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REDUCTION PRINCIPLE FOR PARTIAL FUNCTIONAL DIFFERENTIAL EQUATION WITHOUT COMPACTNESS

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Abstract. This article establishes a reduction principle for partial functional differential equation without compactness of the semigroup generated by the linear part. Under conditions more general than the compactness of the C_0-semigroup generated by the linear part, we establish the quasi-compactness of the C_0-semigroup associated to the linear part of the partial functional differential equation. This result allows as to construct a reduced system that is posed by an ordinary differential equation posed in a finite dimensional space. Through this result we study the existence of almost automorphic and almost periodic solutions for partial functional differential equations. For illustration, we study a transport model.

1. Introduction

The theory of functional differential equations with delay has emerged as an important branch of nonlinear analysis because it has wide range of application in various fields of pure and applied mathematics as well as in other fields like physics, chemistry, population dynamics, biology, engineering, economics, and so on. One of the theories related to functional differential equations with delay is the one of almost automorphy. This last notion has been introduced by Bochner in 1950, as a generalization of almost periodicity [3].

The problem of the existence of periodic and almost periodic solutions of functional differential equations with delay have received the attention of many authors. We refer the reader to the book [13] and to the papers [5, 20]. More recently, the existence of almost automorphic solutions to ordinary as well as abstract differential equations has been intensively studied. For information of the reader, we refer to N'Guérékata's book [15]. In [14], the author proved the existence of almost automorphic solution for the ordinary differential equation

$$x'(t) = Hx(t) + e(t) \quad t \in \mathbb{R},$$

where H is a constant $(n \times n)$-matrix and $e : \mathbb{R} \to \mathbb{R}^n$ is almost automorphic. He proved that the existence of a bounded solution on $\mathbb{R}^+$ implies the existence of an

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almost automorphic solution. In [16], the author studied the existence of almost automorphic solutions for the semilinear abstract differential equation

$$x'(t) = \mathcal{C}x(t) + \theta(t, x(t)) \quad t \geq 0, \tag{1.1}$$

where $\mathcal{C}$ generates an exponentially stable C_0-semigroup on a Banach space E and θ is an almost automorphic function from $\mathbb{R}$ to E. The author proved that the only bounded mild solution of (1.1) on $\mathbb{R} \times E$ is almost automorphic. Recently Ezzinbi and N'Guérékata [11] established the existence of an almost automorphic solution for the partial functional differential equation

$$\begin{gathered} \frac{d}{dt}u(t) = \tilde{A}u(t) + \tilde{L}(u_t) + \tilde{h}(t) \quad \text{for } t \geq 0, \\ u_0 = \varphi \in C([-r, 0], F), \end{gathered} \tag{1.2}$$

where $\tilde{A}$ is a linear operator on a Banach space F not necessarily densely defined and satisfies the Hille-Yosida condition, see [11]. $\tilde{L}$ is a bounded linear operator from $C([-r, 0], F)$ to F with $C([-r, 0], F)$ is the space of continuous functions from $[-r, 0]$ to F endowed with the uniform norm topology and $\tilde{h}$ is an almost automorphic function from $\mathbb{R}$ to F, the history function $u_t \in C([-r, 0], F)$ is defined by

$$u_t(\theta) = u(t + \theta), \quad \text{for } \theta \in [-r, 0].$$

By developing new fundamental results about the spectral analysis of the solutions and a new reduction principle, the authors proved that the existence of a bounded solution on $\mathbb{R}^+$ of (1.2) is equivalent to the existence of an almost automorphic solution.

In this work we are interested in investigating the existence of almost automorphic and almost periodic solutions for the partial functional differential equation

$$x'(t) = Ax(t) + L(x_t) + f(t) \quad \text{for } t \in \mathbb{R}, \tag{1.3}$$

where A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$ on a Banach space X. $x(t) \in X$, L is a bounded linear operator from $C([-r, 0], X)$ to X with $C([-r, 0], X)$ is the space of continuous functions from $[-r, 0]$ to X endowed with the uniform norm topology and $r > 0$. The history function $x_t \in C([-r, 0], X)$ is defined by $x_t : [-r, 0] \to X$,

$$x_t(\theta) = x(t + \theta).$$

The function $f : \mathbb{R} \to X$ is a Stepanov almost automorphic function, which is a weaker notion of almost automorphy. For more details on Stepanov almost automorphic functions, we refer the reader to [17]. The usual condition for studying the existence of almost periodic and almost automorphic solutions for this problem is that the C_0-semigroup $(T(t))_{t\geq 0}$ is compact. For example, the problem of existence of almost automorphic solutions has been studied recently by Benkhalti, Es-sebbar and Ezzinbi in [2], using this condition. Our aim in this paper is to establish the existence of almost automorphic and almost periodic solutions for a class of equations in which the C_0-semigroup $(T(t))_{t\geq 0}$ is not necessarily compact. In the last direction, we refer the reader to [12], where the authors studied the existence of almost periodic solutions of (1.3) when the C_0-semigroup $(T(t))_{t\geq 0}$ is not compact but the operator $T(t)L$ is compact for $t > 0$ and the input term f is almost periodic. As an extension of the work [12], we prove that the (1.3) has an almost periodic solution under the hypothesis that the operator $T(t)L$ is compact for $t > 0$ and the input term f is only Stepanov almost periodic. For more details on Stepanov almost periodic functions, we refer the reader to [8]. also extend our results to the almost automorphic case. We prove that the (1.3) has an almost automorphic solution if the function f is just Stepanov almost automorphic. To achieve this goal, we use the formula for the variation of constants and the reduction principle developed by Ezzinbi and N'Guérékata in [11].

This work is organized as follows: In Section 2, we recall some results on partial functional differential equations and we establish fundamental results about the spectral decomposition of solutions of (1.3). In Section 3, we develop a new fundamental reduction principle. Section 4 is devoted to almost automorphic and almost periodic functions in both the Bochner and the Stepanov senses. In Section 5, we study the existence of almost automorphic and almost periodic solutions of (1.3) through a new reduction principle. In Section 6, we illustrate our results to the transportation equation. The last one is a conclusion.

2. Variation of constants formula and spectral decomposition

Throughout this work, $(X, \|\cdot\|)$ is a Banach space and $C = C([-r, 0], X)$ is the space of all continuous functions from $[-r, 0]$ to X endowed with the uniform norm topology. Let $\mathcal{L}(X)$ be the space of linear and bounded maps from X to X and $\mathcal{K}(X)$ be the space of all compact operators on X. We assume that the operator A satisfies the following condition:

- (H1) A is the infinitesimal generator of a C_0-semigroup of bounded linear operators $(T(t))_{t\geq 0}$ on a Banach space X. $L : C \to X$ is a bounded linear operator on C and $f : \mathbb{R} \to X$ is a continuous function from $\mathbb{R}$ to X.

