + \eta_1\psi) - \lambda f(u_\lambda - \varepsilon\psi) \\
&= \lambda\Big(- f(u_\lambda - \varepsilon\psi) + f(u_\lambda) - \varepsilon f'(u_\lambda)\psi\Big) - \varepsilon\eta_1\psi \\
&= \lambda o_\varepsilon(\varepsilon\psi) - \varepsilon\eta_1\psi \\
&= \varepsilon\psi(\lambda o_\varepsilon(1) - \eta_1).
\end{aligned}$$

Since $\eta_1(\lambda, u_\lambda) < 0$, for $\varepsilon > 0$ small enough, we have

$$\Delta^2 u^\varepsilon - \operatorname{div}(c(x)\nabla u^\varepsilon) - \lambda f(u^\varepsilon) \geq 0 \quad \text{in } \Omega.$$

Then, for $\varepsilon > 0$ small enough, we use the strong maximum principle to deduce that $u^\varepsilon \geq 0$ is a super-solution of (1.2). As before, we obtain a solution u such that $u \leq u^\varepsilon$ and since $u^\varepsilon < u_\lambda$, then we contradict the minimality of u_λ.

Now, we show that (1.2) has at most one stable solution. Assume the existence of another stable solution $v \neq u_\lambda$ of problem (1.2). Then the function $w := v - u_\lambda$ satisfies

$$\begin{aligned}
\lambda\int_\Omega f'(v)w^2 &\leq \int_\Omega |\Delta w|^2 + \int_\Omega c(x)|\nabla w|^2 \\
&\leq \int_\Omega \Delta^2 w w - \int_\Omega \operatorname{div}(c(x)\nabla w)w
\end{aligned}$$

$$\begin{aligned}&\leq \int_\Omega \Big[\Delta^2 v - \operatorname{div}(c(x)\nabla v) - \Delta^2 u_\lambda + \operatorname{div}(c(x)\nabla u_\lambda)\Big] w\\ &\leq \lambda \int_\Omega \Big[f(v) - f(u_\lambda)\Big] w.\end{aligned}$$

Therefore

$$\int_\Omega \Big[f(v) - f(u_\lambda) - f'(v)(v-u_\lambda)\Big] w \geq 0.$$

By the maximum principle, we deduce that $w > 0$ in Ω. Thanks to the convexity of f, the term in the brackets is nonpositive, hence

$$f(v) - f(u_\lambda) - f'(v)(v-u_\lambda) = 0 \quad \text{in } \Omega,$$

which implies that f is affine over $[u_\lambda, v]$ in Ω. So, there exists two real numbers $\bar{a}$ and b such that

$$f(x) = \bar{a}x + b \quad \text{in } [0, \max_\Omega v].$$

Finally, since u_λ and v are two solutions to $\Delta^2 w - \operatorname{div}(c(x)\nabla w) = \lambda \bar{a} w + \lambda b$, we obtain

$$0 = \int_\Omega \Big(u_\lambda \Delta^2 v - v\Delta^2 u_\lambda\Big) - \int_\Omega \Big(u_\lambda \operatorname{div}(c(x)\nabla v) - v \operatorname{div}(c(x)\nabla u_\lambda)\Big) = \lambda b \int_\Omega (u_\lambda - v).$$

This is impossible since $b = f(0) > 0$ and $w = v - u_\lambda$ is positive in Ω. $\square$

Proof of (ii) of Theorem 1.5. Recall that λ_1 is defined in (1.3). By the convexity of f, we deduce that $a = \sup_{\mathbb{R}_+} f'(t)$. Let u be a solution to (1.2) for $\lambda \in (0, \lambda_1/a)$, we suppose that u is unstable. Then, we can take $\varphi = v_1 \in H^2(\Omega) \cap H_0^1(\Omega)$ which satisfy

$$\lambda a \int_\Omega \varphi^2 \geq \lambda \int_\Omega f'(u)\varphi^2 > \int_\Omega |\Delta\varphi|^2 + \int_\Omega c(x)|\nabla\varphi|^2 = \lambda_1 \int_\Omega \varphi^2,$$

which shows that

$$(\lambda a - \lambda_1) \int_\Omega \varphi^2 > 0.$$

That is impossible for $\lambda \in (0, \lambda_1/a)$. So, $\eta_1(\lambda, u) \geq 0$ and by (i), we obtain the uniqueness of u.

