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GEVREY REGULARITY OF THE SOLUTIONS OF INHOMOGENEOUS NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. In this article, we are interested in the Gevrey properties of the formal power series solutions in time of some inhomogeneous nonlinear partial differential equations with analytic coefficients at the origin of $\mathbb{C}^{n+1}$. We systematically examine the cases where the inhomogeneity is s-Gevrey for any $s \geq 0$, in order to carefully distinguish the influence of the data (and their degree of regularity) from that of the equation (and its structure). We thus prove that we have a noteworthy dichotomy with respect to a nonnegative rational number s_c fully determined by the Newton polygon of a convenient associated linear partial differential equation: for any $s \geq s_c$, the formal solutions and the inhomogeneity are simultaneously s-Gevrey; for any $s < s_c$, the formal solutions are generically s_c-Gevrey. In the latter case, we give an explicit example in which the solution is s'-Gevrey for no $s' < s_c$. As a practical illustration, we apply our results to the generalized Burgers-Korteweg-de Vries equation.

1. Problem setting

The nonlinear evolution equations are often used to represent the motion of the isolated waves, localized in a small part of space in many fields such as optical fibers, neural physics, solid state physics, hydrodynamics, diffusion process, plasma physics and nonlinear optics (nonlinear heat equation, nonlinear Klein-Gordon equation, nonlinear Euler-Lagrange equation, Burgers equation, Korteweg-de Vries equation, Boussinesq equation, etc.).

When studying such equations, one of the major challenges is the determination of exact solutions, if any exists, and the precise analysis of their properties (dynamic, asymptotic behavior, etc.) in order to have a better understanding of the mechanism of the underlying physical phenomena and dynamic processes. To do that, one possible way is given by the summation theory which allows to construct analytic solutions from formal ones.

The present work is devoted to the study of these formal solutions (existence, unicity, Gevrey properties), which is the first step towards this approach. More precisely, we consider an inhomogeneous nonlinear partial differential equation with

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a 1-dimensional time variable $t \in \mathbb{C}$ and a n-dimensional spatial variable $x = (x_1, \dots, x_n) \in \mathbb{C}^n$ of the form

$$\begin{gathered} \partial_t^\kappa u - \sum_{i\in\mathcal{K}} \sum_{q\in Q_i} \Big(\sum_{p\in P_{i,q}} t^{v_{i,q,p}} a_{i,q,p}(t,x) u^p \Big) \partial_t^i \partial_x^q u = \widetilde{f}(t,x) \\ \partial_t^j u(t,x)_{|t=0} = \varphi_j(x), \quad j = 0,\dots,\kappa-1 \end{gathered} \tag{1.1}$$

where

- $\kappa \geq 1$ is a positive integer;
- $\mathcal{K}$ is a nonempty subset of $\{0, \dots, \kappa - 1\}$;
- Q_i is a nonempty finite subset of $\mathbb{N}^n$ for all $i \in \mathcal{K}$ ($\mathbb{N}$ denotes the set of the nonnegative integers);
- $P_{i,q}$ is a nonempty finite subset of $\mathbb{N}$ for all $i \in \mathcal{K}$ and $q \in Q_i$;
- $v_{i,q,p} \geq 0$ is a nonnegative integer for all $i \in \mathcal{K}$, $q \in Q_i$ and $p \in P_{i,q}$;
- the coefficients $a_{i,q,p}(t, x)$ are analytic on a polydisc $D_{\rho_0,\dots,\rho_n} := D_{\rho_0} \times D_{\rho_1} \times \dots \times D_{\rho_n}$ centered at the origin of $\mathbb{C}^{n+1}$ (D_ρ denotes the disc with center $0 \in \mathbb{C}$ and radius $\rho > 0$) and satisfy $a_{i,q,p}(0, x) \not\equiv 0$ for all $i \in \mathcal{K}$, $q \in Q_i$ and $p \in P_{i,q}$;
- ∂_x^q denotes the derivative $\partial_{x_1}^{q_1} \dots \partial_{x_n}^{q_n}$ while $q := (q_1, \dots, q_n) \in \mathbb{N}^n$;
- the inhomogeneity $\widetilde{f}(t, x)$ is a formal power series in t with analytic coefficients in $D_{\rho_1,\dots,\rho_n}$ (we denote by $\widetilde{f}(t, x) \in \mathcal{O}(D_{\rho_1,\dots,\rho_n})[[t]]$) which may be smooth, or not; (We denote $\widetilde{f}$ with a tilde to emphasize the possible divergence of the series)
- the initial conditions $\varphi_j(x)$ are analytic on $D_{\rho_1,\dots,\rho_n}$ for $j = 0, \dots, \kappa - 1$.

Looking for a formal solution $\widetilde{u}(t, x) \in \mathcal{O}(D_{\rho_1,\dots,\rho_n})[[t]]$, and writing any element $\widetilde{g}(t, x) \in \mathcal{O}(D_{\rho_1,\dots,\rho_n})[[t]]$ in the form

$$\widetilde{g}(t,x) = \sum_{j\geq 0} g_{j,*}(x) \frac{t^j}{j!} \quad \text{with } g_{j,*}(x) \in \mathcal{O}(D_{\rho_1,\dots,\rho_n}) \text{ for all } j,$$

we easily get that the coefficients $u_{j,*}(x)$ of $\widetilde{u}(t, x)$ are *uniquely* determined by the recurrence relations

$$\begin{aligned} &u_{j+\kappa,*}(x) \\ &= f_{j,*}(x) \\ &+ \sum_{i\in\mathcal{K}} \sum_{q\in Q_i} \sum_{p\in P_{i,q}} \sum_{\substack{\ell_0+\ell_1+\dots+\ell_p+ \\ \ell_{p+1}=j-v_{i,q,p}}} \frac{j! a_{i,q,p;\ell_0,*}(x) u_{\ell_1,*}(x) \dots u_{\ell_p,*}(x) \partial_x^q u_{\ell_{p+1}+i,*}(x)}{\ell_0! \ell_1! \dots \ell_p! \ell_{p+1}!} \end{aligned} \tag{1.2}$$

together with the initial conditions $u_{j,*}(x) = \varphi_j(x)$ for $j = 0, \dots, \kappa - 1$. As usual, we use the classical conventions that the fourth sum is zero as soon as $j - v_{i,q,p} < 0$, and that the product $\frac{u_{\ell_1,*}(x)\dots u_{\ell_p,*}(x)}{\ell_1!\dots\ell_p!}$ is 1 as soon as $p = 0$.

The purpose of the paper is to answer to the question:

What relationship exists between the Gevrey order of the formal solution $\widetilde{u}(t, x)$ and the Gevrey order of the inhomogeneity $\widetilde{f}(t, x)$?

Indeed, according to the algebraic structure of the s-Gevrey spaces $\mathcal{O}(D_{\rho_1,\dots,\rho_n})[[t]]_s$ (see Section 2 for the exact definition of theses spaces), it is classical one has

$$\widetilde{u}(t,x) \in \mathcal{O}(D_{\rho_1,\dots,\rho_n})[[t]]_s \Rightarrow \widetilde{f}(t,x) \in \mathcal{O}(D_{\rho_1,\dots,\rho_n})[[t]]_s.$$

But, what can we say about the converse?

A first answer was provided by the author in [37] (see also [30, 35] for some particular examples) in the linear case (case $P_{i,q}=\{0\}$ for all i and q) and in the semilinear case with a polynomial nonlinearity in u (case $P_{0,0}\cap(\mathbb{N}\backslash\{0\})\neq\emptyset$ and $P_{i,q}=\{0\}$ for all $(i,q)\neq(0,0)$); that is for the equations of the form

$$L(u)-P(u)=\widetilde{f}(t,x), \tag{1.3}$$

where L is a linear partial differential operator of the form

$$L=\partial_t^{\kappa}-\sum_{i\in\mathcal{K}}\sum_{q\in Q_i}a_{i,q}(t,x)\partial_t^i\partial_x^q,$$

and where $P(X)$ is either zero or a polynomial with valuation ≥ 2 in X and with analytic coefficients at the origin of $\mathbb{C}^{n+1}$. In particular, he proved that the Gevrey orders of the formal solution $\widetilde{u}(t,x)$ and the inhomogeneity $\widetilde{f}(t,x)$ are closely related through a special value, called *the critical value of* (1.3) and denoted in the sequel by s_c, which is fully determined by the Newton polygon at $t=0$ of the linear operator L.

Proposition 1.1 ([37]). *Let s_c denote the nonnegative rational number equal to the inverse of the smallest positive slope of the Newton polygon at $t=0$ of the linear operator L if any exists, and equal to 0 otherwise. Then*

(1) *$\widetilde{u}(t,x)$ and $\widetilde{f}(t,x)$ are simultaneously s-Gevrey for any $s\geq s_c$;*
(2) *$\widetilde{u}(t,x)$ is generically s_c-Gevrey while $\widetilde{f}(t,x)$ is s-Gevrey with $s<s_c$.*

Remark 1.2. When the inhomogeneity $\widetilde{f}(t,x)$ is s-Gevrey with $s<s_c$, the hypotheses made on (1.3) do not allow in general to specify the exact Gevrey order of the formal solution $\widetilde{u}(t,x)$ as in the opposite case $s\geq s_c$ (Point 1). However, the second point of Proposition 1.1 asserts that this order is always less or equal to s_c (This is obvious due to the filtration of the Gevrey spaces (see Section 3.1) and the first point of Proposition 1.1.) and that this inequality is the *best* possible. Indeed, one can easily find cases for which $\widetilde{u}(t,x)$ is exactly s_c-Gevrey (see [37, Prop. 4.11] for instance).

In this article, we propose to extend the results of Proposition 1.1 to the very general (1.1). Similarly as the equations of the form (1.3), the critical value s_c of this equation is fully determined by the Newton polygon at $t=0$ of a convenient linear partial differential operator, called *the associated linear operator*. Section 3 is devoted to the study of this operator: after a heuristic approach using two simple examples (Section 3.1), we describe the general geometric structure of its Newton polygon at $t=0$, and we derive from this the value of s_c as well as some fundamental associated inequalities (Section 3.2). In Section 4, we state our main result (Theorem 4.1) and some direct consequences. The proof of Theorem 4.1 is developed in Sections 5 and 6.

Before remembering some results about the Gevrey formal series (Section 2), let us mention here that a similar problem has already been studied by Tahara in [46] in the case of *real* variables. However, the calculations we develop in this paper are based on a very different approach.

Let us also mention that other slightly different works have also been done for several years by many authors towards the convergence [21, 23, 43] and the Gevrey order [14, 15, 22, 39, 40, 41, 42, 44, 45] of the formal power series solutions of some singular nonlinear partial differential equations, and towards the summability [17, 19, 26, 34, 36, 38] of the formal power series solution of some nonlinear partial differential equations. Furthermore, in [12, 13], Lastra and Malek considered some parametric nonlinear partial differential equations; in [18], Malek investigated the Gevrey properties of some nonlinear integro-differential equations. Of course, given the technical and computational difficulties inherent in the nonlinearity, the known results, especially in the framework of the summability or the multisummability, are currently far fewer than in the linear case.

2. Gevrey formal series

All along the article, we consider t as the variable and x as a parameter. Thereby, to define the notion of *Gevrey classes of formal power series in* $\mathcal{O}(D_{\rho_1,\dots,\rho_n})[[t]]$, one extends the classical notion of *Gevrey classes of formal power series in* $\mathbb{C}[[t]]$ to families parametrized by x in requiring similar conditions, the estimates being however uniform with respect to x. Doing that, any formal power series of $\mathcal{O}(D_{\rho_1,\dots,\rho_n})[[t]]$ can be seen as a formal power series in t with coefficients in a convenient Banach space defined as the space of functions that are holomorphic on a polydisc $D_{\rho,\dots,\rho}$ $(0<\rho\le\min(\rho_1,\dots,\rho_n))$ and continuous up to its boundary, equipped with the usual supremum norm. For a general study of the formal power series with coefficients in a Banach space, we refer for instance to [2].

In the sequel, we endow $\mathbb{C}^n$ with the maximum norm: for $x=(x_1,\dots,x_n)\in\mathbb{C}^n$,

$$\|x\|=\max_{\ell\in\{1,\dots,n\}}|x_\ell|.$$

Definition 2.1. Let $s\ge 0$. A formal power series

$$\widetilde{u}(t,x)=\sum_{j\ge 0}u_{j,*}(x)\frac{t^j}{j!}\in\mathcal{O}(D_{\rho_1,\dots,\rho_n})[[t]]$$

is said to be *Gevrey of order* s (in short, s-Gevrey) if there exist three positive constants $0<\rho<\min(\rho_1,\dots,\rho_n)$, $C>0$ and $K>0$ such that the inequalities

$$\sup_{\|x\|\le\rho}|u_{j,*}(x)|\le CK^j\Gamma(1+(s+1)j)$$

hold for all $j\ge 0$.

In other words, Definition 2.1 means that $\widetilde{u}(t,x)$ is s-Gevrey in t, uniformly in x on a neighborhood of $x=(0,\dots,0)\in\mathbb{C}^n$.

We denote by $\mathcal{O}(D_{\rho_1,\dots,\rho_n})[[t]]_s$ the set of all the formal series in $\mathcal{O}(D_{\rho_1,\dots,\rho_n})[[t]]$ which are s-Gevrey.

