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PRINCIPAL EIGENVALUES FOR THE FRACTIONAL p-LAPLACIAN WITH UNBOUNDED SIGN-CHANGING WEIGHTS

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Abstract. Let Ω be a bounded regular domain of $\mathbb{R}^N$, $N \geqslant 1$, $p \in (1, +\infty)$, and $s \in (0, 1)$. We consider the eigenvalue problem

$$(-\Delta_p)^s u + V|u|^{p-2}u = \lambda m(x)|u|^{p-2}u \quad \text{in } \Omega$$
$$u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

where the potential V and the weight m are possibly unbounded and are sign-changing. After establishing the boundedness and regularity of weak solutions, we prove that this problem admits principal eigenvalues under certain conditions. We also show that when such eigenvalues exist, they are simple and isolated in the spectrum of the operator.

1. Introduction

For $p \in (1, +\infty)$ and $s \in (0, 1)$, the fractional (s, p)-Laplacian is an extension of the s-fractional Laplacian and it is defined, for a regular function $u : \mathbb{R}^N \to \mathbb{R}$, as

$$(-\Delta_p)^s u(x) := 2\mathcal{K}(1-s)\,\mathrm{P.V.}\Big(\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}\big(u(x)-u(y)\big)}{|x-y|^{N+sp}}dy\Big)$$

for all $x \in \mathbb{R}^N$ with

$$\mathcal{K} = p\Big(\int_{S^{N-1}} |\langle \omega, e\rangle|^p d\mathscr{H}^{N-1}(\omega)\Big)^{-1}, \quad e \in S^{N-1},$$

where $\mathscr{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure of the unit sphere S^{N-1} of $\mathbb{R}^N$. Let us recall that for all measurable function on a subset D of $\mathbb{R}^N$ and for all $x \in \mathbb{R}^N$, the principal value function on the integral $\int_D \Psi(x, y)dy$ is denoted by

$$\mathrm{P.V.}\Big(\int_D \Psi(x,y)dy\Big) := \lim_{\varepsilon\to 0} \int_{D\setminus B_\varepsilon(x)} \Psi(x,y)dx,$$

where $B_\varepsilon(x)$ is a ball centered at $x \in \mathbb{R}^N$ with radius $\varepsilon > 0$.

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In this article, we study the conditions under which the principal eigenvalues of the following homogeneous Dirichlet problem exist

$$\begin{gathered}(-\Delta_p)^s u + V|u|^{p-2}u = \lambda m(x)|u|^{p-2}u \quad \text{in } \Omega,\\ u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,\end{gathered} \tag{1.1}$$

where Ω is a bounded regular domain of $\mathbb{R}^N$, V and m are indefinite sign-changing functions and satisfying the following conditions:

(C1) V, $m \in L^r(\Omega)$ with $r \in (1, +\infty) \cap (\frac{N}{sp}, +\infty)$,

(C2) $m^+ = \max(m, 0) \not\equiv 0$.

Our aim is to extend some results obtained by Del-Pezzo et al. in [12] for the eigenvalue problem (1.1). These authors studied, among other issues, the existence of eigenvalues, the positivity of the eigenfunctions associated with the first eigenvalue of (1.1) with $m \equiv 1$ and V satisfying (C1). We want here to address the question of existence of principal eigenvalue in a wide range of weights, precisely when m and V changing sign. The presence of such weights in problem (1.1) brings us to proceed by a considerably different approach called "eigencurve arguments" which requires the construction of some equivalent problem.

To illustrate this eigencurve argument, let us mention the work of Fleckinger et al. [15], where the following eigenvalue problem is considered.

$$-\Delta u + a_0(x)u = \lambda m(x)u, \quad \text{in } \Omega,\ u = 0 \quad \text{on } \partial\Omega \tag{1.2}$$

with Ω a bounded smooth domain, $a_0, m \in L^r(\Omega)$, $r > \frac{N}{2}$ are indefinite and m is unbounded. After separating the positive and negative parts of a_0 and m one find equation (1.2) as

$$-\Delta u + a_0^+(x)u + \lambda m^-(x)u = \lambda m^+(x)u + a_0^-(x)u. \tag{1.3}$$

So, for any fixed λ, they were led to study the following eigenvalue problem of eigenvalue parameter $\sigma(\lambda)$,

$$\begin{gathered}-\Delta u + (a_0^+(x)+1)u + \lambda m^-(x)u = \sigma(\lambda)\Big(m^+(x) + \frac{a_0^-(x)+1}{\lambda}\Big)u \quad \text{in } \Omega,\\ u = 0 \quad \text{on } \partial\Omega\,.\end{gathered}$$

It is clear that $\lambda > 0$ is an eigenvalue of (1.2) if and only if $\sigma(\lambda) = \lambda$. For this purpose, they studied the properties of continuity, concavity and monotonicity of the curve $\lambda \mapsto \sigma(\lambda)$ and they proved that, under certain conditions, the existence of $\lambda > 0$ satisfies $\sigma(\lambda) = \lambda$. For more details see [15].

Our construction of the equivalent problem is different from the one made in [15] and it is closer to the one used by Binding and Huang [3]. These authors considered, for bounded potential V and bounded weight m, the principal eigencurve $\mu_1(\lambda)$, that is, $\mu_1(\lambda)$ is the principal eigenvalue of

$$-\Delta_p u + (V(x) - \lambda m(x))|u|^{p-2}u = \mu_1(\lambda)|u|^{p-2}u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

and deduced the existence of $\lambda \in \mathbb{R}$ such that $\mu_1(\lambda) = 0$ under some conditions on V and m. This technique has generated several results which have enriched the scientific literature (see for example [2, 3, 9, 19, 21]). For instance, recently [8] made use of such arguments when solving the above problem for a potential V and a weight function m that may change sign and may be unbounded. They looked and established additional conditions on V and m that guarantee the existence of principal eigenvalues. In this work, our main results extend those of [8] and references therein to the fractional p-Laplacian.

This article is organized as follows. We start by recalling some basic properties of essential the fractional Sobolev spaces in Section 2. In Section 3 we prove the boundedness and regularity of the weak solutions. Section 4 is devoted to the existence of principal eigenvalues. In Section 5, we show that when principal eigenvalues exist, they are isolated in the spectrum and we give a lower bound of the measure of the nodal domains for changing sign eigenfunctions. Finally in Section 6 we prove some sort of continuity of the principal eigenvalues when varying s. We collect in appendix the proof of a discrete version of some well known identity as well as a regularity result for more general equations involving the fractional p-Laplacian with unbounded terms.

2. Preliminaries

The Lebesgue measure of a Lebesgue measurable set $Z \subset \mathbb{R}^N$ is denoted by $|Z|$.

2.1. **Basic results about fractional Sobolev spaces.**

Let $p \in [1, +\infty)$, $s \in (0,1)$ and let $\Omega \subset \mathbb{R}^N$ be an open set.

- The (s,p)-fractional Sobolev space, denoted by $W^{s,p}(\Omega)$, is defined by
$$W^{s,p}(\Omega) := \Big\{ u \in L^p(\Omega) : \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\, dx\, dy < +\infty \Big\}.$$
The space $W^{s,p}(\Omega)$ is a separable Banach space endowed with the norm
$$\|u\|_{W^{s,p}(\Omega)} := \Big(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\, dx\, dy + \int_\Omega |u|^p dx \Big)^{1/p}.$$
$W^{s,p}(\Omega)$ is reflexive if $p > 1$.
- For any function u of $W^{s,p}(\Omega)$ we denote the Gagliardo semi-norm by
$$\big[u\big]_{W^{s,p}(\Omega)} := \Big(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\, dx\, dy \Big)^{1/p}.$$
- The space $\widetilde{W}^{s,p}(\Omega)$ is defined as the space of all $u \in W^{s,p}(\Omega)$ such that $\tilde{u} \in W^{s,p}(\mathbb{R}^N)$, where $\tilde{u}$ is the extension by zero of u, outside of Ω. $\widetilde{W}^{s,p}(\Omega)$ is a Banach space endowed with the norm
$$\|u\|_{\widetilde{W}^{s,p}(\Omega)} := \|\tilde{u}\|_{W^{s,p}(\mathbb{R}^N)}$$
and it is a reflexive space if $p > 1$.

Let us quote some properties of these spaces that will be used later. Here we will denote by $C(N,p)$ any positive constant depending only on N and p.

Proposition 2.1 ([10])**.** *Let Ω be a bounded open set of $\mathbb{R}^N$.*

(1) *There exists $C(N,p)$ such that, for any $u \in \widetilde{W}^{s,p}(\Omega)$, it holds*
$$\|u\|^p_{L^p(\Omega)} \le C(N,p)(\operatorname{diam}(\Omega))^{sp}(1-s)\big[u\big]^p_{W^{s,p}(\mathbb{R}^N)}. \tag{2.1}$$
Thus, the Gagliardo semi-norm $[\cdot]_{W^{s,p}(\mathbb{R}^N)}$ is a norm in $\widetilde{W}^{s,p}(\Omega)$ equivalent to the previous norm $\|\cdot\|_{\widetilde{W}^{s,p}(\Omega)}$ (c.f. [10, Lemma 2.5]*).*

(2) *Let $0 < s \le s' < 1$. Then there exists a positive constant $C(N,p)$ such that*
$$[u]^p_{W^{s,p}(\mathbb{R}^N)} \le [u]^p_{W^{s',p}(\mathbb{R}^N)} + C(N,p)\Big(\frac{1}{sp} - \frac{1}{s'p}\Big)\|u\|^p_{L^p(\mathbb{R}^N)}$$
for any $u \in W^{s',p}(\mathbb{R}^N)$ (cf. [10, Lemma 2.3]*).*

Proposition 2.2 ([7, 13, 17])**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary. Then*

(1) *$C_0^\infty(\Omega)$ is dense in $\widetilde{W}^{s,p}(\Omega)$ (c.f.* [17, Theorem 1.4.2.2]*).*
(2) *If $u \in \widetilde{W}^{s,p}(\Omega)$ and f is a Lipschitz function then $f(u) \in \widetilde{W}^{1,p}(\Omega)$.*
(3) *Let $0 < s \le s' < 1$. Then there exists a positive constant $C(N,p)$ such that*
$$(1-s)[u]^p_{W^{s,p}(\Omega)} \le 2^{(1-s)p}\operatorname{diam}(\Omega)^{(s'-s)p}(1-s')[u]^p_{W^{s',p}(\Omega)}$$
for any $u \in W^{s',p}(\Omega)$ (c.f. [7, Lemma 2]*;* [13, Lemmas 4.3 and 4.4]*).*
(4) *For any $u \in W^{1,p}(\Omega)$,*
$$\lim_{s\to 1^-}(1-s)[u]^p_{W^{s,p}(\Omega} = \int_\Omega |\nabla u|^p\,dx.$$
(c.f. [7, Corollary 2]*).*

2.2. **Embeddings.** Let the fractional critical exponent of Sobolev be defined by
$$p_s^* = \begin{cases} \frac{Np}{N-sp} & \text{if } sp < N,\\ +\infty & \text{if } sp \geqslant N. \end{cases}$$
The following results are versions of the classical Sobolev injection theorem in the case of fractional Sobolev spaces (c.f. [14, pages 218 and 219]).

Theorem 2.3. [14] *Let Ω be an open set with a Lipschitz boundary. We have the following continuous injections:*

(1) *If $sp < N$, $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q \in [p, p_s^*]$.*
(2) *If $sp = N$, $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q \in [p, +\infty)$.*
(3) *If $sp > N$, $W^{s,p}(\Omega) \hookrightarrow C^{0,\alpha}(\bar\Omega)$ with $\alpha \in \Big(0, s - \frac{N}{p}\Big]$.*

Furthermore we have the following compact injections when Ω is an open bounded domain of $\mathbb{R}^N$ with a Lipschitz boundary:

4. *If $sp \leqslant N$, then $W^{s,p}(\Omega) \hookrightarrow_c L^q(\Omega)$ for all $q \in [1, p_s^*)$.*
5. *If $sp > N$, then $W^{s,p}(\Omega) \hookrightarrow_c C^{0,\alpha}(\bar\Omega)$ with $\alpha \in (0, s - \frac{N}{p})$.*
6. *$W^{s,p}(\Omega) \hookrightarrow_c L^{pq'}(\Omega)$ with $\max\{1, \frac{N}{sp}\} < q < +\infty$ and $\frac{1}{q} + \frac{1}{q'} = 1$.*

Throughout this work we will assume that Ω is a bounded domain of $\mathbb{R}^N$ with a Lipschitz boundary.

3. Weak solutions of the eigenvalue problem and regularity results

For simplicity, from now on we will denote by u, instead of $\tilde u$, the extension by 0 of any function $u \in \widetilde{W}^{s,p}(\Omega)$.

Definition 3.1. (1) We will say that a function $u \in \widetilde{W}^{s,p}(\Omega)$ is a weak solution of (1.1) if
$$\mathcal{H}(u,v) + \int_\Omega V(x)|u|^{p-2}uv dx = \lambda \int_\Omega m(x)|u|^{p-2}uv\,dx \tag{3.1}$$

for all $v \in \widetilde{W}^{s,p}(\Omega)$, where

$$\mathcal{H}(u,v) := \mathcal{K}(1-s)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}\big(u(x)-u(y)\big)}{|x-y|^{N+sp}}\big(v(x)-v(y)\big)\,dx\,dy. \tag{3.2}$$

It should be noted that for all $u \in \widetilde{W}^{s,p}(\Omega)$, we have

$$\mathcal{H}(u,u) = \mathcal{K}(1-s)\big[u\big]^p_{W^{s,p}(\mathbb{R}^N)}.$$

(2) We will say that a real number λ is an eigenvalue of (1.1) if there exist $u \not\equiv 0$ satisfying (3.1). In this case, we say that u is an eigenfunction associated with λ.
(3) Moreover, if the eigenfunction u has a constant sign on Ω, then λ is called a principal eigenvalue of the problem (1.1).
(4) Finally, the eigenvalue λ is said to be simple if any two eigenfunctions u and v associated with λ are such that $u = cv$ for some real constant c.

