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EXISTENCE FOR A NONLOCAL PENROSE-FIFE TYPE PHASE FIELD SYSTEM WITH INERTIAL TERM

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Abstract. This article presents a nonlocal Penrose-Fife type phase field system with inertial term. We do not know whether we can prove the existence of solutions to the problem as in Colli-Grasselli-Ito [3] or not. In this article we introduce a time discretization scheme, then pass to the limit as the time step h approaches 0, and obtain an error estimate for the difference between the continuous solution and the discrete solution.

1. Introduction

Colli-Grasselli-Ito [3] derived the existence of solutions to the parabolic hyperbolic Penrose-Fife phase field system

$$
\begin{gathered}
\big(-\frac{1}{u}\big)_t + (\lambda(\varphi))_t - \Delta u = f \quad \text{in } \Omega\times(0,T),\\
\varphi_{tt} + \varphi_t - \Delta\varphi + \beta(\varphi) + \pi(\varphi) = \lambda'(\varphi)u \quad \text{in } \Omega\times(0,T),\\
\partial_\nu u + u = g \quad \text{on } \partial\Omega\times(0,T),\\
(-\frac{1}{u})(0) = -\frac{1}{u_0}, \quad \varphi(0)=\varphi_0, \quad \varphi_t(0)=v_0 \quad \text{in } \Omega,
\end{gathered}
\tag{1.1}
$$

where $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $T > 0$, $\lambda : \mathbb{R} \to \mathbb{R}$ is a smooth function which may have quadratic growth, $\beta : \mathbb{R} \to \mathbb{R}$ is a maximal monotone function, $\pi : \mathbb{R} \to \mathbb{R}$ is an anti-monotone function, ∂_ν denotes differentiation with respect to the outward normal of $\partial\Omega$, $u_0 : \Omega \to \mathbb{R}$, $\varphi_0 : \Omega \to \mathbb{R}$ and $v_0 : \Omega \to \mathbb{R}$ are given functions. Moreover, in the case that $\lambda(\varphi) = \varphi$, they have proved the uniqueness of solutions to (1.1). Assuming that $|\beta(r)| \le c_1|r|^3 + c_2$ for all $r \in \mathbb{R}$, where $c_1, c_2 > 0$ are some constants, we can obtain an estimate for $\beta(\varphi)$ by establishing the $L^\infty(0,T;H^1(\Omega))$-estimate for φ and by using the continuity of the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$.

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The existence of solutions to the singular nonlocal phase field system with inertial term

$$\begin{gathered}
(\ln u)_t + \varphi_t - \Delta u = f \quad \text{in } \Omega \times (0,T), \\
\varphi_{tt} + \varphi_t + a(\cdot)\varphi - J * \varphi + \beta(\varphi) + \pi(\varphi) = u \quad \text{in } \Omega \times (0,T), \\
\partial_\nu u = 0 \quad \text{on } \partial\Omega \times (0,T), \\
(\ln u)(0) = \ln u_0,\ \varphi(0) = \varphi_0,\ \varphi_t(0) = v_0 \quad \text{in } \Omega
\end{gathered} \tag{1.2}$$

has been studied in [7], where $J : \mathbb{R}^d \to \mathbb{R}$ is an interaction kernel, $a(x) := \int_\Omega J(x-y)\,dy$ and $(J*\varphi)(x) := \int_\Omega J(x-y)\varphi(y)\,dy$ for $x \in \Omega$. To derive the $L^\infty(0,T;H^2(\Omega))$-estimate for $\int_0^t u(s)\,ds$ is a key to establish an estimate for $\beta(\varphi)$. Indeed, it holds that

$$\frac{1}{2}|\varphi(x,t)|^2 = \frac{1}{2}|\varphi_0(x)|^2 + \int_0^t \varphi_t(x,s)\varphi(x,s)\,ds$$

and

$$\begin{aligned}
&\frac{1}{2}|\varphi_t(x,t)|^2 + \int_0^t |\varphi_t(x,s)|^2\,ds + \widehat{\beta}(\varphi(x,t)) \\
&= \int_0^t u(x,s)\varphi_t(x,s)\,ds + \frac{1}{2}|v_0(x)|^2 + \widehat{\beta}(\varphi_0(x)) \\
&\quad - \int_0^t (a(x)\varphi(x,s) - (J*\varphi(s))(x))\varphi_t(x,s)\,ds,
\end{aligned}$$

where $\widehat{\beta}(r) = \int_0^r \beta(s)\,ds$. Moreover, since $u > 0$ in $\Omega \times (0,T)$, we see that

$$\int_0^t u(x,s)\varphi_t(x,s)\,ds \le \|\varphi_t\|_{L^\infty(\Omega\times(0,T))} \int_0^t u(x,s)\,ds.$$