Observe that the set $\mathbb{C}\{t,x\}$ of germs of analytic functions at the origin of $\mathbb{C}^{n+1}$ coincides with the union $\cup_{\rho_1>0,\dots,\rho_n>0}\mathcal{O}(D_{\rho_1,\dots,\rho_n})[[t]]_0$. In particular, any element of $\mathcal{O}(D_{\rho_1,\dots,\rho_n})[[t]]_0$ is convergent and $\mathbb{C}\{t,x\}\cap\mathcal{O}(D_{\rho_1,\dots,\rho_n})[[t]]=\mathcal{O}(D_{\rho_1,\dots,\rho_n})[[t]]_0$.

Observe also that the sets $\mathcal{O}(D_{\rho_1,\dots,\rho_n})[[t]]_s$ are filtered as follows

$$\mathcal{O}(D_{\rho_1,\dots,\rho_n})[[t]]_0\subset\mathcal{O}(D_{\rho_1,\dots,\rho_n})[[t]]_s\subset\mathcal{O}(D_{\rho_1,\dots,\rho_n})[[t]]_{s'}\subset\mathcal{O}(D_{\rho_1,\dots,\rho_n})[[t]]$$

for all s and s' satisfying $0<s<s'<+\infty$. The following proposition specifies the algebraic structure of $\mathcal{O}(D_{\rho_1,\dots,\rho_n})[[t]]_s$.

Proposition 2.2. *Let* $s\ge 0$. *Then the set* $(\mathcal{O}(D_{\rho_1,\dots,\rho_n})[[t]]_s,\partial_t,\partial_{x_1},\dots,\partial_{x_n})$ *is a* $\mathbb{C}$*-differential algebra.*

Proof. Since $(\mathcal{O}(D_{\rho_1,\dots,\rho_n})[[t]], \partial_t, \partial_{x_1}, \dots, \partial_{x_n})$ is a $\mathbb{C}$-differential algebra, it is sufficient to prove that $\mathcal{O}(D_{\rho_1,\dots,\rho_n})[[t]]_s$ is stable under multiplication and derivations.

The proof of the stability under the multiplication and the derivation ∂_t is similar to the one already detailed in [31, Prop. 1] (see also [2, p. 64]) in the case $n = 1$.

To prove the stability under the derivation ∂_{x_ℓ} with $\ell \in \{1, \dots, n\}$, we proceed as follows. Let $\widetilde{u}(t,x) \in \mathcal{O}(D_{\rho_1,\dots,\rho_n})[[t]]_s$ as in Definition 2.1 and $\widetilde{w}(t,x) = \partial_{x_\ell}\widetilde{u}(t,x)$. For a given $0 < \rho' < \rho$, the Cauchy Integral Formula gives us, for all $j \geq 0$ and all $\|x\| \leq \rho'$:

$$w_{j,*}(x) = \partial_{x_\ell} u_{j,*}(x) = \frac{1}{(2i\pi)^n} \int_{\gamma(x)} \frac{u_{j,*}(x')}{(x'_\ell - x_\ell)^2 \prod_{\substack{k=1\\k\neq\ell}}^{n} (x'_k - x_k)} dx',$$

where $\gamma(x) := \{x' = (x'_1, \dots, x'_n) \in \mathbb{C}^n; |x'_k - x_k| = \rho - \rho' \text{ for all } k \in \{1, \dots, n\}\}$. Hence, the inequalities

$$\sup_{\|x\|\leq\rho'} |w_{j,*}(x)| \leq C' K^j \Gamma(1 + (s+1)j) \quad \text{with } C' = \frac{C}{\rho - \rho'} \text{ for all } j \geq 0.$$

Indeed, the definition of the path $\gamma(x)$ implies $\|x'\| \leq \rho$. The proof is complete. $\square$

Observe that the stability under the derivation ∂_{x_ℓ} would not be guaranteed without the condition "there exist $0 < \rho < \min(\rho_1, \dots, \rho_n)$ ..." in Definition 2.1.

3. Associated linear operator

As we said in Section 1, the critical value of (1.1) is fully determined by the Newton polygon at $t = 0$ of a convenient linear partial differential operator. In Section 3.1 below, we investigate two simple examples to heuristically introduce this operator.

3.1. **Two preliminaries examples.**

Example 1: a fundamental equation. In this first example, we consider the equation

$$\begin{gathered} \partial_t u = t^v u^p \partial_x^q u, \quad v, p \geq 0,\ q \geq 2 \\ u(0,x) = \frac{1}{1-x} \in \mathcal{O}(D_1) \end{gathered} \tag{3.1}$$

in two variables $(t,x) \in \mathbb{C}^2$. When $p = 0$, (3.1) is linear and the Newton polygon at $t = 0$ of its associated linear operator $\partial_t - t^v \partial_x^q$ (see Figure 1) has a unique positive slope, which is equal to $k = \frac{1+v}{q-1}$ [30, 37].

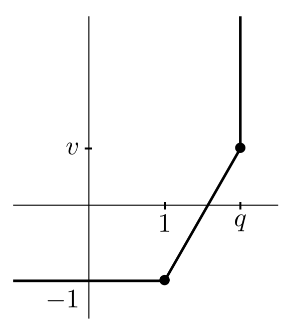


FIGURE 1. Newton polygon at $t = 0$ of $\partial_t - t^v \partial_x^q$

Hence, applying Proposition 1.1, its formal solution $\widetilde{u}(t,x)$ is generically s_c-Gevrey with $s_c = \frac{1}{k} = \frac{q-1}{1+v}$. In fact, using a method similar to the one developed in [30, Prop. 4.11], one can even show that $\widetilde{u}(t,x)$ is exactly s_c-Gevrey.

When $p \geq 1$, (3.1) is no longer linear, but the particular value s_c continues to play an important role as shown by the following result.

Proposition 3.1. *The formal solution $\widetilde{u}(t,x)$ of (3.1) is exactly s_c-Gevrey for any $p \geq 0$.*

Proof. According to the general identities (1.2), the coefficients $u_{j,*}(x)$ of $\widetilde{u}(t,x)$ are recursively determined from the initial conditions $u_{0,*}(x) = \frac{1}{1-x}$ by the relations

$$u_{j+1,*}(x) = j! \sum_{\substack{\ell_1+\dots+\ell_p+\\ \ell_{p+1}=j-v}} \frac{u_{\ell_1,*}(x)\dots u_{\ell_p,*}(x)\partial_x^q u_{\ell_{p+1},*}(x)}{\ell_1!\dots\ell_p!\ell_{p+1}!}.$$

From this, we first derive that

- the terms $u_{j(1+v)+\ell,*}(x)$ are zero for all $j \geq 0$ and all $\ell \in \{1,\dots,v\}$;
- the terms $u_{j(1+v),*}(x)$ read for all $j \geq 0$ in the form
$$u_{j(1+v),*}(x) = \frac{a_j}{(1-x)^{j(q+p)+1}},$$
where the coefficients a_j is positive and satisfies the inequalities
$$A_j \leq a_j \leq C_{j,p+1}A_j$$
with
$$A_j = \frac{(j(q+p))!}{\prod_{\ell=1}^{j}\prod_{m=1}^{p}(\ell q+(\ell-1)p+m)} \prod_{\ell=1}^{j}\prod_{m=1}^{v}((\ell-1)(1+v)+m);$$
and $C_{0,p+1} = 1$ and, for all $j \geq 1$, $C_{j,p+1} =$ the number of the nonzero terms in the sum
$$\sum_{\substack{\ell_1+\dots+\ell_p+\\ \ell_{p+1}=(j-1)(1+v)}} \frac{u_{\ell_1,*}(x)\dots u_{\ell_p,*}(x)\partial_x^q u_{\ell_{p+1},*}(x)}{\ell_1!\dots\ell_p!\ell_{p+1}!}.$$

As previously, we use in the definition of A_j the classical convention that the products are 1 as soon as j (or p, or v) is zero.

The constants A_j's can be bound for all $j \geq 0$ by applying technical Lemmas 3.2 and 3.3. To bound the constants $C_{j,p+1}$'s, it suffices to observe that the property

$$u_{\ell,*}(x) \not\equiv 0 \Leftrightarrow \ell \text{ is a multiple of } 1+v$$

implies that the $C_{j,p+1}$'s can be recursively determined from $C_{0,p+1} = 1$ by the relation

$$C_{j,p+1} = \sum_{\ell'_1+\dots+\ell'_p+\ell'_{p+1}=j-1} C_{\ell'_1,p+1}\dots C_{\ell'_p,p+1}C_{\ell'_{p+1},p+1}. \tag{3.2}$$

Thereby, the $C_{j,p+1}$'s are the generalized Catalan numbers of order $p+1$ and we have

$$C_{j,p+1} = \frac{1}{jp+1}\binom{j(p+1)}{j} \leq 2^{j(p+1)}$$

for all $j \geq 0$ (see [10, 11, 27] for instance). These numbers were named in honor of the mathematician Eugène Charles Catalan (1814-1894). They appear in many probabilist, graphs and combinatorial problems. For example, they can be seen as the number of $(p+1)$-ary trees with j source-nodes, or as the number of ways of associating j applications of a given $(p+1)$-ary operation, or as the number of ways of subdividing a convex polygon into j disjoint $(p+2)$-gons by means of non-intersecting diagonals. They also appear in theoretical computers through the generalized Dyck words. See for instance [10] and the references inside. Therefore,

$$\frac{(j(q+p))!(jv)!}{(q+p)^{jp}(j!)^p} \le a_j \le (2^{p+1}(1+v)^v)^j \frac{(j(q+p))!(j!)^v}{(j!)^p}. \tag{3.3}$$

We are now able to prove Proposition 3.1. Choosing $0<r<1$, we first derive from the second inequaliy of (3.3) the relations

$$|u_{j(1+v),*}(x)| \le \frac{1}{1-r}\Big(\frac{2^{p+1}(1+v)^v}{(1-r)^{q+p}}\Big)^j \frac{(j(q+p))!(j!)^v}{(j!)^p}$$

for all $j\ge 0$ and $|x|\le r$. On the other hand, applying the Stirling's Formula, we obtain the equivalence

$$\frac{(j(q+p))!(j!)^v}{(j!)^p\Gamma(1+j(s_c+1)(1+v))} \underset{j\to+\infty}{\sim} \sqrt{2\pi}^{v-p}\sqrt{\frac{q+p}{q+v}}j^{\frac{v-p}{2}}\Big(\frac{(q+p)^{q+p}}{(q+v)^{q+v}}\Big)^j$$

when j tends to infinity (we have $(s_c+1)(1+v)=q+v$). Consequently, there exist two convenient positive constants $C,K>0$ such that

$$|u_{j(1+v),*}(x)| \le CK^j\Gamma(1+j(s_c+1)(1+v))$$

for all $j\ge 0$ and $|x|\le r$, and we can conclude that $\widetilde{u}(t,x)$ is s_c-Gevrey (recall indeed that the coefficients $u_{\ell,*}(x)$ are zero as soon as ℓ is not a multiple of $1+v$).

It is left to prove that $\widetilde{u}(t,x)$ is s-Gevrey for no $s<s_c$. To do that, let us suppose the opposite. Then, Definition 2.1 and the first inequality of (3.3) imply the relations

$$\frac{(j(q+p))!(jv)!}{(q+p)^{jp}(j!)^p} \le u_{j(1+v),*}(0) \le CK^j\Gamma(1+j(s+1)(1+v))$$

and, consequently, the inequalities

$$1 \le C(K(q+p)^p)^j \frac{(j!)^p\Gamma(1+j(s+1)(1+v))}{(j(q+p))!(jv)!}$$

for all $j\ge 0$ and some convenient positive constants C and K independent of j. The result follows since such inequalities are impossible. Indeed, applying the Stirling's formula, we obtain

$$C(K(q+p)^p)^j \frac{(j!)^p\Gamma(1+j(s+1)(1+v))}{(j(q+p))!(jv)!} \underset{j\to+\infty}{\sim} C'j^{\frac{p-1}{2}}\Big(\frac{K'}{j^\sigma}\Big)^j \tag{3.4}$$

with

- $C' = C\sqrt{\frac{(s+1)(1+v)(2\pi)^{p-1}}{(q+p)v}}$;
- $K' = \frac{Ke^\sigma((s+1)(1+v))^{(s+1)(1+v)}}{(q+p)^q v^v}$;
- $\sigma = q+v-(s+1)(1+v)$,

and the right hand-side of (3.4) goes to 0 when j tends to infinity (we have indeed $\sigma > q+v-(s_c+1)(1+v)=0$ by assumption on s). This completes the proof. $\square$

Lemma 3.2. *Let $j \geq 0$. Then*

$$(j!)^p \leq \prod_{\ell=1}^{j} \prod_{m=1}^{p} (\ell q + (\ell-1)p + m) \leq (q+p)^{jp} (j!)^p.$$

Proof. The statement is clear when $j=0$ or $p=0$, and it stems from the inequalities

$$\prod_{\ell=1}^{j} \ell^p \leq \prod_{\ell=1}^{j} \prod_{m=1}^{p} (\ell q + (\ell-1)p + m) \leq \prod_{\ell=1}^{j} (\ell q + \ell p)^p$$

when $j, p \geq 1$. $\square$

Lemma 3.3. *Let $j \geq 0$. Then*

$$(jv)! \leq \prod_{\ell=1}^{j} \prod_{m=1}^{v} ((\ell-1)(1+v)+m) \leq (1+v)^{jv} (j!)^v. \tag{3.5}$$

Proof. Since Lemma 3.3 is obvious for $v=0$, we assume $v \geq 1$ in calculations below.