Definition 3.2. For each $u \in \widetilde{W}^{s,p}(\Omega)$, let the energy associated with the problem (1.1) be

$$E_V(u) := \mathcal{H}(u,u) + \int_\Omega V(x)|u|^p\,dx = \mathcal{K}(1-s)\big[u\big]^p_{W^{s,p}(\mathbb{R}^N)} + \int_\Omega V(x)|u|^p\,dx. \tag{3.3}$$

It is clear that E_V is of class $\mathscr{C}^1$ on $\widetilde{W}^{s,p}(\Omega)$ with

$$\langle E_V'(u), v\rangle = p\mathcal{H}(u,v) + p\int_\Omega V(x)|u|^{p-2}uv\,dx \quad \forall (u,v) \in \widetilde{W}^{s,p}(\Omega)\times\widetilde{W}^{s,p}(\Omega).$$

Let us now state the main result of this section. Let us consider the homogeneous problem

$$\begin{gathered} (-\Delta_p)^s u + V'|u|^{p-2}u = 0 \quad \text{in } \Omega,\\ u = 0 \quad \text{in } \mathbb{R}^N\setminus\Omega, \end{gathered} \tag{3.4}$$

where V' satisfies condition (C1).

Theorem 3.3. *If $u \in \widetilde{W}^{s,p}(\Omega)$ is a weak solution of (3.4), then $u \in L^\infty(\Omega)\cap C(\Omega)$. Furthermore, there exists a positive constant $C = C(s,p,N,\Omega,\|V'\|_{L^r(\Omega)})$ such that*

$$\|u\|_{L^\infty(\Omega)} \le C\|u\|_{L^{r'p}(\Omega)}. \tag{3.5}$$

The proof of this theorem will follow from Lemma 3.4 below, based on the De Giorgi-Stampacchia iteration technique (see for instance [11, 16, 22], where the case $V' \equiv 1$ has been considered).

Lemma 3.4. *Assume that $sp \le N$. Let u be a weak solution of (3.4) admitting a positive part $u^+ \not\equiv 0$. Let us define the sequence $(w_k)_k$ by*

$$w_k := \Big(u - \Big(1 - \frac{1}{2^k}\Big)\Big)^+.$$

Then there exists a positive constant $\sigma = \sigma(s,p,N,\Omega,\|V'\|_{L^r(\Omega)})$ such that, if $\|u^+\|_{L^{r'p}(\Omega)} < \sigma$, then $u \le 1$ a.e.

Proof. Let us denote $W_k = \|w_k\|^p_{L^{r'p}(\Omega)}$. The conclusion of the lemma will follow from the following results that we prove below:

(1) $\lim_{k\to+\infty} W_k = \|(u-1)^+\|^p_{L^{r'p}(\Omega)}$.

(2) $\lim_{k\to+\infty} W_k = 0$.

Notice that, by definition, $w_k \in W^{s,p}(\Omega)$ and $w_k = 0$ a.e. in Ω^c.
1. Trivially the sequence $(w_k)_k$ is decreasing so, for all $k \in \mathbb{N}$ we have $|w_k|^{r'p} \le |w_0|^{r'p} = |u^+|^{r'p} \in L^1(\Omega)$. Moreover the sequence $(|w_k|^{r'p})_k$ converges to $((u-1)^+)^{r'p}$ almost everywhere in Ω. Hence $W_k \to \|(u-1)^+\|^p_{L^{r'p}(\Omega)}$ by the Lebesgue's dominated convergence theorem.
2. Let us first prove two claims.
Claim 1. For all $k \in \mathbb{N}$,

$$\||u|^{p-1} w_{k+1}\|_{L^{r'}(\Omega)} \le 2^{(p-1)(k+1)} W_k. \tag{3.6}$$

Indeed, first observe that if $w_{k+1}(x) > 0$, that is, if $u(x) > \frac{2^{k+1}-1}{2^{k+1}}$, then

$$w_k(x) = w_{k+1}(x) + \frac{1}{2^{k+1}} \ge \frac{1}{2^{k+1}}, \quad w_k(x) \ge \frac{u(x)}{2^{k+1}-1}$$

and

$$\begin{aligned}
\int_\Omega |u|^{r'(p-1)} w_{k+1}^{r'} dx &= \int_{\{w_{k+1}>0\}} |u|^{r'(p-1)} w_{k+1}^{r'} dx \\
&\le \int_{\{w_{k+1}>0\}} (2^{k+1}-1)^{r'(p-1)} w_k^{r'(p-1)} w_k^{r'}(x) dx \\
&\le (2^{k+1}-1)^{r'(p-1)} \|w_k\|^{r'p}_{L^{r'p}(\Omega)} \\
&\le 2^{r'(p-1)(k+1)} W_k^{r'}.
\end{aligned} \tag{3.7}$$

Claim 2. There exist $D > 1$ and $\beta > 0$ such that for all $k \in \mathbb{N}$,

$$W_{k+1} \le D^k W_k^{1+\beta}.$$

To prove this claim, let us quote the following (trivial) inequality:

$$\forall (a,b) \in \mathbb{R}^2, \quad |a^+ - b^+|^p \le |a-b|^{p-2}(a-b)(a^+ - b^+). \tag{3.8}$$

By taking

$$a = u(x) - \Big(1 - \frac{1}{2^{k+1}}\Big), \quad b = u(y) - \Big(1 - \frac{1}{2^{k+1}}\Big)$$

in (3.8) for all $(x,y) \in \mathbb{R}^N$, we obtain

$$\big[w_{k+1}\big]^p_{W^{s,p}(\mathbb{R}^N)} \le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} (w_{k+1}(x) - w_{k+1}(y))\, dx\, dy.$$

Besides, by taking w_{k+1} in the weak formulation of (3.4), we obtain from the previous inequality

$$\mathcal{K}(1-s)\big[w_{k+1}\big]^p_{W^{s,p}(\mathbb{R}^N)} \le \int_\Omega |V'(x)||u|^{p-1} w_{k+1} dx,$$

and therefore, using Claim 1,

$$\begin{aligned}
\mathcal{K}(1-s)\big[w_{k+1}\big]^p_{W^{s,p}(\mathbb{R}^N)} &\le \|V'\|_{L^r(\Omega)} \Big[\int_\Omega \big(|u|^{p-1} w_{k+1}\big)^{r'} dx\Big]^{1/r'} \\
&\le C 2^{(p-1)(k+1)} W_k
\end{aligned} \tag{3.9}$$

for some positive constant C depending on $\|V'\|_{L^r(\Omega)}, s, p, N$, and Ω. On the other hand, using Hölder's inequality with the exponents $q := \frac{N}{r'(N-sp)}$ if $q < Ns$ (or any $q > 1$ if $N = ps$), by Sobolev's embedding we have

$$\begin{aligned} W_{k+1} &= \|w_{k+1}\|^p_{L^{r'p}(\Omega)} \\ &\le \|w_{k+1}\|^p_{L^{r'pq}(\Omega)} |\{w_{k+1} > 0\}|^{\frac{q-1}{r'q}} \\ &\le C\big[w_{k+1}\big]^p_{W^{s,p}(\mathbb{R}^N)} |\{w_{k+1} > 0\}|^{\frac{q-1}{r'q}} \end{aligned} \tag{3.10}$$

for some $0 < C = C(N, p, s, \Omega)$. Moreover, since $w_k = w_{k+1} + \frac{1}{2^{k+1}}$, then

$$|\{w_{k+1} > 0\}| \le |\{w_k > 2^{-k-1}\}| \le 2^{r'p(k+1)} W_k^{r'} \tag{3.11}$$

and hence, using (3.9), (3.10) and (3.11)

$$\begin{aligned} W_{k+1} &\le C2^{(p-1)(k+1)} W_k \times |\{w_{k+1} > 0\}|^{\frac{q-1}{r'q}} \\ &\le C2^{(p-1)(k+1)} W_k \times \big(2^{r'p(k+1)} W_k^{r'}\big)^{\frac{q-1}{r'q}} \\ &\le C(2^{p-1} \times 2^{\frac{p(q-1)}{q}})^{k+1} W_k^{1+\frac{q-1}{q}} \\ &\le D^k W_k^{1+\beta}, \end{aligned}$$

with $D = \big\{ [1 + C]\, 2^{p-1} \times 2^{\frac{p(q-1)}{q}} \big\}^2 > 1$ and $\beta = \frac{q-1}{q} > 0$. Claim 2 is proved.

Now we complete the proof of 2. Let $\sigma = D^{-\frac{q^2}{p(q-1)^2}}$, denote $\rho = \|u^+\|^p_{L^{r'p}(\Omega)}$ and assume that $\rho^{1/p} < \sigma$. Choose $\eta \in (\rho^{\frac{q-1}{q}}, D^{-\frac{q}{q-1}})$. It should be noted that $\eta \in]0, 1[$, $\rho^{\frac{q-1}{q}} \le \eta$, and $D\eta^{\frac{q-1}{q}} \le 1$. Let us prove by induction that for all $k \in \mathbb{N}$

$$W_k \le \rho\eta^k. \tag{3.12}$$

By definition

$$W_0 := \|w_0\|^p_{L^{r'p}(\Omega)} = \|u_+\|^p_{L^{r'p}(\Omega)} = \rho \le \rho\eta^0.$$

Assume that (3.12) holds at order k and let us show that it holds at order $k + 1$. By Claim 2,

$$W_{k+1} \le D^k W_k^{1+\frac{q-1}{q}} \le D^k (\rho\eta^k)^{1+\frac{q-1}{q}} = \rho\big(\eta^{\frac{q-1}{q}}\big)^k \rho^{\frac{q-1}{q}} \eta^k \le \rho\eta^{k+1}.$$

Thus by passing to the limit in (3.12), we finally obtain that $W_k \to 0$. □

Proof of Theorem 3.3. Take $v = \frac{\sigma u}{2\|u^+\|_{L^{r'p}(\Omega)}}$, where $\sigma = \sigma(s, p, N, \Omega, \|V'\|_{L^r(\Omega)})$ is given by Lemma 3.4. Since v is a weak solution of (1.1) and satisfies $\|v^+\|_{L^{r'p}(\Omega)} = \frac{\sigma}{2}$ then $v \le 1$ a.e., which gives

$$u \le \frac{2}{\sigma}\|u^+\|_{L^{r'p}(\Omega)} \quad a.e.$$

If u^- is not identically zero, we apply the same argument to $-u$, which is a weak solution of (1.1), to find that

$$u \ge -\frac{2}{\sigma}\|u^-\|_{L^{r'p}(\Omega)} \quad \text{a.e.}$$

and estimate (3.5) follows.

Finally, the continuity of u results from [6, Theorem 3.13], which derives from [23, Theorem 1.5]. □

Remark 3.5. To our knowledge, it is not known if the solutions are continuous up to the boundary of Ω or class $C^{0,\alpha}$ in the case $sp \le N$ and V' unbounded. Indeed, when V' is bounded then $f = V'|u|^{p-2}u \in L^\infty(\Omega)$ and, by the results by Iannizzotto et al. [20], $u \in C^\alpha(\overline{\Omega})$ for some $\alpha > 0$.

4. Existence of principal eigenvalues with indefinite weights

Let us assume in this section that V and m satisfy conditions (C1) and (C2) and consider the eigenvalue problem

$$\begin{gathered} (-\Delta_p)^s u + V'|u|^{p-2}u = \mu|u|^{p-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega, \end{gathered} \tag{4.1}$$

with $V' = (V - \lambda m)$ and μ an eigenvalue parameter depending on the real λ. According to [12], problem (4.1) admits a unique principal eigenvalue which we will denote $\mu(\lambda)$. Moreover, $\mu(\lambda)$ is simple and can be characterized as

$$\begin{aligned} \mu(\lambda) = \inf \Big\{ & \mathcal{K}(1-s)[u]^p_{W^{s,p}(\mathbb{R}^N)} + \int_\Omega (V(x) - \lambda m(x))|u|^p \, dx : u \in \widetilde{W}^{s,p}(\Omega), \\ & \|u\|_{L^p(\Omega)} = 1 \Big\}. \end{aligned} \tag{4.2}$$

Note that λ_0 is a principal eigenvalue of our problem (1.1) if and only if $\mu(\lambda_0) = 0$. Our aim here is to give reasonable assumptions on V and m so that the curve of the function $\lambda \mapsto \mu(\lambda)$ intersects the x-axis.

We introduce the sets

$$\begin{gathered} G_0 := \big\{ u \in \widetilde{W}^{s,p}(\Omega) : \|u\|_{L^p(\Omega)} = 1, \int_\Omega m(x)|u|^p dx = 0 \big\}, \\ G := \big\{ u \in \widetilde{W}^{s,p}(\Omega) : \|u\|_{L^p(\Omega)} = 1 \big\} . \end{gathered} \tag{4.3}$$

The following proposition gives useful properties on the function $\lambda \mapsto \mu(\lambda)$. We will denote here

$$\Omega^+ := \{x \in \Omega,\, m(x) > 0\}, \quad \Omega^- := \{x \in \Omega,\, m(x) < 0\}, \quad \Omega^0 := \{x \in \Omega,\, m(x) = 0\},$$

and φ_λ the unique positive eigenfunction of $L^p(\Omega)$-norm equal to 1 associated with $\mu(\lambda)$.

Proposition 4.1. (i) *$\mu : \mathbb{R} \to \mathbb{R}$ is concave and differentiable, with*

$$\mu'(\lambda) = -\int_\Omega m(x)\varphi_\lambda^p dx \quad \forall \lambda \in \mathbb{R}.$$

(ii) (a) $\lim_{\lambda\to+\infty} \mu(\lambda) = -\infty$.
(b) *If* $|\Omega^-| > 0$ *then* $\lim_{\lambda\to-\infty} \mu(\lambda) = -\infty$.