To (1.3), we associate the problem

$$\begin{gathered} x'(t) = Ax(t) + L(x_t) + f(t) \quad \text{for } t \geq \sigma, \\ x_\sigma = \varphi \in C, \end{gathered} \tag{2.1}$$

we refer to Engel and Nagel [10], and to Wu [19] for the basic properties of the problem (2.1). We only mention here that (2.1) with the initial condition $x_\sigma = \varphi$, has a unique mild solution $x(\cdot, \sigma, \varphi, f)$. This signifies that $x : [\sigma - r, \infty) \to X$ is a continuous function and the restriction of $x(\cdot)$ on $[\sigma, \infty)$ satisfies the integral equation

$$x(t) = T(t-\sigma)x(\sigma) + \int_\sigma^t T(t-s)(L(x_s) + f(s))\, ds \quad t \geq \sigma.$$

To develop new fundamental results about the spectral analysis of the solutions, we need to introduce some preliminary results.

Definition 2.1 ([10])**.** A C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X is called quasi-compact if

$$\lim_{t\to+\infty} d(T(t), \mathcal{K}(X)) = 0.$$

Definition 2.2. [19]/ If B is a bounded set in a Banach space X, the Kuratowski measure of noncompactness is defined by

$$\alpha(B) = \inf\{d : B \text{ has a finite cover of radius less than } d\}.$$

Theorem 2.3 ([19])**.** *Assume that X is a Banach space and $\alpha(\cdot)$is the the Kuratowski measure of noncompactness of a bounded set B of X. Then*

(i) $\alpha(B) = 0$ *if and only if the closure of B is compact.*

(ii) $\alpha(A \cup B) = \max(\alpha(A), \alpha(B))$.
(iii) $\alpha(A + B) \le \alpha(A) + \alpha(B)$.
(iv) $\alpha(\overline{co}B) = \alpha(B)$ *where* $\overline{co}B$ *is the closed convex hull of* B.

Definition 2.4 ([10]). Let $S \in \mathcal{L}(X)$. The essential norm is defined by

$$|S|_{\text{ess}} = \inf\{c > 0 : \alpha(S(B)) \le c\alpha(B) \text{ for any bounded } B \in X\}.$$

Definition 2.5 ([10]). *The essential growth bound of a* C_0*-semigroup* $(T(t))_{t\ge 0}$ *is defined by*

$$w_{\text{ess}} = \inf\{w \in \mathbb{R} : \sup_{t\ge 0} e^{-wt}|T(t)|_{\text{ess}} < \infty\}.$$

Theorem 2.6 ([10]). *For a* C_0*-semigroup* $(T(t))_{t\ge 0}$ *on a Banach space* X*, the following assertions are equivalent:*
(i) $(T(t))_{t\ge 0}$ *is quasi-compact.*
(ii) $w_{\text{ess}} < 0$.
(iii) $\|T(t_0) - K\| < 1$*, for some* $t_0 > 0$ *and* $K \in \mathcal{K}(X)$.

Now, we consider the linear problem

$$\begin{gathered} x'(t) = Ax(t) + L(x_t), \quad \text{for } t \ge 0, \\ x_0 = \varphi \in C. \end{gathered} \tag{2.2}$$

The solution operator $V(t)$ is defined by $V(t)\varphi = x_t(\cdot, \varphi)$, where $x(\cdot, \varphi)$ is the mild solution of (2.2). For more details see [19].

Theorem 2.7 ([12]). $(V(t))_{t\ge 0}$ *is a* C_0*-semigroup of bounded linear operators on* C*, the infinitesimal generator* $\mathcal{A}$ *is given by*

$$\begin{gathered} D(\mathcal{A}) = \{\varphi \in C^1([-r, 0], X) : \varphi(0) \in D(A) \text{and } \varphi'(0) = A\varphi(0) + L(\varphi)\} \\ \mathcal{A}\varphi = \varphi'. \end{gathered}$$

Lemma 2.8 ([12]). *Assume that* (H1) *holds. Then*

$$[V(t)\varphi](\theta) = \begin{cases} [V(t+\theta)\varphi](0), & t + \theta \ge 0, \\ \varphi(t+\theta), & t + \theta \le 0. \end{cases}$$

Let $W(t)$ the solution operator corresponding to $L = 0$. Then $W(t)$ is given by

$$[W(t)\varphi](\theta) = \begin{cases} T(t+\theta)\varphi(0), & -t \le \theta \le 0, \\ \varphi(t+\theta), & -r \le \theta \le -t. \end{cases}$$

We establish the first result on the asymptotic behavior of the semigroup $(V(t))_{t\ge 0}$ by the following Theorem.

Theorem 2.9 ([12]). *Assume that the semigroup* $(T(t))_{t\ge 0}$ *is exponentially stable and that the operator* $T(t)L : C \to X$ *is compact for all* $t > 0$*. Then, the semigroup* $(V(t))_{t\ge 0}$ *is quasi-compact.*

Remark 2.10.
(i) The operator $T(t)L$ is compact if the semigroup $(T(t))_{t\ge 0}$ is compact or the linear delay operator L is compact. If for example $(T(t))_{t\ge 0}$ is not necessarily compact and L is given by $L(\varphi) = \sum_{i=1}^{k} B_i\varphi(-r_i)$, where $B_i : X \to X$, for $i = 1, \dots, k$ are compact linear operators on X, then of course $T(t)L$ is compact.

(ii) If the semigroup $(T(t))_{t\geq 0}$ is not exponentially stable, we can substitute the operator A by $A-\alpha I$, where α is an enough large constant. Then, we obtain that the semigroup $e^{-\alpha t}(T(t))_{t\geq 0}$ is exponentially stable and we assume that the operator $L+\alpha I$ is compact.

To give the variation of constants formula, we need to recall some notation and results which are taken from [11]. Let $\langle X_0\rangle$ be the space defined by

$$\langle X_0\rangle = \{X_0 c : c\in X\},$$

where

$$(X_0 c)(\theta) = \begin{cases} 0, & \text{if } \theta\in[-r,0] \\ c, & \text{if } \theta = 0. \end{cases}$$

The space $C\oplus\langle X_0\rangle$ is equipped with the norm

$$|\phi + X_0 c| = |\phi|_C + |c| \quad \text{for } (\phi,c)\in C\times X,$$

is a Banach space and consider the extension $\tilde{\mathcal{A}}$ of the operator $\mathcal{A}$ defined on $C\oplus\langle X_0\rangle$ by

$$D(\tilde{\mathcal{A}}) = \{\varphi\in C^1([-r,0],X),\ \varphi(0)\in D(A) \text{ and } \varphi'(0)\in\overline{D(A)}\},$$
$$\tilde{\mathcal{A}}\varphi = \varphi' + X_0(A\varphi(0) + L(\varphi) - \varphi'(0)).$$