For the existence, we consider the minimization problem

$$\min_{u \in H^2(\Omega)\cap H_0^1(\Omega)} \mathfrak{J}(u),$$

where

$$\mathfrak{J}(u) := \frac{1}{2}\int_\Omega |\Delta u|^2 + \frac{1}{2}\int_\Omega c(x)|\nabla u|^2 - \lambda \int_\Omega \mathfrak{F}(u),$$

for all $u \in H^2(\Omega) \cap H_0^1(\Omega)$ with

$$u^+ := \max(u, 0) \quad \text{and} \quad \mathfrak{F}(u) := \int_0^{u^+} f(s)ds.$$

If $\lambda \in (0, \lambda_1/a)$, there exist $\varepsilon > 0$ and $A > 0$ depending on λ such that

$$2\lambda \mathfrak{F}(t) \leq (\lambda_1 - \varepsilon)t^2 + A, \qquad \forall\, t \in \mathbb{R}.$$

Standard arguments imply that $\mathfrak{J}(u)$ is coercive, bounded from below and weakly lower semi-continuous in $H^2(\Omega) \cap H_0^1(\Omega)$. Hence, the minimum of $\mathfrak{J}$ is attained by some function $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and also by u^+ since $\mathfrak{J}(u^+) \leq \mathfrak{J}(u)$. So, the critical point u of $\mathfrak{J}$ gives a solution of (1.2). $\square$

Proof of (iii) and (iv) in Theorem 1.5. By sub- and super-solution method, see Lemma 2.2, we obtain that the mapping $\lambda \longmapsto u_\lambda$ is increasing and this proves (iii).

Now we consider the nonlinear operator $G : (0, +\infty) \times C^{4,\alpha}(\overline{\Omega}) \cap E \to C^{0,\alpha}(\overline{\Omega})$,

$$(\lambda, u) \longmapsto \Delta^2 u - \operatorname{div}(c(x)\nabla u) - \lambda f(u),$$

where $\alpha \in (0,1)$ and E is the function space

$$E := \{u \in W^{4,2}(\Omega) : \Delta u = u = 0 \text{ on } \partial\Omega\}. \tag{2.2}$$

Assume that (1.2) with λ in stead of λ has a solution u. Then for any $\lambda \in (0, \lambda^*)$, $u_\lambda \le u$ in Ω. Using the monotonicity of u_λ, we deduce that the function

$$u^* = \lim_{\lambda \to \lambda^*} u_\lambda$$

is well defined in Ω and is a stable solution of problem (1.2) with λ in stead of λ. Assuming that the first eigenvalue $\eta_1(\lambda^*, u^*)$ is positive, we can apply the implicit function theorem to the operator G. It follows that problem (1.2) has a solution for λ in a neighborhood of λ^*. But this contradicts the definition of λ^*. So, $\eta_1(\lambda^*, u^*) = 0$ and this completes the proof of Theorem 1.5. □

Remark 2.4. Thanks to Lemma 2.1 and (ii) of Theorem 1.5, the critical value λ^* satisfies

$$\lambda_1/a \le \lambda^* \le \lambda_1/r_0.$$

3. Proof of Theorem 1.6

To prove this theorem, we show that the three assertions are equivalent. And finally, we prove that one hoolds. We first recall the following result which is due to Hörmander [11].

Lemma 3.1. *Let Ω be an open bounded subset of $\mathbb{R}^n$, $n \ge 2$ with smooth boundary. Let (u_n) be a sequence of super-harmonic nonnegative functions defined on Ω. Then the following alternative holds:*

(i) *either $\lim_{n\to\infty} u_n = \infty$ uniformly on compact subsets of Ω,*
(ii) *or (u_n) contains a subsequence which converges in $L^1_{\rm loc}(\Omega)$ to some function u.*

Remark 3.2. The result by Hörmander is also true if (u_n) is a sequence of a super-biharmonic nonnegative functions.

First, we assume that $\lambda^* = \lambda_1/a$. If (1.2) with λ in stead of λ has a solution u^*, then, as we have already observed in (iv) of Theorem 1.5, $\eta_1(\lambda^*, u^*) = 0$. Thus, there exists $\psi \in H^4(\Omega)$ satisfying:

$$\begin{gathered} \Delta^2\psi - \operatorname{div}(c(x)\nabla\psi) - \lambda^* f'(u^*)\psi = 0 \quad \text{in } \Omega \\ \psi > 0 \quad \text{in } \Omega \\ \Delta\psi = \psi = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

Using v_1, given in (1.3), as a test function and integrating by parts, we obtain

$$\int_\Omega \Big(\Delta^2 v_1 - \operatorname{div}(c(x)\nabla v_1)\Big)\psi - \lambda^* \int_\Omega f'(u^*)\psi v_1 = 0;$$

therefore

$$\int_\Omega \big(\lambda_1 - \lambda^* f'(u^*)\big)\psi v_1 = 0.$$

Since $\lambda_1 - \lambda^* f'(u^*) \geq 0$, the above equation forces $\lambda_1 - \lambda^* f'(u^*) = 0$. Hence

$$f'(u^*) \equiv a \quad \text{in} \quad \Omega.$$

This implies that $f(t) = at + b$ in $[0, \max_\Omega u^*]$ for some scalar $b > 0$. But there is no positive function in Ω such that $u = \Delta u = 0$ on $\partial\Omega$ and

$$\Delta^2 u - \operatorname{div}(c(x)\nabla u) = \lambda^* a u + \lambda^* b \quad \text{in } \Omega.$$