(iii) *If* $|\Omega^-| = 0$ *then* μ *is strictly decreasing on* $\mathbb{R}$ *and, if moreover* $|\Omega^- \cup \Omega^0| = 0$*, then*

$$\lim_{\lambda\to-\infty} \mu(\lambda) = +\infty.$$

(iv) $\sup_{\lambda\in\mathbb{R}} \mu(\lambda) = \alpha(V, m)$ *where*

$$\alpha(V, m) := \inf \{E_V(u), u \in G_0\} . \tag{4.4}$$

Moreover, $\alpha(V, m)$ *is finite if and only if* $|\Omega^+| < |\Omega|$.

Proof. (i) We prove that $\mu : \mathbb{R} \to \mathbb{R}$ is concave. Let λ and β be two distinct real numbers. Let $t \in [0,1]$ and set $\theta_t = t\lambda + (1-t)\beta$. Let $u \in G$. Since

$$-E_{V-\theta_t m}(u) = t(-E_{V-\lambda m}(u)) + (1-t)(-E_{V-\beta m}(u)),$$

we have

$$-\mu(\theta_t) \leqslant t\,(-\mu(\lambda)) + (1-t)\,(-\mu(\beta))\,,$$

which means that the function $-\mu$ is convex.

Let $\lambda \in \mathbb{R}$ and $(\lambda_k)_k$ be a sequence converging to λ. Let φ_k and φ_λ be eigenfunctions associated with $\mu(\lambda_k)$ and $\mu(\lambda)$ respectively with $L^p(\Omega)$-norm equal to 1. By the results of [12, Theorem 2.9], the eigenfunction φ_λ is > 0 a.e. in Ω (see Remark 4.3). By Lemma 4.4 below we have, for some $C_1 > 0$ and $C_2 > 0$,

$$\big[\varphi_k\big]^p_{W^{s,p}(\mathbb{R}^N)} \leq C_1 E_{V-\lambda_k m}(\varphi_k) + C_2 \int_\Omega |\varphi_k|^p dx = C_1\mu(\lambda_k) + C_2,$$

so

$$\limsup_{k\to\infty} \big[\varphi_k\big]^p_{W^{s,p}(\mathbb{R}^N)} \leq C_1\mu(\lambda) + C_2$$

and therefore the sequence $(\varphi_k)_k$ is bounded in $\widetilde{W}^{s,p}(\Omega)$. Hence there exists $\varphi_0 \in \widetilde{W}^{s,p}(\Omega)$ and some subsequence, written again $(\varphi_k)_k$, such that $\varphi_k \rightharpoonup \varphi_0$ in $\widetilde{W}^{s,p}(\Omega)$, $\varphi_k \to \varphi_0$ in $L^p(\Omega)$ and in $L^{r'p}(\Omega)$. In particular $\|\varphi_0\|_{L^p(\Omega)} = 1$. Since $\mu(\lambda) = \lim_{k\to+\infty}\mu(\lambda_k)$, it follows that

$$\mu(\lambda) \geqslant \lim_{k\to+\infty} E_{V-\lambda_k m}(\varphi_k) \geqslant E_{V-\lambda m}(\varphi_0) \geqslant \mu(\lambda)$$

and hence $\mu(\lambda) = E_{V-\lambda m}(\varphi_0)$. Using the simplicity of the principal eigenvalue of problem (4.1) and the fact that $\|\varphi_0\|_{L^p(\Omega)} = 1$ and $\varphi_0 \geq 0$, we conclude that $\varphi_0 = \varphi_\lambda$. Moreover,

$$\begin{aligned}
\mu(\lambda_k) &= E_{V-\lambda_k m}(\varphi_k) \\
&= E_{V-\lambda m}(\varphi_k) + (\lambda - \lambda_k)\int_\Omega m(x)|\varphi_k|^p dx \\
&\geqslant \mu(\lambda) + (\lambda - \lambda_k)\int_\Omega m(x)|\varphi_k|^p dx
\end{aligned}$$

and, by replacing λ_k by λ and φ_k by φ_λ in the inequality above, we obtain:

$$\mu(\lambda) \geqslant \mu(\lambda_k) + (\lambda_k - \lambda)\int_\Omega m(x)|\varphi_\lambda|^p dx.$$

Putting together this two inequalities we obtain

$$(\lambda - \lambda_k)\int_\Omega m(x)|\varphi_k|^p dx \leqslant \mu(\lambda_k) - \mu(\lambda) \leqslant (\lambda - \lambda_k)\int_\Omega m(x)|\varphi_\lambda|^p dx$$

from which we conclude that

$$\mu'(\lambda) = -\int_\Omega m(x)|\varphi_\lambda|^p dx.$$

(ii) Since $|\Omega^+| > 0$ by (C1), there exists a function $\xi \in \widetilde{W}^{s,p}(\Omega)$ such that $\int_\Omega m(x)|\xi|^p dx > 0$, $\int_\Omega |\xi|^p dx = 1$ and therefore

$$\mu(\lambda) \leqslant \mathcal{K}(1-s)\int_{\mathbb{R}^{2N}} \frac{|\xi(x)-\xi(y)|^p}{|x-y|^{N+sp}}\,dx\,dy + \int_\Omega V(x)|\xi|^p dx - \lambda\int_\Omega m(x)|\xi|^p dx.$$

Thus $\lim_{\lambda\to+\infty}\mu(\lambda) = -\infty$. Similarly, if $|\Omega^-| > 0$ then $\lim_{\lambda\to-\infty}\mu(\lambda) = -\infty$.

(iii) If $|\Omega^-|=0$, then μ is strictly decreasing on $\mathbb{R}$ because $-\mu'(\lambda)=\int_\Omega m(x)|\varphi_\lambda|^p dx>0$ for all $\lambda\in\mathbb{R}$. If $|\Omega^-\cup\Omega^0|=0$ assume by contradiction that the function $\lambda\mapsto\mu(\lambda)$ is bounded. Let $(\lambda_k)_k$ be such that $\lambda_k\to-\infty$ and write $\varphi_k=\varphi_{\lambda_k}$. Using Lemma 4.4, we have

$$\begin{aligned}\sup_{\lambda\in\mathbb{R}}\mu(\lambda)\ge\mu(\lambda_k)&=E_{(V-\lambda_k m)}(\varphi_k)\\&=E_V(\varphi_k)-\underbrace{\lambda_k\int_\Omega m(x)|\varphi_k|^p dx}_{\leqslant 0}\\&\ge E_V(\varphi_k)\geqslant\frac{1}{C_1}\Big(\big[\varphi_k\big]^p_{W^{s,p}(\mathbb{R}^N)}-C_2\Big)\end{aligned}$$

so $(\varphi_k)_k$ is a bounded sequence in $\widetilde{W}^{s,p}(\Omega)$. Thus, there exist $\varphi\in\widetilde{W}^{s,p}(\Omega)$ and some subsequence $(\varphi_k)_k$ such that $\varphi_k\rightharpoonup\varphi$ in $\widetilde{W}^{s,p}(\Omega)$ and $\varphi_k\to\varphi$ in $L^{pr'}(\Omega)$ and in $L^p(\Omega)$. As φ_k is L^p-normalized, then $\|\varphi\|_{L^p(\Omega)}=1$ and $\int_\Omega m(x)|\varphi|^p\,dx=\lim_{k\to\infty}\int_\Omega m(x)|\varphi_k|^p\,dx>0$. Then

$$-\infty=\lim_{k\to\infty}\lambda_k\int_\Omega m(x)|\varphi_k|^p\,dx\ge E_V(\varphi)-\sup_{\lambda\in\mathbb{R}}\mu(\lambda)>-\infty,$$

a contradiction.

(iv). If $|\Omega^-\cup\Omega^0|=0$ then $G_0=\emptyset$ and using (ii) and (iii) we obtain

$$\alpha(Vm)=+\infty=\lim_{\lambda\to-\infty}\mu(\lambda)=\sup_{\lambda\in\mathbb{R}}\mu(\lambda).$$

If $|\Omega^-\cup\Omega^0|>0$, as $G_0\subset G$ and

$$\mu(\lambda)\leqslant E_{V-\lambda m}(u)=E_V(u)-\lambda\int_\Omega m(x)|u|^p dx=E_V(u)\quad\forall u\in G_0.$$

then $\sup_{\lambda\in\mathbb{R}}\mu(\lambda)\leqslant\alpha(V,m)$. To obtain the reverse inequality observe that, by (i) and (ii), the function μ possesses a global maximum, that is, $\sup_{\lambda\in\mathbb{R}}\mu(\lambda)$ is reached at some $\lambda_0\in\mathbb{R}$, which in particular implies that

$$0=\mu'(\lambda_0)=\int_\Omega m(x)|\varphi_{\lambda_0}|^p dx.$$

Consequently, $\varphi_{\lambda_0}\in G_0$ and then $\alpha(V,m)\leqslant E_V(\varphi_{\lambda_0})$. But

$$\mu(\lambda_0)=E_{V-\lambda_0 m}(\varphi_{\lambda_0})=E_V(\varphi_{\lambda_0})\quad\text{and}\quad\mu(\lambda_0)=\sup_{\lambda\in\mathbb{R}}\mu(\lambda),$$

then $\alpha(V,m)\leqslant\sup_{\lambda\in\mathbb{R}}\mu(\lambda)$. Thus we obtain

$$\alpha(V,m)=\sup_{\lambda\in\mathbb{R}}\mu(\lambda).$$

The proof that $\alpha(V,m)$ is achieved whenever it is finite, is standard and we omit it. □

As a consequence of this proposition we have the following result.

Theorem 4.2. *Assume that V and m satisfy the hypotheses* (C1) *and* (C2).

(i) *If $|\Omega^-|=0$, then* (1.1) *admits a principal eigenvalue if and only if $\alpha(V,m)>0$. In this case the principal eigenvalue is unique and it is characterized by the equation $\lambda_1(V,m)=\min_{\mathcal{M}}E_V$, where $\mathcal{M}:=\big\{u\in\widetilde{W}^{s,p}(\Omega):\int_\Omega m(x)|u|^p dx=1\big\}$.*

(ii) *If $|\Omega^-| > 0$, then (1.1) admits a principal eigenvalue if and only if $\alpha(V,m) \geqslant 0$. More precisely,*

(a) *if $\alpha(V,m) > 0$, then (1.1) admits exactly two principal eigenvalues*
$$\lambda_{-1}(V,m) = -\min_{\mathcal{M}^-} E_V < \lambda_1(V,m) = \min_{\mathcal{M}} E_V,$$
where
$$\mathcal{M}^- := \big\{u \in \widetilde{W}^{s,p}(\Omega) : \int_\Omega m|u|^p dx = -1\big\};$$

(b) *if $\alpha(V,m) = 0$, then (1.1) admits a unique principal eigenvalue $\lambda_1(V,m)$ given by*
$$\lambda_1(V,m) = \inf_{\mathcal{M}} E_V = -\inf_{\mathcal{M}^-} E_V.$$
These infima are not achieved. In addition, any function $u \in \widetilde{W}^{s,p}(\Omega)\setminus\{0\}$ such that $E_V(u) = \int_\Omega m|u|^p dx = 0$ is an eigenfunction associated with $\lambda_1(V,m)$.

(iii) *In the case $\alpha(V,m) > 0$ any function $u \in \mathcal{M}$ satisfying $E_V(u) = \lambda_1(V,m)$ is an eigenfunction associated with $\lambda_1(V,m)$ and it is sign definite. Same result for $u \in \mathcal{M}^-$ satisfying $E_V(u) = \lambda_{-1}(V,m)$.*

Proof. The proof given in [8] can be easily adapted here as a corollary of Proposition 4.1. We only give the proof (b) of ii. to show how to use Picone's inequality stated in Lemma 7.1. If $\alpha(V,m) = 0$, then there exists a real λ_0 such that $\mu(\lambda_0) = 0$ so λ_0 is a principal eigenvalue of (1.1). Let us show that
$$\lambda_0 = \inf_{\mathcal{M}} E_V = -\inf_{\mathcal{M}^-} E_V.$$
We only give the proof of the first identity, the proof of the second one is similar. As $\alpha(V,m) = \sup_{\lambda\in\mathbb{R}} \mu(\lambda)$ then $\mu'(\lambda_0) = 0 = -\int_\Omega m(x)|\varphi_{\lambda_0}|^p dx$.

