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PSEUDO ALMOST PERIODICITY FOR STOCHASTIC DIFFERENTIAL EQUATIONS IN INFINITE DIMENSIONS

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ABSTRACT. In this article, we introduce the concept of p-mean θ-pseudo almost periodic stochastic processes, which is slightly weaker than p-mean pseudo almost periodic stochastic processes. Using the operator semigroup theory and stochastic analysis theory, we obtain the existence and uniqueness of square-mean θ-pseudo almost periodic mild solutions for a semilinear stochastic differential equation in infinite dimensions. Moreover, we prove that the obtained solution is also pseudo almost periodic in path distribution. It is noteworthy that the ergodic part of the obtained solution is not only ergodic in square-mean but also ergodic in path distribution. Our main results are even new for the corresponding stochastic differential equations (SDEs) in finite dimensions.

1. INTRODUCTION

The theory of almost periodic functions was introduced by Bohr [5, 6, 7] in 1924-1926. Since then, many interesting generalizations of almost periodic functions appeared. The concept of pseudo almost periodic functions is among these, which was first introduced by Zhang [20, 21]. In the last two decades, pseudo almost periodic functions have been extensively investigated and have many applications in the theory of differential equations (see [1, 2, 4, 13, 16, 17, 18, 22] for example).

Recently, pseudo almost periodicity of stochastic differential equations (SDEs) has attracted more and more attention. In the random case, there are several different ways to define pseudo almost periodicity for stochastic processes, such as square-mean pseudo almost periodicity, pseudo almost periodicity in distribution (in various senses), and so on. It is difficult to study square-mean pseudo almost periodicity for SDEs since some SDEs never have square-mean pseudo almost periodic solutions (cf. [2, Example 3.1]), and thus it is reasonable to consider pseudo almost periodicity in distribution for SDEs. However, there are seldom results on pseudo almost periodicity in distribution of SDEs (cf. [16, 17, 18]); except for [16] almost all earlier works were concerned with pseudo almost periodic in distribution solutions whose ergodic part are only ergodic in p-mean rather than in distribution.

In this article, we aim to study square-mean pseudo almost periodicity and pseudo almost periodicity in distribution for the following semilinear stochastic

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differential equations in a separable Hilbert space H:

$$dX(t) = AX(t)dt + F(t, X(t))dt + G(t, X(t))dW(t), \quad t \in \mathbb{R} \tag{1.1}$$

where $A : D(A) \subset H \to H$ is a linear operator, $F : \mathbb{R} \times H \to H$, and $G : \mathbb{R} \times H \to L(H)$ are continuous.

Motivated by a recent work [15], where Raynaud de Fitte proposed a new method to study almost periodicity of equation (1.1), we introduce the concept of p-mean θ-pseudo almost periodicity and obtain the existence and uniqueness of square-mean θ-pseudo almost periodic solution to equation (1.1). Moreover, using the maximal inequality of stochastic convolution for semigroups, we show that the solution is also pseudo almost periodic in path distribution. Note that the ergodic part of pseudo almost periodic solution we obtain is not only ergodic in square-mean but also ergodic in path distribution (see Definition 2.4).

The article is organized as follows. In section 2 we introduce some notions and properties of pseudo almost periodic processes, including p-mean θ-pseudo almost periodic processes and pseudo almost periodic in distribution processes. In section 3 we first establish a convolution theorem of square-mean θ-pseudo almost periodic stochastic processes, and with its help, we obtain the existence of square-mean θ-pseudo almost periodic and pseudo almost periodic in distribution solutions.

2. Preliminaries

Let $(\mathbb{X}, d_{\mathbb{X}})$ be a complete metric space. A set $\mathbb{J} \subset \mathbb{R}$ is said to be relatively dense in $\mathbb{R}$ if there exists a constant $l > 0$ such that for any $a \in \mathbb{R}$, we have $[a, a+l] \cap \mathbb{J} \neq \emptyset$.

Definition 2.1. [12] A continuous function $f : \mathbb{R} \to \mathbb{X}$ is called almost periodic if for every $\epsilon > 0$, the set

$$P(\epsilon, f) := \{\tau \in \mathbb{R} : \sup_{t\in\mathbb{R}} d_{\mathbb{X}}(f(t+\tau), f(t)) < \epsilon\}$$

is relatively dense in $\mathbb{R}$. Denote by $AP(\mathbb{R}, \mathbb{X})$ the set of all such functions.

Let $(\mathbb{B}, \|\cdot\|)$ be a Banach space. Denote by $BC(\mathbb{R}, \mathbb{B})$ be the Banach space of all continuous and bounded functions $f : \mathbb{R} \to \mathbb{B}$ equipped with the norm $\|f\|_\infty := \sup_{s\in\mathbb{R}} \|f(s)\|$. Define

$$PAP_0(\mathbb{B}) := \big\{f \in BC(\mathbb{R}, \mathbb{B}) : \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|f(t)\| dt = 0\big\}.$$

Definition 2.2 ([19])**.** A function $f \in BC(\mathbb{R}, \mathbb{B})$ is called pseudo almost periodic if it can be expressed as $f = g + \phi$, where $g \in AP(\mathbb{R}, \mathbb{B})$ and $\phi \in PAP_0(\mathbb{B})$. Denote by $PAP(\mathbb{B})$ the set of all such functions.

The functions g and ϕ are called the almost periodic component and ergodic perturbation of the function f respectively.