The first inequality of (3.5) is proved by induction on j. It is clear for $j=0$ and $j=1$. Let us now suppose that it holds for a certain $j \geq 1$. Then

$$\prod_{\ell=1}^{j+1} \prod_{m=1}^{v} ((\ell-1)(1+v)+m) \geq (jv)! \prod_{m=1}^{v} (j(1+v)+m) = \frac{((j+1)v+j)!}{(jv+j)!} (jv)!$$

and the conclusion follows from the inequality

$$\binom{(j+1)v+j}{j} \geq \binom{jv+j}{j}.$$

As for the second inequality of (3.5), it is clear when $j=0$, and it stems from the inequality

$$\prod_{\ell=1}^{j} \prod_{m=1}^{v} ((\ell-1)(1+v)+m) \leq \prod_{\ell=1}^{j} (\ell(1+v))^v$$

when $j \geq 1$. $\square$

Proposition 3.1 tells us that the power u^p does not affect the Gevrey order of the formal solution $\widetilde{u}(t,x)$, that is it is the same for any $p \geq 0$. In particular, it is fully determined by the Newton polygon at $t=0$ of the linear operator $\partial_t - t^v \partial_x^q$. Changing besides the initial condition $u(0,x) = \frac{1}{1-x}$ by another simple initial condition (e.g. e^x, or any other condition provided that the estimates on the coefficients $u_{j,*}(x)$ of $\widetilde{u}(t,x)$ are sufficiently simple to calculate), one can check that this property remains valid, that is the Gevrey order of $\widetilde{u}(t,x)$ is the same for any $p \geq 0$ (here, generically s_c-Gevrey according to Proposition 1.1). It is then reasonable to think that this is also true for any arbitrary analytic initial condition at the origin $x=0$.

Let us now look at another fundamental equation.

Example 2: the gBKdV equation. In this second example, we focus on the generalized Burgers-Korteweg-de Vries equation (in short, the gBKdV equation)

$$\begin{gathered} \partial_t u = t^{v_1} u^{p_1} \partial_x^{q_1} u + t^{v_2} u^{p_2} \partial_x^{q_2} u, \quad v_1, v_2, p_1, p_2, q_2 \geq 0,\ q_1 \geq 2 \\ u(0,x) = \varphi(x) \end{gathered} \tag{3.6}$$

in two variables $(t,x) \in \mathbb{C}^2$ and with analytic initial condition.

When $(v_1, p_1, q_1, v_2, p_2, q_2) = (0,0,2,0,1,1)$, we have the Burgers equation, and when $(v_1, p_1, q_1, v_2, p_2, q_2) = (0,0,3,0,1,1)$, we have the Korteweg-de Vries equation. (3.6) is fundamental in many physical, mechanical and chemical problems. For instance, it allows to model nonlinear waves in dispersive-dissipative media with instabilities, waves arising in thin films flowing down an inclined surface, moderate-amplitude shallow-water surface waves, changes of the concentration of substances in chemical reactions, etc.

Various investigations in real variables were already done by many authors towards the Gevrey properties of the formal solution $\widetilde{u}(t,x)$ of (3.6) under more or less generic assumptions (see [5, 6, 7, 8, 9] for instance, and the references inside). In particular, it was proved for the Korteweg-de Vries equation that $\widetilde{u}(t,x)$ is 2-Gevrey. In our present study, we can prove in a similar way of (3.1) the following result.

Proposition 3.4. *Let us assume $\varphi(x) = \frac{1}{1-x}$ and let us set*

$$s_c = \max\Big(\frac{q_1 - 1}{1+v_1}, \frac{q_2 - 1}{1+v_2}\Big).$$

Then, the formal solution $\widetilde{u}(t,x)$ of (3.6) is exactly s_c-Gevrey for any $p_1, p_2 \geq 0$.

A much more general statement including variable coefficients, general analytic initial condition and inhomogeneous part will be given later in Example 4.3.

As in the previous example, Proposition 3.4 tells us that the Gevrey order of $\widetilde{u}(t,x)$ does not depend on either the power u^{p_1} or the power u^{p_2}, and that it is fully determined by the smallest positive slope of the Newton polygon at $t = 0$ of a convenient linear operator; namely, the operator $\partial_t - t^{v_1}\partial_x^{q_1} - t^{v_2}\partial_x^{q_2}$ [37].

Remark 3.5. In the case where $q_1 = q_2 = q$, the value s_c given in Proposition 3.4 is more precisely equal to

$$s_c = \frac{q-1}{1 + \min(v_1, v_2)},$$

and we can actually choose the operator $\partial_t - t^{\min(v_1,v_2)}\partial_x^q$ as the associated linear operator.

3.2. **Associated linear operator.** According to the above calculations, it is natural to introduce the following definition.

Definition 3.6. We call linear operator associated with (1.1) the operator

$$\partial_t^\kappa - \sum_{i \in \mathcal{K}} \sum_{q \in Q_i} t^{v_{i,q,p^*}} \partial_t^i \partial_x^q, \tag{3.7}$$

where $v_{i,q,p^*} = \min_{p \in P_{i,q}} v_{i,q,p}$ for all $i \in \mathcal{K}$ and $q \in Q_i$.

Let us now describe the general structure of the Newton polygon at $t = 0$ of the operator (3.7). We choose here the definition of Miyake [20] (see also Yonemura [49] or Ouchi [25]) which is an analogue to the one given by Ramis [29] for the linear ordinary differential equations. Recall that, Tahara and Yamazawa use in [47] a slightly different one.

Then, denoting by $C(a,b)$ the domain

$$C(a,b)=\{(x,y)\in\mathbb{R}^2 : x\le a \text{ and } y\ge b\}$$

for any $(a,b)\in\mathbb{R}^2$, the Newton polygon at $t=0$ of the operator (3.7) is defined as the convex hull of

$$C(\kappa,-\kappa)\bigcup\bigcup_{i\in\mathcal{K}}\bigcup_{q\in Q_i} C(\lambda(q)+i, v_{i,q,p^*}-i),$$

where $\lambda(q)=q_1+\dots+q_n$ denotes the length of $q=(q_1,\dots,q_n)\in\mathbb{N}^n$.

The geometric structure of this domain is specified as follows.

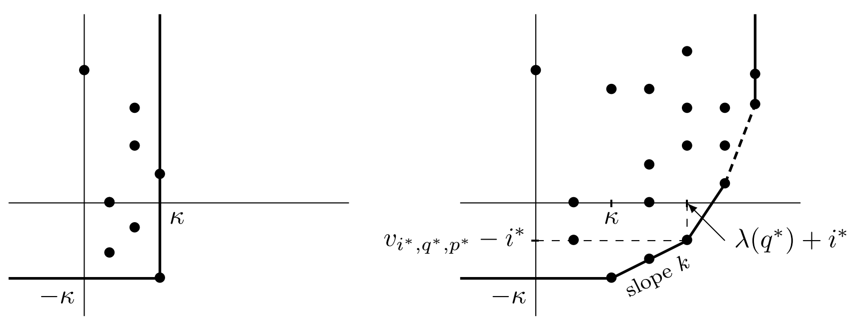
Proposition 3.7. *Let $\mathcal{S}=\{(i,q)$ such that $i\in\mathcal{K}$, $q\in Q_i$ and $\lambda(q)>\kappa-i\}$.*

(1) *Suppose $\mathcal{S}=\emptyset$. Then, the Newton polygon at $t=0$ of the operator* (3.7) *is reduced to the domain $C(\kappa,-\kappa)$. In particular, it has no side with a positive slope (see Figure 2(a)).*

(2) *Suppose $\mathcal{S}\neq\emptyset$. Then, the Newton polygon at $t=0$ of the operator* (3.7) *has at least one side with a positive slope. Moreover, its smallest positive slope k is given by*

$$k=\min_{(i,q)\in\mathcal{S}}\Big(\frac{\kappa-i+v_{i,q,p^*}}{\lambda(q)-\kappa+i}\Big)=\frac{\kappa-i^*+v_{i^*,q^*,p^*}}{\lambda(q^*)-\kappa+i^*},$$

where the pair $(i^*,q^*)\in\mathcal{S}$ stands for any convenient pair, chosen and fixed once and for all, so that the edge with slope k be the segment with end points $(\kappa,-\kappa)$ and $(\lambda(q^*)+i^*, v_{i^*,q^*,p^*}-i^*)$ (see Figure 2(b)).



(a) Case $\mathcal{S}=\emptyset$ (b) Case $\mathcal{S}\neq\emptyset$

FIGURE 2. Newton polygon at $t=0$ of the operator (3.7)

Proof. The first point is obvious from the fact that $\mathcal{S}=\emptyset$ implies $C(\lambda(q)+i, v_{i,q,p^*}-i)\subset C(\kappa,-\kappa)$ for all $i\in\mathcal{K}$ and $q\in Q_i$. As for the second point, it suffices to remark, on one hand, that $C(\lambda(q)+i,v_{i,q,p^*}-i)\subset C(\kappa,-\kappa)$ for all pairs $(i,q)\notin\mathcal{S}$, and, on the other hand, that the segment with the two end points $(\kappa,-\kappa)$ and $(\lambda(q)+i, v_{i,q,p^*}-i)$ has a positive slope equal to $(\kappa-i+v_{i,q,p^*})/(\lambda(q)-\kappa+i)$ for all pairs $(i,q)\in\mathcal{S}$. $\square$

The critical value of (1.1) is defined as in [37] for the case of the equations of the form (1.3).

Definition 3.8. Let $\mathcal{S} = \{(i,q) : \text{such that } i \in \mathcal{K},\ q \in Q_i \text{ and } \lambda(q) > \kappa - i\}$. Then, *the critical value* s_c *of* (1.1) is the nonnegative rational number

$$s_c := \begin{cases} 0 & \text{if } \mathcal{S} = \emptyset \\ \dfrac{1}{k} = \dfrac{\lambda(q^*) - \kappa + i^*}{\kappa - i^* + v_{i^*,q^*,p^*}} & \text{if } \mathcal{S} \neq \emptyset \end{cases}$$

In other words, the critical value of (1.1) is equal to the inverse of the smallest positive slope of the Newton polygon at $t = 0$ of its associated linear operator (3.7) if any exists, and 0 otherwise.

According to the definition of s_c, we derive in particular from Proposition 3.7 the following inequalities which will play a fundamental role in the proof of our main theorem (see Section 5).

Lemma 3.9. *Let $s \geq s_c$. Then*

$$(s+1)(\kappa - i + v_{i,q,p}) \geq \lambda(q) + v_{i,q,p}$$

for all $i \in \mathcal{K}$, all $q \in Q_i$ and all $p \in P_{i,q}$.

Proof. Let us first assume $\mathcal{S} = \emptyset$. Then, since $s \geq s_c = 0$, we have

$$(s+1)(\kappa - i + v_{i,q,p}) \geq \kappa - i + v_{i,q,p}$$

and the result follows from the inequality $\lambda(q) \leq \kappa - i$.

Let us now assume $\mathcal{S} \neq \emptyset$. From the definition of s_c and the definition of the v_{i,q,p^*}'s, we first have

$$s \geq s_c \geq \frac{\lambda(q) - \kappa + i}{\kappa - i + v_{i,q,p^*}} \geq \frac{\lambda(q) - \kappa + i}{\kappa - i + v_{i,q,p}} > 0$$

for all $(i,q) \in \mathcal{S}$, and next

$$s \geq \frac{\lambda(q) - \kappa + i}{\kappa - i + v_{i,q,p}} \tag{3.8}$$

for all i and q, since $\lambda(q) - \kappa + i \leq 0$ while $(i,q) \notin \mathcal{S}$. Lemma 3.9 follows by first adding $+1$ to both sides of (3.8) and then by multiplying by the positive term $\kappa - i + v_{i,q,p}$. □

4. Main result

We are now able to state the main result of the article.

Theorem 4.1. *Let s_c be the critical value of* (1.1) *(see Definition 3.8). Then*

1. *$\widetilde{u}(t,x)$ and $\widetilde{f}(t,x)$ are simultaneously s-Gevrey for any $s \geq s_c$;*
2. *$\widetilde{u}(t,x)$ is generically s_c-Gevrey while $\widetilde{f}(t,x)$ is s-Gevrey with $s < s_c$.*

Observe that Theorem 4.1 coincides with Proposition 1.1 in the case of (1.3). It generalizes therefore the results stated in [37], but also those already obtained by the author in the linear case [31, 32, 33].

Observe also that Theorem 4.1 yields a result similar to the Maillet-Ramis Theorem for the ordinary linear differential equations [28, 29] (see also [16, Thm. 4.2.7]).

Corollary 4.2. *Assume that the inhomogeneity $\widetilde{f}(t,x)$ is convergent. Then, the formal solution $\widetilde{u}(t,x)$ of* (1.1) *is either convergent or $1/k$-Gevrey, where k stands for the smallest positive slope of the Newton polygon at $t = 0$ of the associated linear operator* (3.7).

Before starting the proof of Theorem 4.1, let us illustrate it with a simple example.