Thus, deriving the $L^\infty(0,T;H^2(\Omega))$-estimate for $\int_0^t u(x,s)\,ds$ from the first equation in (1.2), using the continuity of the embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, applying the Young inequality and the Gronwall lemma, we can establish the $L^\infty(\Omega \times (0,T))$-estimates for φ_t and φ, whence we can obtain the $L^\infty(\Omega\times(0,T))$-estimate for $\beta(\varphi)$ by assuming that β is continuous.

It seems that this is the first study of nonlocal Penrose-Fife type phase field systems with inertial term. So we verify the existence of solutions to the problem

$$\begin{gathered}
\big(-\frac{1}{u}\big)_t + \varphi_t - \Delta u = f \quad \text{in } \Omega \times (0,T), \\
\varphi_{tt} + \varphi_t + a(\cdot)\varphi - J * \varphi + \beta(\varphi) + \pi(\varphi) = u \quad \text{in } \Omega \times (0,T), \\
\partial_\nu u + u = g \quad \text{on } \partial\Omega \times (0,T), \\
(-\frac{1}{u})(0) = -\frac{1}{u_0},\ \varphi(0) = \varphi_0,\ \varphi_t(0) = v_0 \quad \text{in } \Omega,
\end{gathered} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^d$ $(d = 1,2,3)$ is a bounded domain with smooth boundary $\partial\Omega$. Moreover, we assume the following conditions:

- (A1) $J(-x) = J(x)$ for all $x \in \mathbb{R}^d$ and $\sup_{x\in\Omega} \int_\Omega |J(x-y)|\,dy < +\infty$.
- (A2) $\beta : \mathbb{R} \to \mathbb{R}$ is a single-valued maximal monotone function such that there exists a proper lower semicontinuous convex function $\widehat{\beta} : \mathbb{R} \to [0,+\infty)$ satisfying that $\widehat{\beta}(0) = 0$ and $\beta = \partial\widehat{\beta}$, where $\partial\widehat{\beta}$ is the subdifferential of $\widehat{\beta}$. Moreover, $\beta : \mathbb{R} \to \mathbb{R}$ is local Lipschitz continuous.
- (A3) $\pi : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function.

(A4) $f \in L^2(\Omega \times (0,T))$, $g \in L^2(0,T;H^{1/2}(\partial\Omega))$, $g \le 0$ a.e. on $\partial\Omega \times (0,T)$, $\theta_0 := -\frac{1}{u_0} \in L^2(\Omega)$, $\theta_0 > 0$ a.e. in Ω, $\ln\theta_0 \in L^1(\Omega)$, $\varphi_0, v_0 \in L^\infty(\Omega)$.

Definition 1.1. A pair (u,φ) with

$$u \in L^2(0,T;H^1(\Omega)), \quad -\frac{1}{u} \in H^1(0,T;(H^1(\Omega))^*) \cap L^\infty(0,T;L^2(\Omega)),$$
$$\varphi \in W^{2,2}(0,T;L^2(\Omega)) \cap W^{1,\infty}(0,T;L^\infty(\Omega))$$

is called a *weak solution* of (1.3) if (u,φ) satisfies

$$\begin{aligned}
&\big\langle (-\frac{1}{u})_t, w\big\rangle_{(H^1(\Omega))^*,H^1(\Omega)} + (\varphi_t, w)_{L^2(\Omega)} + \int_\Omega \nabla u \cdot \nabla w + \int_{\partial\Omega} (u-g)w \\
&= (f,w)_{L^2(\Omega)} \quad \text{a.e. in } (0,T) \text{ for all } w \in H^1(\Omega),\\
&\varphi_{tt} + \varphi_t + a(\cdot)\varphi - J*\varphi + \beta(\varphi) + \pi(\varphi) = u \quad \text{a.e. in } \Omega\times(0,T),\\
&(-\frac{1}{u})(0) = \theta_0, \quad \varphi(0) = \varphi_0, \quad \varphi_t(0) = v_0 \quad \text{a.e. in } \Omega.
\end{aligned}$$

Theorem 1.2. *Assume that* (A1)–(A4) *hold. Then there exists a unique weak solution* (u,φ) *of* (1.3).