Let (E, d) be a Polish space and $\mathcal{P}(E)$ be the set of all probability measures onto σ-Borel field of E. Denote by $BC(E, \mathbb{R})$ the space of bounded continuous functions $f : E \to \mathbb{R}$ with the norm $\|f\|_\infty = \sup_{x\in E} |f(x)|$. Let $f \in BC(E, \mathbb{R})$ be Lipschitz continuous, we define

$$\|f\|_L := \sup_{x\neq y} \big\{\frac{|f(x) - f(y)|}{d(x, y)}\big\}, \quad \|f\|_{BL} := \|f\|_\infty + \|f\|_L.$$

Then $(\mathcal{P}(E), d_{BL})$ is a complete metric space where

$$d_{BL}(\mu,\nu) := \sup\big\{\big|\int_E f d\mu - \int_E f d\nu\big| : \|f\|_{BL} \le 1\big\}, \quad \mu,\nu \in \mathcal{P}(E).$$

We denote by $C(\mathbb{R}, E)$ the space of all continuous functions $f : \mathbb{R} \to E$ equipped with the distance

$$d_{C(\mathbb{R},E)}(f,g) := \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\sup_{t\in[-k,k]} d(f(t),g(t))}{1 + \sup_{t\in[-k,k]} d(f(t),g(t))}.$$

Then $(C(\mathbb{R}, E), d_{C(\mathbb{R},E)})$ is a complete metric space.

Definition 2.3. [3] Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X : \mathbb{R} \times \Omega \to E$ be a stochastic process.

(a) We call that X is almost periodic in one-dimensional distribution if the mapping $t \mapsto \mathrm{law}(X(t))$ from $\mathbb{R}$ to $\mathcal{P}(E)$ is almost periodic.

(b) We call that X is almost periodic in finite-dimensional distribution, if for every finite sequence $(t_1, \dots, t_n)$, the mapping $\mathbb{R} \to \mathcal{P}(E^n)$ given by

$$t \mapsto \mathrm{law}(X(t_1 + t), \dots, X(t_n + t))$$

is almost periodic.

(c) Assume that X has continuous trajectories. We call that X is almost periodic in path distribution if the mapping $t \mapsto \mathrm{law}(X(t + \cdot))$ from $\mathbb{R}$ to $\mathcal{P}(C(\mathbb{R}, E))$ is almost periodic, where $C(\mathbb{R}, E)$ is endowed with the distance $d_{C(\mathbb{R},E)}$ and $\mathcal{P}(C(\mathbb{R}, E))$ is endowed with the distance d_{BL}.

Definition 2.4. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X : \mathbb{R} \times \Omega \to E$ be a stochastic process.

(a) We call that X is pseudo almost periodic in one-dimensional distribution if the mapping $t \mapsto \mathrm{law}(X(t))$ is continuous and there exists a stochastic process $Y : \mathbb{R} \times \Omega \to E$ which is almost periodic in one-dimensional distribution such that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} d_{BL}(\mathrm{law}(X(t)), \mathrm{law}(Y(t)))dt = 0.$$

(b) We say that X is pseudo almost periodic in finite-dimensional distribution, if there exists a stochastic process $Y : \mathbb{R} \times \Omega \to E$ which is almost periodic in finite-dimensional distribution such that, for every finite sequence $(t_1, \dots, t_n)$, the mapping $t \mapsto \mathrm{law}(X(t + t_1, \dots, X(t + t_n)))$ is continuous and

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} d_{BL}(\mathrm{law}(X(t+t_1), \dots, X(t+t_n)), \mathrm{law}(Y(t+t_1), \dots, Y(t+t_n)))dt = 0.$$

(c) Assume that X has continuous trajectories. We call that X is pseudo almost periodic in path distribution if the mapping $t \mapsto \mathrm{law}(X(t + \cdot))$ is continuous and there exists a stochastic process $Y : \mathbb{R} \times \Omega \to E$ which is almost periodic in path distribution such that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} d_{BL}(\mathrm{law}(X(t + \cdot)), \mathrm{law}(Y(t + \cdot)))dt = 0.$$

In this case, the function $Z : \mathbb{R} \to \mathbb{R}, Z(t) := d_{BL}(\mathrm{law}(X(t + \cdot)), \mathrm{law}(Y(t + \cdot))$, is called the ergodic part of X and we say that Z is ergodic in the distribution sense.

For $p \geq 1$, we denote by $L^p(\Omega, \mathbb{B})$ the space of all $\mathbb{B}$-valued random variables X such that $E\|X\|^p = \int_\Omega \|X\|^p dP < \infty$. For $X \in L^p(\Omega, \mathbb{B})$, let

$$\|X\|_{L^p} = (E\|X\|^p)^{1/p}.$$

Then $(L^p(\Omega, \mathbb{B}), \|\cdot\|_{L^p})$ is a Banach space.

Definition 2.5 ([9]). Let $(\Omega, \mathcal{F}, P)$ be a probability space. A family of measurable mappings on the sample space, $\theta_t : \Omega \to \Omega$, $t \in \mathbb{R}$, is called a measurable dynamical system if the following conditions are satisfied:

(i) identity property: $\theta_0 = Id_\Omega$,
(ii) flow property: $\theta_t \theta_s = \theta_{t+s}$, for $t, s \in \mathbb{R}$,
(iii) measurability: $(\omega, t) \mapsto \theta_t \omega$ is measurable.

It is called a measure-preserving dynamical system if, furthermore,

(iv) measure-preserving property: $P(\theta_t A) = P(A)$, for every $A \in \mathcal{F}$ and $t \in \mathbb{R}$.

In the sequel, we always assume that $\theta = (\theta_t)_{t \in \mathbb{R}}$ is a measure-preserving dynamical system.

Definition 2.6. [15] Let $X : \mathbb{R} \times \Omega \to \mathbb{B}$ be a stochastic process. Assume that $X(t) \in L^p(\Omega, \mathbb{B})$ for every $t \in \mathbb{R}$. We say that X is p-mean θ-almost periodic (or simply θ_p-almost periodic) if conditions (i) and (ii) below are satisfied:

(i) the mapping $\mathbb{R} \times \mathbb{R} \to L^p(\Omega, \mathbb{B})$ defined by $(t, s) \mapsto X(t + s, \theta_{-s}\cdot)$ is continuous,
(ii) for every $\epsilon > 0$, the set

$$P_\theta(\epsilon, X) := \left\{\tau : \sup_{t \in \mathbb{R}} (E\|X(t + \tau, \theta_{-\tau}\cdot) - X(t, \cdot)\|^p)^{1/p} \leq \epsilon\right\}$$

is relatively dense in $\mathbb{R}$.

We denote by $AP_\theta(\mathbb{R}, L^p(\Omega, \mathbb{B}))$ the set of all such processes.