Lemma 2.11 ([11]). *Assume that* (H1) *holds. Then, $\tilde{\mathcal{A}}$ satisfies the Hille-Yosida condition on $C\oplus\langle X_0\rangle$: there exist $\tilde{M}\geq 0$ and $\tilde{\omega}\in\mathbb{R}$ such that*

$$(\tilde{\omega},+\infty)\subset\rho(\tilde{\mathcal{A}}) \text{ and } \|(\lambda I_d - \tilde{\mathcal{A}})^{-n}\| \leq \frac{\tilde{M}}{(\lambda-\tilde{\omega})^n} \quad \text{for } n\in\mathbb{N} \text{ and } \lambda>\tilde{\omega}.$$

Theorem 2.12 ([11]). *Assume that* (H1) *holds. Then, for all $\varphi\in C$ the solution x of* (1.3) *is given by the variation of constants formula*

$$x_t = V(t)\varphi + \lim_{\lambda\to+\infty}\int_0^t V(t-s)\tilde{B}_\lambda(X_0 f(s))ds \quad \text{for } t\geq 0,$$

where $\tilde{B}_\lambda = \lambda(\lambda I_d - \tilde{A}_V)^{-1}$ for $\lambda>\tilde{\omega}$.

Using the quasi-compactness of the semigroup $(V(t))_{t\geq 0}$, we obtain the following spectral decomposition result.

Theorem 2.13. *Assume that the semigroup $(T(t))_{t\geq 0}$ is exponentially stable and that the operator $T(t)L : C\to X$ is compact for $t>0$. Then, the space C is decomposed as:*

$$C = \mathcal{S}\oplus\mathcal{V},$$

where $\mathcal{S}$ and $\mathcal{V}$ are spaces invariant under $V(t)$ and there are constants $\alpha>0$ and $M\geq 1$ such that

$$\|V(t)\varphi\|_C \leq Me^{-\alpha t}\|\varphi\|_C, \quad \text{for each } t\geq 0 \text{ and } \quad \varphi\in\mathcal{S}.$$

$\mathcal{V}$ is a finite dimensional space and the restriction $V(t)$ on $\mathcal{V}$ is a groupe.

3. Reduction principle

If the semigroup $(V(t))_{t\geq 0}$ is quasi-compact, we can apply the properties and notation introduced in Theorem 2.13. In particular, we set $\mathcal{V}$ a space of finite dimension d with a vector basis $\Phi = \{\phi_1, \phi_2 \dots, \phi_d\}$. Then there exist d-elements $(\psi_1, \psi_2, \dots, \psi_d)$ in C^* such that

$$\begin{gathered} \langle \psi_i, \phi_j \rangle = \delta_{ij}, \\ \langle \psi_i, \phi \rangle = 0, \quad \forall \phi \in \mathcal{S} \text{ and } i \in \{1, \dots, d\}, \end{gathered} \tag{3.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between C and C^*, and

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Let $\Psi = col\{\psi_1, \psi_2, \dots, \psi_d\}$, $\langle \Psi, \Phi \rangle$ is a $(d \times d)$-matrix, where the (i, j)-component is $\langle \psi_i, \varphi_i \rangle$ and denote by $\Pi^{\mathcal{V}}$ and $\Pi^{\mathcal{S}}$ the projections respectively on $\mathcal{V}$ and $\mathcal{S}$. For each $\varphi \in C$ we have

$$\Pi^{\mathcal{V}} \varphi = \Phi \langle \Psi, \varphi \rangle.$$

In fact, for $\varphi \in C$, we have $\varphi = \Pi^{\mathcal{V}} \varphi + \Pi^{\mathcal{S}} \varphi$ with $\Pi^{\mathcal{V}} \varphi = \sum_{i=1}^{d} \alpha_i \phi_i$ and $\alpha_i \in \mathbb{R}$. From (3.1) we conclude that

$$\alpha_i = \langle \psi_i, \varphi \rangle.$$

Hence

$$\Pi^{\mathcal{V}} \varphi = \sum_{i=1}^{d} \langle \psi_i, \varphi \rangle \phi_i = \Phi \langle \Psi, \varphi \rangle.$$

Since $V^{\mathcal{V}}(t)$ is a group on $\mathcal{V}$, then there exists a $(d \times d)$-matrix G such that $V^{\mathcal{V}}(t)\Phi = \Phi e^{Gt}$, for $t \geq 0$. Moreover, for each $n, n_0 \in \mathbb{N}$ such that $n \geq n_0 \geq \tilde{\omega}$ and $i \in \{1, \dots, d\}$, we define the linear operator $x^*_{i,n}$ by

$$x^*_{i,n}(a) = \langle \psi_i, \tilde{B}_n X_0 a \rangle, \quad \text{for } a \in X.$$

Since $|\tilde{B}_n| \leq \frac{n}{n - \tilde{\omega}} \tilde{M}$ for any $n \geq n_0$, then $x^*_{i,n}$ is a bounded linear operator from X to $\mathbb{R}$ such that

$$|x^*_{i,n}| \leq \frac{n}{n - n_0} \tilde{M} |\psi_i| \quad \text{for all } n \geq n_0.$$

Define the d-column vector $x^*_n = col(x^*_{1,n}, \dots, x^*_{d,n})$ then

$$\langle x^*_n, a \rangle = \langle \Psi, \tilde{B}_n X_0 a \rangle, \quad \forall a \in X,$$

with

$$\langle x^*_n, a \rangle_i = \langle \psi_i, \tilde{B}_n X_0 a \rangle, \quad \text{for } i = 1, \dots, d \text{ and } a \in X.$$

Consequently, $\sup_{n \geq n_0} |x^*_n| < \infty$, which implies that $(x^*_n)_{n \geq n_0}$ is a bounded sequence in $\mathcal{L}(X, \mathbb{R}^d)$. As a result, we obtain the following important result.

Theorem 3.1 ([11]). *There exists $x^* \in \mathcal{L}(X, \mathbb{R}^d)$, such that $(x^*_n)_{n \geq n_0}$ converges weakly to x^*:*

$$\langle x^*_n, x \rangle \underset{n \to \infty}{\to} \langle x^*, x \rangle \quad \textit{for all } x \in X.$$

As a consequence, we conclude that

Corollary 3.2 ([11]). *For any continuous function $h : \mathbb{R} \to X$, we have*

$$\lim_{n \to \infty} \int_\sigma^t V^{\mathcal{V}}(t - \xi) \Pi^{\mathcal{V}} (\tilde{B}_n X_0 h(\xi))\, d\xi = \Phi \int_\sigma^t e^{(t-\xi)G} \langle x^*, h(\xi) \rangle\, d\xi, \quad \textit{for } \; t, \sigma \in \mathbb{R}.$$

Theorem 3.3 ([11]). *Assume that the semigroup $(T(t))_{t\geq 0}$ is compact. Moreover, f is continuous and x is a solution of* (1.3) *on $\mathbb{R}$, then $z(t) = \langle \Psi, x_t\rangle$ is a solution of the ordinary differential equation*

$$z'(t) = Gz(t) + \langle x^*, f(t)\rangle, \quad \text{for } t \in \mathbb{R}. \tag{3.2}$$

Conversely, if $z(\cdot)$ is a solution of (3.2) *on $\mathbb{R}$ and f is bounded then the function*

$$x(t) = \big[\Phi\, z(t) + \lim_{n\to\infty}\int_{-\infty}^{t} V^{\mathcal{S}}(t-\xi)\Pi^{\mathcal{S}}(\tilde{B}_n X_0 f(\xi))\, d\xi\big](0), \quad \text{for } t \in \mathbb{R},$$

is a mild solution of (1.3) *on $\mathbb{R}$.*

As a consequence of the above, we establish the following fundamental reduction principle which allows us to prove the existence of an almost automorphic and almost periodic solution of the (1.3).