If not, Using v_1 and integrating by parts, we have

$$\int_\Omega \Delta^2 u v_1 - \int_\Omega \operatorname{div}(c(x)\nabla u) v_1 = \lambda^* a \int_\Omega u v_1 + \lambda^* b \int_\Omega v_1$$

then

$$\int_\Omega \Big(\Delta^2 v_1 - \operatorname{div}(c(x)\nabla v_1)\Big) u = \lambda_1 \int_\Omega u v_1 + \lambda^* b \int_\Omega v_1$$

i.e.

$$0 = \lambda^* b \int_\Omega v_1 \text{ which is impossible.}$$

Hence, problem (1.2) with λ in stead of λ has no solution and (i) implies (ii).

Next, we assume that (ii) occurs and we claim that $\lim_{\lambda\to\lambda^*} u_\lambda = \infty$ uniformly on compact subsets of Ω. If not, by Lemma 3.1 and up to a subsequence, (u_λ) converges locally in $L^1(\Omega)$ to u^* as $\lambda \to \lambda^*$. If u_λ is not bounded in $L^2(\Omega)$, we define

$$u_\lambda := l_\lambda w_\lambda,$$

with

$$\|w_\lambda\|_2 = 1 \quad \text{and} \quad l_\lambda \to +\infty \quad \text{as } \lambda \to \lambda^*.$$

Since $f(t) \leq at + f(0)$, we have

$$\begin{aligned}
\int_\Omega |\Delta w_\lambda|^2 &\leq \int_\Omega |\Delta w_\lambda|^2 + \int_\Omega c(x)|\nabla w_\lambda|^2 \\
&= \int_\Omega \Delta^2 w_\lambda w_\lambda - \int_\Omega \operatorname{div}(c(x)\nabla w_\lambda) w_\lambda = \int_\Omega \frac{\lambda f(u_\lambda)}{l_\lambda} w_\lambda \\
&\leq \lambda^* \int_\Omega \Big(a\, w_\lambda^2 + \frac{f(0)}{l_\lambda} w_\lambda\Big) \leq \lambda^* a + c_\lambda \int_\Omega w_\lambda \\
&\leq \lambda^* a + c_\lambda \sqrt{|\Omega|},
\end{aligned}$$

where c_λ is a positive constant independent on λ.

Recall that w_λ satisfies $\Delta^2 w_\lambda - \operatorname{div}(c(x)\nabla w_\lambda) = \frac{\lambda f(l_\lambda w_\lambda)}{l_\lambda}$ and f is quasilinear. These facts imply that (w_λ) is bounded in $H^4(\Omega)$. Hence, up to a subsequence, we have

$$w_\lambda \rightharpoonup w \text{ weakly in } H^4(\Omega) \text{ and } w_\lambda \to w \text{ strongly in } H^3(\Omega) \text{ as } \lambda \to \lambda^*.$$

Moreover, by the trace theorem,

$$w = \Delta w = 0 \quad \text{on } \partial\Omega. \tag{3.1}$$

We deduce that

$$\Delta^2 w_\lambda - \operatorname{div}(c(x)\nabla w_\lambda) = \frac{\lambda f(u_\lambda)}{l_\lambda} \to 0 \quad \text{in } L^1_{\rm loc}(\Omega) \quad \text{as } \lambda \to \lambda^*.$$

This implies $\Delta^2 w - \operatorname{div}(c(x)\nabla w) = 0$ in $\mathcal{D}'(\Omega)$. So, by (3.1), we deduce that $w \equiv 0$ in Ω. This contradicts the fact that $\|w\|_2 = \lim_{\lambda\to\lambda^*} \|w_\lambda\|_2 = 1$. Hence, (u_λ) is bounded in $L^2(\Omega)$ and by the same arguments as above, it is bounded in $H^4(\Omega)$.

This shows that (ii) implies (iii). Moreover, this simply shows that (ii) and (iii) are equivalent.

Now, if (1.2) with λ in stead of λ has a solution u^*, then the sequence (u_λ) converges to u^* as λ tends to λ^*, which cannot happen in the case where $\lim_{\lambda\to\lambda^*} u_\lambda = \infty$. Hence, (iii) implies (i).