If $p = 2$, then we say that X is square-mean θ-almost periodic.

Proposition 2.7 (Equicontinuity and uniform continuity [15]). Let $X : \mathbb{R} \times \Omega \to \mathbb{B}$ be a θ_p-almost periodic random process. Then

(a) the mapping $t \mapsto X(t + s, \theta_{-s}\cdot)$ is continuous from $\mathbb{R}$ to $L^p(\Omega, \mathbb{B})$, uniformly with respect to $s \in \mathbb{R}$.
(b) the mapping $s \mapsto X(t + s, \theta_{-s}\cdot)$ is uniformly continuous from $\mathbb{R}$ to $L^p(\Omega, \mathbb{B})$, uniformly with respect to $t \in \mathbb{R}$.

Proposition 2.8 (Compactness [15]). *Let $X : \mathbb{R} \times \Omega \to \mathbb{B}$ be a θ_p-almost periodic random process, and let J be a compact interval of $\mathbb{R}$. Then*

(i) *the set $\mathcal{L}_J = \{X(s + t, \theta_{-t}\cdot) : s \in J, t \in \mathbb{R}\}$ is relatively compact in $L^p(\Omega, \mathbb{B})$,*
(ii) *the set $\mathcal{S} = \{\mathrm{law}(X(t, \cdot)) : t \in \mathbb{R}\}$ is uniformly tight, that is, for each $\epsilon > 0$, there exists a compact subset K of $\mathbb{B}$ such that*

$$\sup_{t \in \mathbb{R}} P(\{\omega \in \Omega : X(t, \omega) \notin K\}) \leq \epsilon.$$

Definition 2.9. A stochastic process $X \in BC(\mathbb{R}, L^p(\Omega, \mathbb{B}))$ is said to be p-mean θ-pseudo almost periodic if it can be expressed as

$$X = Y + Z,$$

where $Y \in AP_\theta(\mathbb{R}, L^p(\Omega, \mathbb{B}))$ and $Z \in PAP_0(L^p(\Omega, \mathbb{B}))$. The processes Y and Z are called the almost periodic part and ergodic part of X respectively. Denote by

$PAP_\theta(\mathbb{R}, L^p(\Omega, \mathbb{B}))$ of all such processes. If $p = 2$, then we say that X is square-mean θ-pseudo almost periodic.

Proposition 2.10. *If $X \in PAP_\theta(\mathbb{R}, L^p(\Omega, \mathbb{B}))$, then X is pseudo almost periodic in finite-dimensional distribution.*

Proof. Without loss of generality, assume $p = 1$. Let $(t_1, \dots, t_n)$ be a finite sequence in $\mathbb{R}$. Let us endow $\mathbb{B}^n$ with the norm

$$\|(x_1, \dots, x_n)\|_n = \sum_{k=1}^{n} \|x_k\|.$$

Let $X = Y + Z$ where $Y \in AP_\theta(\mathbb{R}, L^1(\Omega, \mathbb{B}))$ and $Z \in PAP_0(L^1(\Omega, \mathbb{B}))$. Furthermore, it follows from [15, Theorem 4.1] that Y is almost periodic in finite-dimensional distribution. Since $Z = X - Y \in PAP_0(L^1(\Omega, \mathbb{B}))$ and $PAP_0(L^1(\Omega, \mathbb{B}))$ is translation invariant, we have

$$\begin{aligned}
&\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} d_{BL}(\mathrm{law}(X(t+t_1), \dots, X(t+t_n)), \mathrm{law}(Y(t+t_1), \dots, Y(t+t_n)))dt \\
&\le \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} E\|(X(t+t_1), \dots, X(t+t_n)) - (Y(t+t_1), \dots, Y(t+t_n))\|_n dt \\
&\le \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \sum_{k=1}^{n} E\|X(t+t_k) - Y(t+t_k)\| dt = 0.
\end{aligned}$$

Similarly, one can show that the mapping $t \mapsto \mathrm{law}(X(t+t_1), \dots, X(t+t_n))$ is continuous. $\square$

Proposition 2.11. Let $X_n \in PAP_\theta(\mathbb{R}, L^p(\Omega, \mathbb{B}))$, $n = 1, 2, \dots$. Assume further that there exists a stochastic process X such that

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \|X_n(t) - X(t)\|_{L^p} = 0. \tag{2.1}$$

Then $X \in PAP_\theta(\mathbb{R}, L^p(\Omega, \mathbb{B}))$.

Proof. Let $\Psi \in PAP_\theta(\mathbb{R}, L^p(\Omega, \mathbb{B}))$ and $\Psi = Y + Z$, where $Y \in AP_\theta(\mathbb{R}, L^p(\Omega, \mathbb{B}))$ and $Z \in PAP_0(L^p(\Omega, \mathbb{B}))$. Let us show that $\|Y\|_\infty \le \|\Psi\|_\infty$.

To obtain a contradiction, assume that $\|Y\|_\infty > \|\Psi\|_\infty$. Then there exists $t_0 \in \mathbb{R}$ such that $\|Y(t_0)\|_{L^p} > \sup_{t\in\mathbb{R}} \|\Psi(t)\|_{L^p}$. Let

$$\lambda = \|Y(t_0)\|_{L^p} - \sup_{t\in\mathbb{R}} \|\Psi(t)\|_{L^p} > 0. \tag{2.2}$$

By Proposition 2.7 and Definition 2.6, we deduce that there are numbers $l > 0$ and $\delta > 0$ such that any interval in $\mathbb{R}$ of length l contains a subinterval of length δ whose numbers belong to $P_\theta(\frac{\lambda}{2}, Y)$. Thus, for every $a \in \mathbb{R}$, there exists some number b such that $[b, b+\delta] \subset [a, a+l]$ and

$$\|Y(t_0 + \tau, \theta_{-\tau}\cdot) - Y(t_0)\|_{L^p} \le \frac{\lambda}{2}, \quad \tau \in [b, b+\delta].$$