Example 4.3 (Back to the gBKdV equation)**.** Let us consider the general inhomogeneous gBKdV equation

$$\begin{gathered}\partial_t u - t^{v_1} a_1(t,x) u^{p_1} \partial_x^{q_1} u - t^{v_2} a_2(t,x) u^{p_2} \partial_x^{q_2} u = \widetilde{f}(t,x) \\ v_1, v_2, p_1, p_2, q_2 \geq 0,\ q_1 \geq 2\end{gathered} \tag{4.1}$$

in two variables $(t,x) \in \mathbb{C}^2$ and with analytic initial condition $u(0,x) = \varphi(x)$. Let us also consider the following two particular cases:

- the inhomogeneous Burgers equation:
$$\partial_t u - t^{v_1} a_1(t,x) \partial_x^2 u - t^{v_2} a_2(t,x) u \partial_x u = \widetilde{f}(t,x); \tag{4.2}$$
- the inhomogeneous Korteweg-de Vries equation:
$$\partial_t u - t^{v_1} a_1(t,x) \partial_x^3 u - t^{v_2} a_2(t,x) u \partial_x u = \widetilde{f}(t,x). \tag{4.3}$$

Applying successively Definitions 3.6 and 3.8, the linear operator associated with (4.1) is defined as

$$\begin{gathered}\partial_t - t^{v_1} \partial_x^{q_1} - t^{v_2} \partial_x^{q_2} \quad \text{if } q_1 \neq q_2 \\ \partial_t - t^{\min(v_1,v_2)} \partial_x^{q} \quad \text{if } q_1 = q_2 = q,\end{gathered}$$

and consequently the critical value of (4.1) is given by

$$s_c = \begin{cases} \max\left(\frac{q_1-1}{1+v_1}, \frac{q_2-1}{1+v_2}\right) & \text{if } q_1 \neq q_2 \\ \frac{q-1}{1+\min(v_1,v_2)} & \text{if } q_1 = q_2 = q \end{cases}$$

Theorem 4.1 yields the characterization of the Gevrey order of the formal solution $\widetilde{u}(t,x)$ of (4.1) in terms of the one of the inhomogeneity $\widetilde{f}(t,x)$.

(1) Assume that $\widetilde{f}(t,x)$ is s-Gevrey with $s \geq s_c$. Then $\widetilde{u}(t,x)$ is also s-Gevrey.
(2) Assume that $\widetilde{f}(t,x)$ is s-Gevrey with $s < s_c$. Then $\widetilde{u}(t,x)$ is generically s_c-Gevrey.

In the two special cases of (4.2) and (4.3), these two points apply respectively with $s_c = \frac{1}{1+v_1}$ and $s_c = \frac{2}{1+v_1}$. In particular, for the classical analytic initial condition Korteweg-de Vries equation

$$\begin{gathered}\partial_t u + \partial_x^3 u + 6u\partial_x u = 0 \\ u(0,x) = \varphi(x)\end{gathered}$$

we find the well-known fact that the formal solution $\widetilde{u}(t,x)$ is 2-Gevrey.

Let us now turn to the proof of Theorem 4.1. This one is detailed in the following two sections. The first point is the most technical and the most complicated. Its proof is based on the Nagumo norms, a technique of majorant series and a fixed point procedure (see Section 5). As for the second point, it stems both from the first one and from Proposition 6.1 that gives an explicit example for which $\widetilde{u}(t,x)$ is s'-Gevrey for no $s' < s_c$ while $\widetilde{f}(t,x)$ is s-Gevrey with $s < s_c$ (see Section 6).

5. Proof of the first item of Theorem 4.1

According to Proposition 2.2, it is clear that

$$\widetilde{u}(t,x) \in \mathcal{O}(D_{\rho_1,\dots,\rho_n})[[t]]_s \Rightarrow \widetilde{f}(t,x) \in \mathcal{O}(D_{\rho_1,\dots,\rho_n})[[t]]_s.$$

Reciprocally, let us fix $s \geq s_c$ and let us suppose that the inhomogeneity $\widetilde{f}(t,x)$ is s-Gevrey. By assumption, its coefficients $f_{j,*}(x) \in \mathcal{O}(D_{\rho_1,\dots,\rho_n})$ satisfy the following condition (see Definition 2.1): there exist three positive constants $0 < \rho < \min(\rho_1,\dots,\rho_n)$, $C > 0$ and $K > 0$ such that the inequalities

$$|f_{j,*}(x)| \leq CK^j\Gamma(1+(s+1)j) \tag{5.1}$$

for all $j \geq 0$ and all $\|x\| \leq \rho$.

We must prove that the coefficients $u_{j,*}(x) \in \mathcal{O}(D_{\rho_1,\dots,\rho_n})$ of $\widetilde{u}(t,x)$ satisfy similar inequalities. The approach we present below is analogous to the ones already developed in [3, 31, 32, 33] in the framework of linear partial and integro-differential equations, and in [30, 35, 37] in the case of semilinear equations of the form (1.3). It is based on the Nagumo norms [4, 24, 48] and on a technique of majorant series. However, as we shall see, our calculations appear to be much more technical and complicated. Furthermore, the nonlinear polynomial terms associated with each derivation $\partial_t^i\partial_x^q$ generate several new technical combinatorial situations.

Before starting the calculations, let us first recall for the convenience of the reader the definition of the Nagumo norms and some of their properties which are needed in the sequel.

5.1. Nagumo norms.

Definition 5.1. Let $f \in \mathcal{O}(D_{\rho_1,\dots,\rho_n})$, $p \geq 0$ and $0 < r < \min(\rho_1,\dots,\rho_n)$. Then the Nagumo norm, with indices (p,r), of f is

$$\|f\|_{p,r} := \sup_{\|x\|\leq r} |f(x)d_r(x)^p|,$$

where $d_r(x)$ denotes the Euclidian distance $d_r(x) := r - \|x\|$.

Proposition 5.2. *Let $f, g \in \mathcal{O}(D_{\rho_1,\dots,\rho_n})$, $p, p' \geq 0$ and $0 < r < \min(\rho_1,\dots,\rho_n)$. Then*

(1) $\|\cdot\|_{p,r}$ *is a norm on* $\mathcal{O}(D_{\rho_1,\dots,\rho_n})$.
(2) $|f(x)| \leq \|f\|_{p,r}d_r(x)^{-p}$ *for all* $\|x\| < r$.
(3) $\|f\|_{0,r} = \sup_{\|x\|\leq r}|f(x)|$ *is the usual sup-norm on the polydisc* $D_{r,\dots,r}$.
(4) $\|fg\|_{p+p',r} \leq \|f\|_{p,r}\|g\|_{p',r}$.
(5) $\|\partial_{x_\ell}f\|_{p+1,r} \leq e(p+1)\|f\|_{p,r}$ *for all* $\ell \in \{1,\dots,n\}$.

Proof. Properties 1–4 are straightforward and are left to the reader. To prove Property 5, we proceed as follows. Let $\ell \in \{1,\dots,n\}$ and $x \in \mathbb{C}^n$ such that $\|x\| < r$ and $0 < R < d_r(x)$. Using the Cauchy Integral Formula, we have

$$\partial_{x_\ell}f(x) = \frac{1}{(2i\pi)^n}\int_{\gamma(x)} \frac{f(x')}{(x'_\ell - x_\ell)^2\prod_{\substack{k=1\\k\neq\ell}}^n(x'_k - x_k)}dx',$$

where $\gamma(x) := \{x' = (x'_1,\dots,x'_n) \in \mathbb{C}^n; |x'_k - x_k| = R \text{ for all } k \in \{1,\dots,n\}\}$. Since

$$x' \in \gamma(x) \Rightarrow \|x'\| < r,$$

we can apply Property 2 of Proposition 5.2; hence

$$|\partial_{x_\ell} f(x)| \le \frac{1}{R} \max_{x'\in\gamma(x)} |f(x')| \le \frac{1}{R}\|f\|_{p,r} \max_{x'\in\gamma(x)} d_r(x')^{-p} = \frac{1}{R}\|f\|_{p,r}(d_r(x)-R)^{-p}.$$

Observe that the last equality stems from the relations

$$d_r(x') = r - \|x'\| = r - \|x + x' - x\| \ge d_r(x) - \|x' - x\| = d_r(x) - R > 0.$$

When $p = 0$, the choice $R = \frac{d_r(x)}{e}$ implies the inequality

$$|\partial_{x_\ell} f(x)| \le e\|f\|_{0,r} d_r(x)^{-1};$$

hence,

$$|\partial_{x_\ell} f(x)| d_r(x) \le e\|f\|_{0,r}. \tag{5.2}$$

When $p > 0$, the choice $R = \frac{d_r(x)}{p+1}$ and

$$\Big(1 - \frac{1}{p+1}\Big)^{-p} = \Big(1 + \frac{1}{p}\Big)^{p} < e,$$

bring us to the inequalities

$$|\partial_{x_\ell} f(x)| \le \|f\|_{p,r} d_r(x)^{-p-1}(p+1)\Big(1 - \frac{1}{p+1}\Big)^{-p} \le e(p+1)\|f\|_{p,r} d_r(x)^{-p-1}$$

and then to the inequality

$$|\partial_{x_\ell} f(x)| d_r(x)^{p+1} \le e(p+1)\|f\|_{p,r}. \tag{5.3}$$

Property 5 follows since inequalities (5.2) and (5.3) remain valid when $\|x\| = r$. This completes the proof. □

Remark 5.3. Inequalities 4–5 of Proposition 5.2 are the most important properties. Also observe that the same index r occurs on both of their sides, allowing thus to get estimates for the product fg in terms of f and g, and for the derivatives $\partial_{x_\ell} f$ for any $\ell \in \{1, \dots, n\}$ in terms of f without having to shrink the polydisc $D_{r,\dots,r}$.

Let us now turn to the proof of the first item of Theorem 4.1.

5.2. **Some inequalities.** Let σ_s denote the positive rational number defined by

$$\sigma_s = (s+1)(\kappa + v),$$

where $v \ge 0$ stands for the maximum of the $v_{i,q,p}$'s. The purpose of this section is to construct from the recurrence relations (1.2) some inequalities relating the terms

$$\frac{\|u_{\ell,*}\|_{\ell\sigma_s,\rho}}{\Gamma(1+(s+1)\ell)}$$

for $\ell = 0, \dots, j + \kappa$, in order to apply a technique of majorant series in Section 5.3 below.

Preliminary inequalities. From the recurrence relations (1.2), the relations

$$\begin{aligned}
&\frac{u_{j+\kappa,*}(x)}{\Gamma(1+(s+1)(j+\kappa))}\\
&= \frac{f_{j,*}(x)}{\Gamma(1+(s+1)(j+\kappa))}\\
&\quad + \sum_{i\in\mathcal{K}}\sum_{q\in Q_i}\sum_{p\in P_{i,q}}\sum_{\substack{\ell_0+\ell_1+\dots+\ell_p+\\ \ell_{p+1}=j-v_{i,q,p}}} \frac{j!a_{i,q,p;\ell_0,*}(x)u_{\ell_1,*}(x)\dots u_{\ell_p,*}(x)\partial_x^q u_{\ell_{p+1}+i,*}(x)}{\ell_0!\ell_1!\dots\ell_p!\ell_{p+1}!\Gamma(1+(s+1)(j+\kappa))},
\end{aligned}$$

starting with $u_{\ell,*}(x)=\varphi_\ell(x)$ for $\ell=0,\dots,\kappa-1$, hold for all $j\geq 0$. As usual, we use the classical conventions that the fourth sum is zero when $j-v_{i,q,p}<0$, and that the product $\frac{u_{\ell_1,*}(x)\dots u_{\ell_p,*}(x)}{\ell_1!\dots\ell_p!}$ is 1 when $p=0$.

Applying then the Nagumo norm of indices $((j+\kappa)\sigma_s,\rho)$, we derive successively from Property 1 and from Properties $4-5$ of Proposition 5.2 the relations

$$\begin{aligned}
&\frac{\|u_{j+\kappa,*}\|_{(j+\kappa)\sigma_s,\rho}}{\Gamma(1+(s+1)(j+\kappa))}\\
&\leq \frac{\|f_{j,*}\|_{(j+\kappa)\sigma_s,\rho}}{\Gamma(1+(s+1)(j+\kappa))}\\
&\quad + \sum_{i\in\mathcal{K}}\sum_{q\in Q_i}\sum_{p\in P_{i,q}}\sum_{\substack{\ell_0+\ell_1+\dots+\ell_p+\\ \ell_{p+1}=j-v_{i,q,p}}} \frac{j!\|a_{i,q,p;\ell_0,*}u_{\ell_1,*}\dots u_{\ell_p,*}\partial_x^q u_{\ell_{p+1}+i,*}\|_{(j+\kappa)\sigma_s,\rho}}{\ell_0!\ell_1!\dots\ell_p!\ell_{p+1}!\Gamma(1+(s+1)(j+\kappa))}
\end{aligned}$$

and, next, the inequalities

$$\begin{aligned}
&\frac{\|u_{j+\kappa,*}\|_{(j+\kappa)\sigma_s,\rho}}{\Gamma(1+(s+1)(j+\kappa))}\\
&\leq \frac{\|f_{j,*}\|_{(j+\kappa)\sigma_s,\rho}}{\Gamma(1+(s+1)(j+\kappa))}\\
&\quad + \sum_{i\in\mathcal{K}}\sum_{q\in Q_i}\sum_{p\in P_{i,q}}\sum_{\ell_0+\ell_1+\dots+\ell_p+\ell_{p+1}=j-v_{i,q,p}} A_{i,q,p,j,\ell_0,\ell_1,\dots\ell_p,\ell_{p+1},s}\\
&\quad \times \frac{\|u_{\ell_1,*}\|_{\ell_1\sigma_s,\rho}\dots\|u_{\ell_p,*}\|_{\ell_p\sigma_s,\rho}\|u_{\ell_{p+1}+i,*}\|_{(\ell_{p+1}+i)\sigma_s,\rho}}{\Gamma(1+(s+1)\ell_1)\dots\Gamma(1+(s+1)\ell_p)\Gamma(1+(s+1)(\ell_{p+1}+i))}
\end{aligned}\tag{5.4}$$

for all $j\geq 0$, where $A_{i,q,p,j,\ell_0,\ell_1,\dots\ell_p,\ell_{p+1},s}$ is nonnegative and defined by

$$\begin{aligned}
&A_{i,q,p,j,\ell_0,\ell_1,\dots\ell_p,\ell_{p+1},s}\\
&= \frac{\Gamma(1+(s+1)\ell_1)\dots\Gamma(1+(s+1)\ell_p)}{\ell_0!\ell_1!\dots\ell_p!\ell_{p+1}!}\\
&\quad \times \frac{j!e^{\lambda(q)}\prod_{\ell=0}^{\lambda(q)-1}((\ell_{p+1}+i)\sigma_s+\lambda(q)-\ell)\Gamma(1+(s+1)(\ell_{p+1}+i))}{\Gamma(1+(s+1)(j+\kappa))}\\
&\quad \times \|a_{i,q,p;\ell_0,*}\|_{(\ell_0+\kappa-i+v_{i,q,p})\sigma_s-\lambda(q),\rho}.
\end{aligned}$$

for all $j\geq v_{i,q,p}$.