Indeed, clearly if (ii) and (iii) occur, we have $\lim_{\lambda\to\lambda^*} \|u_\lambda\|_2 = \infty$. Set

$$u_\lambda = l_\lambda w_\lambda \quad \text{with } \|w_\lambda\|_2 = 1.$$

Then, up to a subsequence, we obtain

$$w_\lambda \rightharpoonup w \text{ weakly in } H^4(\Omega) \text{ and } w_\lambda \to w \text{ strongly in } H^3(\Omega) \text{ as } \lambda \to \lambda^*.$$

Moreover,

$$\Delta^2 w_\lambda - \operatorname{div}(c(x)\nabla w_\lambda) \to \Delta^2 w - \operatorname{div}(c(x)\nabla w) \quad \text{in } \mathcal{D}'(\Omega) \quad \text{as } \lambda \to \lambda^*$$

and

$$\frac{\lambda}{l_\lambda} f(l_\lambda w_\lambda) \to \lambda^* a w \quad \text{in } L^2(\Omega) \quad \text{as } \lambda \to \lambda^*.$$

Then

$$\begin{gathered}\Delta^2 w - \operatorname{div}(c(x)\nabla w) = \lambda^* a w \quad \text{in } \Omega,\\ \Delta w = w = 0 \quad \text{on } \partial\Omega.\end{gathered}$$

Multiplying by v_1, which is defined in (1.3), we obtain

$$\begin{aligned}\int_\Omega \lambda^* a w v_1 &= \int_\Omega \Delta^2 w v_1 - \operatorname{div}(c(x)\nabla w) v_1\\ &= \int_\Omega \Delta^2 v_1 w - \operatorname{div}(c(x)\nabla v_1) w = \int_\Omega \lambda_1 v_1 w.\end{aligned}$$

This proves (i).

To finish the proof of Theorem 1.6, we need only to show that (1.2) with λ_1/a in stead of λ has no solution. Indeed, assume that u is a solution of (1.2) with λ_1/a in stead of λ. Since $f(t) - at \geq 0$, we have

$$\Delta^2 u - \operatorname{div}(c(x)\nabla u) = \frac{\lambda_1}{a} f(u) \geq \lambda_1 u \quad \text{in } \Omega.$$

Multiplying the previous equation by v_1 and integrating by parts, we obtain $f(u) = au$ in Ω, which contradicts $f(0) > 0$. This concludes the proof of Theorem 1.6.

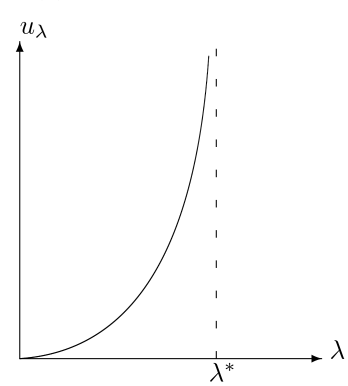

FIGURE 1. Behavior of the minimal solution.

Finally, we see that the branch containing the minimal solution has the behavior shown in Figure 1.

Remark 3.3. Observe that the equivalence of the assertions of Theorem 1.6 does not depend on the sign of l.

4. Proof of Theorem 1.7

For the first part of Theorem 1.7, we have already seen in Remark 2.4 that $\lambda_1/a \le \lambda^* \le \lambda_1/r_0$. Hence it suffices to prove that $\lambda^* \ne \lambda_1/a$ and $\lambda^* \ne \lambda_1/r_0$.

First, assume that $\lambda^* = \lambda_1/a$. Let u_λ be the minimal solution to (1.2). Then, multiplying (1.2) by v_1 given in (1.3) and integrating, we obtain

$$\begin{aligned} 0 = \int_\Omega \Big(\lambda_1 u_\lambda - \lambda f(u_\lambda)\Big) v_1 &= \int_\Omega \Big((\lambda_1 - a\lambda)u_\lambda - \lambda(f(u_\lambda) - au_\lambda)\Big) v_1 \\ &\ge -\lambda \int_\Omega \big(f(u_\lambda) - au_\lambda\big) v_1. \end{aligned}$$

Passing to the limit in the last inequality as λ tends to λ^*, we find

$$0 \ge -l\lambda^* \int_\Omega v_1 > 0,$$

which is impossible.

Now, assume that $\lambda^* = \lambda_1/r_0$ and let u be a solution of problem (1.2) with λ in stead of λ. Multiplying (1.2) with λ in stead of λ by v_1 and integrating by parts, we have

$$\lambda_1 \int_\Omega u v_1 = \frac{\lambda_1}{r_0} \int_\Omega f(u) v_1 \ge \lambda_1 \int_\Omega u v_1,$$

which forces $f(u) = r_0 u$ in Ω, so that $f(t) = r_0 t$ in $[0, \max_\Omega u]$. As above, this contradicts the fact that $f(0) > 0$.