Then

$$\|Y(t_0)\|_{L^p} - \frac{\lambda}{2} \le \|Y(t_0 + \tau, \theta_{-\tau}\cdot)\|_{L^p} \le \|Y(t_0)\|_{L^p} + \frac{\lambda}{2}. \tag{2.3}$$

Since θ is measure-preserving, we have $\|Y(t_0+\tau)\|_{L^p} = \|Y(t_0+\tau, \theta_{-\tau}\cdot)\|_{L^p}$. Using (2.2) and (2.3), we have

$$\|Y(t_0+\tau)\|_{L^p} - \sup_{t\in\mathbb{R}} \|\Psi(t)\|_{L^p} \geq \frac{\lambda}{2}.$$

Hence

$$\|Z(t_0+\tau)\|_{L^p} = \|\Psi(t_0+\tau) - Y(t_0+\tau)\|_{L^p} \geq \frac{\lambda}{2}.$$

This implies that

$$\limsup_{n\to\infty} \frac{1}{2nl} \int_{-nl}^{nl} \|Z(t)\|_{L^p} dt \geq \frac{\lambda\delta}{2l}.$$

But $Z \in PAP_0(L^p(\Omega, \mathbb{B}))$, and we have a contradiction.

Now, assume that $X_n = Y_n + Z_n$, $n = 1, 2, \dots$, where $Y_n \in AP_\theta(\mathbb{R}, L^p(\Omega, \mathbb{B}))$ and $Z_n \in PAP_0(L^p(\Omega, \mathbb{B}))$. By (2.1), we obtain that $(X_n)_{n=1}^\infty$ is a Cauchy sequence in $BC(\mathbb{R}, L^p(\Omega, \mathbb{B}))$. Thus we deduce that $(Y_n)_{n=1}^\infty$ and $(Z_n)_{n=1}^\infty$ are Cauchy sequences in $BC(\mathbb{R}, L^p(\Omega, \mathbb{B}))$ since

$$\|Y_n - Y_m\|_\infty \leq \|X_n - X_m\|_\infty$$

and $Z_n = X_n - Y_n$ for $n, m \in \mathbb{N}^+$. Denote by $\widetilde{Y}$ and $\widetilde{Z}$ the limits of $(Y_n)_{n=1}^\infty$ and $(Z_n)_{n=1}^\infty$, respectively. Then $\widetilde{Y} \in AP_\theta(\mathbb{R}, L^p(\Omega, \mathbb{B}))$ and $\widetilde{Z} \in PAP_0(L^p(\Omega, \mathbb{B}))$ because $AP_\theta(\mathbb{R}, L^p(\Omega, \mathbb{B}))$ and $PAP_0(L^p(\Omega, \mathbb{B}))$ is closed in $BC(\mathbb{R}, L^p(\Omega, \mathbb{B}))$. It is easy to see that $X = \widetilde{Y} + \widetilde{Z}$, hence $X \in PAP_\theta(\mathbb{R}, L^p(\Omega, \mathbb{B}))$. $\square$

3. Main results

In this section, H is a separable Hilbert space, $\Omega = \mathrm{C}(\mathbb{R}, H)$ is endowed with the compact-open topology, $\mathcal{F}$ is the Borel σ-algebra of Ω, and P is the Wiener measure on Ω with trace class covariance operator Q, and the process W with values in H defined by

$$W(t, \omega) = \omega(t), \quad \omega \in \Omega, t \in \mathbb{R}$$

is a Brownian motion with covariance operator Q. Let $(\mathcal{F}_t)_{t\in\mathbb{R}}$ be the augmented natural filtration of W. We refer to [8, 14] for more information about stochastic integration and stochastic equations in Hilbert spaces. Let $L(H)$ be the Banach space of continuous linear operators from H to itself with the operator norm $\|\cdot\|_{L(H)}$. Define $\theta = (\theta_t)_{t\in\mathbb{R}}$ by

$$\theta_\tau(\omega)(t) = \omega(t+\tau) - \omega(\tau) = W(t+\tau, \omega) - W(\tau, \omega)$$

for all $\tau, t \in \mathbb{R}$ and $\omega \in \Omega$. Then by Definition 2.5, $\theta = (\theta_t)_{t\in\mathbb{R}}$ is a measure-preserving dynamical system.

To study equation (1.1), we first list our assumptions:

(H1) A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ and there exists $\delta > 0$ such that

$$\|T(t)\|_{L(H)} \leq e^{-\delta t}, \quad t \geq 0.$$

(H2) There exists a constant $K > 0$ such that $F : \mathbb{R} \times H \to H$ and $G : \mathbb{R} \times H \to L(H)$ satisfy, for every $t \in \mathbb{R}$ and $x, y \in H$,

$$\|F(t,x)\| + \|G(t,x)\|_{L(H)} \leq K(1 + \|x\|),$$
$$\|F(t,x) - F(t,y)\| + \|G(t,x) - G(t,y)\|_{L(H)} \leq K\|x - y\|.$$

(H3) The functions F, G is pseudo almost periodic in $t \in \mathbb{R}$ for each $x \in H$, that is, the mappings $F(\cdot, x) : \mathbb{R} \to H$ and $G(\cdot, x) : \mathbb{R} \to L(H)$ are pseudo almost periodic for every $x \in H$.

Definition 3.1. A H-valued $\mathcal{F}_t$-progressively measurable stochastic process $X(t)$, $t \in \mathbb{R}$ is called the mild solution of equation (1.1) if it satisfies

$$X(t) = T(t-s)X(s) + \int_s^t F(r, X(r))\, dr + \int_s^t G(r, X(r))\, dW(r)$$

for $t, s \in \mathbb{R}$ with $t \geq s$.

Next, we give some technical lemmas for later use.

Lemma 3.2 ([2])**.** *Let $h \in PAP_0(\mathbb{R})$. Then the function*

$$t \mapsto \Big(\int_{-\infty}^t e^{-\delta(t-s)} h^2(s) ds\Big)^{1/2}$$

is also in $PAP_0(\mathbb{R})$.

Lemma 3.3. *Let $g \in PAP_0(\mathbb{R})$. Then for every $T > 0$, the function*

$$t \mapsto \int_{t-T}^{t+T} g(\sigma) d\sigma$$

is also in $PAP_0(\mathbb{R})$.