This article is organized as follows. In Section 2 we introduce a time discretization of (1.3) and set precisely the approximate problem. In Section 3 we prove the existence for the discrete problem. In Section 4 we establish some uniform estimates for the approximate problem. Section 5 obtains Cauchy's criterion for solutions of the approximate problem and is devoted to the proofs of the existence and uniqueness of weak solutions to (1.3) and an error estimate between the solution of (1.3) and the solution of the approximate problem.

2. Time discretization

To prove the existence of weak solutions to (1.3) we deal with the discrete problem

$$\begin{gathered}
\frac{\theta_{n+1}-\theta_n}{h} + \frac{\varphi_{n+1}-\varphi_n}{h} - \Delta u_{n+1} = f_{n+1} \quad \text{in } \Omega,\\
z_{n+1} + v_{n+1} + a(\cdot)\varphi_n - J*\varphi_n + \beta(\varphi_{n+1}) + \pi(\varphi_{n+1}) = u_{n+1} \quad \text{in } \Omega,\\
z_{n+1} = \frac{v_{n+1}-v_n}{h}, \quad v_{n+1} = \frac{\varphi_{n+1}-\varphi_n}{h} \quad \text{in } \Omega,\\
\partial_\nu u_{n+1} + u_{n+1} = g_{n+1} \quad \text{on } \partial\Omega
\end{gathered} \tag{2.1}$$

for $n = 0,\dots,N-1$, where $h = \frac{T}{N}$, $N \in \mathbb{N}$,

$$\theta_j := -\frac{1}{u_j}$$

for $j = 0,1,\dots,N$, and $f_k := \frac{1}{h}\int_{(k-1)h}^{kh} f(s)\,ds$, $g_k := \frac{1}{h}\int_{(k-1)h}^{kh} g(s)\,ds$ for $k = 1,\dots,N$. Indeed, we can show the existence for (2.1).

Theorem 2.1. *Assume that* (A1)-(A4) *hold. Then there exists* $h_0 \in (0,1]$ *such that for all* $h \in (0,h_0)$ *there exists a unique solution of* (2.1) *satisfying*

$$u_{n+1} \in H^2(\Omega), \quad \varphi_{n+1} \in L^\infty(\Omega) \quad \textit{for } n = 0,\dots,N-1.$$

Putting

$$\widehat{\theta}_h(t) := \theta_n + \frac{\theta_{n+1} - \theta_n}{h}(t - nh), \tag{2.2}$$

$$\widehat{\varphi}_h(t) := \varphi_n + \frac{\varphi_{n+1} - \varphi_n}{h}(t - nh), \tag{2.3}$$

$$\widehat{v}_h(t) := v_n + \frac{v_{n+1} - v_n}{h}(t - nh) \tag{2.4}$$

for $t \in [nh, (n+1)h]$, $n = 0, \dots, N-1$, and

$$\overline{u}_h(t) := u_{n+1}, \quad \overline{\theta}_h(t) := \theta_{n+1}, \quad \overline{\varphi}_h(t) := \varphi_{n+1}, \quad \underline{\varphi}_h(t) := \varphi_n, \tag{2.5}$$

$$\overline{v}_h(t) := v_{n+1}, \quad \overline{z}_h(t) := z_{n+1}, \quad \overline{f}_h(t) := f_{n+1} \tag{2.6}$$

for $t \in (nh, (n+1)h]$, $n = 0, \dots, N-1$, we can rewrite (2.1) as

$$\begin{gathered}
(\widehat{\theta}_h)_t + (\widehat{\varphi}_h)_t - \Delta \overline{u}_h = \overline{f}_h \quad \text{in } \Omega \times (0,T), \\
\overline{z}_h + \overline{v}_h + a(\cdot)\underline{\varphi}_h - J * \underline{\varphi}_h + \beta(\overline{\varphi}_h) + \pi(\overline{\varphi}_h) = \overline{u}_h \quad \text{in } \Omega \times (0,T), \\
\overline{z}_h = (\widehat{v}_h)_t, \ \overline{v}_h = (\widehat{\varphi}_h)_t \quad \text{in } \Omega \times (0,T), \\
\overline{\theta}_h = -\frac{1}{\overline{u}_h} \quad \text{in } \Omega \times (0,T), \\
\partial_\nu \overline{u}_h + \overline{u}_h = \overline{g}_h \quad \text{on } \partial\Omega \times (0,T), \\
\widehat{\theta}_h(0) = \theta_0, \ \widehat{\varphi}_h(0) = \varphi_0, \ \widehat{v}_h(0) = v_0 \quad \text{in } \Omega.
\end{gathered} \tag{2.7}$$