Since $\lambda^* > \lambda_1/a$, the existence of a solution to (1.2) with λ in stead of λ is assured by Remark 3.3. Then, it remains to prove the uniqueness. Assume that u is another solution to (1.2) with λ in stead of λ and let $w := u - u^*$. Since $u_\lambda < u$ and $\lim_{\lambda \to \lambda^*} u_\lambda = u^*$, we have $w > 0$. Then by convexity of f we have

$$\Delta^2 w - \operatorname{div}(c(x)\nabla w) = \lambda^*(f(u) - f(u^*)) \ge \lambda^* f'(u^*) w \quad \text{in } \Omega.$$

Recall that $\eta_1(\lambda^*, u^*) = 0$, so let ψ be the corresponding eigenfunction. Multiplying the last inequality by ψ and integrating by parts, we find

$$0 = \int_\Omega \lambda^* \Big(f(u) - f(u^*) - f'(u^*) w\Big) \psi \ge 0.$$

Therefore, we must have equality $f(u) - f(u^*) = f'(u^*) w$ in Ω, which implies that f is linear in $[0, \max_\Omega u]$ and this leads a contradiction as in the proof of Theorem 1.6.

The second part of Theorem 1.7 concerning the existence of a non stable solution v_λ of (1.2) will be proved by using the mountain pass theorem of Ambrosetti and Rabinowitz [2] in the following form.

Theorem 4.1. *Let E be a real Banach space and $J \in C^1(E, \mathbb{R})$. Assume that J satisfies the Palais-Smale condition and the following geometric assumptions:*

(*) *there exist positive constants R and ρ such that*

$$J(u) \ge J(u_0) + \rho, \ \textit{for all } u \in E \textit{ with } \|u - u_0\| = R.$$

(**) *there exists* $v_0 \in E$ *such that* $\|v_0 - u_0\| > R$ *and* $J(v_0) \le J(u_0)$.

Then the functional J *possesses at least a critical point. The critical value is characterized by*

$$c := \inf_{g\in\Gamma} \max_{u\in g([0,1])} J(u),$$

where

$$\Gamma := \big\{g \in C([0,1], E) : g(0) = u_0,\, g(1) = v_0\big\}$$

and satisfies $c \ge J(u_0) + \rho$.

In our case, $J : E \to \mathbb{R}$

$$u \longmapsto \frac{1}{2}\int_\Omega |\Delta u|^2 + \frac{1}{2}\int_\Omega c(x)|\nabla u|^2 - \int_\Omega F(u),$$

where E is the function space defined in (2.2) and

$$F(t) = \lambda \int_0^t f(s)ds, \quad \text{for all } t \ge 0.$$

We take u_0 as the stable solution u_λ for each $\lambda \in (\lambda_1/a, \lambda^*)$.

Remark 4.2. The energy functional J belongs to $C^1(E,\mathbb{R})$ and

$$\langle J'(u), v\rangle = \int_\Omega \Delta u \cdot \Delta v + \int_\Omega c(x)\nabla u \cdot \nabla v - \lambda \int_\Omega f(u)v, \quad \text{for all } u, v \in E.$$

Since $\eta_1(\lambda, u_\lambda) > 0$, the function u_λ is a strict local minimum for J, we apply the mountain pass theorem for J.

Using the same arguments of Mironescu and Rădulescu in [15, Lemma 9], we show in the next lemma that J satisfies the Palais-Smale compactness condition.

Lemma 4.3. *Let* $(u_n) \subset E$ *be a Palais-Smale sequence; that is,*

$$\sup_{n\in\mathbb{N}} |J(u_n)| < +\infty, \tag{4.1}$$

$$\|J'(u_n)\|_{E^*} \to 0 \quad \text{as } n \to \infty. \tag{4.2}$$

Then (u_n) *is relatively compact in* E.

Proof. Since any subsequence of (u_n) verifies (4.1) and (4.2) it is enough to prove that (u_n) contains a convergent subsequence. It suffices to prove that (u_n) contains a bounded subsequence in E. Indeed, suppose we have proved this. Then, up to a subsequence, $u_n \to u$ weakly in E, strongly in $L^2(\Omega)$. Now (4.2) gives

$$\Delta^2 u_n - \operatorname{div}(c(x)\nabla u_n) - \lambda f(u_n) \to 0 \quad \text{in } \mathcal{D}'(\Omega)$$

Note that $f(u_n) \to f(u)$ in $L^2(\Omega)$ because $|f(u_n) - f(u)| \le a|u_n - u|$. This shows that

$$\Delta^2 u_n - \operatorname{div}(c(x)\nabla u_n) \to \lambda f(u) \quad \text{in } \mathcal{D}'(\Omega).$$