Proof. Since $PAP_0(\mathbb{R})$ is translation invariant, $g(\cdot + t) \in PAP_0(\mathbb{R})$ for every $t \in \mathbb{R}$. Then, by Lebesgue's dominated convergence theorem, we obtain

$$\begin{aligned} \frac{1}{2r} \int_{-r}^{r} \int_{t-T}^{t+T} g(\sigma) d\sigma dt &= \frac{1}{2r} \int_{-r}^{r} \int_{-T}^{T} g(\sigma + t) d\sigma dt \\ &= \frac{1}{2r} \int_{-T}^{T} \int_{-r}^{r} g(\sigma + t) dt d\sigma \\ &= \int_{-T}^{T} \Big(\frac{1}{2r} \int_{-r}^{r} g(\sigma + t) dt\Big) d\sigma \to 0 \quad \text{as } r \to \infty. \end{aligned}$$ □

Define the operator $\psi : BC(\mathbb{R}, L^p(\Omega, H)) \to BC(\mathbb{R}, L^p(\Omega, H))$ by

$$(\psi X)(t) = \int_{-\infty}^t T(t-s)F(s, X(s))\, ds + \int_{-\infty}^t T(t-s)G(s, X(s)) dW(s).$$

It is easy to see that ψ is well defined.

Theorem 3.4. *Assume that conditions* (H1)–(H3) *hold. Then the operator ψ maps $PAP_\theta(\mathbb{R}, L^2(\Omega, H))$ into itself.*

Proof. Let $F = F_1 + F_2$, where $F_1(\cdot, x)$ and $F_2(\cdot, x)$ are the almost periodic component and ergodic perturbation of the function $F(\cdot, x)$ for every $x \in H$. Let $G = G_1 + G_2$, where $G_1(\cdot, x)$ and $G_2(\cdot, x)$ are the almost periodic component and ergodic perturbation of the function $G(\cdot, x)$ for every $x \in H$. By Lemma 5.2 in [19, Page 57], the functions F_1, G_1 satisfy assumption (H2). Consequently, the functions F_2, G_2 satisfy assumption (H2) with constant $2K$.

Let $X \in PAP_\theta(\mathbb{R}, L^2(\Omega, H))$ and $X = Y + Z$, where $Y \in AP_\theta(\mathbb{R}, L^2(\Omega, H))$ and $Z \in PAP_0(L^2(\Omega, H))$. By [15, Proposition 5.1], we obtain $\psi_1 Y \in AP_\theta(\mathbb{R}, L^2(\Omega, H))$, where

$$(\psi_1 Y)(t) = \int_{-\infty}^t T(t-s)F_1(s, Y(s))\, ds + \int_{-\infty}^t T(t-s)G_1(s, Y(s))\, dW(s).$$

Now, we prove that $\psi X - \psi_1 Y \in PAP_0(L^2(\Omega, H))$. By definition of ψ and ψ_1, we have

$$\begin{aligned}
&\psi X(t) - \psi_1 Y(t)\\
&= \int_{-\infty}^t T(t-s)[F(s, X(s)) - F_1(s, Y(s))]\, ds\\
&\quad + \int_{-\infty}^t T(t-s)[G(s, X(s) - G_1(s, Y(s))]\, dW(s)\\
&= \int_{-\infty}^t T(t-s)[F(s, X(s)) - F(s, Y(s))] ds + \int_{-\infty}^t T(t-s)F_2(s, Y(s))\, ds\\
&\quad + \int_{-\infty}^t T(t-s)[G(s, X(s) - G(s, Y(s))]\, dW(s)\\
&\quad + \int_{-\infty}^t T(t-s)G_2(s, Y(s))\, dW(s)\\
&=: I_1(t) + I_2(t) + I_3(t) + I_4(t).
\end{aligned}$$

Since $Z \in PAP_0(L^2(\Omega, H))$, we have that $s \mapsto (E\|Z(s)\|^2)^{1/2}$ is in $PAP_0(\mathbb{R})$. Moreover, using Lemma 3.2, we deduce that $t \mapsto (\int_{-\infty}^t e^{-\delta(t-s)} E\|Z(s)\|^2 ds)^{1/2}$ is also in $PAP_0(\mathbb{R})$. Then, using conditions (H1), (H2), and Hölder's inequality, we obtain

$$\begin{aligned}
&\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^r \big(E\|I_1(t)\|^2\big)^{1/2}\, dt\\
&\le \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^r \Big(E(\int_{-\infty}^t e^{-\delta(t-s)} K\|X(s) - Y(s)\| ds)^2\Big)^{1/2} dt\\
&\le K\frac{1}{\sqrt{\delta}} \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^r \Big(E\int_{-\infty}^t e^{-\delta(t-s)} \|X(s) - Y(s)\|^2 ds\Big)^{1/2} dt\\
&\le K\frac{1}{\sqrt{\delta}} \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^r \Big(\int_{-\infty}^t e^{-\delta(t-s)} E\|Z(s)\|^2 ds\Big)^{1/2} dt = 0.
\end{aligned}$$

This implies that $I_1 \in PAP_0(L^2(\Omega, H))$.

Furthermore, using Itô's isometry, we have

$$\begin{aligned}
&\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^r \big(E\|I_3(t)\|^2\big)^{1/2}\, dt\\
&\le (trQ)^{1/2} \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^r \Big(E\int_{-\infty}^t e^{-2\delta(t-s)} \|G(s, X(s)) - G(s, Y(s))\|_{L(H)}^2 ds\Big)^{1/2} dt\\
&\le K(trQ)^{1/2} \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^r \Big(\int_{-\infty}^t e^{-2\delta(t-s)} E\|X(s) - Y(s)\|^2 ds\Big)^{1/2} dt\\
&\le K(trQ)^{1/2} \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^r \Big(\int_{-\infty}^t e^{-2\delta(t-s)} E\|Z(s)\|^2 ds\Big)^{1/2} dt = 0.
\end{aligned}$$

Thus $I_3 \in PAP_0(L^2(\Omega, H))$.