Here we can check directly the following identities by (2.2)-(2.6):

$$\|\widehat{\theta}_h\|_{L^\infty(0,T;L^2(\Omega))} = \max\{\|\theta_0\|_{L^2(\Omega)}, \|\overline{\theta}_h\|_{L^\infty(0,T;L^2(\Omega))}\}, \tag{2.8}$$

$$\|\widehat{\varphi}_h\|_{L^\infty(0,T;L^\infty(\Omega))} = \max\{\|\varphi_0\|_{L^\infty(\Omega)}, \|\overline{\varphi}_h\|_{L^\infty(0,T;L^\infty(\Omega))}\}, \tag{2.9}$$

$$\|\widehat{v}_h\|_{L^\infty(0,T;L^\infty(\Omega))} = \max\{\|v_0\|_{L^\infty(\Omega)}, \|\overline{v}_h\|_{L^\infty(0,T;L^\infty(\Omega))}\}, \tag{2.10}$$

$$\|\overline{\theta}_h - \widehat{\theta}_h\|^2_{L^2(0,T;(H^1(\Omega))^*)} = \frac{h^2}{3}\|(\widehat{\theta}_h)_t\|^2_{L^2(0,T;(H^1(\Omega))^*)}, \tag{2.11}$$

$$\|\overline{\varphi}_h - \widehat{\varphi}_h\|_{L^\infty(0,T;L^\infty(\Omega))} = h\|(\widehat{\varphi}_h)_t\|_{L^\infty(0,T;L^\infty(\Omega))} = h\|\overline{v}_h\|_{L^\infty(0,T;L^\infty(\Omega))}, \tag{2.12}$$

$$\|\overline{v}_h - \widehat{v}_h\|^2_{L^2(0,T;L^2(\Omega))} = \frac{h^2}{3}\|(\widehat{v}_h)_t\|^2_{L^2(0,T;L^2(\Omega))} = \frac{h^2}{3}\|\overline{z}_h\|^2_{L^2(0,T;L^2(\Omega))}, \tag{2.13}$$

$$\underline{\varphi}_h = \overline{\varphi}_h - h(\widehat{\varphi}_h)_t. \tag{2.14}$$

We can prove Theorem 1.2 by passing to the limit in (2.7) as $h \searrow 0$. Moreover, we can obtain the following theorem which asserts an error estimate between the solution of (1.3) and the solution of (2.7).

Theorem 2.2. *Let h_0 be as in Theorem 1.2. Assume that* (A1)-(A4) *hold. Assume further that $f \in W^{1,1}(0,T;L^2(\Omega))$ and $g \in W^{1,1}(0,T;L^2(\partial\Omega))$. Then there exist constants $h_{00} \in (0,h_0)$ and $M > 0$ depending on the data such that*

$$\begin{aligned}
&\|1 \star (\overline{u}_h - u)\|_{C([0,T];H^1(\Omega))} + \|\widehat{\varphi}_h - \varphi\|_{C([0,T];L^2(\Omega))} + \|\widehat{v}_h - \varphi_t\|_{C([0,T];L^2(\Omega))} \\
&\le M h^{1/2}
\end{aligned}$$

for all $h \in (0,h_{00})$, where $(1 \star w)(t) := \int_0^t w(s)\,ds$ for vector-valued functions w summable in $(0,T)$.

3. Existence for the discrete problem

In this section we will show Theorem 2.1.

Lemma 3.1. *For all $h > 0$, $G \in L^2(\Omega)$, $G_{\partial\Omega} \in H^{1/2}(\partial\Omega)$, if $G_{\partial\Omega} \le 0$ a.e. on $\partial\Omega$, then there exists a unique function $u \in H^2(\Omega)$ satisfying*

$$u < 0 \text{ a.e. in } \Omega, \quad -\frac{1}{u} - h\Delta u = G \text{ a.e. in } \Omega, \quad \partial_\nu u + u = G_{\partial\Omega} \text{ a.e. on } \partial\Omega.$$

Proof. We set the operator $\mathcal{A} : D(\mathcal{A}) \subset L^2(\Omega) \to L^2(\Omega)$ as

$$\mathcal{A}u := -\Delta u - cu \quad \text{for } u \in D(\mathcal{A}) := \{u \in H^2(\Omega) : \partial_\nu u + u = G_{\partial\Omega} \text{ a.e. on } \partial\Omega\}.$$