Remark 5.4. Observe that all the norms written in the inequality (5.4), and especially the norms $\|a_{i,q,p;\ell_0,*}\|_{(\ell_0+\kappa-i+v_{i,q,p})\sigma_s-\lambda(q),\rho}$ are well-defined. Indeed, due to Lemma 3.9, we have the inequality

$$(\ell_0+\kappa-i+v_{i,q,p})\sigma_s-\lambda(q)\geq(\kappa-i+v_{i,q,p})(s+1)-\lambda(q)\geq v_{i,q,p}\geq 0.$$

Bounds of $A_{i,q,p,j,\ell_0,\ell_1,\dots\ell_p,\ell_{p+1},s}$. We shall prove the following.

Proposition 5.5. *Let* $i\in\mathcal{K}$, $q\in Q_i$, $p\in P_{i,q}$, $j\geq v_{i,q,p}$, *and* $\ell_0,\ell_1,\dots,\ell_p,\ell_{p+1}\in\mathbb{N}$ *such that* $\ell_0+\ell_1+\dots+\ell_p+\ell_{p+1}=j-v_{i,q,p}$. *Then*

$$A_{i,q,p,j,\ell_0,\ell_1,\dots\ell_p,\ell_{p+1},s}\leq(e(\kappa+v))^{\lambda(q)}\frac{\|a_{i,q,p;\ell_0,*}\|_{(\ell_0+\kappa-i+v_{i,q,p})\sigma_s-\lambda(q),\rho}}{\Gamma(1+(s+1)\ell_0)}.$$

Proof. From the definition of $A_{i,q,p,j,\ell_0,\ell_1,\dots\ell_p,\ell_{p+1},s}$ and Lemma 5.6, we first derive the inequality

$$\begin{aligned}
&A_{i,q,p,j,\ell_0,\ell_1,\dots\ell_p,\ell_{p+1},s}\\
&\leq\frac{\Gamma(1+(s+1)(j-v_{i,q,p}))\Gamma(1+(s+1)(\ell_{p+1}+i))}{\Gamma(1+(s+1)\ell_{p+1})}\\
&\quad\times\frac{j!e^{\lambda(q)}\prod_{\ell=0}^{\lambda(q)-1}((\ell_{p+1}+i)\sigma_s+\lambda(q)-\ell)}{(j-v_{i,q,p})!\Gamma(1+(s+1)(j+\kappa))}\frac{\|a_{i,q,p;\ell_0,*}\|_{(\ell_0+\kappa-i+v_{i,q,p})\sigma_s-\lambda(q),\rho}}{\Gamma(1+(s+1)\ell_0)}.
\end{aligned}$$

Next, applying successively Lemmas 5.8 and 5.9, we obtain respectively

$$\begin{aligned}
&A_{i,q,p,j,\ell_0,\ell_1,\dots\ell_p,\ell_{p+1},s}\\
&\leq\frac{\Gamma(1+(s+1)(j-v_{i,q,p}))\Gamma(1+(s+1)(\ell_{p+1}+i))}{\Gamma(1+(s+1)\ell_{p+1})}\\
&\quad\times\frac{e^{\lambda(q)}\prod_{\ell=0}^{\lambda(q)-1}((\ell_{p+1}+i)\sigma_s+\lambda(q)-\ell)}{\Gamma(1+(s+1)(j+\kappa)-v_{i,q,p})}\frac{\|a_{i,q,p;\ell_0,*}\|_{(\ell_0+\kappa-i+v_{i,q,p})\sigma_s-\lambda(q),\rho}}{\Gamma(1+(s+1)\ell_0)},
\end{aligned}$$

and

$$\begin{aligned}
&A_{i,q,p,j,\ell_0,\ell_1,\dots\ell_p,\ell_{p+1},s}\\
&\leq\frac{\Gamma(1+(s+1)(j-v_{i,q,p}))\Gamma(1+(s+1)(\ell_{p+1}+i))}{\Gamma(1+(s+1)(j-v_{i,q,p}+i))\Gamma(1+(s+1)\ell_{p+1})}\\
&\quad\times(e(\kappa+v))^{\lambda(q)}\frac{\|a_{i,q,p;\ell_0,*}\|_{(\ell_0+\kappa-i+v_{i,q,p})\sigma_s-\lambda(q),\rho}}{\Gamma(1+(s+1)\ell_0)},
\end{aligned}$$

Proposition 5.5 follows then from Lemma 5.10 and that $\ell_{p+1}\leq j-v_{i,q,p}$. □

Lemma 5.6. *The inequality*

$$\frac{\Gamma(1+(s+1)k_1)\dots\Gamma(1+(s+1)k_m)}{\Gamma(1+(s+1)(k_1+\dots+k_m))}\leq\frac{k_1!\dots k_m!}{(k_1+\dots+k_m)!}$$

holds for all $m\geq 1$ *and* $k_1,\dots,k_m\in\mathbb{N}$.

Proof. It suffices to prove that the function $f_{k_1,\dots,k_m}$ defined on $[1,+\infty[$ by

$$f_{k_1,\dots,k_m}(x)=\frac{\Gamma(1+k_1x)\dots\Gamma(1+k_mx)}{\Gamma(1+(k_1+\dots+k_m)x)}$$

is decreasing. Computing its derivative with respect to x, we have

$$f'_{k_1,\dots,k_m}(x)=f_{k_1,\dots,k_m}(x)\Big(\sum_{\ell=1}^{m}k_\ell\psi(1+k_\ell x)-\Big(\sum_{\ell=1}^{m}k_\ell\Big)\psi\Big(1+\Big(\sum_{\ell=1}^{m}k_\ell\Big)x\Big)\Big)$$

for all $x \geq 1$, where $\psi := \Gamma'/\Gamma$ is the Digamma function (or Psi function). Applying then the classical relation (see [1, p. 259] for instance)

$$\psi(1+q) = -\gamma + \sum_{h=1}^{+\infty} \frac{q}{h(h+q)},$$

where $q \geq 0$ and γ is the Euler's constant, we obtain that for all $x \geq 1$,

$$f'_{k_1,\dots,k_m}(x) = x f_{k_1,\dots,k_m}(x) \sum_{h=1}^{+\infty} \Big(\sum_{\ell=1}^{m} \frac{k_\ell^2}{h(h+k_\ell x)} - \frac{\left(\sum_{\ell=1}^m k_\ell\right)^2}{h\Big(h + \left(\sum_{\ell=1}^m k_\ell\right)x\Big)} \Big).$$

Lemma 5.7 below complete the proof by showing that $f'_{k_1,\dots,k_m}(x) \leq 0$ for all $x \geq 1$. □

Lemma 5.7. *Let $x, h \geq 1$ be. Then*

$$\sum_{\ell=1}^{m} \frac{k_\ell^2}{h+k_\ell x} \leq \frac{\left(\sum_{\ell=1}^m k_\ell\right)^2}{h + \left(\sum_{\ell=1}^m k_\ell\right)x}$$

for all $m \geq 1$ and all $k_1,\dots,k_m \in \mathbb{N}$.

Proof. We proceed by induction on m. Lemma 5.7 is obvious for $m = 1$. For $m = 2$, we clearly have

$$\frac{k_1^2}{h+k_1 x} + \frac{k_2^2}{h+k_2 x} - \frac{(k_1+k_2)^2}{h+(k_1+k_2)x} = \frac{-k_1k_2h(k_1x+k_2x+2h)}{(h+k_1x)(h+k_2x)(h+(k_1+k_2)x)} \leq 0.$$

Let us now suppose that Lemma 5.7 is true for all $k \in \{1,\dots,m\}$ for a certain $m \geq 1$. Then, the successive relations

$$\begin{aligned} \sum_{\ell=1}^{m+1} \frac{k_\ell^2}{h+k_\ell x} &\leq \frac{\left(\sum_{\ell=1}^m k_\ell\right)^2}{h + \left(\sum_{\ell=1}^m k_\ell\right)x} + \frac{k_{m+1}^2}{h+k_{m+1}x} \\ &\leq \frac{\left(\sum_{\ell=1}^m k_\ell + k_{m+1}\right)^2}{h + \left(\sum_{\ell=1}^m k_\ell + k_{m+1}\right)x} \\ &= \frac{\left(\sum_{\ell=1}^{m+1} k_\ell\right)^2}{h + \left(\sum_{\ell=1}^{m+1} k_\ell\right)x} \end{aligned}$$

hold for any $k_1,\dots,k_{m+1} \in \mathbb{N}$, which completes the proof. □

Lemma 5.8. *Let $i \in \mathcal{K}$, $q \in Q_i$, $p \in P_{i,q}$ and $j \geq v_{i,q,p}$ be. Then*

$$\frac{j!}{(j-v_{i,q,p})!\Gamma(1+(s+1)(j+\kappa))} \leq \frac{1}{\Gamma(1+(s+1)(j+\kappa)-v_{i,q,p})}.$$

Proof. It is clear that the inequality holds when $v_{i,q,p} = 0$. Let us now assume $v_{i,q,p} \geq 1$ and let us write the quotient $j!/(j-v_{i,q,p})!$ in the form

$$\frac{j!}{(j-v_{i,q,p})!} = \prod_{\ell=0}^{v_{i,q,p}-1} (j-\ell). \tag{5.5}$$

On the other hand, applying $v_{i,q,p}$ times the recurrence relation $\Gamma(1+z) = z\Gamma(z)$ to $\Gamma(1+(s+1)(j+\kappa))$, we obtain

$$\Gamma(1+(s+1)(j+\kappa)) = \Gamma(1+(s+1)(j+\kappa)-v_{i,q,p}) \prod_{\ell=0}^{v_{i,q,p}-1} ((s+1)(j+\kappa)-\ell). \tag{5.6}$$

Combining then the identities (5.5) and (5.6), we obtain

$$\frac{j!}{(j-v_{i,q,p})!\Gamma(1+(s+1)(j+\kappa))}=\frac{\prod_{\ell=0}^{v_{i,q,p}-1}\frac{j-\ell}{(s+1)(j+\kappa)-\ell}}{\Gamma(1+(s+1)(j+\kappa)-v_{i,q,p})},$$

and Lemma 5.8 follows from the inequalities $0<j-\ell<(s+1)(j+\kappa)-\ell$. $\square$

Lemma 5.9. *Let $i\in\mathcal{K}$, $q\in Q_i$, $p\in P_{i,q}$, $j\geq v_{i,q,p}$, and $k\in\{0,\dots,j-v_{i,q,p}\}$. Then*

$$\frac{\prod_{\ell=0}^{\lambda(q)-1}((k+i)\sigma_s+\lambda(q)-\ell)}{\Gamma(1+(s+1)(j+\kappa)-v_{i,q,p})}\leq\frac{(\kappa+v)^{\lambda(q)}}{\Gamma(1+(s+1)(j-v_{i,q,p}+i))}.$$

Proof. Let us first assume $j=v_{i,q,p}$ and $i=0$. Then, we have $k=0$, and we must prove the inequality

$$\frac{\prod_{\ell=0}^{\lambda(q)-1}(\lambda(q)-\ell)}{\Gamma(1+(s+1)(v_{i,q,p}+\kappa)-v_{i,q,p})}\leq(\kappa+v)^{\lambda(q)}. \tag{5.7}$$

Let us note that

$$\prod_{\ell=0}^{\lambda(q)-1}(\lambda(q)-\ell)=(\lambda(q))!=\Gamma(1+\lambda(q))$$

for all $\lambda(q)$, including the case $\lambda(q)=0$ since the product is 1 by convention.

On the other hand, Lemma 3.9 implies in the case $\lambda(q)>0$ the inequalities

$$1+(s+1)(v_{i,q,p}+\kappa)-v_{i,q,p}\geq 1+\lambda(q)\geq 2;$$

hence,

$$\Gamma(1+(s+1)(v_{i,q,p}+\kappa)-v_{i,q,p})\geq\Gamma(1+\lambda(q))$$

since the Gamma function is increasing on $[2,+\infty[$. In the special case $\lambda(q)=0$, we observe that the increase of the Gamma function applied to the inequalities

$$1+(s+1)(v_{i,q,p}+\kappa)-v_{i,q,p}\geq 1+\kappa\geq 2$$

implies

$$\Gamma(1+(s+1)(v_{i,q,p}+\kappa)-v_{i,q,p})\geq\Gamma(2)=\Gamma(1)=\Gamma(1+\lambda(q)).$$

Consequently, the left hand-side of (5.7) is ≤ 1 and Lemma 5.9 follows then from the inequality $\kappa+v\geq 1$.