Let $u \in \mathcal{M}$ be such that $u \geqslant 0$. For any $T > 0$ define $u_T := \min\{u, T\}$ and take $\varphi_{\lambda_0} + \varepsilon$ with $\varepsilon > 0$ Let us prove that $z := \frac{u_T^p}{\left(\varphi_{\lambda_0}+\varepsilon\right)^{p-1}} \in \widetilde{W}^{s,p}(\Omega)$. Indeed, for any $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$ we have
$$|z(x) - z(y)| \leq \Big|\frac{u_T(x)^p - u_T(y)^p}{(\varphi_{\lambda_0}(x)+\varepsilon)^{p-1}}\Big| + |u_T(y)|^p\Big|\frac{(\varphi_{\lambda_0}(y)+\varepsilon)^{p-1} - (\varphi_{\lambda_0}(x)+\varepsilon)^{p-1}}{(\varphi_{\lambda_0}(y)+\varepsilon)^{p-1}(\varphi_{\lambda_0}(x)+\varepsilon)^{p-1}}\Big| \tag{4.5}$$
and using for all $(a,b) \in \mathbb{R}^+ \times \mathbb{R}^+$ and $q > 0$ the trivial inequality
$$|a^q - b^q| \leq q\left(|a|^{q-1} + |b|^{q-1}\right)|a - b|$$
with $q = p$ or $q = p-1$ we have
$$|z(x) - z(y)| \leq \frac{2pT^{p-1}}{\varepsilon^{p-1}}|u(x) - u(y)| + \frac{2(p-1)T^p}{\varepsilon^p}|\varphi_{\lambda_0}(x) - \varphi_{\lambda_0}(y)| \tag{4.6}$$
and therefore
$$[z]^p_{W^{s,p}(\mathbb{R}^N)} \leq C\Big([u]^p_{W^{s,p}(\mathbb{R}^N)} + [\varphi_{\lambda_0}]^p_{W^{s,p}(\mathbb{R}^N)}\Big) < \infty.$$
An application of Picone's inequality to functions u_T and $\varphi_{\lambda_0} + \varepsilon$, and the fact that $\frac{u_T^p}{\left(\varphi_{\lambda_0}+\varepsilon\right)^{p-1}} \in \widetilde{W}^{s,p}(\Omega)$, imply that
$$0 \leqslant \int_{\mathbb{R}^{2N}} \frac{L(u_T,\, \varphi_{\lambda_0}+\varepsilon)(x,y)}{|y-x|^{N+sp}}\,dx\,dy$$

$$
\begin{aligned}
&= \big[u_T\big]^p_{W^{s,p}(\mathbb{R}^N)} - \int_{\mathbb{R}^{2N}} \frac{\big|\varphi_{\lambda_0}(y)-\varphi_{\lambda_0}(x)\big|^{p-2}\big(\varphi_{\lambda_0}(y)-\varphi_{\lambda_0}(x)\big)}{|x-y|^{N+sp}} \\
&\quad \times \Big(\frac{u_T^p(y)}{\big(\varphi_{\lambda_0}+\varepsilon\big)^{p-1}(y)} - \frac{u_T^p(x)}{\big(\varphi_{\lambda_0}+\varepsilon\big)^{p-1}(x)}\Big)\,dx\,dy \\
&= \big[u_T\big]^p_{W^{s,p}(\mathbb{R}^N)} + \frac{1}{\mathcal{K}(1-s)}\Big(-\lambda_0\int_\Omega m(x)\big|\varphi_{\lambda_0}(x)\big|^{p-1}\frac{u_T^p}{\big(\varphi_{\lambda_0}+\varepsilon\big)^{p-1}}dx \\
&\quad + \int_\Omega V(x)\big|\varphi_{\lambda_0}\big|^{p-1}\frac{u_T^p}{\big(\varphi_{\lambda_0}+\varepsilon\big)^{p-1}}dx\Big).
\end{aligned}
$$

So when $\varepsilon \to 0$, by the Lebesgue convergence theorem

$$
0 \le \mathcal{K}(1-s)[u_T]^p_{W^{s,p}(\mathbb{R}^N)} - \lambda_0\int_\Omega m(x)|u_T|^p dx + \int_\Omega V(x)u_T^p\,dx
$$

for all $T>0$. Moreover, since as $T\to+\infty$ we obtain $u_T = u$. Then by Fatou's lemma,

$$
0 \leqslant \mathcal{K}(1-s)[u]^p_{W^{s,p}(\mathbb{R}^N)} - \lambda_0 + \int_\Omega V(x)|u|^p dx. \tag{4.7}
$$

So $\lambda_0 \leqslant \inf_{\mathcal{M}} E_V$. To prove the reverse inequality let us show that there exists a sequence of functions of $\mathcal{M}$ whose energy E_V converges to λ_0. Let $\psi\in C^\infty(\Omega)$ such that $\psi>0$, $\int_\Omega m(x)\psi^p dx>0$ and $\int_\Omega m(x)\varphi_{\lambda_0}^{p-1}\psi dx>0$. Let the sequence $(u_k)_k$ be of the form

$$
u_k = \frac{\varphi_{\lambda_0}+\frac{\psi}{k}}{\Big(\int_\Omega m(x)\big|\varphi_{\lambda_0}+\frac{\psi}{k}\big|^p dx\Big)^{1/p}}.
$$

It is straightforward that all elements of this sequence are in manifold $\mathcal{M}$, and when k is big enough $u_k>0$. Furthermore, because the functions $t\mapsto E_V\big(\varphi_{\lambda_0}+t\psi\big)$ and $s\mapsto \big|\varphi_{\lambda_0}+s\psi\big|^p$ are continuous and at least once differentiable on $\big[0,\,\frac{1}{k}\big]$, then there exist $0<t_k,\ s_k<1/k$ such that

$$
\begin{gathered}
E_V\big(\varphi_{\lambda_0}+\frac{\psi}{k}\big) = \frac{1}{k}\langle E_V'\big(\varphi_{\lambda_0}+t_k\psi\big),\psi\rangle, \\
\int_\Omega m(x)\big|\varphi_{\lambda_0}+\frac{1}{k}\psi\big|^p dx = \frac{p}{k}\int_\Omega m(x)\big|\varphi_{\lambda_0}+s_k\psi\big|^{p-1}\psi dx.
\end{gathered}
$$

As a result,

$$
\begin{aligned}
E_V(u_k) &= \frac{1}{\int_\Omega m(x)\big|\varphi_{\lambda_0}+\frac{1}{k}\psi\big|^p dx}E_V\big(\varphi_{\lambda_0}+\frac{\psi}{k}\big) \\
&= \frac{k}{p\int_\Omega m(x)\big|\varphi_{\lambda_0}+s_k\psi\big|^{p-1}\psi dx} \times \frac{\langle E_V'\big(\varphi_{\lambda_0}+t_k\psi\big),\psi\rangle}{k}\,.
\end{aligned}
$$

So when k tends to infinity we find that $E_V(u_k)\to\lambda_0$. Thus we can conclude that

$$
\lambda_0 = \inf_{\mathcal{M}} E_V. \tag{4.8}
$$

But φ_{λ_0} is an eigenfunction associated with $\mu(\lambda_0)=0$, that means that $E_V(\varphi_0) = \lambda_0\int_\Omega m(x)\big|\varphi_{\lambda_0}\big|^p dx = 0$ which implies λ_0 is not achieved. Finally, if $u\in\widetilde{W}^{s,p}(\Omega)\setminus\{0\}$ satisfies $E_V(u)=\int_\Omega m(x)|u|^p dx = 0$ we have

$$
\sup_{\lambda\in\mathbb{R}}\mu(\lambda) = 0 = E_V(u) = E_{V-\lambda_0 m}(u) \geqslant \mu(\lambda_0)\int_\Omega |u|^p dx = 0, \tag{4.9}
$$

and therefore u is a function where the infimum $\mu(\lambda_0)$ of equation (4.2) is achieved. Then, as the eigenvalue $\mu(\lambda_0)$ is simple, there exists $c > 0$ such that $u = c\phi_{\lambda_0}$ so then $\lambda_0 = \lambda_1(V, m)$. $\square$

Remark 4.3. One can prove, as at the beginning of the previous proof, that if $0 \le u \in \widetilde{W}^{s,p}(\Omega) \cap L^\infty(\Omega)$ and $v \in \widetilde{W}^{s,p}(\Omega)$ satisfies $v \ge c > 0$ a.e. for some $c > 0$ then $\frac{u^p}{v^{p-1}} \in \widetilde{W}^{s,p}(\Omega) \cap L^\infty(\Omega)$.

Lemma 4.4. *Let ω be a function satisfying* (C1) *and let Z be a bounded subset of $L^r(\Omega)$. If $\omega > 0$ a.e. is a function on $L^r(\Omega)$ for some $1 \le r < p_s^*$, then there are two strictly positive constants C_1 and C_2 such that*

$$\big[u\big]^p_{W^{s,p}(\mathbb{R}^N)} \leqslant C_1 E_V(u) + C_2 \int_\Omega \omega(x)|u|^p dx \tag{4.10}$$

for all functions $V \in Z$ and for all $u \in \widetilde{W}^{s,p}(\Omega)$.

Proof. This proof is a partial adaptation of Lemma 2 of [8]. Let T be a positive real such that $\|V\|_{L^r(\Omega)} \leqslant T$ for all $V \in Z$. Let $\varepsilon > 0$ fixed such that $\varepsilon < \frac{\mathcal{K}(1-s)}{T}$. According to Hölder inequality and the hypothesis (C1), we can write

$$\Big|\int_\Omega V(x)|u|^p dx\Big| \leqslant \|V\|_{L^r(\Omega)} \|u\|^p_{L^{pr'}(\Omega)}.$$

Claim. For all $\varepsilon > 0$, there exists $M_\varepsilon > 0$ such that

$$\|u\|^p_{L^{pr'}(\Omega)} \leqslant \varepsilon \big[u\big]^p_{W^{s,p}(\mathbb{R}^N)} + M_\varepsilon \int_\Omega \omega(x)|u|^p dx \tag{4.11}$$

for all $u \in \widetilde{W}^{s,p}(\Omega)$.

Indeed, suppose by contradiction that there exists $\varepsilon_0 > 0$, and sequence $(u_k)_k$ of $\widetilde{W}^{s,p}(\Omega)$ such that

$$\|u_k\|_{L^{pr'}(\Omega)} = 1 \quad \text{and} \quad \varepsilon_0 \big[u\big]^p_{W^{s,p}(\mathbb{R}^N)} + k \int_\Omega \omega(x)|u_k|^p dx < 1.$$

Then $(u_k)_k$ is bounded $\widetilde{W}^{s,p}(\Omega)$, so there exists $u_0 \in \widetilde{W}^{s,p}(\Omega)$ and sub-sequence also denoted by $(u_k)_k$ of $\widetilde{W}^{s,p}(\Omega)$ such that $u_k \rightharpoonup u_0$ in $\widetilde{W}^{s,p}(\Omega)$ and $u_k \to u_0$ in $L^{pr'}(\Omega)$ (see [10, Theorem 2.16]). So, we have on one hand

$$\lim_{k \to +\infty} \|u_k\|_{L^{pr'}(\Omega)} = \|u_0\|_{L^{pr'}(\Omega)} = 1,$$

and therefore $u_0 \not\equiv 0$ in Ω. Moreover, using once again the inequality of the hypothesis we have

$$\int_\Omega \omega(x)|u_k|^p dx < \frac{1}{k}.$$

Then passing to the limit we find by Fatou's lemma that

$$\int_\Omega \omega(x)|u_0|^p dx \leqslant 0,$$

which is a contradiction since $\omega > 0$ in Ω and $u_0 \not\equiv 0$ in Ω. We have proved the claim.

By applying the inequality (4.11) for $0 < \varepsilon < \frac{\mathcal{K}(1-s)}{T}$ there is a positive real M_ε such that

$$\Big|\int_\Omega V(x)|u|^p dx\Big| \leqslant \varepsilon \|V\|_{L^r(\Omega)} \big[u\big]^p_{W^{s,p}(\mathbb{R}^N)} + \|V\|_{L^r(\Omega)} M_\varepsilon \int_\Omega \omega(x)|u|^p dx$$

$$\leqslant \varepsilon T\big[u\big]^p_{W^{s,p}(\mathbb{R}^N)} + TM_\varepsilon \int_\Omega \omega(x)|u|^p dx.$$

So we obtain

$$\big[u\big]^p_{W^{s,p}(\mathbb{R}^N)} \leqslant \frac{1}{\mathcal{K}(1-s)-\varepsilon T} E_V(u) + \frac{TM_\varepsilon}{\mathcal{K}(1-s)-\varepsilon T}\int_\Omega \omega(x)|u|^p dx.$$

The lemma follows by setting

$$C_1 = \frac{1}{\mathcal{K}(1-s)-\varepsilon T} \quad \text{and} \quad C_2 = \frac{TM_\varepsilon}{\mathcal{K}(1-s)-\varepsilon T}.$$

□

As an application of Picone's inequality of Lemma 7.1 we can prove the simplicity and the uniqueness of the principal eigenvalues $\lambda_{\pm 1}(V,m)$.

Proposition 4.5. *Assume that $\alpha(V,m) \geq 0$. Let $u > 0$ a.e. be an eigenfunction of problem* (1.1) *associated with $\lambda_1(V,m)$ and let $v \geq 0$ a.e. be an eigenfunction associated with an eigenvalue $\lambda \geq \lambda_1(V,m)$. Then there exists $c \in \mathbb{R}$ such that $u = cv$ a.e. and $\lambda = \lambda_1(V,m)$.*

Similarly, if u is an eigenfunction of problem (1.1) *associated with $\lambda_{-1}(V,m)$ with $u > 0$ a.e. and v is eigenfunction associated with an eigenvalue $\lambda \leq \lambda_1(V,m)$ with $v > 0$ a.e. then there exists $c \in \mathbb{R}$ such that $u = cv$ a.e. and $\lambda = \lambda_{-1}(V,m)$.*

Proof. Let us apply Picone's inequality given in Lemma 7.1 to the functions u and $v+\varepsilon$ with $\varepsilon > 0$. By Remark 4.3,

$$\begin{aligned}
0 &\leqslant \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{L(u,v+\varepsilon)(x,y)}{|x-y|^{N+sp}}\,dx\,dy \\
&= [u]^p_{W^{s,p}(\mathbb{R}^N)} - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \\
&\quad\times \frac{|v(y)-v(x)|^{p-2}(v(y)-v(x))}{|x-y|^{N+sp}}\Big(\frac{u^p(y)}{(v(y)+\varepsilon)^{p-1}} - \frac{u^p(x)}{(v(x)+\varepsilon)^{p-1}}\Big)\,dx\,dy \\
&= \frac{1}{\mathcal{K}(1-s)}\Big(\lambda_1(V,m)\int_\Omega m(x)|u|^p dx - \int_\Omega V(x)|u|^p dx\Big) \\
&\quad - \frac{1}{\mathcal{K}(1-s)}\Big(\lambda\int_\Omega m(x)|v|^{p-1}\frac{u^p}{(v+\varepsilon)^{p-1}}dx - \int_\Omega V(x)|v|^{p-1}\frac{u^p}{(v+\varepsilon)^{p-1}}dx\Big).
\end{aligned}$$

By using the Lebesgue dominated convergence theorem and passing to the limit we have

$$0 \leqslant \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{L(u,v)(x,y)}{|x-y|^{N+sp}}\,dx\,dy \leqslant \frac{\lambda_1(V,m)-\lambda}{\mathcal{K}(1-s)}\int_\Omega m(x)|u|^p dx.$$

Therefore if $\alpha(V,m) > 0$ and $\lambda > \lambda_1(V,m)$, as we have $\int_\Omega m(x)|u|^p dx > 0$, we conclude from the previous inequality that $\lambda_1(V,m) = \lambda$ and $L(u,v) = 0$. Hence, by Picone's inequality there is a constant $c > 0$ such that $u = cv$. In the case $\alpha(V,m) = 0$ we have $\int_\Omega m(x)|u|^p dx = 0$ and we can conclude from the previous calculation that $L(u,v) = 0$. Hence, by the conclusions of Picone's inequality, there is a constant $c > 0$ such that $u = cv$ from which we deduce that $E_V(v) = \int_\Omega m(x)|v|^p dx = 0$. Thus, according to the result (ii)(b) of Theorem 4.2, v is an eigenfunction associated with $\lambda_1(V,m)$, and therefore one must have $\lambda = \lambda_1(V,m)$. □

5. Nodal domains and isolation of the principal eigenvalues

5.1. Measure of the nodal domains of non principal eigenvalues. By a *nodal domain* of a function $v \in \widetilde{W}^{s,p}(\Omega) \cap C(\Omega)$, $v \not\equiv 0$, we mean a maximal connected open subset of either $\{x \in \Omega : v(x) > 0\}$ or $\{x \in \Omega : v(x) < 0\}$.