Then this operator is maximal monotone for some constant $c > 0$. Also, we define the operator $\mathcal{B} : D(\mathcal{B}) \subset L^2(\Omega) \to L^2(\Omega)$ as

$$\mathcal{B}u := -\frac{h^{-1}}{u} \quad \text{for } u \in D(\mathcal{B}) := \{u \in L^2(\Omega) : u < 0 \text{ a.e. in } \Omega\}.$$

Then this operator is maximal monotone. Now we set the function $b : D(b) \subset \mathbb{R} \to \mathbb{R}$ as $b(r) := -\frac{h^{-1}}{r}$ for $r \in D(b) := \{r \in \mathbb{R} : r < 0\}$. Let $\lambda > 0$, let $\mathcal{B}_\lambda$ be the Yosida approximation of $\mathcal{B}$ and let b_λ be the Yosida approximation of b on $\mathbb{R}$. Then, noting that b_λ is monotone, $u = \lambda b_\lambda(u) + (1 + \lambda b)^{-1}(u)$, $b_\lambda(u) = -\frac{h^{-1}}{(1+\lambda b)^{-1}(u)} > 0$, and $G_{\partial\Omega} \le 0$ a.e. on $\partial\Omega$, we can confirm that

$$\begin{aligned}
(\mathcal{A}u, \mathcal{B}_\lambda u)_{L^2(\Omega)} &= \int_\Omega b'_\lambda(u)|\nabla u|^2 + \int_{\partial\Omega} u b_\lambda(u) - \int_{\partial\Omega} G_{\partial\Omega} b_\lambda(u) - c\int_\Omega u b_\lambda(u) \\
&\ge \lambda \|b_\lambda(u)\|^2_{L^2(\partial\Omega)} - h^{-1}|\partial\Omega| - c\lambda\|b_\lambda(u)\|^2_{L^2(\Omega)} + ch^{-1}|\Omega| \\
&\ge -\max\{c, h^{-1}|\partial\Omega|\}(\lambda\|\mathcal{B}_\lambda(u)\|^2_{L^2(\Omega)} + 1)
\end{aligned}$$

for all $u \in D(\mathcal{A})$ and all $\lambda > 0$. Therefore we can conclude that the operator $\mathcal{A} + \mathcal{B}$ is maximal monotone (see e.g., Barbu [2, Theorem 2.7]). $\square$

Lemma 3.2. *For all $G \in L^2(\Omega)$ and all $h \in (0, \min\{1, 1/\|\pi'\|_{L^\infty(\mathbb{R})}\})$ there exists a unique solution $\varphi \in L^2(\Omega)$ of the equation*

$$\varphi + h\varphi + h^2\beta(\varphi) + h^2\pi(\varphi) = G \quad \text{a.e. in } \Omega.$$

The above lemma can be proved as in [6, Lemma 2.1].

Proof of Theorem 2.1. We can rewrite (2.1) as

$$\begin{gathered}
-\frac{1}{u_{n+1}} - h\Delta u_{n+1} = -\varphi_{n+1} + hf_{n+1} + \varphi_n + \theta_n, \\
\partial_\nu u_{n+1} + u_{n+1} = g_{n+1}, \\
\varphi_{n+1} + h\varphi_{n+1} + h^2\beta(\varphi_{n+1}) + h^2\pi(\varphi_{n+1}) \\
= h^2 u_{n+1} + \varphi_n + hv_n + h\varphi_n - h^2 a(\cdot)\varphi_n + h^2 J * \varphi_n.
\end{gathered} \tag{3.1}$$

To prove Theorem 2.1 it suffices to establish the existence and uniqueness of solutions to (3.1) in the case that $n = 0$. Let $h \in (0, \min\{1, 1/\|\pi'\|_{L^\infty(\mathbb{R})}\})$. Then, owing to Lemma 3.1, for all $\varphi \in L^2(\Omega)$ there exists a unique function $\overline{u} \in H^2(\Omega)$ such that

$$-\frac{1}{\overline{u}} - h\Delta\overline{u} = -\varphi + hf_1 + \varphi_0 + \theta_0, \quad \partial_\nu\overline{u} + \overline{u} = g_1. \tag{3.2}$$