Let us now assume $(j,i)\neq(v_{i,q,p},0)$. According to the definition of σ_s, we first have the identity

$$\begin{aligned}&\prod_{\ell=0}^{\lambda(q)-1}((k+i)\sigma_s+\lambda(q)-\ell)\\&=(\kappa+v)^{\lambda(q)}\prod_{\ell=0}^{\lambda(q)-1}\Big((s+1)(k+i)+\frac{\lambda(q)-\ell}{\kappa+v}\Big).\end{aligned} \tag{5.8}$$

On the other hand, applying $\lambda(q)$ times the recurrence relation $\Gamma(1+z)=z\Gamma(z)$ to $\Gamma(1+(s+1)(j+\kappa)-v_{i,q,p})$, we have

$$\begin{aligned}&\Gamma(1+(s+1)(j+\kappa)-v_{i,q,p})\\&=\Gamma(1+(s+1)(j+\kappa)-v_{i,q,p}-\lambda(q))\prod_{\ell=0}^{\lambda(q)-1}((s+1)(j+\kappa)-v_{i,q,p}-\ell).\end{aligned} \tag{5.9}$$

Observe that this identity makes sense since the inequality $j \geq v_{i,q,p}$ and Lemma 3.9 imply

$$\begin{aligned}(s+1)(j+\kappa) - v_{i,q,p} - \lambda(q) &\geq (s+1)(v_{i,q,p}+\kappa) - v_{i,q,p} - \lambda(q)\\ &\geq (s+1)(v_{i,q,p}+\kappa-i) - v_{i,q,p} - \lambda(q) \geq 0.\end{aligned}$$

Observe also that

$$(s+1)(k+i) + \frac{\lambda(q)-\ell}{\kappa+v} \leq (s+1)(j+\kappa) - v_{i,q,p} - \ell$$

for all $\ell \in \{0,\dots,\lambda(q)-1\}$ when $\lambda(q)>0$. Indeed, according to the inequalities $k \leq j - v_{i,q,p}$ and $\kappa + v \geq 1$, and Lemma 3.9, we have

$$\begin{aligned}&(s+1)(j+\kappa) - v_{i,q,p} - \ell - (s+1)(k+i) - \frac{\lambda(q)-\ell}{\kappa+v}\\ &= (s+1)(j+\kappa-i-k) - v_{i,q,p} - \ell - \frac{\lambda(q)-\ell}{\kappa+v}\\ &\geq (s+1)(\kappa-i+v_{i,q,p}) - v_{i,q,p} - \ell - \frac{\lambda(q)-\ell}{\kappa+v}\\ &\geq (\lambda(q)-\ell)\Big(1-\frac{1}{\kappa+v}\Big) \geq 0.\end{aligned}$$

Consequently, combining the identities (5.8) and (5.9), we finally obtain

$$\frac{\prod_{\ell=0}^{\lambda(q)-1}((k+i)\sigma_s+\lambda(q)-\ell)}{\Gamma(1+(s+1)(j+\kappa)-v_{i,q,p})} \leq \frac{(\kappa+v)^{\lambda(q)}}{\Gamma(1+(s+1)(j+\kappa)-v_{i,q,p}-\lambda(q))}$$

for all $\lambda(q)$, including the case $\lambda(q)=0$ since the product is 1 by convention. Lemma 5.9 follows then from

$$\begin{aligned}1+(s+1)(j+\kappa)-v_{i,q,p}-\lambda(q) &\geq 1+(s+1)(j+\kappa)-(s+1)(\kappa-i+v_{i,q,p})\\ &= 1+(s+1)(j-v_{i,q,p}+i) \geq 2\end{aligned}$$

and from Gamma function on $[2,+\infty[$ being increasing. Observe that the first inequality stems again from Lemma 3.9. Observe also that, without the condition $(j,i) \neq (v_{i,q,p},0)$, the second inequality is no longer valid. This completes the proof. $\square$

Lemma 5.10. *Let $a \geq 0$. The function f defined on $[0,+\infty[$ by*

$$f(x) = \frac{\Gamma(1+a+x)}{\Gamma(1+x)}$$

is increasing.

Proof. Denoting by f' the derivative of f with respect to x, we have

$$f'(x) = f(x)(\psi(1+a+x)-\psi(1+x))$$

for all $x \geq 0$, where ψ denotes as in the proof of Lemma 5.6 the Digamma function. The latter being increasing on $[0,+\infty[$ (see [1, pp. 258-260] for instance), we deduce that $f'(x) \geq 0$ for all $x \geq 0$, which completes the proof. $\square$

More precise inequalities. Let us apply Proposition 5.5 to the inequalities (5.4) to obtain that for all $j \geq 0$,

$$\begin{aligned}
&\frac{\|u_{j+\kappa,*}\|_{(j+\kappa)\sigma_s,\rho}}{\Gamma(1+(s+1)(j+\kappa))} \\
&\leq g_{j,s} + \sum_{i\in\mathcal{K}} \sum_{q\in Q_i} \sum_{p\in P_{i,q}} \sum_{\substack{\ell_0+\ell_1+\dots+\ell_p+ \\ \ell_{p+1}=j-v_{i,q,p}}} \alpha_{i,q,p,\ell_0,s} \\
&\quad\times \frac{\|u_{\ell_1,*}\|_{\ell_1\sigma_s,\rho}\dots\|u_{\ell_p,*}\|_{\ell_p\sigma_s,\rho}\|u_{\ell_{p+1}+i,*}\|_{(\ell_{p+1}+i)\sigma_s,\rho}}{\Gamma(1+(s+1)\ell_1)\dots\Gamma(1+(s+1)\ell_p)\Gamma(1+(s+1)(\ell_{p+1}+i))},
\end{aligned} \tag{5.10}$$

where the terms $g_{j,s}$ and $\alpha_{i,q,p,\ell_0,s}$ are nonnegative and defined by

$$g_{j,s} = \frac{\|f_{j,*}\|_{(j+\kappa)\sigma_s,\rho}}{\Gamma(1+(s+1)(j+\kappa))},$$

$$\alpha_{i,q,p,\ell_0,s} = (e(\kappa+v))^{\lambda(q)} \frac{\|a_{i,q,p;\ell_0,*}\|_{(\ell_0+\kappa-i+v_{i,q,p})\sigma_s-\lambda(q),\rho}}{\Gamma(1+(s+1)\ell_0)}.$$

The following result provides some bounds on these terms, and will be useful in the next section.

Lemma 5.11. *Let $i \in \mathcal{K}$, $q \in Q_i$ and $p \in P_{i,q}$. Then there exist four positive constants $C_1, C_2, K_1, K_2 > 0$ such that*

$$0 \leq g_{j,s} \leq C_1 K_1^j \text{ and } 0 \leq \alpha_{i,q,p,j,s} \leq C_2 K_2^j$$

for all $j \geq 0$.

Proof. According to the analyticity of the function $a_{i,q,p}(t,x)$ on $D_{\rho_0,\dots,\rho_n}$, and the hypothesis on the coefficients $f_{j,*}(x)$ of the inhomogeneity $\widetilde{f}(t,x)$ (see inequality (5.1)), we first have

$$|f_{j,*}(x)| \leq CK^j\Gamma(1+(s+1)j) \quad \text{and} \quad |a_{i,q,p;j,*}(x)| \leq C'K'^j j!$$

for all $j \geq 0$ and all $\|x\| \leq \rho$, the constants $C, K, C', K' > 0$ being independent of j and x. Hence, applying Proposition 5.2, we have

$$0 \leq g_{j,s} \leq \frac{CK^j\Gamma(1+(s+1)j)\rho^{(j+\kappa)\sigma_s}}{\Gamma(1+(s+1)(j+\kappa))} \leq C\rho^{\kappa\sigma_s}(K\rho^{\sigma_s})^j,$$

and

$$\begin{aligned}
0 \leq \alpha_{i,q,p,j,s} &\leq \frac{(e(\kappa+v))^{\lambda(q)}C'K'^j j!\rho^{(j+\kappa-i+v_{i,q,p})\sigma_s-\lambda(q)}}{\Gamma(1+(s+1)j)} \\
&\leq (e(\kappa+v))^{\lambda(q)}C'\rho^{(\kappa-i+v_{i,q,p})\sigma_s-\lambda(q)}(K'\rho^{\sigma_s})^j.
\end{aligned}$$

This completes the proof. □

We shall now bound the Nagumo norms $\|u_{j,*}\|_{j\sigma_s,\rho}$ for any $j \geq 0$. To do that, we shall proceed similarly as in [3, 30, 31, 32, 33, 35, 37] by using a technique of majorant series. However, as we shall see, the calculations are much more complicated because of the nonlinear polynomial terms associated with each derivation $\partial_t^i\partial_x^q$.

5.3. **A majorant series.** Let us consider the formal power series $v(X)=\sum_{j\geq 0} v_j X^j$, the coefficients of which are recursively determined for all $j\geq 0$ by the relations

$$v_{j+\kappa} = g_{j,s} + \sum_{i\in\mathcal{K}} \sum_{q\in Q_i} \sum_{p\in P_{i,q}} \sum_{\substack{\ell_0+\ell_1+\dots+\ell_{d+1} \\ =j+i-v_{i,q,p}}} \alpha_{i,q,p,\ell_0,s} v_{\ell_1}\dots v_{\ell_{d+1}} \tag{5.11}$$

starting with the initial condition $v_0 = 1+\|\varphi_0\|_{0,\rho}$, and, for $j=1,\dots,\kappa-1$ (if $\kappa\geq 2$):

$$v_j = \frac{\|\varphi_j\|_{j\sigma_s,\rho}}{\Gamma(1+(s+1)j)} + \sum_{(i,q,p)\in V_j} \sum_{\substack{\ell_0+\ell_1+\dots+\ell_{d+1} \\ =j-\kappa+i-v_{i,q,p}}} \alpha_{i,q,p,\ell_0,s} v_{\ell_1}\dots v_{\ell_{d+1}},$$

where

$$d = \max\Big(p\in \bigcup_{i\in\mathcal{K}} \bigcup_{q\in Q_i} P_{i,q}\Big),$$

and

$$V_j = \big\{(i,q,p)\in\mathcal{K}\times Q_i\times P_{i,q} \text{ such that } j-\kappa+i-v_{i,q,p}\geq 0\big\}.$$

Observe that the condition $\kappa>i$ for all $i\in\mathcal{K}$ implies $j-\kappa+i-v_{i,q,p}<j$; hence, the initial conditions on the v_j's with $j=1,\dots,\kappa-1$ make sense. Observe also that the set V_j may be empty (this is particularly the case when $\mathcal{K}=\{0\}$, or when $v_{i,q,p}\geq i$ for all i,q,p).

Proposition 5.12. *The inequalities*

$$0\leq \frac{\|u_{j,*}\|_{j\sigma_s,\rho}}{\Gamma(1+(s+1)j)} \leq v_j \tag{5.12}$$

hold for all $j\geq 0$.

Proof. According to the initial conditions on the u_j's and on the v_j's, the inequalities in (5.12) hold for all $j=0,\dots,\kappa-1$. Let us now suppose that these inequalities are true for all $k\leq j-1+\kappa$ for a certain $j\geq 0$, and let us prove them for $j+\kappa$.

First of all, applying our hypotheses to relations (5.10), we have

$$\begin{aligned} 0 &\leq \frac{\|u_{j+\kappa,*}\|_{(j+\kappa)\sigma_s,\rho}}{\Gamma(1+(s+1)(j+\kappa))} \\ &\leq g_{j,s} + \sum_{i\in\mathcal{K}} \sum_{q\in Q_i} \sum_{p\in P_{i,q}} \sum_{\substack{\ell_0'+\ell_1'+\dots+\ell_p'+ \\ \ell_{p+1}'=j-v_{i,q,p}}} \alpha_{i,q,p,\ell_0',s} v_{\ell_1'}\dots v_{\ell_p'} v_{\ell_{p+1}'+i} \end{aligned} \tag{5.13}$$

and then

$$\begin{aligned} 0 &\leq \frac{\|u_{j+\kappa,*}\|_{(j+\kappa)\sigma_s,\rho}}{\Gamma(1+(s+1)(j+\kappa))} \\ &\leq g_{j,s} + \sum_{i\in\mathcal{K}} \sum_{q\in Q_i} \sum_{p\in P_{i,q}} \sum_{\substack{\ell_0+\ell_1+\dots+\ell_p+ \\ \ell_{p+1}=j+i-v_{i,q,p}}} \alpha_{i,q,p,\ell_0,s} v_{\ell_1}\dots v_{\ell_p} v_{\ell_{p+1}} \end{aligned} \tag{5.14}$$

since all the tuples $(\ell_0',\ell_1',\dots,\ell_{p+1}'+i)\in\mathbb{N}^{p+2}$ in (5.13) satisfy the identity $\ell_0'+\ell_1'+\dots+\ell_{p+1}'+i = j+i-v_{i,q,p}$, and since all the terms $\alpha_{i,q,p,\ell_0,s} v_{\ell_1}\dots v_{\ell_p} v_{\ell_{p+1}}$ in (5.14) are nonnegative.