That is

$$\Delta^2 u - \operatorname{div}(c(x)\nabla u) - \lambda f(u) = 0.$$

The above equality multiplied by u gives

$$\int_\Omega |\Delta u|^2 + \int_\Omega c(x)|\nabla u|^2 - \lambda \int_\Omega f(u)u = 0. \tag{4.3}$$

Now (4.2) multiplied by (u_n) gives

$$\int_\Omega |\Delta u_n|^2 + \int_\Omega c(x)|\nabla u_n|^2 - \lambda \int_\Omega f(u_n)u_n \to 0 \tag{4.4}$$

in view of the boundedness of (u_n) and the $L^2(\Omega)$-convergence of u_n and $f(u_n)$, we have

$$\lambda \int_\Omega f(u_n)u_n \to \lambda \int_\Omega f(u)u$$

Hence, (4.3) and (4.4) give

$$\int_\Omega |\Delta u_n|^2 \to \int_\Omega |\Delta u|^2 \quad \text{and} \quad \int_\Omega c(x)|\nabla u_n|^2 \to \int_\Omega c(x)|\nabla u|^2$$

which insures us that $u_n \to u$ in E. Actually, it is enough to prove that (u_n) is (up to a subsequence) bounded in $L^2(\Omega)$. Indeed, the $L^2(\Omega)$-boundedness of (u_n) implies that E-boundedness of (u_n) as it can be seen by examining (4.1).

We shall conclude the proof obtaining a contradiction from the supposition that $\|u_n\|_2 \to \infty$. Let $u_n = k_n w_n$ with $k_n > 0$, $k_n \to \infty$ and $\|w_n\|_2 = 1$. Then

$$0 = \lim_{n\to\infty} \frac{J(u_n)}{k_n^2} = \lim_{n\to\infty} \Big[\frac{1}{2}\int_\Omega |\Delta w_n|^2 + \frac{1}{2}\int_\Omega c(x)|\nabla w_n|^2 - \frac{1}{k_n^2}\int_\Omega F(u_n)\Big]$$

However, since $|f(t)| \le a|t| + b$, we have

$$|F(u_n)| = |F(k_n w_n)| \le \frac{a\lambda}{2} k_n^2 w_n^2 + b\lambda |k_n w_n|.$$

This shows that

$$\frac{1}{k_n^2}\int_\Omega F(u_n) \le \frac{a\lambda}{2}\int_\Omega w_n^2 + \frac{b\lambda}{k_n}\int_\Omega w_n < \infty.$$

We claim that

$$\Delta^2 w - \operatorname{div}(c(x)\nabla w) = a\lambda w^+ \quad \text{where } w^+ := \max\{0, w\}. \tag{4.5}$$

Indeed, (4.2) divided by k_n gives

$$\int_\Omega \Delta w_n \cdot \Delta v + \int_\Omega c(x)\nabla w_n \cdot \nabla v - \lambda \int_\Omega \frac{f(u_n)}{k_n} v \to 0 \tag{4.6}$$

for each $v \in E$. Now

$$\int_\Omega \Delta w_n \cdot \Delta v + \int_\Omega c(x)\nabla w_n \cdot \nabla v \to \int_\Omega \Delta w \cdot \Delta v + \int_\Omega c(x)\nabla w \cdot \nabla v$$

Hence (4.5) can be concluded from (4.6) if we show that $1/k_n f(u_n)$ converges (up to a subsequence) to aw^+ in $L^2(\Omega)$. Now $1/k_n f(u_n) = 1/k_n f(k_n w_n)$ and it is easy to see that the required limit is equal to aw in the set $\{x \in \Omega : w_n(x) \to w(x) \neq 0\}$.

If $w(x) = 0$ and $w_n(x) \to w(x)$, let $\varepsilon > 0$ and n_0 be such that $|w_n(x)| < \varepsilon$ for $n \ge n_0$. Then

$$\frac{f(k_n w_n)}{k_n} \le a\varepsilon + \frac{b}{k_n} \quad \text{for such} n,$$

that is the required limit is 0. Thus, $f(u_n)/k_n \to aw^+$ a.e. Here $b = f(0)$. Now $w_n \to w$ in $L^2(\Omega)$ and, thus, up to a subsequence, w_n is dominated in $L^2(\Omega)$ (see [5, Theorem IV.9]).