Let us estimate the second term I_2. By [19, Lemma 5.10], $I_2 \in PAP_0(L^2(\Omega, H))$ if and only if the mapping

$$t \mapsto E\| \int_{-\infty}^{t} T(t-s)F_2(s, Y(s))ds\|^2$$

is in $PAP_0(\mathbb{R})$. Using condition (H1) and Hölder's inequality, we have

$$\begin{aligned}
E\| \int_{-\infty}^{t} T(t-s)F_2(s, Y(s))ds\|^2 &\le E(\int_{-\infty}^{t} e^{-\delta(t-s)}\|F_2(s, Y(s))\|ds)^2 \\
&\le \frac{1}{\delta}\int_{-\infty}^{t} e^{-\delta(t-s)}E\|F_2(s, Y(s))\|^2 ds.
\end{aligned}$$

Using Proposition 2.8, we have that the family $(Y(s, \theta_{-s}\cdot))_{s\in\mathbb{R}}$ is uniformly square integrable. Thus the family $(Y(s,\cdot))_{s\in\mathbb{R}}$ is uniformly square integrable since θ is measure-preserving. Then for every $\epsilon > 0$, there exists a constant $\vartheta \in (0, \epsilon)$ such that for every $A \in \mathcal{F}$ with $P(A) < \vartheta$, we have

$$\sup_{s\in\mathbb{R}} E\|Y(s)\|^2 1_A < \epsilon. \tag{3.1}$$

Moreover, the set $\{\mathrm{law}(Y(s,\cdot)) : s \in \mathbb{R}\}$ is uniformly tight. Then there exists a compact subset $\mathcal{K} \subset \mathbb{B}$ such that

$$\sup_{s\in\mathbb{R}} P(Y(s) \in \mathcal{K}) > 1 - \vartheta. \tag{3.2}$$

Since $\mathcal{K}$ is compact, there are points $x_1, x_2, \dots, x_J \in \mathcal{K}$ such that $\mathcal{K}$ is covered by the balls of radius ϵ with centers at the points $x_i, i = 1, 2, \dots, J$. We denote the set $\{Y(s,\cdot) \in \mathcal{K}\}$ by A_s and $\{Y(s,\cdot) \notin \mathcal{K}\}$ by A_s^c. Then using (3.1) and (3.2), we have

$$\begin{aligned}
\Delta_1 &:= \frac{1}{\delta}\int_{-\infty}^{t} e^{-\delta(t-s)}E\left(\|F_2(s, Y(s))\|^2 1_{A_s}\right)ds \\
&\le \frac{1}{\delta}\int_{-\infty}^{t} e^{-\delta(t-s)}(2K\epsilon + \sum_{i=1}^{J}\|F_2(s, x_i)\|)^2 ds \\
&\le \frac{2}{\delta}\int_{-\infty}^{t} e^{-\delta(t-s)}(4K^2\epsilon^2 + J\sum_{i=1}^{J}\|F_2(s, x_i)\|^2)ds \\
&\le \frac{8K^2}{\delta^2}\epsilon^2 + \frac{2J}{\delta}\sum_{i=1}^{J}\int_{-\infty}^{t} e^{-\delta(t-s)}\|F_2(s, x_i)\|^2 ds
\end{aligned}$$

and

$$\begin{aligned}
\Delta_2 &:= \frac{1}{\delta}\int_{-\infty}^{t} e^{-\delta(t-s)}E\left(\|F_2(s, Y(s))\|^2 1_{A_s^c}\right)ds \\
&\le 4K^2\frac{1}{\delta}\int_{-\infty}^{t} e^{-\delta(t-s)}E\left((1 + \|Y(s)\|)^2 1_{A_s^c}\right)ds \\
&\le 8K^2\frac{1}{\delta}\int_{-\infty}^{t} e^{-\delta(t-s)}E\left((1 + \|Y(s)\|^2)1_{A_s^c}\right)ds \\
&\le 8K^2\frac{1}{\delta^2}\epsilon + 8K^2\frac{1}{\delta^2}\epsilon = 16K^2\frac{1}{\delta^2}\epsilon.
\end{aligned}$$

For every $x_i, i = 1, 2, \dots, J$, $F_2(\cdot, x_i) \in PAP_0(H)$. By Lemma 3.2 and [19, Lemma 5.10], the function

$$t \mapsto \int_{-\infty}^{t} e^{-\delta(t-s)} \|F_2(s, x_i)\|^2 ds$$

is in $PAP_0(\mathbb{R})$. Thus

$$\begin{aligned}
&\limsup_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} E\| \int_{-\infty}^{t} T(t-s)F_2(s, Y(s))ds\|^2 dt \\
&\leq \limsup_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} (\Delta_1 + \Delta_2) dt \\
&\leq C\epsilon + \frac{2J}{\delta} \sum_{i=1}^{J} \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \int_{-\infty}^{t} e^{-\delta(t-s)} \|F_2(s, x_i)\|^2 ds dt \\
&= C\epsilon,
\end{aligned}$$

where C is a constant. Since ϵ is arbitrary, we have

$$\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} E\| \int_{-\infty}^{t} T(t-s)F_2(s, Y(s))\, ds\|^2 dt = 0.$$

Hence $I_2 \in PAP_0(L^2(\Omega, H))$.

By [19, Lemma 5.10] again, $I_4 \in PAP_0(L^2(\Omega, H))$ if and only if the mapping

$$t \mapsto E\| \int_{-\infty}^{t} T(t-s)G_2(s, Y(s))dW(s)\|^2$$

is in $PAP_0(\mathbb{R})$. Using Itô's isometry, we obtain

$$E\| \int_{-\infty}^{t} T(t-s)G_2(s, Y(s))dW(s)\|^2 \leq (trQ) \int_{-\infty}^{t} e^{-2\delta(t-s)} E\|G_2(s, Y(s))\|^2_{L(H)} ds.$$

For the same reason as for I_2, we have $I_4 \in PAP_0(L^2(\Omega, H))$.