Theorem 3.4. *Assume that the semigroup $(T(t))_{t\geq 0}$ is exponentially stable and that the operator $T(t)L$ is compact for $t > 0$. Moreover, f is locally integrable and x is a solution of* (1.3) *on $\mathbb{R}$, then $z(t) = \langle \Psi, x_t\rangle$ is a solution of the ordinary differential equation*

$$z'(t) = Gz(t) + \langle x^*, f(t)\rangle, \quad \text{for } t \in \mathbb{R}. \tag{3.3}$$

Conversely, if $z(\cdot)$ is a solution of (3.3) *on $\mathbb{R}$ and*

$$\sup_{t\in\mathbb{R}}\Big(\int_t^{t+1} \|f(s)\|^p ds\Big)^{1/p} < \infty$$

then the function

$$x(t) = \big[\Phi\, z(t) + \lim_{n\to\infty}\int_{-\infty}^{t} V^{\mathcal{S}}(t-\xi)\Pi^{\mathcal{S}}(\tilde{B}_n X_0 f(\xi))\, d\xi\big](0), \quad \text{for } t \in \mathbb{R}, \tag{3.4}$$

is a solution of (1.3) *on $\mathbb{R}$.*

Proof. The proof is similar to that of Theorem 3.3. We only prove that

$$\lim_{n\to\infty}\int_{-\infty}^{t} V^{\mathcal{S}}(t-\xi)\Pi^{\mathcal{S}}(\tilde{B}_n X_0 f(\xi))\, d\xi,$$

exists in C. For $t \in \mathbb{R}$ and for n sufficiently large, we have

$$\|\lim_{n\to\infty}\int_{-\infty}^{t} V^{\mathcal{S}}(t-\xi)\Pi^{\mathcal{S}}(\tilde{B}_n X_0 f(\xi))\, d\xi\| \leq K,$$

where

$$K = 2\tilde{M}N\|\Pi^s\|\sup_{t\in\mathbb{R}}\Big(\int_t^{t+1}\|f(s)\|ds\Big)\frac{1}{1-e^{-\alpha}}.$$

Let

$$H(n,s,t) = V^{\mathcal{S}}(t-\xi)\Pi^{\mathcal{S}}(\tilde{B}_n X_0 f(\xi)), \quad \text{for } n \in \mathbb{N} \text{ and } s \leq t.$$

For n and m sufficiently large and $\sigma \leq t$, we have

$$\begin{aligned}
&\|\int_{-\infty}^{t} H(n,s,t)ds - \int_{-\infty}^{t} H(m,s,t)ds\| \\
&\leq 2Ke^{-\alpha(t-\sigma)} + \|\int_{\sigma}^{t} H(n,s,t)ds - \int_{\sigma}^{t} H(m,s,t)ds\|.
\end{aligned}$$

Since $\lim_{n\to\infty}\int_\sigma^t H(n,s,t)ds$ exists, it follows that

$$\limsup_{n,m\to\infty}\|\int_{-\infty}^t H(n,s,t)ds-\int_{-\infty}^t H(m,s,t)ds\|\le 2Ke^{-\alpha(t-\sigma)}.$$

By letting $\sigma\to-\infty$, we obtain

$$\limsup_{n,m\to\infty}\|\int_{-\infty}^t H(n,s,t)ds-\int_{-\infty}^t H(m,s,t)ds\|=0.$$

Thus, by the completeness of the phase space C, we deduce that

$$\lim_{n\to\infty}\|\int_{-\infty}^t H(n,s,t)ds\|$$

exists in C. □

Remark 3.5. This principal result was established when the semigroup $(T(t))_{t\ge0}$ is compact. We establish the same result even if the semigroup is not necessary compact but the operator $T(t)L$ is compact for all $t>0$.

4. Almost periodicity and almost automorphy

In what follows, we recall some results on almost automorphic functions and almost periodic functions. Let $BC(\mathbb{R},X)$ be the space of bounded continuous functions from $\mathbb{R}$ to X, provided with the uniform norm topology. Let $x\in BC(\mathbb{R},X)$ and $\tau\in\mathbb{R}$, we define the translation function

$$x_\tau(s)=x(\tau+s)\quad\text{for } s\in\mathbb{R}.$$

Definition 4.1 ([6]). A bounded continuous function $x:\mathbb{R}\to X$ is said to be almost periodic if $\{x_\tau,\tau\in\mathbb{R}\}$ is relatively compact in $BC(\mathbb{R},X)$.

Theorem 4.2 ([4]). *A function $f:\mathbb{R}\to X$ is almost periodic if and only if for every sequence of real numbers $(s'_n)_n$, there exist a subsequence $(s_n)_n$ of $(s'_n)_n$ and a function g such that,*

$$f(t+s_n)\to g(t)\quad\text{as } n\to\infty$$

uniformly on $\mathbb{R}$

We denote by $AP(\mathbb{R},X)$ the set of all such functions. For some preliminary results on almost periodic functions, we refer the reader to [20].

Definition 4.3 ([17]). The Bochner transform f^b of a function $f\in L^p_{\rm loc}(\mathbb{R},X)$ is the function $f^b:\mathbb{R}\to L^p_{\rm loc}([0,1],X)$, defined for each $t\in\mathbb{R}$ by

$$(f^b(t))(s)=f(t+s)\quad\text{for } s\in[0,1].$$

Definition 4.4 ([17]). Let $p\ge1$. The space $BS^p(\mathbb{R},X)$ consists of all functions $f\in L^p_{\rm loc}(\mathbb{R},X)$ such that $f^b:\mathbb{R}\to L^p_{\rm loc}([0,1],X)$, is bounded; that is,

$$\sup_{t\in\mathbb{R}}\Big(\int_t^{t+1}\|f(s)\|^p ds\Big)^{1/p}<\infty.$$

This is a normed space when equipped with the norm

$$\|f\|_{BS^p}=\sup_{t\in\mathbb{R}}\Big(\int_t^{t+1}\|f(s)\|^p ds\Big)^{1/p}$$

Note that the functions of $BS^p(\mathbb{R},X)$ may not be bounded.