Theorem 5.1. *Let v be an eigenfunction of (1.1) associated with an eigenvalue λ different from $\lambda_1(V,m)$ and $\lambda_{-1}(V,m)$. Then there exists constant $C = C(s,p,N,\Omega) > 0$ such that, if $\mathcal{N}$ is a nodal domain of v, then*

$$|\mathcal{N}| \geq \big(C\|V - \lambda m\|_{L^r(\Omega)}\big)^{-\gamma} > 0, \tag{5.1}$$

for

$$\gamma = \frac{r'q}{q - r'p} \quad \text{with} \quad \begin{cases} q = \infty & \text{if } N < sp \\ q \geq p & \text{if } N = sp \\ q = p_s^* & \text{if } N > sp. \end{cases}$$

Proof. Let $\mathcal{N}$ be a nodal domain, and assume for instance that $v < 0$ on $\mathcal{N}$. Let us take $\varphi = v^-.\chi_{\mathcal{N}}$ as test function in (1.1). Notice that trivially $\varphi \in \widetilde{W}^{(s,p)}(\Omega)$. Thus

$$\begin{aligned} &\mathcal{K}(1-s)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(y)-v(x)|^{p-2}\,(v(y)-v(x))\,(\varphi(y)-\varphi(x))}{|x-y|^{N+sp}}\,dx\,dy \\ &= \int_{\mathcal{N}} (\lambda m - V)|v^-|^p dx \end{aligned}$$

so

$$\mathcal{K}(1-s)\big[\varphi\big]^p_{W^{s,p}(\mathbb{R}^N)} = \int_{\mathcal{N}} (V - \lambda m)|v^-|^p dx \leq \|V - \lambda m\|_{L^r(\Omega)} \Big(\int_{\mathcal{N}} |v^-|^{pr'} dx\Big)^{1/r'}.$$

Let us start with the case $N > ps$. By the previous Sobolev embedding theorem, for some constant $c > 0$, we have

$$c\|\varphi\|^p_{L^{p_s^*}(\Omega)} \leq \big[\varphi\big]^p_{W^{s,p}(\mathbb{R}^N)}.$$

Hence

$$\begin{aligned} c\mathcal{K}(1-s)\|\varphi\|^p_{L^{p_s^*}(\Omega)} &\leq \mathcal{K}(1-s)\big[\varphi\big]^p_{W^{s,p}(\mathbb{R}^N)} \\ &\leq \|V - \lambda m\|_{L^r(\Omega)} \Big(\int_{\mathcal{N}} |v^-|^{pr'} dx\Big)^{1/r'} \\ &\leq \|V - \lambda m\|_{L^r(\Omega)} \|v^-\|^p_{L^{p_s^*}(\Omega)} |\mathcal{N}|^{\frac{1}{r'} - \frac{p}{p_s^*}}, \end{aligned}$$

and the estimate (5.1) follows.

If $N = sp$ there exists some $c > 0$ such that for all $q \geq p$,

$$\begin{aligned} c\mathcal{K}(1-s)\|\varphi\|^p_{L^q(\Omega)} &\leq \mathcal{K}(1-s)\big[\varphi\big]^p_{W^{s,p}(\mathbb{R}^N)} \\ &\leq \|V - \lambda m\|_{L^r(\Omega)} \Big(\int_{\mathcal{N}} |v^-|^{pr'} dx\Big)^{1/r'} \\ &\leq \|V - \lambda m\|_{L^r(\Omega)} \|v^-\|^p_{L^q(\Omega)} |\mathcal{N}|^{\frac{1}{r'} - \frac{p}{q}}, \end{aligned}$$

and the estimate (5.1) follows.

In the case $N < sp$, there exists some $c > 0$ such that

$$c\mathcal{K}(1-s)\|\varphi\|^p_{L^\infty(\Omega)} \leq \mathcal{K}(1-s)\big[\varphi\big]^p_{W^{s,p}(\mathbb{R}^N)}$$

$$
\begin{aligned}
&\le \|V-\lambda m\|_{L^r(\Omega)}\Big(\int_{\mathcal{N}}|v^-|^{pr'}\,dx\Big)^{1/r'}\\
&\le \|V-\lambda m\|_{L^r(\Omega)}\|v^-\|^p_{L^\infty(\Omega)}|\mathcal{N}|^{1/r'},
\end{aligned}
$$

and the estimate (5.1) follows. □

The following statement is a straightforward consequence of the above theorem.

Corollary 5.2. *Any weak solution of* (1.1) *has a finite number of nodal domains.*

Proof. Let $\mathcal{N}_j$ be a nodal domain of a certain eigenfunction associated with an eigenvalue. Let us assume by contradiction that there exists an infinity of nodal domains $(\mathcal{N}_j)_{j\ge 1}$ of this eigenfunction. We know that according to (5.1) there exists a positive constant $c>0$ such that we have

$$
|\mathcal{N}_j| > c \quad \forall j\,.
$$

Thus

$$
|\Omega| \geqslant \sum_j |\mathcal{N}_j| > c\sum_j 1\,,
$$

which is a contradiction. □

5.2. **Isolation of** $\lambda_1(V,m)$ **and** $\lambda_{-1}(V,m)$**.** The following theorem states that the eigenvalues $\lambda_{\pm1}(V,m)$ are isolated provided $\alpha(V,m)\ge 0$. Notice that if $\alpha(V,m)>0$, there are no eigenvalues in the interval $(\lambda_{-1}(V,m),\lambda_1(V,m))$.

Theorem 5.3. *Let* $\alpha(V,m)\ge 0$. *The eigenvalues* $\lambda_{\pm1}(V,m)$ *are isolated in the spectrum of* (1.1), *that is to say that there exists* $\delta_\pm>0$ *such that there are no eigenvalues in the intervals* $(\lambda_1(V,m),\lambda_1(V,m)+\delta_+)$ *and* $(\lambda_{-1}(V,m)-\delta_-,\,\lambda_{-1}(V,m))$.

Proof. We only prove the result for $\lambda_1(V,m)$ by arguing by contradiction. Let us assume that there exists a sequence $(\lambda_k)_k$ of eigenvalues such that

$$
\lambda_k > \lambda_1(V,m) \quad \text{and} \quad \lim_{k\to\infty}\lambda_k = \lambda_1(V,m).
$$

Denote by u_k a positive eigenfunction associated with λ_k. Replacing u_k by $u_k/[u_k]_{\widetilde{W}^{s,p}(\Omega)}$ if necessary, we can assume that the sequence $(u_k)_k$ is bounded. By the results on compact embeddings, there exists a subsequence (still denoted $(u_k)_k$) converging to some $u\in\widetilde{W}^{s,p}(\Omega)$ weakly in $\widetilde{W}^{s,p}(\Omega)$, strongly in $L^{r'p}(\Omega)$, a.e. and in measure in Ω such that

$$
\lim_{k\to\infty}\int_\Omega V(x)|u_k|^p\,dx = \int_\Omega V(x)|u|^p\,dx, \quad \lim_{k\to\infty}\int_\Omega m(x)|u_k|^p\,dx = \int_\Omega m(x)|u|^p\,dx.
$$

Since u_k is an eigenfunction associated with λ_k we have

$$
\begin{aligned}
E_V(u_k) &= \mathcal{K}(1-s)\big[u_k\big]^p_{W^{s,p}(\mathbb{R}^N)} + \int_\Omega V(x)|u_k|^p\,dx\\
&= \mathcal{K}(1-s) + \int_\Omega V(x)|u_k|^p\,dx = \lambda_k\int_\Omega m(x)|u_k|^p\,dx.
\end{aligned}
$$

Thus passing to the limit and using that E_V is weakly lower semi-continuous we obtain

$$
E_V(u) = \mathcal{K}(1-s)[u]^p_{W^{s,p}(\mathbb{R}^N)} + \int_\Omega V(x)|u|^p\,dx
$$

$$\le \mathcal{K}(1-s) + \int_\Omega V(x)|u|^p\,dx = \lambda_1(V,m)\int_\Omega m(x)|u|^p\,dx.$$

In particular $u \not\equiv 0$ and

$$E_V(u) \le \lambda_1(V,m)\int_\Omega m(x)|u|^p\,dx. \tag{5.2}$$

Assume first that $\alpha(V,m) > 0$. Then (5.2) implies that $\int_\Omega m(x)|u|^p\,dx \ne 0$. In fact we have $\int_\Omega m(x)|u|^p\,dx > 0$ otherwise, by taking $v = u/\big(-\int_\Omega m(x)|u|^p\,dx\big)^{1/p} \in \mathcal{M}^-$ we will have from the definition of $\lambda_{-1}(V,m)$ that

$$-\lambda_{-1}(V,m) \le E_V(v) = \frac{E_V(u)}{-\int_\Omega m(x)|u|^p\,dx} \Longrightarrow \lambda_{-1}(V,m)\int_\Omega m(x)|u|^p\,dx \le E_V(u)$$

which, jointly with the inequality (5.2) will give $\lambda_{-1}(V,m) \ge \lambda_1(v,M)$, a contradiction. Since we have proved that $\int_\Omega m(x)\,|u|^p\,dx > 0$ we then have, by definition of $\lambda_1(V,m)$, that

$$\lambda_1(V,m)\int_\Omega m(x)\,|u|^p\,dx \le E_V(u)$$

and therefore $\lambda_1(V,m)\int_\Omega m(x)\,|u|^p\,dx = E_V(u)$. Thus, u is an eigenfunction associated with the principal eigenvalue $\lambda_1(V,m)$ and it must be either positive a.e. or negative a.e. in Ω. On the other hand, if for each k we denote $\mathcal{N}_k^+ := \{x \in \Omega :\ u_k(x) > 0\}$ and $\mathcal{N}_k^- := \{x \in \Omega :\ u_k(x) < 0\}$, by Theorem 5.1, we obtain the existence of a constant $c > 0$ such that $|\mathcal{N}_k^+| > c$ and $|\mathcal{N}_k^-| > c$. However, if we assume that $u > 0$ (the case $u < 0$ is analogous) it follows from the convergence in measure that $|\mathcal{N}_k^-| \to 0$, which is a contradiction.

Assume now that $\alpha(V,m) = 0$. We claim that $\int_\Omega m(x)|u|^p\,dx = 0$. Indeed, if for instance $\int_\Omega m(x)|u|^p\,dx > 0$ then we will have, by definition of $\lambda_1(V,m)$, that

$$\lambda_1(V,m)\int_\Omega m(x)|u|^p\,dx \le E_V(u)$$

that, jointly with equation (5.2) will give that the infimum $\lambda_1(V,m)$ is achieved, a contradiction. If $\int_\Omega m(x)|u|^p\,dx < 0$ then we will have instead

$$\lambda_{-1}(V,m)\int_\Omega m(x)\,|u|^p\,dx \le E_V(u)$$

and, since $\lambda_1(V,m) = \lambda_{-1}(V,m)$, we again get a contradiction. We have just proved that $\int_\Omega m(x)|u|^p\,dx = 0$. Hence, by equation (5.2) $E_V(u) \le 0$, it must be $E_V(u) = 0$ by the definition of $\alpha(V,m) = 0$. Thus u is an eigenfunction associated with $\lambda_1(V,m)$ so u must be either > 0 a.e. or < 0 a.e. in Ω and we obtain a contradiction as in the previous case. □

6. Regularity of the principal eigenvalues with respect to s

Now we study the behaviour of the first eigenvalues $\lambda_{\pm 1}(V,m)$ with respect to s. As we want to vary s, then to simplify the study, we now impose conditions on V and m which are independent of s. So we assume V, m in $L^r(\Omega)$ with $r > \max\{1, \frac{N}{p}\}$.

We start by proving a lemma in the behaviour of sequences $(1-s)[u_s]^p_{W^{s,p}(\mathbb{R}^N)}$ as s varies.