Gathering the estimates for I_1-I_4, we deduce that $\psi X - \psi_1 Y \in PAP_0(L^2(\Omega, H))$. Thus $\psi X \in PAP_\theta(\mathbb{R}, L^2(\Omega, H))$. □

Theorem 3.5. *Assume that conditions* (H1)–(H3) *hold. If* $\eta = \frac{2K^2}{\delta^2} + \frac{K^2}{\delta} trQ < 1$, *then* (1.1) *has a unique* L^2*-bounded mild solution* X *which satisfies, for every* $t \in \mathbb{R}$,

$$X(t) = \int_{-\infty}^{t} T(t-s)F(s, X(s))ds + \int_{-\infty}^{t} T(t-s)G(s, X(s))dW(s).$$

Moreover, $X \in PAP_\theta(\mathbb{R}, L^2(\Omega, H))$ *and* X *is pseudo almost periodic in path distribution.*

Proof. The proof of the existence and uniqueness of a mild solution to (1.1) in the functions space $BC(\mathbb{R}, L^2(\Omega, H))$ is based on Banach fixed point theorem, which is the same as that of Theorem 3.1 in [11]. Using the factorization method (see section 5.3 in [8] or section 3.2 in [10]), we deduce that X has a continuous version. Moreover, we have

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}} E\|X_n(t) - X(t)\|^2 = 0,$$

where $X_0 = 0$ and $X_n = \psi X_{n-1}, n = 1, 2, \dots$. By Proposition 2.11 and Theorem 3.4, we deduce that $X \in PAP_\theta(\mathbb{R}, L^2(\Omega, H))$.

Now, let us prove that X is also pseudo almost periodic in path distribution. Let $F = F_1 + F_2$, where $F_1(\cdot, x)$ and $F_2(\cdot, x)$ are the almost periodic component and ergodic perturbation of the function $F(\cdot,x)$ for every $x\in H$. Let $G=G_1+G_2$, where $G_1(\cdot,x)$ and $G_2(\cdot,x)$ are the almost periodic component and ergodic perturbation of the function $G(\cdot,x)$ for every $x\in H$.

By [19, Lemma 5.2], the functions F_1,G_1 satisfy assumption (H2). Consequently, the functions F_2,G_2 satisfy assumption (H2) with constant $2K$. By [15, Theorem 5.1], there exists a stochastic process Y which is the mild solution of

$$dY(t)=AY(t)dt+F_1(t,Y(t))dt+G_1(t,Y(t))dW(t).$$

In other words, Y satisfies for every $t,s\in\mathbb{R}$ with $t\geq s$,

$$Y(t)=T(t-s)Y(s)+\int_s^t T(t-\sigma)F_1(\sigma,Y(\sigma))\,d\sigma+\int_s^t T(t-\sigma)G_1(\sigma,Y(\sigma))dW(\sigma).$$

Moreover, $Y\in AP_\theta(\mathbb{R},L^2(\Omega,H))$ and Y is also almost periodic in path distribution.

Step 1. For every positive integer N, the mapping

$$t\mapsto E\sup_{s\in[t-N,t+N]}\|X(s)-Y(s)\|^2$$

is in $PAP_0(\mathbb{R})$. By the definition of mild solutions and condition (H1), we have

$$\begin{aligned}
&E\sup_{s\in[t-N,t+N]}\|X(s)-Y(s)\|^2\\
&\leq 3E\sup_{s\in[t-N,t+N]}\|T(s-(t-N))X(t-N)-T(s-(t-N))Y(t-N)\|^2\\
&\quad+3E\sup_{s\in[t-N,t+N]}\|\int_{t-N}^s T(s-\sigma)F(\sigma,X(\sigma))-T(s-\sigma)F_1(\sigma,Y(\sigma))d\sigma\|^2\\
&\quad+3E\sup_{s\in[t-N,t+N]}\|\int_{t-N}^s T(s-\sigma)G(\sigma,X(\sigma))-T(s-\sigma)G_1(\sigma,Y(\sigma))dW(\sigma)\|^2\\
&\leq 3E\|X(t-N)-Y(t-N)\|^2\\
&\quad+3E\Big(\int_{t-N}^{t+N}\|F(\sigma,X(\sigma))-F_1(\sigma,Y(\sigma))\|d\sigma\Big)^2\\
&\quad+3E\sup_{s\in[t-N,t+N]}\|\int_{t-N}^s T(s-\sigma)G(\sigma,X(\sigma))-T(s-\sigma)G_1(\sigma,Y(\sigma))dW(\sigma)\|^2\\
&=:\Sigma_1(t)+\Sigma_2(t)+\Sigma_3(t).
\end{aligned}$$

By Theorem 3.4, we have $Z=X-Y\in PAP_0(L^2(\Omega,H))$. Then by [19, Lemma 5.10], we have $E\|Z(\cdot)\|^2\in PAP_0(\mathbb{R})$. Since $PAP_0(\mathbb{R})$ is translation invariant, we deduce that $\Sigma_1\in PAP_0(\mathbb{R})$.

Using condition (H2) and Hölder's inequality, we obtain

$$\begin{aligned}
\Sigma_2(t)&=3E\Big(\int_{t-N}^{t+N}\|F(\sigma,X(\sigma))-F_1(\sigma,Y(\sigma))\|\,d\sigma\Big)^2\\
&\leq 12N^2E\int_{t-N}^{t+N}\|F(\sigma,X(\sigma))-F_1(\sigma,Y(\sigma))\|^2d\sigma\\
&\leq 24N^2E\int_{t-N}^{t+N}\|F(\sigma,X(\sigma))-F(\sigma,Y(\sigma))\|^2+\|F_2(\sigma,Y(\sigma))\|^2d\sigma
\end{aligned}$$

$$\leq 24K^2N^2E\int_{t-N}^{t+N}\|X(\sigma)-Y(\sigma)\|^2d\sigma+24N^2E\int_{t-N}^{t+N}\|F_2(\sigma,Y(\sigma))\|^2d\sigma.$$