Definition 4.5 ([8]). A function $f \in L^p_{\rm loc}(\mathbb{R}, X)$ is S^p-almost periodic if for every sequence of real numbers $(s'_n)_n$, there exist a subsequence $(s_n)_n$ of $(s'_n)_n$ and a function $g \in L^p_{\rm loc}(\mathbb{R}, X)$ such that, for each $t \in \mathbb{R}$,

$$\sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} \|f(s+s_n) - g(s)\|^p ds\Big)^{1/p} \to 0 \quad \text{as} \quad n \to \infty$$

Let $SAP^p(\mathbb{R}, X)$ denote this class of functions. Using the Bochner characterization in Theorem 4.2 and the completeness of the space $BS^p(\mathbb{R}, X)$, we can see that $f \in AP^p(\mathbb{R}, X)$ if and only if $f^b \in AP^p(\mathbb{R}, L^p([0, 1]X))$. Moreover, for all $p \geq 1$, $AP(\mathbb{R}, X)$ is a subset of $\subset SAP^p(\mathbb{R}, X)$. If $p \geq q$, then $SAP^p(\mathbb{R}, X) \subset SAP^q(\mathbb{R}, X)$.

Definition 4.6 ([6]). A continuous function $x : \mathbb{R} \to X$ is said to be almost automorphic if for any sequence of real numbers $(t'_n)_n$, there exists a subsequence $(t_n)_n$ of $(t'_n)_n$ such that

$$y(t) = \lim_{n\to+\infty} x(t + t_n), \tag{4.1}$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to+\infty} y(t - t_n) = x(t) \quad \text{for all } t \in \mathbb{R}. \tag{4.2}$$

We denote by $AA(\mathbb{R}, X)$ the space of all almost automorphic X-valued functions. Moreover, if the limits in (4.1) and (4.2) are uniform on any compact subset $K \subset \mathbb{R}$, we say that X is compact almost automorphic. If we denote $AA_c(\mathbb{R}, X)$ the space of all compact almost automorphic X-valued functions, then we have

$$AP(\mathbb{R}, X) \subset AA_c(\mathbb{R}, X) \subset AA(\mathbb{R}, X) \subset BC(\mathbb{R}, X).$$

Example 4.7 ([15]). $h(t) = \sin\big(\frac{1}{2+\cos t+\cos\sqrt{2}t}\big)$ is an almost automorphic function but it is not almost periodic. Since it is not uniformly continuous.

Definition 4.8 ([9]). A function $f \in L^p_{\rm loc}(\mathbb{R}, X)$ is said to be S^p-almost automorphic for some $p \geq 1$ if the function $f^b : \mathbb{R} \to L^p_{\rm loc}([0, 1], X)$ is almost automorphic.

The following characterization of almost automorphy in the sense of Stepanov is essential for the remainder of this work.

Proposition 4.9 ([17]). *A function $f \in L^p_{\rm loc}(\mathbb{R}, X)$ is S^p-almost automorphic if and only if, for every sequence of real numbers $(s'_n)_n$, there exist a subsequence $(s_n)_n$ of $(s'_n)_n$ and a function $g \in L^p_{\rm loc}(\mathbb{R}, X)$ such that, for each $t \in \mathbb{R}$,*

$$\Big(\int_t^{t+1} \|f(s+s_n) - g(s)\|^p ds\Big)^{1/p} \to 0 \quad \text{as } n \to \infty,$$
$$\Big(\int_t^{t+1} \|g(s-s_n) - f(s)\|^p ds\Big)^{1/p} \to 0 \quad \text{as } n \to \infty$$

Let $SAA^p(\mathbb{R}, X)$ denote the space of S^p-almost automorphic X-valued functions on $\mathbb{R}$. Then, for all $p \geq 1$, we have $AA(\mathbb{R}, X) \subset SAA^p(\mathbb{R}, X)$. Moreover, if $p \geq q$, then $SAA^p(\mathbb{R}, X) \subset SAA^q(\mathbb{R}, X)$. If $h \in AA(\mathbb{R}, \mathbb{C})$ and $f \in SAA^p(\mathbb{R}, \mathbb{C})$, then $hf \in SAA^p(\mathbb{R}, \mathbb{C})$.

5. Existence of almost automorphic and almost periodic solutions

The aim of this section is to study the existence of an almost automorphic and almost periodic solution of (1.3). In the rest of this section, we assume that A, L and f satisfy the conditions in Section 2. We consider the ordinary differential equation

$$z'(t) = Bz(t) + g(t), \quad \text{for } t \in \mathbb{R}, \tag{5.1}$$

where B is a matrix and $g : \mathbb{R} \to \mathbb{R}^d$.

Theorem 5.1 ([1]). *Assume that g is a S^1-almost periodic function. If (5.1) has a bounded solution on $\mathbb{R}^+$ then it admits an almost periodic solution on $\mathbb{R}$.*

Now, we are able to establish one of the main results of this work.

Theorem 5.2. *Assume that the semigroup $(T(t))_{t\geq 0}$ is exponentially stable and that the operator $T(t)L$ is compact for $t > 0$. If $f : \mathbb{R} \to X$ is a S^1-almost periodic function and (1.3) has a bounded solution on $\mathbb{R}^+$, then it has an almost periodic solution.*

Proof. Let x be the mild solution of (1.3) given by (3.4). Since $z(t)$ satisfies (3.3), $z(\cdot)$ is bounded on $\mathbb{R}^+$. Moreover, the function $g(t) = \langle x^*, f(t)\rangle$ is S^1-almost periodic. By Theorem 5.1, we obtain that $z(\cdot)$ is almost periodic on $\mathbb{R}$ and $\Phi\, z(\cdot)$ is an almost periodic function on $\mathbb{R}$. Let

$$Y(t) = \lim_{n\to\infty} \int_{-\infty}^{t} V^{\mathcal{S}}(t-\xi)\Pi^{\mathcal{S}}(\tilde{B}_n X_0 f(\xi))d\xi, \quad \text{for } t \in \mathbb{R}. \tag{5.2}$$

Since f is S^1-almost periodic, for any sequence of real numbers $(s'_p)_p$ there exists a subsequence $(s_p)_p$ of $(s'_p)_p$ and a function $g \in L^p_{\rm loc}(\mathbb{R}, X)$ such that, for each $t \in \mathbb{R}$,

$$\sup_{t\in\mathbb{R}} \int_{t}^{t+1} \|f(s+s_p) - g(s)\| ds \to 0 \quad \text{as } p \to \infty \tag{5.3}$$

On the other hand, let

$$Y_k(t) = \lim_{n\to\infty} \int_{k-1}^{k} V^{\mathcal{S}}(\xi)\Pi^{\mathcal{S}}(\tilde{B}_n X_0 f(t-\xi))d\xi, \quad \text{for } t \in \mathbb{R}.$$