Lemma 6.1. *Let $s_0 \in (0,1]$ and $(s_n)_n$ be a sequence in $(0,1)$ converging to s_0. Let (u_n) be a sequence of functions such that, for all $n \in \mathbb{N}$, $u_n \in \widetilde{W}^{s_n,p}(\Omega)$ and*

$$(1-s_n)[u_n]^p_{W^{s_n,p}(\mathbb{R}^N)} \le L$$

for some $L \ge 0$. Let $q \in [1,p^*)$. Then there exists a function $u \in \widetilde{W}^{s_0,p}(\Omega)$ such that, up to a subsequence,

1. *$[u]^p_{W^{s_0,p}(\mathbb{R}^N)} \le \liminf_{n\to\infty}[u_n]^p_{W^{s_n,p}(\mathbb{R}^N)}$ if $s_0 < 1$ and $\int_\Omega |\nabla u|^p\,dx \le \liminf_{n\to\infty}(1-s_n)\mathcal{K}[u_n]^p_{W^{s_n,p}(\mathbb{R}^N)}$ if $s_0 = 1$.*
2. *$u_n \to u$ in $L^q(\Omega)$ and $u_n \to u$ in $L^p(\Omega)$.*

Proof. First of all, by Poincaré's inequality,

$$\|u_n\|_p^p \le C(N,p)(diam(\Omega))^{s_np}(1-s_n)[u_n]^p_{W^{s_n,p}(\mathbb{R}^N)} \le C(N,p)(diam(\Omega)^{s_np}L \le C$$

for some constant depending only $N, p, diam(\Omega), s_0$ and L. Assume first that $s_0 < 1$ and let $\varepsilon > 0$ be small enough. Observe that since $q < p^*$ it follows that $q < p^*_{s_0-\varepsilon}$ if ε is small enough. Hence, if $s_0 - \varepsilon < s_n$, using property 2 of Proposition 2.1, and the previous estimate we have

$$[u_n]^p_{W^{s_0-\varepsilon,p}(\mathbb{R}^N)} \le [u_n]^p_{W^{s_n,p}(\mathbb{R}^N)} + C(N,p)\Big(\frac{1}{(s_0-\varepsilon)p} - \frac{1}{s_np}\Big) \le C \tag{6.1}$$

for some C independent of n. Then there exists $u \in \widetilde{W}^{s_0-\varepsilon,p}(\Omega)$ and a subsequence, still denoted by $(u_n)_n$, such that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } W^{s_0-\varepsilon,p}(\mathbb{R}^N),\\ u_n &\to u \quad \text{in } L^q(\Omega),\\ u_n &\to u \quad \text{in } L^p(\Omega), \end{aligned}$$

where we have used the compact imbedding of $W^{1-\varepsilon,p}(\Omega)$ into $L^q(\Omega)$ and into $L^p(\Omega)$. Hence for all $\varepsilon > 0$, using (6.1),

$$\begin{aligned} [u]^p_{W^{s_0-\varepsilon,p}(\mathbb{R}^N)} &\le \liminf_{n\to\infty}[u_n]^p_{W^{s_0-\varepsilon,p}(\mathbb{R}^N)}\\ &\le \liminf_{n\to\infty}[u_n]^p_{W^{s_n,p}(\mathbb{R}^N)} + C(N,p)\Big(\frac{1}{(s_0-\varepsilon)p} - \frac{1}{s_0p}\Big). \end{aligned}$$

Letting $\varepsilon \to 0$ and using Fatou's lemma the conclusion 1 is reached.

If $s_0 = 1$, by Lemma 3.10 of [5] we infer the existence of $u \in W^{1,p}_0(\Omega)$ such that, up to a subsequence, $u_n \to u$ in $L^p(\Omega)$. Moreover, using property 3 of Proposition 2.2 and the hypothesis we obtain that, since $1 - \varepsilon < s_n$ if n is large enough,

$$\varepsilon[u_n]^p_{W^{1-\varepsilon,p}(\Omega)} \le (1-s_n)2^{(1-s_n+\varepsilon)p}\operatorname{diam}(\Omega)^{(s_n-1+\varepsilon)p}[u_n]^p_{W^{s_n,p}(\Omega)} \le C \tag{6.2}$$

for some C independent of n. Thus, the sequence (u_n) is bounded in $W^{1-\varepsilon,p}(\Omega)$. Hence there exists $u \in \widetilde{W}^{1-\varepsilon,p}(\Omega)$ and a subsequence, still denoted by $(u_n)_n$, such that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } W^{1-\varepsilon,p}(\Omega),\\ u_n &\to u \quad \text{in } L^q(\Omega),\\ u_n &\to u \quad \text{in } L^p(\Omega). \end{aligned}$$

Thus, letting $n \to \infty$ in (6.2) and using that $u_n \rightharpoonup u$ in $W^{1-\varepsilon,p}(\mathbb{R}^N)$ we obtain

$$\varepsilon[u]^p_{W^{1-\varepsilon,p}(\Omega)} \le \liminf_{n\to\infty}\varepsilon[u_n]^p_{W^{1-\varepsilon,p}(\Omega)} \le 2^{\varepsilon p}\operatorname{diam}(\Omega)^{\varepsilon p}\liminf_{n\to\infty}(1-s_n)[u_n]^p_{W^{s_n,p}(\mathbb{R}^N)}.$$

Finally, letting $\varepsilon \to 0$ and using Corollary 2 of [7] we obtain the result of 1. $\square$

Our next result concerns the hypothesis on $\alpha(V,m)$ that allow us to have principal eigenvalues. As we want to study the sign of $\alpha(V,m)$ as s varies, for $s \in (0,1]$ let us write

$$\alpha(s) = \inf\Big\{(1-s)\mathcal{K}[u]^p_{W^{s,p}(\mathbb{R}^N)} + \int_\Omega V(x)|u|^p\,dx : u \in \widetilde{W}^{s,p}(\Omega),\ \|u\|_p = 1,\ \text{and}\ \int_\Omega m(x)|u|^p\,dx = 0\Big\}$$

if $s \neq 1$, and

$$\alpha(s) = \inf\Big\{\int_\Omega |\nabla u|^p\,dx + \int_\Omega V(x)|u|^p\,dx : u \in W^{1,p}_0(\Omega),\ \|u\|_p = 1 \text{ and } \int_\Omega m(x)|u|^p\,dx = 0\Big\}$$

if $s = 1$.

Proposition 6.2. *Let $s_0 \in (0,1]$ and assume that $\alpha(s_0) > 0$. Then there exists $\varepsilon > 0$ such that $\alpha(s) > 0$ for all $s \in (s_0-\varepsilon, s_0+\varepsilon) \cap (0,1]$.*

Proof. Assume by contradiction that there exists a sequence $s_n \to s_0$ and a function $u_n \in \widetilde{W}^{s_n,p}(\Omega)$ such that

$$(1-s_n)\mathcal{K}[u_n]^p_{W^{s_n,p}(\mathbb{R}^N)} + \int_\Omega V(x)|u_n|^p dx \le 0, \quad \|u_n\|_p = 1, \quad \int_\Omega m(x)|u_n|^p\,dx = 0.$$

Let $t_n = \|u_n\|_{r'p}$ and distinguish two cases.
Case (a): the sequence $(t_n)_n$ is bounded. Then the sequence $(1-s_n)\mathcal{K}[u_n]^p_{W^{s_n,p}(\mathbb{R}^N)}$ is bounded.
Case (b): the sequence $(t_n)_n$ tends to $+\infty$. Then taking $v_n = u_n/t_n$ we have

$$(1-s_n)\mathcal{K}[v_n]^p_{W^{s_n,p}(\mathbb{R}^N)} + \int_\Omega V(x)|v_n|^p\,dx \le 0$$

and the sequence $(1-s_n)\mathcal{K}[v_n]^p_{W^{s_n,p}(\mathbb{R}^N)}$ is bounded. Let us write $z_n = u_n$ if case (a) occurs and $z_n = v_n$ if case (b) occurs. Let us now distinguish the cases $0 < s_0 < 1$ and the case $s_0 = 1$.
1. Case $0 < s_0 < 1$. It follows from Lemma 6.1 with $q = r'p$ that there exists $z \in \widetilde{W}^{s_0,p}(\Omega)$ such that, in case (a),

$$\begin{aligned}
&(1-s_0)\mathcal{K}[z]^p_{W^{s_0,p}(\mathbb{R}^N)} + \int_\Omega V(x)|z|^p dx \\
&\le \liminf_{n\to\infty}\Big((1-s_n)\mathcal{K}[z_n]^p_{W^{s_n,p}(\mathbb{R}^N)} + \int_\Omega V(x)|z_n|^p dx\,dx\Big) \le 0, \quad \|z\|_p = 1,
\end{aligned}$$

and the same inequality holds in case (b) with $\|z\|_{r'p} = 1$. Since $\int_\Omega m(x)|z|^p = 0$ we have a contradiction with $\alpha(s_0) > 0$.
2. Case $s_0 = 1$. We obtain similarly that the sequence $(1-s_n)\mathcal{K}[z_n]^p_{W^{s_n,p}(\mathbb{R}^N)}$ is bounded, with either $\|z_n\|_p = 1$ or $\|z_n\|_{r'p} = 1$. By Lemma 6.1 there exists $z \in W^{1,p}_0(\Omega)$ such that

$$\int_\Omega |\nabla u|^p\,dx + \int_\Omega V(x)|z|^p\,dx \le \liminf_{n\to\infty}(1-s_n)\mathcal{K}[z_n]^p_{W^{s_n,p}(\mathbb{R}^N)} + \int_\Omega V(x)|z_n|^p\,dx \le 0.$$

Notice that again $\|z\|_q = 1$, with either $q = p$ or $q = r'p$, and $\int_\Omega m|(x)z|^p = 0$. Thus $\alpha(1) \le 0$, a contradiction. $\square$

Let us now write

$$\lambda_{\pm 1}(s) := \lambda_{\pm 1}(V, m) = \pm \inf \Big\{ \mathcal{K}(1-s) \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p \, dx \, dy}{|x-y|^{N+sp}} + \int_\Omega V|u|^p dx : \\ u \in \widetilde{W}^{s,p}(\Omega), \int_\Omega m|u|^p dx = \pm 1 \Big\}$$

and

$$\lambda_{\pm 1} := \pm \inf \Big\{ \int_\Omega \Big(|\nabla u|^p + V(x)|u|^p \Big) dx : u \in W_0^{1,p}(\Omega), \int_\Omega m(x)|u|^p dx = \pm 1 \Big\}.$$

We have the following result that generalizes, for indefinite weights, [10, Lemma 4.12].

Proposition 6.3. *Assume that for some $s_0 \in (0,1]$, $\alpha(s_0) > 0$. Then*

$$\lim_{s \to s_0} \lambda_{\pm 1}(s) = \lambda_{\pm 1}(s_0).$$

Proof. We only give the proof for $\lambda_1(s)$. By Proposition 6.2 $\alpha(s) > 0$ for s close to s_0, so $\lambda_1(s)$ is a principal eigenvalue associated with problem (1.1). Let $(s_n)_n$ be a sequence in $(0,1]$ converging to $s_0 \in (0,1]$. Let us show that

$$\lim_{n \to +\infty} \lambda_1(s_n) = \lambda_1(s_0). \tag{6.3}$$

1. Case $s_0 \in (0,1)$. By definition of the first eigenvalue, we know that if $\varphi \in C_0^\infty(\Omega)$ and $\int_\Omega m(x)|\varphi|^p = 1$, then

$$\lambda_1(s_n) \le \mathcal{K}(1-s_n) \int_{\mathbb{R}^{2N}} \frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{N+s_n p}} \, dx \, dy + \int_\Omega V(x)|\varphi|^p \, dx$$

for all $n \in \mathbb{N}$. Therefore, by dominated convergence theorem, we obtain $\limsup_{n \to +\infty} \lambda_1(s_n) \le \lambda_1(s_0)$. To prove the reverse inequality let $(s_{n_k})_k$ be a subsequence of $(s_n)_n$ such that

$$\lim_{k \to +\infty} \lambda_1(s_{n_k}) = \liminf_{n \to +\infty} \lambda_1(s_n).$$

Let $0 \le u_{n_k} \in \widetilde{W}^{s_k,p}(\Omega)$ be an eigenfunction associated with $\lambda_1(s_{n_k})$ such that $\int_\Omega m|u_{n_k}|^p \, dx > 0$ and

$$[u_{n_k}]^p_{W^{s_{n_k},p}(\mathbb{R}^N)} = 1,$$

then in particular, using u_{n_k} as test function in equation (1.1) for $\lambda = \lambda(s_{n_k})$, we have

$$\lambda_1(s_{n_k}) \int_\Omega m(x)|u_{n_k}|^p dx = \mathcal{K}(1-s_{n_k}) + \int_\Omega V(x)|u_{n_k}|^p dx. \tag{6.4}$$

By Lemma 6.1 there exists $u \in \widetilde{W}^{s_0,p}(\Omega)$ such that, up to a subsequence,

$$[u]^p_{W^{s_0,p}(\mathbb{R}^N)} \le \liminf_{k \to \infty} [u_{n_k}]^p_{W^{s_{n_k},p}(\mathbb{R}^N)} = 1, \quad u_{n_k} \to u \text{ in } L^{r'p}(\Omega) \text{ and } u_{n_k} \to u \text{ in } L^p(\Omega). \tag{6.5}$$

Hence, using (6.4) we find on the one hand that

$$\liminf_{k \to \infty} \lambda_1(s_{n_k}) \int_\Omega m(x)|u|^p dx = \mathcal{K}(1-s_0) + \int_\Omega V(x)|u|^p \, dx \tag{6.6}$$

and, on the other hand using (6.4) and (6.6),

$$\begin{aligned}
&\mathcal{K}(1-s_0)[u]^p_{W^{s_0,p}(\mathbb{R}^N)} + \int_\Omega V(x)|u|^p\,dx \\
&\le \mathcal{K}(1-s_0) + \liminf_{k\to\infty}\int_\Omega V(x)|u_{n_k}|^p\,dx \\
&= \liminf_{k\to\infty}\lambda_1(s_{n_k})\int_\Omega m(x)u|^p\,dx.
\end{aligned} \tag{6.7}$$

It remains to prove that $\int_\Omega m(x)|u|^p\,dx > 0$ to conclude from the previous inequality that

$$\lambda_1(s_0) \le \liminf_{k\to\infty}\lambda(s_{n_k})$$

(notice that the function $v = u/\big(\int_\Omega m(x)|u|^p\,dx\big)^{1/p}$ will be then admissible in the definition of $\lambda_1(s_0)$) and the proof of the proposition is completed.

To prove that $\int_\Omega m(x)|u|^p\,dx > 0$, remember first that $\int_\Omega m(x)|u_{n_k}|^p\,dx > 0$ for all $k \in \mathbb{N}$ and assume by contradiction that $\int_\Omega m(x)|u|^p\,dx = 0$. Using (6.6) we infer that $u \not\equiv 0$ and, using (6.7) we obtain

$$\mathcal{K}(1-s_0)[u]^p_{W^{s_0,p}(\mathbb{R}^N)} + \int_\Omega V(x)|u|^p\,dx \le 0,$$

a contradiction with the hypothesis $\alpha(s_0) > 0$.