For every $\epsilon>0$, let $\vartheta,\mathcal{K},x_1,x_2,\dots,x_J$ be the same as in Theorem 3.4. Then by a similar argument of I_2 in Theorem 3.4, we have

$$24N^2E\int_{t-N}^{t+N}\|F_2(\sigma,Y(\sigma))\|^2d\sigma\leq\widetilde{C}\epsilon+2J\sum_{i=1}^{J}\int_{t-N}^{t+N}\|F_2(\sigma,x_i)\|^2d\sigma,$$

where $\widetilde{C}$ is a constant. By Lemma 3.3, we have that the mappings

$$t\mapsto E\int_{t-N}^{t+N}\|X(\sigma)-Y(\sigma)\|^2d\sigma\quad\text{and}\quad t\mapsto\int_{t-N}^{t+N}\|F_2(\sigma,x_i)\|^2d\sigma$$

are in $PAP_0(\mathbb{R})$. Since ϵ is arbitrary, we deduce that $\Sigma_2\in PAP_0(\mathbb{R})$.

By [8, Theorem 6.10], for every $L>0$, there exists a constant C_L such that, for every $a\in\mathbb{R}$ and every predictable stochastic process Φ with $E\int_a^{a+L}\|\Phi\|^2ds<\infty$, we have

$$E\sup_{t\in[a,a+L]}\|\int_a^t T(t-s)\Phi(s)\,dW(s)\|^2\leq C_LE\int_a^{a+L}\|\Phi(s)\|^2ds.$$

Then we obtain

$$\Sigma_3(t)\leq 3C_nE\int_{t-N}^{t+N}\|G(\sigma,X(\sigma))-G_1(\sigma,Y(\sigma))\|^2d\sigma.$$

For the same reason as for Σ_2, we have $\Sigma_3\in PAP_0(\mathbb{R})$.

Gathering the estimations for Σ_1-Σ_3, we conclude that the mapping

$$t\mapsto E\sup_{s\in[t-N,t+N]}\|X(s)-Y(s)\|^2$$

is in $PAP_0(\mathbb{R})$.

Step 2. We claim that

$$\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}d_{BL}(\mathrm{law}(X(t+\cdot)),\mathrm{law}(Y(t+\cdot)))dt=0.$$

For every positive integer N, we have

$$\begin{aligned}
&d_{BL}(\mathrm{law}(X(t+\cdot)),\mathrm{law}(Y(t+\cdot)))\\
&\leq\int_\Omega\sum_{k=1}^{\infty}\frac{1}{2^k}\frac{\sup_{s\in[-k,k]}\|X(t+s)-Y(t+s)\|}{1+\sup_{s\in[-k,k]}\|X(t+s)-Y(t+s)\|}dP\\
&\leq\sum_{k=1}^{N}\int_\Omega\frac{1}{2^k}\sup_{s\in[-k,k]}\|X(t+s)-Y(t+s)\|dP+\sum_{k=N+1}^{\infty}\frac{1}{2^k}\\
&\leq N\Big(E\sup_{s\in[t-N,t+N]}\|X(s)-Y(s)\|^2\Big)^{1/2}+\sum_{k=N+1}^{\infty}\frac{1}{2^k}.
\end{aligned}$$

Let $\epsilon>0$. Choose N large enough such that $\sum_{k=N+1}^{\infty}\frac{1}{2^k}<\epsilon$. Then by Step 1 and [19, Lemma 5.10], we have

$$\limsup_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}d_{BL}(\mathrm{law}(X(t+\cdot)),\mathrm{law}(Y(t+\cdot)))dt$$

$$\le \lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r} N\Big(E\sup_{s\in[t-N,t+N]}\|X(s)-Y(s)\|^2\Big)^{1/2}dt+\epsilon=\epsilon.$$

Since ϵ is arbitrary, we obtain

$$\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r} d_{BL}(\mathrm{law}(X(t+\cdot)),\mathrm{law}(Y(t+\cdot)))dt=0.$$

Step 3. Let us show that $t\mapsto \mathrm{law}(X(t+\cdot))$ is continuous. For every positive integer N, every $t_0\in\mathbb{R}$ and every $t\in\mathbb{R}$ with $|t-t_0|<1$,

$$\begin{aligned}
&d_{BL}(\mathrm{law}(X(t+\cdot)),\mathrm{law}(X(t_0+\cdot)))\\
&\le \sum_{k=1}^{N}\int_\Omega \frac{1}{2^k}\sup_{s\in[-k,k]}\|X(t+s)-X(t_0+s)\|dP+\sum_{k=N+1}^{\infty}\frac{1}{2^k}\\
&\le N\int_\Omega \sup_{s\in[-N,N]}\|X(t+s)-X(t_0+s)\|dP+\sum_{k=N+1}^{\infty}\frac{1}{2^k}.
\end{aligned}$$

Let $\epsilon>0$. Choose N large enough such that $\sum_{k=N+1}^{\infty}\frac{1}{2^k}<\frac{\epsilon}{2}$. By a simple calculation we obtain

$$E\sup_{s\in[t_0-N-1,t_0+N+1]}\|X(s)\|^2<\infty.$$

Thus $E\sup_{s\in[t_0-N-1,t_0+N+1]}2\|X(s)\|<\infty$. Since X has continuous trajectories and

$$\sup_{s\in[-N,N]}\|X(t+s)-X(t_0+s)\|\le \sup_{s\in[t_0-N-1,t_0+N+1]}2\|X(s)\|,$$

we have $\int_\Omega \sup_{s\in[-N,N]}\|X(t+s)-X(t_0+s)\|dP\to 0$ as $t\to t_0$ by Dominated convergence theorem. Then there exists a constant $h>0$ such that $|t-t_0|<h$,

$$\int_\Omega \sup_{s\in[-N,N]}\|X(t+s)-X(t_0+s)\|dP<\frac{\epsilon}{2}.$$

Hence $d_{BL}(\mathrm{law}(X(t+\cdot)),\mathrm{law}(X(t_0+\cdot)))<\epsilon$. This implies that $t\mapsto \mathrm{law}(X(t+\cdot))$ is continuous. Then by Definition 2.4, X is pseudo almost periodic in path distribution. □

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