Next, let us observe that any $(p+2)$-tuple $(\ell_0,\dots,\ell_{p+1})\in\mathbb{N}^{p+2}$ such that $\ell_0+\dots+\ell_{p+1}=j+i-v_{i,q,p}$ can be seen as the $(d+2)$-tuple $(\ell_0,\dots,\ell_{p+1},\ell_{p+2},\dots,\ell_{d+1})\in\mathbb{N}^{d+2}$, where $\ell_{p+2}=\dots=\ell_{d+1}=0$. Therefore, using the fact that $v_0\ge 1$, we have

$$\begin{aligned}
0 &\le \alpha_{i,q,p,\ell_0,s}v_{\ell_1}\dots v_{\ell_{p+1}}\\
&\le \alpha_{i,q,p,\ell_0,s}v_{\ell_1}\dots v_{\ell_{p+1}}v_0^{d-p}\\
&= \alpha_{i,q,p,\ell_0,s}v_{\ell_1}\dots v_{\ell_{p+1}}v_{\ell_{p+2}}\dots v_{\ell_{d+1}},
\end{aligned}$$

and, consequently,

$$\begin{aligned}
0 &\le \sum_{\substack{\ell_0+\dots+\ell_{p+1}\\ =j+i-v_{i,q,p}}}\alpha_{i,q,p,\ell_0,s}v_{\ell_1}\dots v_{\ell_{p+1}}\\
&\le \sum_{\substack{\ell_0+\dots+\ell_{p+1}+\\ 0+\dots+0=j+i-v_{i,q,p}}}\alpha_{i,q,p,\ell_0,s}v_{\ell_1}\dots v_{\ell_{d+1}}\\
&\le \sum_{\substack{\ell_0+\dots+\ell_{d+1}\\ =j+i-v_{i,q,p}}}\alpha_{i,q,p,\ell_0,s}v_{\ell_1}\dots v_{\ell_{d+1}},
\end{aligned}$$

since all the terms are nonnegative. Therefore,

$$\begin{aligned}
0 &\le \frac{\|u_{j+\kappa,*}\|_{(j+\kappa)\sigma_s,\rho}}{\Gamma(1+(s+1)(j+\kappa))}\\
&\le g_{j,s}+\sum_{i\in\mathcal{K}}\sum_{q\in Q_i}\sum_{p\in P_{i,q}}\sum_{\substack{\ell_0+\ell_1+\dots+\ell_{d+1}\\ =j+i-v_{i,q,p}}}\alpha_{i,q,p,\ell_0,s}v_{\ell_1}\dots v_{\ell_{d+1}}=v_{j+\kappa}
\end{aligned}$$

which completes the proof. □

Following proposition allows us to bound the v_j's.

Proposition 5.13. *The formal series $v(X)$ is convergent. In particular, there exist two positive constants $C',K'>0$ such that $v_j\le C'K'^j$ for all $j\ge 0$.*

Proof. It is sufficient to prove the convergence of $v(X)$. Let us start by observing that $v(X)$ is the unique formal power series in X solution of the functional equation

$$v(X)=X\alpha(X)(v(X))^{d+1}+h(X), \tag{5.15}$$

where $\alpha(X)$ and $h(X)$ are the two formal power series defined by

$$\begin{gathered}
\alpha(X)=\sum_{i\in\mathcal{K}}\sum_{q\in Q_i}\sum_{p\in P_{i,q}}X^{\kappa-i-1+v_{i,q,p}}\alpha_{i,q,p}(X),\\
h(X)=A_0+A_1X+\dots+A_{\kappa-1}X^{\kappa-1}+X^\kappa\sum_{j\ge0}g_{j,s}X^j
\end{gathered}$$

with

$$\alpha_{i,q,p}(X)=\sum_{j\ge0}\alpha_{i,q,p,j,s}X^j,$$

$A_0=1+\|\varphi_0\|_{0,\rho}$, and, for $j=1,\dots,\kappa-1$ (if $\kappa\ge2$),

$$A_j=\frac{\|\varphi_j\|_{j\sigma_s,\rho}}{\Gamma(1+(s+1)j)}.$$

Next we notice that, according to Lemma 5.11, $\alpha(X)$ and $h(x)$ are actually two convergent power series with nonnegative coefficients. In particular, they respectively define two increasing functions on $[0, r_\alpha[$ and $[0, r_h[$, where $r_\alpha > 0$ and $r_h > 0$ stand respectively for the radius of convergence of $\alpha(X)$ and $h(X)$. Besides, given the assumptions on the functions $a_{i,q,p}(t,x)$ (see page 2) and the fact that $A_0 \geq 1$, we have $\alpha(r) > 0$ (resp. $h(r) > 0$) for all $r \in]0, r_\alpha[$ (resp. $]0, r_h[$).

When $d = 0$, the convergence of $v(X)$ is obvious, since we have the identity $(1 - X\alpha(X))v(X) = h(X)$. When $d \geq 1$, we proceed through a fixed point method as follows. Let us set

$$V(X) = \sum_{m \geq 0} V_m(X)$$

and let us choose the solution of (5.15) given by the system

$$\begin{gathered} V_0(X) = h(X) \\ V_{m+1}(X) = X\alpha(X) \sum_{\ell_1 + \dots + \ell_{d+1} = m} V_{\ell_1}(X) \dots V_{\ell_{d+1}}(X) \quad \text{for } m \geq 0. \end{gathered}$$

By induction on $m \geq 0$, we easily check that

$$V_m(X) = C_{m,d+1} X^m (\alpha(X))^m (h(X))^{md+1}, \tag{5.16}$$

where the $C_{m,d+1}$'s are the positive constants recursively determined from $C_{0,d+1} := 1$ by the relations

$$C_{m+1,d+1} = \sum_{k_1 + \dots + k_{d+1} = m} C_{k_1,d+1} \dots C_{k_{d+1},d+1}.$$

Thereby, all the $V_m(X)$'s are analytic functions on the disc with center $0 \in \mathbb{C}$ and radius $\min(r_\alpha, r_h)$. Moreover, identities (5.16) show us that $V_m(X)$ is of order X^m for all $m \geq 0$. Consequently, the series $V(X)$ makes sense as a formal power series in X and we obtain $V(X) = v(X)$ by unicity.

It is left to prove the convergence of $V(X)$. To do that, let us choose $0 < r < \min(r_\alpha, r_h)$. By construction (see page 6), the constants $C_{m,d+1}$'s are the generalized Catalan numbers of order $d + 1$. We have therefore

$$C_{m,d+1} = \frac{1}{dm+1} \binom{m(d+1)}{m} \leq 2^{m(d+1)}$$

for all $m \geq 0$. On the other hand, according to the remark above on the series $\alpha(X)$ and $h(X)$, we derive from the identities (5.16) the inequalities

$$|V_m(X)| \leq h(r) \left(2^{d+1} \alpha(r) (h(r))^d |X|\right)^m$$

for all $m \geq 0$ and all $|X| \leq r$; hence, the fact that the series $V(X)$ is normally convergent on any disc with center $0 \in \mathbb{C}$ and radius

$$0 < r' < \min\Big(r, \frac{1}{2^{d+1} \alpha(r) (h(r))^d}\Big).$$

This proves the analyticity of $V(X)$ at 0 and achieves thereby the proof of Proposition 5.13. $\square$

According to Propositions 5.12 and 5.13, we can now bound the Nagumo norms $\|u_{j,*}\|_{j\sigma_s,\rho}$.

Corollary 5.14. *Let $C', K' > 0$ be as in Proposition 5.13. Then*

$$\|u_{j,*}\|_{j\sigma_s,\rho} \leq C' K'^j \Gamma(1 + (s+1)j)$$

for all $j \geq 0$.

We are now able to conclude the proof of the first item of Theorem 4.1.

5.4. **Conclusion.** We must prove estimates on the sup-norm of the $u_{j,*}(x)$, similar to the ones on the norms $\|u_{j,*}\|_{j\sigma_s,\rho}$ (see Corollary 5.14). To this end, we proceed by shrinking the closed polydisc $\|x\| \leq \rho$. Let $0 < \rho' < \rho$. Then, for all $j \geq 0$ and all $\|x\| \leq \rho'$, we have

$$|u_{j,*}(x)| = \left|u_{j,*}(x) d_\rho(x)^{j\sigma_s} \frac{1}{d_\rho(x)^{j\sigma_s}}\right| \leq \frac{|u_{j,*}(x) d_\rho(x)^{j\sigma_s}|}{(\rho-\rho')^{j\sigma_s}} \leq \frac{\|u_{j,*}\|_{j\sigma_s,\rho}}{(\rho-\rho')^{j\sigma_s}}$$

and, consequently,

$$\sup_{\|x\| \leq \rho'} |u_{j,*}(x)| \leq C' \Big(\frac{K'}{(\rho-\rho')^{\sigma_s}}\Big)^j \Gamma(1 + (s+1)j)$$

by applying Corollary 5.14. This completes the proof of the first item of Theorem 4.1.

6. Proof of the second item of Theorem 4.1

In this section, we assume $\mathcal{S} \neq \emptyset$ and we fix $0 \leq s < s_c$. (Of course, this case does not occur when $\mathcal{S} = \emptyset$). According to the filtration of the s-Gevrey spaces $\mathcal{O}(D_{\rho_1,\dots,\rho_n})[[t]]_s$ (see Section 3) and the first item of Theorem 4.1, it is clear that we have the following implications:

$$\begin{aligned} \widetilde{f}(t,x) \in \mathcal{O}(D_{\rho_1,\dots,\rho_n})[[t]]_s &\Rightarrow \widetilde{f}(t,x) \in \mathcal{O}(D_{\rho_1,\dots,\rho_n})[[t]]_{s_c} \\ &\Rightarrow \widetilde{u}(t,x) \in \mathcal{O}(D_{\rho_1,\dots,\rho_n})[[t]]_{s_c}. \end{aligned}$$

Therefore, to conclude that we can not say better about the Gevrey order of $\widetilde{u}(t,x)$, that is $\widetilde{u}(t,x)$ is *generically* s_c-Gevrey, we need to find an example for which the formal solution $\widetilde{u}(t,x)$ of (1.1) is s'-Gevrey for no $s' < s_c$. Section 3.1 has already provided us with such two examples in the case $\kappa = 1$. In Proposition 6.1 below, we propose a much more general example.

Proposition 6.1. *Let us consider the equation*

$$\begin{gathered} \partial_t^\kappa u - \sum_{i\in\mathcal{K}} \sum_{q\in Q_i} \Big(\sum_{p\in P_{i,q}} a_{i,q,p} t^{v_{i,q,p}} u^p\Big) \partial_t^i \partial_x^q u = \widetilde{f}(t,x), \quad a_{i,q,p} > 0 \\ \partial_t^j u(t,x)_{|t=0} = \varphi_j(x), \quad j = 0,\dots,\kappa-1 \end{gathered} \tag{6.1}$$

where the initial condition $\varphi_{i^*}(x)$ is the analytic function defined by

$$\varphi_{i^*}(x) = \frac{1}{1 - x_1 - \dots - x_n}$$

on the disc $D_{1/n,\dots,1/n}$, and where the initial conditions $\varphi_j(x)$ for $j \neq i^*$ are analytic functions on $D_{1/n,\dots,1/n}$ satisfying $\partial_x^\ell \varphi_j(0) \geq 0$ for all $\ell \in \mathbb{N}^n$, and $\varphi_0(0) > 0$ when $i^* \neq 0$. Suppose also that the inhomogeneity $\widetilde{f}(t,x)$ satisfies the following two conditions:

- *$\widetilde{f}(t,x)$ is s-Gevrey;*
- *$\partial_x^\ell f_{j,*}(0) \geq 0$ for all $j \geq 0$ and all $\ell \in \mathbb{N}^n$.*

Then, the formal solution $\widetilde{u}(t,x)$ *of* (6.1) *is exactly* s_c*-Gevrey.*

Proof. From the calculations above, it is sufficient to prove that $\widetilde{u}(t,x)$ is s'-Gevrey for no $s' < s_c$.