2. Case $s_0 = 1$. Let $\varphi \in C_0^\infty(\Omega)$ such that $\int_\Omega m(x)|\varphi|^p\,dx = 1$. Then for any $n \in \mathbb{N}$,

$$\lambda_1(s_n) \le \mathcal{K}(1-s_n)\int_{\mathbb{R}^{2N}} \frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{N+s_np}}\,dx\,dy + \int_\Omega V(x)|\varphi|^p dx.$$

Thus, by Proposition 2.2,

$$\limsup_{n\to+\infty}\lambda_1(s_n) \le \int_\Omega |\nabla\varphi|^p dx + \int_\Omega V(x)|\varphi|^p dx.$$

As φ is arbitrary, we have

$$\limsup_{n\to+\infty}\lambda_1(s_n) \le \lambda_1(1).$$

As in the previous case, let us prove that

$$\liminf_{n\to+\infty}\lambda_1(s_n) \ge \lambda_1(1).$$

Let $(s_{n_k})_k$ be a subsequence of $(s_n)_n$ such that

$$\lim_{k\to+\infty}\lambda_1(s_{n_k}) = \liminf_{n\to+\infty}\lambda_1(s_n). \tag{6.8}$$

Let u_k be an eigenfunction associated with $\lambda_1(s_{n_k})$ such that

$$\mathcal{K}(1-s_k)[u_k]^p_{W^{s_k,p}(\mathbb{R}^N)} = 1. \tag{6.9}$$

Then, as u_{n_k} is an eigenfunction we have

$$\begin{aligned}
\lambda_1(s_{n_k})\int_\Omega m(x)|u_{n_k}|^p dx &= \mathcal{K}(1-s_k)[u_{n_k}]^p_{W^{s_k,p}(\mathbb{R}^N)} + \int_\Omega V(x)|u_{n_k}|^p dx \\
&= 1 + \int_\Omega V(x)|u_{n_k}|^p dx.
\end{aligned} \tag{6.10}$$

By Lemma 6.1 there exists $u \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} &\int_\Omega |\nabla u|^p\,dx + \int_\Omega V(x)|u|^p\,dx \\ &\le \liminf_{k\to\infty} \Big((1-s_{n_k})\mathcal{K}[u_{n_k}]^p_{W^{s_k,p}(\mathbb{R}^N)} + \int_\Omega V(x)|u_{n_k}|^p\,dx\Big) \\ &= \liminf_{k\to\infty} \lambda_1(s_{n_k}) \int_\Omega m(x)|u|^p dx. \end{aligned} \tag{6.11}$$

Thus, if $\int_\Omega m(x)|u|^p\,dx > 0$, we can conclude, rescaling the previous inequality, that

$$\lambda_1 \le \liminf_{k\to+\infty} \lambda_1(s_{n_k})$$

and the proof of the proposition is complete. To prove that $\int_\Omega m(x)|u|^p\,dx > 0$ we argue as before using now equations (6.10), (6.11) and that $\alpha(1) > 0$ by hypothesis. □

7. Appendix A

The following two results are, essentially, consequence of the convexity of the function $t \mapsto |t|^{p-2}t$.

Lemma 7.1. *A discrete version of Picone's inequality* [1] *Let $p \in (1,+\infty)$. For all functions ξ and ϕ defined on $\mathbb{R}^N$ such that $\xi \geqslant 0$, and $\phi > 0$, we have*

$$L(\xi,\phi) \geqslant 0 \quad \text{on } \mathbb{R}^N \times \mathbb{R}^N$$

with

$$L(\xi,\phi)(x,y) := |\xi(y)-\xi(x)|^p - |\phi(y)-\phi(x)|^{p-2}\,(\phi(y)-\phi(x))\Big(\frac{\xi^p(y)}{\phi(y)^{p-1}} - \frac{\xi^p(x)}{\phi(x)^{p-1}}\Big),$$

for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$. Moreover, we have

$$L(\xi,\phi) = 0 \iff \exists k \in \mathbb{R} \text{ s.t. } \phi = k\xi.$$

Proof. For sake of completeness we give the proof of this inequality. It uses the following convexity inequality due to [1]. Fix x, y in $\mathbb{R}^N$ and put $a = \xi(y)$, $b = \xi(x)$, $t = \frac{\phi(x)}{\phi(y)}$ and assume that $0 < b < a$. It suffices to prove that for any $p > 1$ and $0 < t < 1$, one has

$$|a-b|^p \ge a^p(1-t)^{p-1} - b^p\big(\frac{1}{t} - 1\big)^{p-1}$$

which is equivalent to say that

$$(1-t)\Big(\frac{|a-b|^p}{(1-t)^p}\Big) + t\frac{b^p}{t^p} > a^p$$

which follows from the convexity of the function $f(x) = |x|^p$. Notice that the equality on this inequalities arrives if and only if $t = b/a$, that is, $L(\xi,\phi)(x,y) = 0$ for all $x, y \in \mathbb{R}^N$ if and only if $\xi/\phi = cte$. □

Let us quote without proof the following second estimate.

Lemma 7.2 ([6, Lemma A_1])**.** *Let $1 < p < \infty$ and $g : \mathbb{R} \to \mathbb{R}$ be a convex function, then*

$$\begin{aligned} &|a-b|^{p-2}(a-b)\big[A|g'(a)|^{p-2}g'(a) - B|g'(b)|^{p-2}g'(b)\big] \\ &\ge |g(a)-g(b)|^{p-2}\big(g(a)-g(b)\big)\big(A-B\big), \end{aligned} \tag{7.1}$$

for every $a, b \in \mathbb{R}$, and every $A, B \geq 0$.

8. Appendix B

For completeness we give the following regularity result for the nonlocal nonhomogeneous problem

$$(-\Delta_p)^s u + V(x)|u|^{p-2}u = f(x) \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^N \setminus \Omega. \tag{8.1}$$

Proposition 8.1. *Assume $sp < N$, $f \in L^q(\Omega)$, $V \in L^q(\Omega)$ for some $q \geq N/sp$, and $u \in \widetilde{W}^{s,p}(\Omega)$ is a solution of (8.1). Then for any $t \in [1, +\infty)$, $u \in L^t(\Omega)$ and there exists a constant C_t depending on t and on Ω, $\|V\|_{N/sp}$, $\|f\|_{N/sp}$, N, s, p such that*

$$\|u\|_{L^t(\Omega)} \leq C_t. \tag{8.2}$$

Proof. We borrow some ideas from [6, 18]. For every $0 < \varepsilon \ll 1$ and any positive function $\varphi \in C_0^\infty(\Omega)$ we define the smooth convex Lipschitz function

$$g_\varepsilon(t) = (\varepsilon^2 + t^2)^{1/2},$$

and choose the test function $\psi = \varphi |g'_\varepsilon(u)|^{p-2} g'_\varepsilon(u)$ in the variational formulation of (8.1). Then we obtain

$$\begin{aligned}
&\mathcal{K}(1-s) \int_{\mathbb{R}^{2N}} \frac{\big|u(x) - u(y)\big|^{p-2}\big(u(x) - u(y)\big)}{|x-y|^{N+sp}} \big(\varphi(x)|g'_\varepsilon\big(u(x)\big)|^{p-2} g'_\varepsilon\big(u(x)\big) \\
&- \varphi(y)|g'_\varepsilon\big(u(y)\big)|^{p-2} g'_\varepsilon\big(u(y)\big)\big)\,dx\,dy \\
&\leq \int_\Omega |f(x)\varphi(x)|g'_\varepsilon\big(u(x)\big)|^{p-2} g'_\varepsilon\big(u(x)\big)|dx \\
&\quad + \int_\Omega |V(x)|u(x)|^{p-1}\varphi(x)|g'_\varepsilon\big(u(x)\big)|^{p-2} g'_\varepsilon\big(u(x)\big)|dx.
\end{aligned}$$

By using (7.1) with $a = u(x)$, $b = u(y)$, $A = \varphi(x)$ and $B = \varphi(y)$ we have

$$\begin{aligned}
&\mathcal{K}(1-s) \int_{\mathbb{R}^{2N}} \frac{\big|g_\varepsilon\big(u(x)\big) - g_\varepsilon\big(u(y)\big)\big|^{p-2}\big(g_\varepsilon\big(u(x)\big) - g_\varepsilon\big(u(y)\big)\big)}{|x-y|^{N+sp}} \big(\varphi(x) - \varphi(y)\big)\,dx\,dy \\
&\leq \int_\Omega |f(x)|\varphi(x)|g'_\varepsilon\big(u(x)\big)|^{p-1}dx + \int_\Omega |V(x)\|u(x)|^{p-1}\varphi(x)|g'_\varepsilon\big(u(x)\big)|^{p-1}dx.
\end{aligned}$$

By observing that g_ε converges to $g(t) := |t|$ as $\varepsilon \to 0$, $|g'_\varepsilon(t)| \leq 1$ and using Fatou's Lemma, we obtain

$$\begin{aligned}
&\mathcal{K}(1-s) \int_{\mathbb{R}^{2N}} \frac{\big||u(x)| - |u(y)|\big|^{p-2}\big(|u(x)| - |u(y)|\big)}{|x-y|^{N+sp}} \big(\varphi(x) - \varphi(y)\big)\,dx\,dy \\
&\leq \int_\Omega |f(x)|\varphi(x)dx + \int_\Omega |V(x)\|u(x)|^{p-1}\varphi(x)dx.
\end{aligned} \tag{8.3}$$

By the density of $C_0^\infty(\Omega)$ in $\widetilde{W}^{s,p}(\Omega)$ (see Proposition 2.2), the same inequality remains true for any positive $\varphi \in \widetilde{W}^{s,p}(\Omega)$.

For $k > 0$ and $t \geq p$ we define u_k and φ_k as follows:

$$u_k := \min\{|u|, k\} \quad \text{and} \quad \varphi_k(u) := \frac{t^p}{p^p(t-p+1)} u_k^{t-p+1}.$$

By definition, $\varphi_k(u) \in \widetilde{W}^{s,p}(\Omega)$, and then by relation (8.3), we can write

$$\begin{aligned} &\mathcal{K}(1-s)\int_{\mathbb{R}^{2N}} \frac{\big||u(x)|-|u(y)|\big|^{p-2}\big(|u(x)|-|u(y)|\big)}{|x-y|^{N+sp}}(\varphi_k(u)(x)-\varphi_k(u)(y))\,dx\,dy \\ &\le \int_\Omega |f(x)|\varphi_k(u(x))dx + \int_\Omega |V(x)||u(x)|^{p-1}\varphi_k(u(x))dx. \end{aligned} \tag{8.4}$$

For $M>0$, set $\Omega_M := \{x\in\Omega : |V(x)|>M\}$. We have

$$\begin{aligned} \int_\Omega |V(x)||u|^{p-1}\varphi_k(u)dx &\le M\int_{\Omega\setminus\Omega_M} |u(x)|^{p-1}|\varphi_k(u)|dx \\ &\quad + \|V\|_{L^{N/sp}(\Omega_M)}\Big(\int_\Omega \||u(x)|^{p-1}\varphi_k(u)|^{N/(N-sp)}dx\Big)^{N-sp/N}. \end{aligned}$$

Moreover, thanks to [4, Lemma C.2] we have

$$\begin{aligned} &\big||u(x)|-|u(y)|\big|^{p-2}\big(|u(x)|-|u(y)|\big)(u_k^{t-p+1}(x)-u_k^{t-p+1}(y)) \\ &\ge \frac{(t-p+1)p^p}{t^p}\big|u_k(x)^{t/p}-u_k(y)^{t/p}\big|^p. \end{aligned} \tag{8.5}$$

Thus, by (8.4) and (8.5), the relation

$$\begin{aligned} &(1-s)\mathcal{K}\int_{\mathbb{R}^{2N}} \frac{\big|u_k(x)^{t/p}-u_k(y)^{t/p}\big|^p}{|x-y|^{N+sp}}\,dx\,dy \\ &\le \frac{Mt^p}{(t-p+1)p^p}\int_\Omega |u(x)|^{p-1}|\varphi_k(u(x))|dx \\ &\quad + \frac{t^p\|V\|_{L^{N/sp}(\Omega_M)}}{(t-p+1)p^p}\Big(\int_\Omega \||u(x)|^{p-1}\varphi_k(u(x))|^{p_s^*/p}dx\Big)^{p/p_s^*} \\ &\quad + \frac{t^p}{(t-p+1)p^p}\int_\Omega |f(x)|\varphi_k(u(x))dx \end{aligned}$$

holds and by the Sobolev's embedding of $\widetilde{W}^{s,p}(\Omega)$ into $L^{p_s^*}(\Omega)$, there exists a constant $S_{N,s,p}$ such that

$$\begin{aligned} &S_{N,s,p}\Big(\int_\Omega u_k(x)^{\frac{tp_s^*}{p}}\Big)^{p/p_s^*} \\ &\le (1-s)\mathcal{K}\int_{\mathbb{R}^{2N}} \frac{\big|u_k(x)^{t/p}-u_k(y)^{t/p}\big|^p}{|x-y|^{N+sp}}\,dx\,dy \\ &\le \frac{Mt^p}{(t-p+1)p^p}\int_\Omega |u|^{p-1}|\varphi_k(u)|dx + \frac{t^p\|V\|_{L^{N/sp}(\Omega_M)}}{(t-p+1)p^p}\Big(\int_\Omega \||u|^{p-1}\varphi_k(u)|^{p_s^*/p}dx\Big)^{p/p_s^*} \\ &\quad + \frac{t^p}{(t-p+1)p^p}\int_\Omega |f(x)|\varphi_k(u)dx. \end{aligned}$$