Since $1/k_n f(u_n) \le a|w_n| + 1/k_n b$, it follows that $1/k_n f(u_n)$ is also dominated. Hence (4.5) is now obtained. Now (4.5) and the maximum principle imply that $w \ge 0$ and (4.5) becomes

$$\begin{gathered}\Delta^2 w - \operatorname{div}(c(x)\nabla w) = \lambda a w \quad \text{in } \Omega,\\ w \ge 0 \quad \text{in } \Omega,\\ \|w\|_2 = 1 \quad \text{in } \Omega.\end{gathered} \tag{4.7}$$

Thus from (1.3), we have $\lambda a = \lambda_1$ and $w = v_1$, which contradicts the fact that $\lambda \ne \lambda_1/a$. This contradiction finishes the proof of the lemma 4.3. $\square$

Now, we need only to check that the two geometric assumptions of theorem 4.1 are fulfilled.

First, since u_λ is a local minimum of J, there exists $R > 0$ such that for all $u \in E$ satisfying $\|u - u_\lambda\| = R$, we have $J(u) \ge J(u_\lambda)$. Then

$$J(u) - J(u_\lambda) = J"(u_\lambda)(u - u_\lambda, u - u_\lambda) + \rho \quad \text{where} \rho > 0.$$

This makes u_λ becomes a strict local minimal for J, which proves $(*)$.

Recall that $\lim_{t\to+\infty}(f(t) - a\,t)$ is finite, then there exists $\beta \in \mathbb{R}$ such that

$$f(t) \ge a\,t + \beta, \quad \forall t > 0.$$

Hence

$$F(t) \ge \frac{a\,\lambda}{2} t^2 + \beta\lambda t, \quad \forall t > 0.$$

This yields, using the definition of v_1 mentioned in (1.3),

$$J(tv_1) = \frac{\lambda_1 - a\lambda}{2} t^2 \int_\Omega v_1^2 - \beta\lambda t \int_\Omega v_1,$$

since $\|v_1\|_2 = 1$, then we have

$$\frac{J_\varepsilon(tv_1)}{t^2} = \frac{\lambda_1 - a\lambda}{2} - \frac{\beta\lambda}{t} \int_\Omega v_1 \tag{4.8}$$

which implies

$$\limsup_{t\to+\infty} \frac{1}{t^2}\, J(tv_1) \le \frac{\lambda_1 - a\lambda}{2} < 0, \quad \forall \lambda > \lambda_1/a.$$

Therefore

$$\lim_{t\to+\infty} J(tv_1) = -\infty.$$

So, there exists $v_0 \in E$ such that $J(v_0) \le J(u_\lambda)$ and $(**)$ is proved.

Finally, let $\tilde v$ (respectively $\tilde c$) be the critical point (respectively critical value) of J, we recall that the function $\tilde v$ belongs to E and satisfies

$$\Delta^2 \tilde v - \operatorname{div}(c(x)\nabla \tilde v) = \lambda f(\tilde v) \quad \text{in } \Omega \quad \text{and} \quad J(\tilde v) = \tilde c.$$

The next lemma states that the limit of a sequence of unstable solutions is also unstable (the proof is similar to that of [15, Lemma 11]).

Lemma 4.4. *Let $u_n \rightharpoonup u$ in $H^2(\Omega) \cap H_0^1(\Omega)$ and $\mu_n \to \mu$ be such that $\eta_1(\mu_n, u_n) < 0$. Then, $\eta_1(\mu, u) < 0$.*

Proof. The fact that $\eta_1(\mu_n, u_n) < 0$ is equivalent to the existence of a $\varphi_n \in H^2(\Omega) \cap H_0^1(\Omega)$ such that

$$\int_\Omega |\Delta\varphi_n|^2 + \int_\Omega c(x)|\nabla\varphi_n|^2 \le \mu_n \int_\Omega f'(u_n)\varphi_n^2 \quad \text{with } \int_\Omega \varphi_n^2 = 1 \tag{4.9}$$

Since $f' \le a$, (4.9) shows that (φ_n) is bounded in $H^2(\Omega) \cap H_0^1(\Omega)$. Let $\varphi \in E$ be such that, up to a subsequence, $\varphi_n \rightharpoonup \varphi$ in $H^2(\Omega) \cap H_0^1(\Omega)$. Then

$$\mu_n \int_\Omega f'(u_n)\varphi_n^2 \to \mu \int_\Omega f'(u)\varphi^2$$

This can be seen by extracting from (φ_n) a subsequence dominated in $L^2(\Omega))$ as in [5, Theorem IV.9]. Now we have

$$\int_\Omega |\Delta\varphi|^2 \le \liminf \int_\Omega |\Delta\varphi_n|^2,$$
$$\int_\Omega c(x)|\nabla\varphi|^2 \le \liminf \int_\Omega c(x)|\nabla\varphi_n|^2$$

finally, since $\|\varphi\|_2 = 1$, we obtain

$$\int_\Omega |\Delta\varphi|^2 + \int_\Omega c(x)|\nabla\varphi|^2 \le \mu \int_\Omega f'(u)\varphi^2.$$

□

Obviously, the fact that the function v belongs to $C^4(\bar{\Omega}) \cap E$ follows from a bootstrap argument.