First,we show that the function $Y_k : t \to Y_k(t)$ is continuous on $\mathbb{R}$. Let

$$Z_k(t) = \lim_{n\to\infty} \int_{k-1}^{k} V^{\mathcal{S}}(\xi)\Pi^{\mathcal{S}}(\tilde{B}_n X_0 g(t-\xi))d\xi, \quad \text{for } t \in \mathbb{R}.$$

We obtain that

$$\begin{aligned}
\|Y_k(t+s_p) - Z_k(t)\| &= \|\lim_{n\to\infty} \int_{k-1}^{k} V^{\mathcal{S}}(\xi)\Pi^{\mathcal{S}}(\tilde{B}_n X_0 (f(t+s_p-\xi) - g(t-\xi)))\, d\xi\| \\
&\leq M\tilde{M}\|\Pi^{\mathcal{S}}\| \int_{k-1}^{k} e^{-\alpha\xi} \|f(t+s_p-\xi) - g(t-\xi)\|\, d\xi \\
&\leq M\tilde{M}\|\Pi^{\mathcal{S}}\| e^{-\alpha(k-1)} \int_{k-1}^{k} \|f(t+s_p-\xi) - g(t-\xi)\|\, d\xi \\
&= M\tilde{M}\|\Pi^{\mathcal{S}}\| e^{-\alpha(k-1)} \int_{t-k}^{t-k+1} \|f(s_p+\xi) - g(\xi)\|\, d\xi \\
&\leq M\tilde{M}\|\Pi^{\mathcal{S}}\| e^{-\alpha(k-1)} \sup_{s\in\mathbb{R}} \int_{s+1}^{s} \|f(s_p+\xi) - g(\xi)\|\, d\xi\,.
\end{aligned}$$

Therefore, $Y_k(t) \in AP(\mathbb{R}, X)$ for $k \geq 1$. On the other hand, for each $t \in \mathbb{R}$ and $k \geq 1$, we have

$$\begin{aligned}
\|Y_k(t)\| &\leq M\tilde{M}\|\Pi^{\mathcal{S}}\| \int_{k-1}^{k} e^{-\alpha\xi}\|f(t-\xi)\|\,d\xi \\
&\leq M\tilde{M}\|\Pi^{\mathcal{S}}\| e^{-\alpha(k-1)} \int_{k-1}^{k} \|f(t-\xi)\|\,d\xi \\
&= M\tilde{M}\|\Pi^{\mathcal{S}}\| e^{-\alpha(k-1)} \int_{t-k}^{t-k+1} \|f(\xi)\|\,d\xi \\
&\leq M\tilde{M}\|\Pi^{\mathcal{S}}\|\,\|f\|_{BS^1} e^{-\alpha(k-1)}.
\end{aligned}$$

From the well-known Weierstrass Theorem we deduce that the series $\sum_{k=1}^{\infty} Y_k(t)$ is uniformly convergent on $\mathbb{R}$. Let

$$H(n,\xi,t) = V^s(t-\xi)\Pi^{\mathcal{S}}(\tilde{B}_n X_0 f(\xi)).$$

We claim that $Y(t) = \sum_{k=1}^{\infty} Y_k(t)$. In fact,

$$\begin{aligned}
\|\sum_{k=1}^{N} Y_k(t) - Y(t)\| &= \|\sum_{k=1}^{N} \lim_{n\to\infty} \int_{t-k}^{t-k+1} H(n,\xi,t)\,d\xi - \lim_{n\to\infty} \int_{-\infty}^{t} H(n,\xi,t)\,d\xi\| \\
&= \|\lim_{n\to\infty} \sum_{k=N+1}^{\infty} \int_{t-k}^{t-k+1} H(n,\xi,t)\,d\xi\| \\
&\leq M\tilde{M}\|\Pi^{\mathcal{S}}\| \sum_{k=N+1}^{\infty} \int_{t-k}^{t-k+1} e^{-\alpha(t-\xi)}\|f(\xi)\|\,d\xi \\
&\leq M\tilde{M}\|\Pi^{\mathcal{S}}\|\,\|f\|_{BS^1} \sum_{k=N+1}^{\infty} e^{-\alpha(k-1)} \quad \to 0 \quad \text{as} \quad N \to \infty.
\end{aligned}$$

Because the convergence of the series $\sum_{k=1}^{\infty} Y_k(t)$ is uniform, $Y(t) \in AP(\mathbb{R}, X)$. $\square$

Now, we extend the results above to the almost automorphic case. In [11], the authors established the following result which ensuring the existence of almost automorphic solutions to (1.3).

Theorem 5.3 ([11]). *Assume that the semigroup $(T(t))_{t\geq 0}$ is compact and f is an almost automorphic function. Moreover, if* (1.3) *has a bounded solution on $\mathbb{R}^+$, then it has an almost automorphic solution.*

We establish the same result even if the semigroup is not necessary compact but the operator $T(t)L$ is compact for all $t > 0$.

Theorem 5.4. *Assume that the semigroup $(T(t))_{t\geq 0}$ is exponentially stable and that the operator $T(t)L$ is compact for $t > 0$. If $f : \mathbb{R} \to X$ is a S^1-almost automorphic function and* (1.3) *has a bounded solution on $\mathbb{R}^+$, then it has an almost automorphic solution.*

To prove this theorem we need the following Lemma.

Lemma 5.5 ([2]). *Assume that g is a S^1-almost automorphic function. If* (5.1) *has a bounded solution on $\mathbb{R}^+$ then it admits an almost automorphic solution on $\mathbb{R}$.*

Proof of Theorem 5.4. Let x be the mild solution of (1.3) given by (3.4). Since $z(t)$ satisfies (3.3), it follows that $z(\cdot)$ is bounded on $\mathbb{R}^+$. Moreover, the function $g(t) = \langle x^*, f(t)\rangle$ is S^1-almost automorphic. By Lemma 5.5, we obtain that $z(\cdot)$ is almost automorphic on $\mathbb{R}$ and $\Phi\, z(\cdot)$ is an almost automorphic function on $\mathbb{R}$. Let

$$Y(t) = \lim_{n\to\infty} \int_{-\infty}^{t} V^{\mathcal{S}}(t-\xi)\Pi^{\mathcal{S}}(\tilde{B}_n X_0 f(\xi))d\xi, \quad \text{for } t \in \mathbb{R}. \tag{5.4}$$

Since f is S^1-almost automorphic, for any sequence of real numbers $(s'_p)_p$ there exists a subsequence $(s_p)_p$ of $(s'_p)_p$ and a function $g \in L^p_{\rm loc}(\mathbb{R}, X)$ such that, for each $t \in \mathbb{R}$,

$$\sup_{t\in\mathbb{R}} \int_{t}^{t+1} \|f(s+s_p) - g(s)\| ds \to 0 \quad \text{as } p \to \infty \tag{5.5}$$

$$\sup_{t\in\mathbb{R}} \int_{t}^{t+1} \|g(s-s_p) - f(s)\| ds \to 0 \quad \text{as } p \to \infty \tag{5.6}$$