First of all, let us rewrite the general relations (1.2) as the identities

$$u_{j+\kappa,*}(x) = \frac{j! a_{i^*,q^*,p^*} u_{0,*}^{p^*}(x)}{(j - v_{i^*,q^*,p^*})!} \partial_x^{q^*} u_{j-v_{i^*,q^*,p^*}+i^*,*}(x) + R_j(x)$$

with

$$\begin{aligned} R_j(x) = f_{j,*}(x) + &\sum_{\substack{\ell_1+\dots+\ell_{p^*}+\ell_{p^*+1}=j-v_{i^*,q^*,p^*} \\ (\ell_1,\dots,\ell_{p^*},\ell_{p^*+1})\neq(0,\dots,0,j-v_{i^*,q^*,p^*})}} \\ &\frac{j! a_{i^*,q^*,p^*} u_{\ell_1,*}(x)\dots u_{\ell_{p^*},*}(x) \partial_x^{q^*} u_{\ell_{p^*+1}+i^*,*}(x)}{\ell_1!\dots\ell_{p^*}!\ell_{p^*+1}!} \\ &+ \sum_{\substack{(i,q,p)\in\mathcal{K}\times Q_i\times P_{i,q} \\ (i,q,p)\neq(i^*,q^*,p^*)}} \sum_{\substack{\ell_1+\dots+\ell_p+ \\ \ell_{p+1}=j-v_{i,q,p}}} \frac{j! a_{i,q,p} u_{\ell_1,*}(x)\dots u_{\ell_p,*}(x) \partial_x^q u_{\ell_{p+1}+i,*}(x)}{\ell_1!\dots\ell_p!\ell_{p+1}!} \end{aligned}$$

for all $j \geq 0$, together with the initial conditions $u_{j,*}(x) = \varphi_j(x)$ for $j = 0,\dots,\kappa-1$. Using then our hypotheses on the coefficients $a_{i,q,p}$, on the initial conditions $\varphi_j(x)$, and on the inhomogeneity $\widetilde{f}(t,x)$, we easily check that, for all $j \geq 0$,

$$\begin{aligned} &u_{j(v_{i^*,q^*,p^*}+\kappa-i^*)+i^*,*}(x) \\ &= \frac{a_{i^*,q^*,p^*}^j \varphi_0^{jp^*}(x)(j\lambda(q^*))!}{(1-x_1-\dots-x_n)^{j\lambda(q^*)+1}} \prod_{\ell=1}^{j} \prod_{m=1}^{v_{i^*,q^*,p^*}} ((\ell-1)(v_{i^*,q^*,p^*}+\kappa-i^*)+m) \\ &\quad + \operatorname{rem}_j(x) \end{aligned}$$

with $\operatorname{rem}_j(0) \geq 0$. Hence, applying technical Lemma 6.2 below, we have the inequality

$$u_{j(v_{i^*,q^*,p^*}+\kappa-i^*)+i^*,*}(0) \geq \left(a_{i^*,q^*,p^*}\varphi_0^{p^*}(0)\right)^j (j\lambda(q^*))!(jv_{i^*,q^*,p^*})!. \tag{6.2}$$

Let us now suppose that $\widetilde{u}(t,x)$ is s'-Gevrey for some $s' < s_c$. Then, Definition 2.1 and inequality (6.2) imply

$$1 \leq C\Big(\frac{K}{a_{i^*,q^*,p^*}\varphi_0^{p^*}(0)}\Big)^j \frac{\Gamma(1+i^*(s'+1)+j(s'+1)(v_{i^*,q^*,p^*}+\kappa-i^*))}{\Gamma(1+j\lambda(q^*))\Gamma(1+jv_{i^*,q^*,p^*})} \tag{6.3}$$

for all $j \geq 0$ and some convenient positive constants C and K independent of j. Proposition 6.1 follows since such inequalities are impossible: applying the Stirling's Formula, the right hand-side of (6.3) is equivalent to

$$C' j^{i^*(s'+1)-\frac{1}{2}} \left(\frac{K'}{j^\sigma}\right)^j, \quad j \to +\infty \tag{6.4}$$

with

$$C' = C((s'+1)(v_{i^*,q^*,p^*}+\kappa-i^*))^{i^*(s'+1)} \sqrt{\frac{(s'+1)(v_{i^*,q^*,p^*}+\kappa-i^*)}{2\pi\lambda(q^*)v_{i^*,q^*,p^*}}},$$

$$K' = \frac{K((s'+1)(v_{i^*,q^*,p^*}+\kappa-i^*))^{(s'+1)(v_{i^*,q^*,p^*}+\kappa-i^*)}e^\sigma}{a_{i^*,q^*,p^*}\varphi_0^{p^*}(0)\lambda(q^*)^{\lambda(q^*)}v_{i^*,q^*,p^*}^{v_{i^*,q^*,p^*}}},$$

$$\sigma := \lambda(q^*) + v_{i^*,q^*,p^*} - (s'+1)(v_{i^*,q^*,p^*} + \kappa - i^*),$$

and (6.4) approaches 0 when j tends to infinity. Indeed, the condition $s' < s_c$ implies

$$\sigma > \lambda(q^*) + v_{i^*,q^*,p^*} - (s_c+1)(v_{i^*,q^*,p^*} + \kappa - i^*) = 0.$$

This completes the proof. □

Lemma 6.2. *Let $j \geq 0$. Then*

$$\prod_{\ell=1}^{j} \prod_{m=1}^{v_{i^*,q^*,p^*}} ((\ell-1)(v_{i^*,q^*,p^*} + \kappa - i^*) + m) \geq (j v_{i^*,q^*,p^*})!. \tag{6.5}$$

Proof. The proof is similar to the one of Lemma 3.3 and is left to the reader. □

This completes the proof of the second item of Theorem 4.1

References

[1] M. Abramowitz, I. A. Stegun; *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, Dover Publications, INC., New-York, 1965.

[2] W. Balser; *Formal power series and linear systems of meromorphic ordinary differential equations*, Springer-Verlag, New-York, 2000.

[3] W. Balser, M. Loday-Richaud; Summability of solutions of the heat equation with inhomogeneous thermal conductivity in two variables, *Adv. Dyn. Syst. Appl.*, **4** (2009) (2), 159-177.

[4] M. Canalis-Durand, J. P. Ramis, R. Schäfke, Y. Sibuya; Gevrey solutions of singularly perturbed differential equations, *J. Reine Angew. Math.*, **518** (2009), 95-129.

[5] I. Fukuda; Asymptotic behavior of solutions to the generalized KdV-Burgers equation with a slowly decaying data, *J. Math. Anal. Appl.*, **480** (2019) (2).

[6] J. Gorsky, A. A. Himonas; Construction of non-analytic solutions for the generalized KdV equation, *J. Math. Anal. Appl.*, **303** (2005) (2), 522-529.

[7] J. Gorsky, A. A. Himonas, C. Holliman, G. Petronilho; The Cauchy problem for a periodic higher order KdV equation in analytic Gevrey spaces, *J. Math. Anal. Appl.*, **405** (2013) (2), 349-361.

[8] , H. Hannah, A. A. Himonas, G. Petronilho; Gevrey regularity in time for generalized KdV type equations, in: Recent Progress on Some Problems in Several Complex Variables and Partial Differential Equations, in: *Contemp. Math.*, **400** (2006), 117-127, Amer. Math. Soc., Providence, RI, 2006.

[9] H. Hannah, A. A. Himonas, G. Petronilho; Gevrey regularity of the periodic gKdV equation, *J. Differential Equations*, **250** (2011) (5), 2581-2600.

[10] P. Hilton, J. Pedersen; Catalan numbers, their generalization, and their uses, *Math. Intelligencer*, **13** (1991) (2), 64-75.

[11] D. A. Klarner; Correspondences between plane trees and binary sequences, *J. Combinatorial Theory*, **9** (1970), 401-411.

[12] A. Lastra, S. Malek; On parametric Gevrey asymptotics for some nonlinear initial value Cauchy problems, *J. Differential Equations*, **259** (2015), 5220-5270.

[13] A. Lastra, S. Malek; On parametric multisummable formal solutions to some nonlinear initial value Cauchy problems, *Adv. Differ. Equ.*, **2015** (2015), 200.

[14] A. Lastra, S. Malek, J. Sanz; On Gevrey solutions of threefold singular nonlinear partial differential equations, *J. Differential Equations*, **255** (2013), 3205-3232.

[15] A. Lastra, H. Tahara; Maillet type theorem for nonlinear totally characteristic partial differential equations, *Math. Ann.*, **377** (2020), 1603-1641.

[16] M. Loday-Richaud; *Divergent Series, Summability and Resurgence II. Simple and Multiple Summability*, Lecture Notes in Math., 2154, Springer-Verlag, 2016.

[17] S. Malek; On the summability of formal solutions of nonlinear partial differential equations with shrinkings, *J. Dyn. Control Syst.*, **13** (2007) (1), 1-13.

[18] S. Malek; On Gevrey asymptotic for some nonlinear integro-differential equations, *J. Dyn. Control Syst.*, **16** (2010) (3), 377-406.

[19] S. Malek; On the summability of formal solutions for doubly singular nonlinear partial differential equations, *J. Dyn. Control Syst.*, **18** (2012) (1), 45-82.
[20] M. Miyake; Newton polygons and formal Gevrey indices in the Cauchy-Goursat-Fuchs type equations, *J. Math. Soc. Japan*, **43** (1991)(2), 305-330.
[21] M. Miyake, A. Shirai; Convergence of formal solutions of first order singular nonlinear partial differential equations in the complex domain, *Ann. Polon. Math.*, **74** (2000) , 215-228.
[22] M. Miyake, A. Shirai; Structure of formal solutions of nonlinear first order singular partial differential equations in complex domain, *Funkcial. Ekvac.*, **48** (2005), 113-136.
[23] M. Miyake, A. Shirai; Two proofs for the convergence of formal solutions of singular first order nonlinear partial differential equations in complex domain, *Surikaiseki Kenkyujo Kokyuroku Bessatsu*, Kyoto Unviversity, **B37** (2013), 137-151.
[24] M. Nagumo; Über das Anfangswertproblem partieller Differentialgleichungen, *Jap. J. Math.*, **18** (1942), 41-47.
[25] S. Ouchi; Multisummability of formal solutions of some linear partial differential equations, *J. Differential Equations*, **185** (2002) (2), 513-549.
[26] M. E. Pliś, B. Ziemian; Borel resummation of formal solutions to nonlinear Laplace equations in 2 variables, *Ann. Polon. Math.*, **67** (1997) (1), 31-41.
[27] G. Pólya, G. Szegö; *Aufgaben und Lehrsätze aus der Analysis*, Vol. I, 125, Springer-Verlag, Berlin, Göttingen, Heidelberg, 1954.
[28] J.-P. Ramis; Dévissage Gevrey, *Astérisque*, Soc. Math. France, Paris, **59-60** (1978), 173-204.
[29] J.-P. Ramis; *Théorèmes d'indices Gevrey pour les équations différentielles ordinaires*, Mem. Amer. Math. Soc., **48**, viii+95 , 1984.
[30] P. Remy; Gevrey regularity of the solutions of some inhomogeneous semilinear partial differential equations with variable coefficients, hal-02263353, submitted for publication
[31] P. Remy; Gevrey order and summability of formal series solutions of some classes of inhomogeneous linear partial differential equations with variable coefficients, *J. Dyn. Control Syst.*, **22** (2016) (4), 693-711.
[32] P. Remy; Gevrey order and summability of formal series solutions of certain classes of inhomogeneous linear integro-differential equations with variable coefficients, *J. Dyn. Control Syst.*, **23** (2017) (4), 853-878.
[33] P. Remy; Gevrey properties and summability of formal power series solutions of some inhomogeneous linear Cauchy-Goursat problems, *J. Dyn. Control Syst.*, **26** (2020) (1), 69-108.
[34] P. Remy; On the summability of the solutions of the inhomogeneous heat equation with a power-law nonlinearity and variable coefficients, *J. Math. Anal. Appl.*, **494** (2021) (2), 124656.
[35] P. Remy; Gevrey index theorem for the inhomogeneous n-dimensional heat equation with a power-law nonlinearity and variable coefficients, *Acta Sci. Math. (Szeged)*, **87** (2021) (1-2), 163-181.
[36] P. Remy; Summability of the formal power series solutions of a certain class of inhomogeneous partial differential equations with a polynomial semilinearity and variable coefficients, *Results Math.*, **76** (2021) (3), 118.
[37] P. Remy; Gevrey regularity of the solutions of the inhomogeneous partial differential equations with a polynomial semilinearity, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM*, **115** (2021) (3), 145.
[38] P. Remy; Summability of the formal power series solutions of a certain class of inhomogeneous nonlinear partial differential equations with a single level, *J. Differential Equations*, **313** (2022), 450-502.
[39] A. Shirai; Maillet type theorem for nonlinear partial differential equations and Newton polygons, *J. Math. Soc. Japan*, **53** (2001), 565-587.
[40] A. Shirai; Convergence of formal solutions of singular first order nonlinear partial differential equations of totally characteristic type, *Funkcial. Ekvac.*, **45** (2002), 187-208.
[41] A. Shirai; A Maillet type theorem for first order singular nonlinear partial differential equations, *Publ. RIMS. Kyoto Univ.*, **39** (2093), 275-296.
[42] A. Shirai; Maillet type theorem for singular first order nonlinear partial differential equations of totally characteristic type, *Surikaiseki Kenkyujo Kokyuroku, Kyoto University*, **1431** (20025), 94-106.

[43] A. Shirai; Alternative proof for the convergence or formal solutions of singular first order nonlinear partial differential equations, *Journal of the School of Education*, Sugiyama Jogakuen University, **1** (2008), 91-102.

[44] A. Shirai; Gevrey order of formal solutions of singular first order nonlinear partial differential equations of totally characteristic type, *Journal of the School of Education*, Sugiyama Jogakuen University, **6** (2013), 159-172.

[45] A. Shirai; Maillet type theorem for singular first order nonlinear partial differential equations of totally characteristic type, part II, *Opuscula Math.*, **35** (2015) (5), 689-712.

[46] H. Tahara; Gevrey regularity in time of solutions to nonlinear partial differential equations, *J. Math. Sci. Univ. Tokyo*, **18** (2011), 67-137.

[47] H. Tahara, H. Yamazawa; Multisummability of formal solutions to the Cauchy problem for some linear partial differential equations, *J. Differential Equations*, **255** (2013) (10), 3592-3637.

[48] W. Walter; An elementary proof of the Cauchy-Kowalevsky theorem, *Amer. Math. Monthly*, **92** (1985) (2), 115-126.

[49] A. Yonemura; Newton polygons and formal Gevrey classes, *Publ. Res. Inst. Math. Sci.*, **26** (190), 197-204.

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