If we choose M such that $\|V\|_{L^{N/sp}(\Omega_M)} \le \frac{(t-p+1)p^pS_{N,s,p}}{2t^p}$ and use the definition of φ_k and Hölder's inequality for the last term of right-hand side of the previous inequality, we obtain

$$\begin{aligned} S_{N,s,p}\Big(\int_\Omega |u_k(x)|^{\frac{tp_s^*}{p}}\Big)^{p/p_s^*} &\le \frac{Mt^p}{(t-p+1)p^p}\int_\Omega |u(x)|^t dx + \frac{S_{N,s,p}}{2}\Big(\int_\Omega |u_k(x)|^{\frac{tp_s^*}{p}}\Big)^{p/p_s^*} \\ &\quad + \frac{t^p}{(t-p+1)p^p}\|f\|_{L^{N/sp}(\Omega)}\Big(\int_\Omega |u(x)|^{p_s^*(t+1-p)/p}dx\Big)^{p/p_s^*} \end{aligned}$$

and

$$\Big(\int_\Omega |u(x)|^{p_s^*(t+1-p)/p}dx\Big)^{p/p_s^*} \le |\Omega|^{p(p-1)/p_s^* t}\Big(\int_\Omega |u(x)|^{tp_s^*/p}\Big)^{p(t+1-p)/tp_s^*}.$$

Then for all $t \geqq p$ we obtain

$$\begin{aligned}
&\frac{(t-p+1)p^p S_{N,s,p}}{2t^p}\Big(\int_\Omega |u_k(x)|^{tN/(N-sp)}\Big)^{(N-sp)/N} \\
&\le M\int_\Omega |u(x)|^t dx \\
&\quad + \|f\|_{L^{N/sp}(\Omega)}|\Omega|^{p(p-1)/p_s^* t}\Big(\int_\Omega |u(x)|^{tN/(N-sp)}\Big)^{\frac{(t+1-p)(N-sp)}{tN}}.
\end{aligned} \tag{8.6}$$

Let us set $t_0 = p$. Since by definition, $u \in L^{t_0}(\Omega)$, it follows that $u \in L^{t_0 N/(N-sp)}(\Omega)$, and thus, thanks to Fatou's Lemma we have

$$\begin{aligned}
&\frac{(t_0-p+1)p^p S_{N,s,p}}{2t_0^p}\|u\|^{t_0}_{L^{t_0 N/(N-sp)}} \\
&\le M\|u\|^{t_0}_{L^{t_0}(\Omega)} + |\Omega|^{p(p-1)/p_s^* t_0}\|f\|_{L^{N/sp}}\|u\|^{t_0+1-p}_{L^{t_0 N/N-sp}}.
\end{aligned}$$

Therefore using Young's inequality we obtain

$$\|u\|_{L^{t_0}} \le C_{1,t_0}\|u\|_{L^{t_0}(\Omega)} + C_{2,t_0}\|f\|^{1/(p-1)}_{L^{N/sp}},$$

where C_{1,t_0} and C_{2,t_0} depend on M, N, s, p, t_0, and $|\Omega|$.

Now if we take $t_1 = \frac{t_0 N}{N-sp} \ge p$ and since $u \in L^{t_1}(\Omega)$, it follows that $u \in L^{t_1 N/(N-sp)}(\Omega)$ and we let $k \to +\infty$, by Fatou's Lemma, and using Young's inequality we obtain

$$\|u\|_{L^{t_1 N/(N-sp)}} \le C_{1,t_1}\|u\|_{L^t(\Omega)} + C_{2,t_1}\|f\|^{1/(p-1)}_{L^{N/sp}},$$

where C_{1,t_1} and C_{2,t_1} depend on M, N, s, p, t_1, and $|\Omega|$.

Thus as a consequence, if we define the sequence $(t_l)_{l\in\mathbb{N}}$ by

$$t_0 = p, \quad t_l = \Big(\frac{N}{N-sp}\Big)^l p, \quad l \in \mathbb{N}^*$$

we find that $u \in L^{t_l}(\Omega)$ for any $l \in \mathbb{N}^*$. Since $1 < p \le t_l$ for all $l \in \mathbb{N}$ and $t_l \underset{l\to+\infty}{\longrightarrow} +\infty$, we conclude that $u \in L^t(\Omega)$ for any $t > 1$, and (8.2) follows. □

When $V \in L^r(\Omega)$ and $f \in L^r(\Omega)$ with $r > \frac{N}{sp}$, a better estimate holds.

Proposition 8.2. *Assume that $V \in L^r(\Omega)$ and $f \in L^r(\Omega)$ with $r > \frac{N}{sp}$. Let $u \in \widetilde{W}^{p,s}(\Omega)$ be a solution of (8.1). Then $u \in L^\infty(\Omega)$ and there exists $C = C\Big(s, p, N, \Omega, \|f\|_{L^r(\Omega)}, \|V\|_{L^r(\Omega)}, \|u\|_{L^{p_s^*}(\Omega)}\Big)$ such that*

$$\|u\|_{L^\infty(\Omega)} \le C. \tag{8.7}$$

Proof. By Proposition 8.1, $u \in L^t(\Omega)$ for any $t > 1$ and therefore $V|u|^{p-2}u \in L^t(\Omega)$ for any $t \in \big(\frac{N}{sp}, r\big) \cap \big(\frac{1}{p-1+\frac{1}{r}}, +\infty\big)$. Moreover, by Holder's inequality,

$$\|V|u|^{p-2}u\|_{L^t(\Omega)} \le \|V\|_{L^r(\Omega)} C_{\frac{(p-1)tr}{r-t}}.$$

Thus, by replacing f by $f + V|u|^{p-2}u$ we can assume that $V \equiv 0$ in equation (8.1). Let us assume first that $u \geq 0$. For any $k > 0$ take u_k defined as above and define now for any $\alpha > 0$,

$$\phi_{\alpha,k} := \big(u_k\big)^{\alpha p+1} \in \widetilde{W}^{s,p}(\Omega) \cap L^\infty(\Omega).$$

Using $\phi_{\alpha,k}$ as test function, one obtains

$$\begin{aligned}
&\mathcal{K}(1-s)\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}\big(u_k^{p\alpha+1}(x)-u_k^{p\alpha+1}(y)\big)\,dx\,dy \\
&= \int_\Omega f(x)u_k^{p\alpha+1}dx.
\end{aligned}$$

Thanks to Lemma 7.2 one has

$$\begin{aligned}
\frac{\mathcal{K}(1-s)(\alpha p+1)}{(\alpha+1)^p}\int_{\mathbb{R}^{2N}} \frac{\big|u_k^{\alpha+1}(x)-u_k^{\alpha+1}(y)\big|^p}{|x-y|^{N+sp}}\,dx\,dy &\leq \int_\Omega |f(x)|u_k^{p\alpha+1}dx \\
&\leq \int_\Omega |f(x)|u^{p\alpha+1}dx.
\end{aligned} \tag{8.8}$$

By Holder's inequality we have

$$\big[u_k^{\alpha+1}\big]_{W^{s,p}(\mathbb{R}^N)} \leq \Big(\frac{(\alpha+1)^p}{\mathcal{K}(1-s)(\alpha p+1)}\Big)^{1/p}\Big(\|f\|_{L^r(\Omega)}\Big(\int_\Omega |u|^{r'(\alpha p+1)}\Big)^{1/r'}\Big)^{1/p}. \tag{8.9}$$

Since $u_k^{\alpha+1} \in \widetilde{W}^{s,p}(\Omega)$, by the Sobolev's embedding theorem there exists $C_1 > 0$ such that

$$\|u_k\|_{L^{p_s^*(\alpha+1)}(\Omega)} = \|u_k^{\alpha+1}\|^{\frac{1}{\alpha+1}}_{L^{p_s^*}(\Omega)} \leq C_1^{\frac{1}{\alpha+1}}\big[u_k^{\alpha+1}\big]^{\frac{1}{\alpha+1}}_{W^{s,p}(\mathbb{R}^N)}.$$

Then, by (8.9), we obtain

$$\begin{aligned}
&\|u_k\|_{L^{p_s^*(\alpha+1)}(\Omega)} \\
&\leq C_1^{\frac{1}{\alpha+1}}\Big(\frac{(\alpha+1)^p}{\mathcal{K}(1-s)(\alpha p+1)}\Big)^{\frac{1}{p(\alpha+1)}}\Big(\|f\|_{L^r(\Omega)}\Big(\int_\Omega u^{r'(\alpha p+1)}\Big)^{1/r'}\Big)^{\frac{1}{p(\alpha+1)}}.
\end{aligned}$$

So, denoting $C_2 = \|f\|_{L^r(\Omega)}$, we have

$$\|u_k\|_{L^{p_s^*(\alpha+1)}(\Omega)} \leq C_1^{\frac{1}{\alpha+1}}\Big(\frac{C_2(\alpha+1)^p}{\mathcal{K}(1-s)(\alpha p+1)}\Big)^{\frac{1}{p(\alpha+1)}}\Big(\int_\Omega u^{r'(\alpha p+1)}\Big)^{\frac{1}{pr'(\alpha+1)}}.$$

On the other hand there exists $C_3 > 0$ such that

$$\Big(\frac{C_2(\alpha+1)^p}{\mathcal{K}(1-s)(\alpha p+1)}\Big)^{\frac{1}{p\sqrt{\alpha+1}}} \leq C_3 \quad \text{for all } \alpha > 0.$$

Consequently, we obtain that

$$\begin{aligned}
\|u_k\|_{L^{p_s^*(\alpha+1)}(\Omega)} &\leq C_1^{\frac{1}{\alpha+1}}C_3^{\frac{1}{\sqrt{\alpha+1}}}\|u\|^{\frac{\alpha p+1}{(\alpha+1)p}}_{L^{(p\alpha+1)r'}(\Omega)} \\
&\leq C_1^{\frac{1}{\alpha+1}}C_3^{\frac{1}{\sqrt{\alpha+1}}}|\Omega|^{\frac{p-1}{p^2(1+\alpha)^2r'}}\|u\|^{\frac{\alpha p+1}{(\alpha+1)p}}_{L^{(\alpha+1)pr'}(\Omega)}.
\end{aligned} \tag{8.10}$$

Choosing $\alpha = \alpha_1$ in (8.10) such that $(\alpha_1+1)pr' = p_s^*$ we obtain

$$\|u_k\|_{L^{p_s^*(\alpha_1+1)}(\Omega)} \leq C_1^{\frac{1}{\alpha_1+1}}C_3^{\frac{1}{\sqrt{\alpha_1+1}}}|\Omega|^{\frac{p-1}{p^2(1+\alpha_1)^2r'}}\|u\|^{\frac{\alpha_1 p+1}{(\alpha_1+1)p}}_{L^{(\alpha_1+1)pr'}(\Omega)}.$$

Next we choose $\alpha = \alpha_2$ in (8.10) such that $(1+\alpha_2)pr' = (1+\alpha_1)p_s^*$ and obtain

$$\|u_k\|_{L^{p_s^*(1+\alpha_2)}(\Omega)} \le C_1^{\frac{1}{1+\alpha_2}} C_3^{\frac{1}{\sqrt{1+\alpha_2}}} |\Omega|^{\frac{p-1}{p^2(1+\alpha_2)^2 r'}} \|u\|_{L^{(\alpha_1+1)p_s^*}(\Omega)}^{\frac{\alpha_2 p+1}{(\alpha_2+1)p}}.$$

By induction, for all $m \in \mathbb{N}^*$ we can show that

$$\|u_k\|_{L^{p_s^*(1+\alpha_m)}(\Omega)} \le C_1^{\frac{1}{1+\alpha_m}} C_3^{\frac{1}{\sqrt{1+\alpha_m}}} |\Omega|^{\frac{p-1}{p^2(1+\alpha_m)^2 r'}} \|u\|_{L^{(1+\alpha_{m-1})p_s^*}(\Omega)}^{\frac{1+\alpha_m p}{p(1+\alpha_m)}}, \tag{8.11}$$

where $(\alpha_m)_{m\in\mathbb{N}}$ is a sequence of positive numbers defined by

$$\alpha_0 = 0 \quad \text{and} \quad (1+\alpha_m)pr' = (1+\alpha_{m-1})p_s^* \qquad \forall\, m \ge 1.$$

One easily see that for all $m \in \mathbb{N}$, $1+\alpha_m = \left(\frac{p_s^*}{pr'}\right)^m$, and then, by hypothesis, $\alpha_m \to +\infty$ as $m \to +\infty$ since $r > \frac{N}{ps}$. Moreover we have

$$\|u_k\|_{L^{\sigma_m}(\Omega)} \le C_1^{\beta_m^2} C_3^{\beta_m} |\Omega|^{\frac{\beta_m^4(p-1)}{p^2 r'}} \|u\|_{L^{\sigma_{m-1}}(\Omega)}^{\delta_m},$$

with $\sigma_m = p_s^*(\alpha_m+1)$, $\beta_m = \frac{1}{\sqrt{\alpha_m+1}}$, and $\delta_m = \frac{p\alpha_m+1}{(\alpha_m+1)p}$. Notice that $\sigma_m \to +\infty$, $\beta_m \to 0$ and $\delta_m \uparrow 1$ as $m \to +\infty$. Letting $k \to +\infty$ and using Fatou's lemma we obtain

$$\|u\|_{L^{\sigma_m}(\Omega)} \le C_4^{\beta_m} \|u\|_{L^{\sigma_{m-1}}(\Omega)}^{\delta_m}, \tag{8.12}$$

for some constant $C_4 > 0$. A simple computation gives

$$\|u\|_{L^{\sigma_m}(\Omega)} \le C_4^{\left(\beta_m + \sum_{i=1}^{m-1} \beta_{m-i} \prod_{k=0}^{i-1} \delta_{m-k}\right)} \|u\|_{L^{\sigma_0}(\Omega)}^{\prod_{i=1}^m \delta_i}.$$

Using that $\delta_m \uparrow 1$ and that $\beta_m = (\frac{r'p}{p_s^*})^{m/2}$ one can find that

$$\sum_{i=1}^{m-1} \Big(\beta_{m-i} \prod_{k=0}^{i-1} \delta_{m-k}\Big) \le \sum_{i=0}^{m} \beta_{m-i} \le \frac{1}{1-(\frac{r'p}{p_s^*})^{1/2}} < \infty$$

so

$$\|u\|_{L^{\sigma_m}(\Omega)} \le C \max\left\{1, \|u\|_{L^{p_s^*}(\Omega)}\right\}$$

and the conclusion follows. If u changes sign one can use instead, as in Proposition 8.1, $u_k = \max\{|u|, k\}$. □

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