On the other hand, Let

$$Y_k(t) = \lim_{n\to\infty} \int_{k-1}^{k} V^{\mathcal{S}}(\xi)\Pi^{\mathcal{S}}(\tilde{B}_n X_0 f(t-\xi))d\xi, \quad \text{for } t \in \mathbb{R},$$

the function $Y_k : t \to Y_k(t)$ is continuous on $\mathbb{R}$. Let

$$Z_k(t) = \lim_{n\to\infty} \int_{k-1}^{k} V^{\mathcal{S}}(\xi)\Pi^{\mathcal{S}}(\tilde{B}_n X_0 g(t-\xi))d\xi, \quad \text{for } t \in \mathbb{R},$$

we obtain that

$$\begin{aligned}
\|Y_k(t+s_p) - Z_k(t)\| &= \|\lim_{n\to\infty} \int_{k-1}^{k} V^{\mathcal{S}}(\xi)\Pi^{\mathcal{S}}(\tilde{B}_n X_0 (f(t+s_p-\xi) - g(t-\xi)))\, d\xi\| \\
&\le M\tilde{M}\|\Pi^{\mathcal{S}}\| \int_{k-1}^{k} e^{-\alpha\xi} \|f(t+s_p-\xi) - g(t-\xi)\|\, d\xi \\
&\le M\tilde{M}\|\Pi^{\mathcal{S}}\|\, e^{-\alpha(k-1)} \int_{k-1}^{k} \|f(t+s_p-\xi) - g(t-\xi)\|\, d\xi \\
&= M\tilde{M}\|\Pi^{\mathcal{S}}\|\, e^{-\alpha(k-1)} \int_{t-k}^{t-k+1} \|f(s_p+\xi) - g(\xi)\|\, d\xi \\
&\le M\tilde{M}\|\Pi^{\mathcal{S}}\|\, e^{-\alpha(k-1)} \sup_{s\in\mathbb{R}} \int_{s+1}^{s} \|f(s_p+\xi) - g(\xi)\|\, d\xi\,.
\end{aligned}$$

Using the same argument as above, we can prove that

$$Z_k(t-s_p) \to \lim_{n\to\infty} \int_{k-1}^{k} V^{\mathcal{S}}(\xi)\Pi^{\mathcal{S}}(\tilde{B}_n X_0 f(t-\xi))d\xi \quad \text{as} \quad n \to \infty.$$

Therefore, $Y_k(t) \in AA(\mathbb{R}, X)$ for $k \ge 1$.

On other hand, for each $t \in \mathbb{R}$ and $k \ge 1$, we have

$$\begin{aligned}
\|Y_k(t)\| &\le M\tilde{M}\|\Pi^{\mathcal{S}}\| \int_{k-1}^{k} e^{-\alpha\xi} \|f(t-\xi)\|\, d\xi \\
&\le M\tilde{M}\|\Pi^{\mathcal{S}}\| e^{-\alpha(k-1)} \int_{k-1}^{k} \|f(t-\xi)\|\, d\xi
\end{aligned}$$

$$
\begin{aligned}
&= M\tilde{M}\|\Pi^{\mathcal{S}}\|e^{-\alpha(k-1)}\int_{t-k}^{t-k+1}\|f(\xi)\|\,d\xi \\
&\le M\tilde{M}\|\Pi^{\mathcal{S}}\|\,\|f\|_{BS^1}e^{-\alpha(k-1)}
\end{aligned}
$$

We deduce from the well-known Weierstrass Theorem that the series $\sum_{k=1}^{\infty}Y_k(t)$ is uniformly convergent on $\mathbb{R}$. Let

$$H(n,\xi,t) = V^s(t-\xi)\Pi^{\mathcal{S}}(\tilde{B}_n X_0 f(\xi)).$$

We claim that $Y(t) = \sum_{k=1}^{\infty}Y_k(t)$. In fact,

$$
\begin{aligned}
\|\sum_{k=1}^{N}Y_k(t) - Y(t)\| &= \|\sum_{k=1}^{N}\lim_{n\to\infty}\int_{t-k}^{t-k+1}H(n,\xi,t)\,d\xi - \lim_{n\to\infty}\int_{-\infty}^{t}H(n,\xi,t)\,d\xi\| \\
&= \|\lim_{n\to\infty}\sum_{k=N+1}^{\infty}\int_{t-k}^{t-k+1}H(n,\xi,t)\,d\xi\| \\
&\le M\tilde{M}\|\Pi^{\mathcal{S}}\|\sum_{k=N+1}^{\infty}\int_{t-k}^{t-k+1}e^{-\alpha(t-\xi)}\|f(\xi)\|\,d\xi \\
&\le M\tilde{M}\|\Pi^{\mathcal{S}}\|\,\|f\|_{BS^1}\sum_{k=N+1}^{\infty}e^{-\alpha(k-1)} \quad \to 0 \quad \text{as } N\to\infty
\end{aligned}
$$

Because the convergence of the series $\sum_{k=1}^{\infty}Y_k(t)$ is uniform, we deduce that $Y(t)\in AA(\mathbb{R},X)$. Hence, the (1.3) has an almost automorphic solution on $\mathbb{R}$. □

6. Application

To apply the abstract results of the previous section, we consider the following transportation equation with delay proposed in [12]:

$$
\begin{gathered}
\frac{\partial\, w(t,\xi)}{\partial\, t} + \frac{\partial\, w(t,\xi)}{\partial \xi} + \alpha w(t,\xi) + \int_{-\infty}^{\infty} g(\xi,\eta)w(t-r,\eta)\,d\eta = \tilde{f}(t,\xi), \quad \xi\in\mathbb{R},\ t\ge 0 \\
w(\theta,\xi) = \varphi(\theta,\xi), \quad \xi\in\mathbb{R},\ -r\le\theta\le 0,
\end{gathered}
\tag{6.1}
$$

where $\alpha, r > 0$ and $g, \tilde{f}, \varphi$ are continuous functions and the function $\tilde{f}:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is given by

$$\tilde{f}(t,\xi) = \sin\Big(\frac{1}{2+\cos t+\cos\sqrt{2}t}\Big)h_0(\xi) + a(t) \tag{6.2}$$

where $h_0\in L^2(\mathbb{R})$ and $a(t) = \sum_{n\ge 1}\beta_n(t)$ such that for each $n\ge 0$,

$$\beta_n(t) = \sum_{i\in P_n} H(n^2(t-i)).$$

Where $P_n = 3^n(2\mathbb{Z}+1)$ and $H\in C^{\infty}(\mathbb{R},\mathbb{R})$ with support in $(\frac{-1}{2},\frac{1}{2})$ such that

$$H\ge 0;\quad H(0)=1,\quad \int_{\frac{-1}{2}}^{\frac{1}{2}} H(s)ds = 1\,.$$

Lemma 6.1. [17] *The function $a\in C^{\infty}(\mathbb{R},\mathbb{R})$ but $a\notin AA(\mathbb{R},\mathbb{R})$ since is not bounded on $\mathbb{R}$. However, $a\in AAS^1(\mathbb{R},\mathbb{R})$.*