Proof of (i) of Theorem 1.7. Thanks to Lemma 3.1, if (i) does not occur, then there is a sequence of positives scalars (μ_n) and a sequence (v_n) of unstable solutions to (P_{μ_n}) such that $v_n \to v$ in $L^1_{\rm loc}(\Omega)$ as $\mu_n \to \lambda_1/a$ for some function v.

We first claim that (v_n) cannot be bounded in E. Otherwise, let $w \in E$ be such that, up to a subsequence,

$$v_n \rightharpoonup w \text{ weakly in } E \quad \text{and} \quad v_n \to w \text{ strongly in } L^2(\Omega).$$

Therefore,

$$\Delta^2 v_n - \operatorname{div}(c(x)\nabla v_n) \to \Delta^2 w - \operatorname{div}(c(x)\nabla w) \text{ in } \mathcal{D}'(\Omega),$$
$$f(v_n) \to f(w) \text{ in } L^2(\Omega),$$

which implies that $\Delta^2 w - \operatorname{div}(c(x)\nabla w) = \frac{\lambda_1}{a} f(w)$ in Ω. It follows that $w \in E$ and solves (1.2) with λ_1/a in stead of λ. From Lemma 4.4, we deduce that

$$\eta_1\Big(\frac{\lambda_1}{a}, w\Big) \le 0. \tag{4.10}$$

Relation (4.10) shows that $w \ne u_{\lambda_1/a}$ which contradicts the fact that (1.2) with λ_1/a in stead of λ has a unique solution. Now, since $\Delta^2 v_n - \operatorname{div}(c(x)\nabla v_n) = \mu_n f(v_n)$, the unboundedness of (v_n) in E implies that this sequence is unbounded in $L^2(\Omega)$, too. To see this, let

$$v_n = k_n w_n, \quad \text{where } k_n > 0, \quad \|w_n\|_2 = 1 \quad \text{and} \quad k_n \to \infty.$$

Then

$$\Delta^2 w_n - \operatorname{div}(c(x)\nabla w_n) = \frac{\mu_n}{k_n} f(v_n) \to 0 \quad \text{in } L^1_{\rm loc}(\Omega).$$

So, we have convergence also in the sense of distributions and (w_n) is seen to be bounded in E with standard arguments. We obtain

$$\Delta^2 w - \operatorname{div}(c(x)\nabla w) = 0 \quad \text{and} \quad \|w\|_2 = 1.$$

The desired contradiction is obtained since $w \in E$. $\square$

Proof of (ii) of Theorem 1.7. As before, it is sufficient to prove the $L^2(\Omega)$ boundedness of v_λ near λ^* and to use the uniqueness property of u^*. Assume that $\|v_n\|_2 \to \infty$ as $\mu_n \to \lambda^*$, where v_n is a solution to (P_{μ_n}). We write again $v_n = l_n w_n$. Then,

$$\Delta^2 w_n - \operatorname{div}(c(x)\nabla w_n) = \frac{\mu_n}{l_n} f(v_n). \tag{4.11}$$

The fact that the right-hand side of (4.11) is bounded in $L^2(\Omega)$ implies that (w_n) is bounded in E. Let (w_n) be such that (up to a subsequence)

$$w_n \rightharpoonup w \text{ weakly in } E \quad \text{and} \quad w_n \to w \text{ strongly in } L^2(\Omega).$$

A computation already done shows that

$$\Delta^2 w - \operatorname{div}(c(x)\nabla w) = \lambda^* a w, \quad w \geq 0 \text{ and } \|w\|_2 = 1,$$

which forces λ^* to be λ_1/a. This contradiction concludes the proof. $\square$

In the end, Figure 2 gives the behavior of the solutions when l is negative.

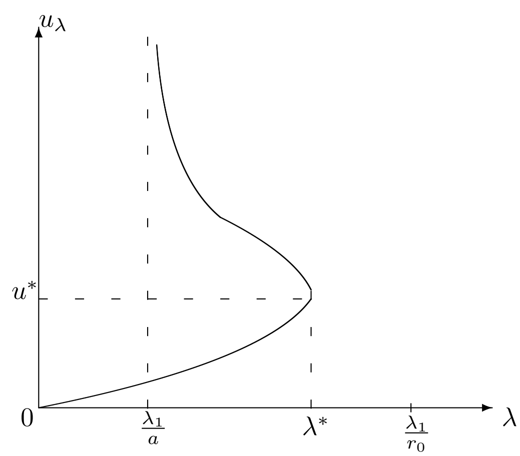

FIGURE 2. Bifurcation branches in the case $l < 0$.

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