To rewrite system (6.1) in the abstract form (2.1), we introduce the space $X = L^2(\mathbb{R})$ and we define the operator A by

$$Az(\xi) = -\frac{dz(\xi)}{d\xi} - \alpha z(\xi)$$

on the domain $D(A) = H^1(\mathbb{R})$. From [12] we obtain operator A is the infinitesimal generator of a strongly continuous group $(T(t))_{t\geq 0}$ on X given by

$$T(t)z(\xi) = e^{-\alpha t} z(\xi - t), \quad \text{for } t \geq 0 \text{ and } \xi \in \mathbb{R}. \tag{6.3}$$

Hence, the semigroup $(T(t))_{t\geq 0}$, is exponentially stable, and the operator $T(t)$ is not compact because $T(t)$ has bounded inverse $T(-t)$.

Let $f : \mathbb{R} \to X$ be $f(t) = \tilde{f}(t, \cdot)$, then f is a continuous function. We assume that $g : \mathbb{R}^2 \to \mathbb{R}$ is continuous and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |g(\xi, \eta)|^2 \, d\eta \, d\xi < \infty.$$

Lemma 6.2 ([12]). *Under the above conditions, the linear operator $N : X \to X$ given by*

$$Nz(\xi) = \int_{-\infty}^{\infty} g(\xi, \eta) z(\eta) \, d\eta$$

is compact.

Let $L : C([-r, 0], X) \to X$ defined by $L(\Psi) = -N\Psi(-r)$. By Lemma 6.2, we obtain that L is a compact linear map. With this construction and by using the notation $x(t) = w(t, \cdot)$, the original system (6.1) is represented by the abstract system the abstract form (2.1). From (6.2) we obtain that f is a S^1-automorphic function. Moreover the semigroup $(T(t))_{t\geq 0}$ given by (6.3) is exponentially stable and L is a compact operator then $T(t)L$ is compact. It remains to show that the (6.1) has bounded mild solution on $\mathbb{R}^+$. To achieve this goal, we need the following Lemma.

Lemma 6.3 ([7]). *If*

$$x(t) \leq h(t) + \int_{t_0}^{t} k(s) x(s) \, ds \quad \textit{for} \quad [t_0, \tau)$$

where all of the functions involved are continuous and nonnegative on $[t_0, \tau)$ and $k(x) \geq 0$, then x satisfies

$$x(t) \leq h(t) + \int_{t_0}^{t} h(s) k(s) e^{\int_s^t k(u) du} \, ds \quad \textit{for } [t_0, \tau)$$

For any initial data $\varphi \in C$, the (6.1) has a solution x, given by

$$x(t) = T(t)\varphi(0) + \int_0^t T(t-s)(L(x_s) + f(s)) \, ds, \quad \text{for } t \geq 0.$$

Then

$$e^{\alpha t} \|x(t)\| \leq \|\varphi\| + \int_0^t e^{\alpha s} (\|L\| \|x_s\| + \|f(s)\|) \, ds \quad t \geq 0. \tag{6.4}$$

Let $\theta \in [-r, 0]$ and $t \geq 0$. If $t + \theta < 0$, then

$$e^{\alpha t} \|x(t + \theta)\| = e^{\alpha t} \|\varphi(t + \theta)\| \leq e^{\alpha r} \|\varphi\| \leq e^{\alpha r} \|\varphi\|$$

If $t+\theta \geq 0$. By using (6.4) and $-\theta \leq r$, we have

$$e^{\alpha t}\|x(t+\theta)\| \leq e^{\alpha r}\|\varphi\| + e^{\alpha r}\int_0^t e^{\alpha s}\|f(s)\|ds + e^{\alpha r}\|L\|\int_0^t e^{\alpha s}\|x_s\|ds.$$

For $t \geq 0$ let $e^{\alpha t}\|x_t\| = \sup_{-r\leq\theta\leq 0} e^{\alpha t}\|x(t+\theta)\|$. Then

$$e^{\alpha t}\|x_t\| \leq e^{\alpha r}\|\varphi\| + M_2\,(e^{\alpha(t+1)}-1) + e^{\alpha r}\|L\|\int_0^t e^{\alpha s}\|x_s\|ds,$$

where

$$M_2 = \frac{e^{\alpha(r+1)}\|f\|_{BS^1}}{e^{\alpha}-1}$$

By Lemma 6.3 we obtain

$$\begin{aligned} e^{\alpha t}\|x_t\| \leq e^{\alpha r}\|\varphi\| + M_2\,(e^{\alpha(t+1)}-1) + e^{\alpha r}\|L\|\int_0^t (e^{\alpha r}\|\varphi\| \\ + M_2(e^{\alpha(s+1)}-1))e^{e^{\alpha r}\|L\|(t-s)}\,ds. \end{aligned}$$

Moreover, if we assume that

$$\|L\| \leq \frac{\alpha}{e^{r\alpha}},$$

then

$$\|x_t\| \leq M_2 e^{\alpha} + e^{\alpha r}\|\varphi\| + \frac{M_2\|L\|e^{\alpha(r+1)}}{\alpha - \|L\|e^{\alpha r}}$$

This shows that x is a bounded solution of (6.1) on $\mathbb{R}^+$. As a consequence of Theorem 5.4, we obtain that the (6.1) has an almost automorphic solution.

7. Conclusions and discussion

In this work, we establish the existence of almost periodic solutions for partial functional differential equations with Stepanov almost periodic forcing functions. More specifically, we improve the assumptions in [12], we prove that the almost periodicity of the coefficients in a weaker sense (Stepanov almost periodicity) of order is enough to obtain solutions that are almost periodic in a strong sense (Bochner almost periodicity). After that, we extend our result to the almost automorphic case. We give sufficient conditions insuring the existence of almost automorphic solutions to equation (1.3) when the input term is only Stepanov almost automorphic.

To arrive at our results, we employ the variation of constant formula and fundamental results on the spectral analysis of the solutions which is the main tool of this work. Under the hypothesis that the operator $T(t)L$ is compact for $t>0$, we develop a new fundamental reduction principle that is different from the one in [11]. Indeed, to establish the reduction principle we take an approach similar to that in [12] and [11] without using the compactness of C_0-semigroup $(T(t))_{t\geq 0}$. Moreover, we prove the fundamental theorem of existence of almost periodic solutions and almost automorphic solutions. At the end, we illustrate our theoretical result to a transportation equation.

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