Also, we see from Lemma 3.2 that for all $u \in L^2(\Omega)$ there exists a unique function $\overline{\varphi} \in L^2(\Omega)$ such that

$$\overline{\varphi} + h\overline{\varphi} + h^2\beta(\overline{\varphi}) + h^2\pi(\overline{\varphi}) = h^2u + \varphi_0 + hv_0 + h\varphi_0 - h^2a(\cdot)\varphi_0 + h^2J * \varphi_0. \quad (3.3)$$

Thus we can set $\Phi : L^2(\Omega) \to L^2(\Omega)$, $\Psi : L^2(\Omega) \to L^2(\Omega)$ and $B : L^2(\Omega) \to L^2(\Omega)$ as

$$\Phi\varphi = \overline{u}, \ \Psi u = \overline{\varphi} \quad \text{for } \varphi, u \in L^2(\Omega),$$
$$B = \Psi \circ \Phi.$$

Moreover, we can obtain that for all $\varphi, \widetilde{\varphi} \in L^2(\Omega)$,

$$\|B\varphi - B\widetilde{\varphi}\|_{L^2(\Omega)} \le \frac{C_1 h}{1 + h - \|\pi'\|_{L^\infty(\mathbb{R})}h^2}\|\varphi - \widetilde{\varphi}\|_{L^2(\Omega)}$$

(cf. [7, Proof of Theorem 1.2]). Then there exists $h_{01} \in (0, \min\{1, 1/\|\pi'\|_{L^\infty(\mathbb{R})}\})$ such that

$$\frac{C_1 h}{1 + h - \|\pi'\|_{L^\infty(\mathbb{R})}h^2} \in (0, 1)$$

for all $h \in (0, h_{01})$. Hence $B : L^2(\Omega) \to L^2(\Omega)$ is a contraction mapping in $L^2(\Omega)$ for all $h \in (0, h_{01})$ and then it follows from the Banach fixed-point theorem that for all $h \in (0, h_{01})$ there exists a unique function $\varphi_1 \in L^2(\Omega)$ such that $\varphi_1 = B\varphi_1 \in L^2(\Omega)$. Thus, for all $h \in (0, h_{01})$, putting $u_1 := \Phi\varphi_1 \in H^2(\Omega)$ implies that there exists a unique pair $(u_1, \varphi_1) \in (L^2(\Omega))^2$ satisfying (3.1) in the case that $n = 0$. Moreover, we can prove that there exists $h_0 \in (0, h_{01})$ such that for all $h \in (0, h_0)$ there exists a constant $C_1 = C_1(h) > 0$ such that $|\varphi_1(x)| \le C_1$ for a.a. $x \in \Omega$ (cf. [7, Proof of Theorem 1.2]). □

4. Uniform estimates for the discrete problem

In this section we derive a priori estimates for (2.7).

Lemma 4.1. *Let h_0 be as in Theorem 2.1. Then there exist constants $h_1 \in (0, h_0)$ and $C > 0$ depending on the data such that*

$$\|\overline{\varphi}_h\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\overline{v}_h\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\overline{u}_h\|^2_{L^2(0,T;H^1(\Omega))} + \|\overline{\theta}_h\|_{L^\infty(0,T;L^1(\Omega))}$$
$$+ \|\ln\overline{\theta}_h\|_{L^\infty(0,T;L^1(\Omega))} \le C$$

for all $h \in (0, h_1)$.

Proof. Multiplying the identity $v_{n+1} = \frac{\varphi_{n+1} - \varphi_n}{h}$ by $h\varphi_{n+1}$ we obtain

$$\frac{1}{2}\|\varphi_{n+1}\|^2_{L^2(\Omega)} - \frac{1}{2}\|\varphi_n\|^2_{L^2(\Omega)} + \frac{1}{2}\|\varphi_{n+1} - \varphi_n\|^2_{L^2(\Omega)} = h(\varphi_{n+1}, v_{n+1})_{L^2(\Omega)}. \quad (4.1)$$

We test the second equation in (2.1) by hv_{n+1} to infer that

$$\begin{aligned}
&\frac{1}{2}\|v_{n+1}\|^2_{L^2(\Omega)} - \frac{1}{2}\|v_n\|^2_{L^2(\Omega)} + \frac{1}{2}\|v_{n+1} - v_n\|^2_{L^2(\Omega)} + h\|v_{n+1}\|^2_{L^2(\Omega)} \\
&+ (\beta(\varphi_{n+1}), \varphi_{n+1} - \varphi_n)_{L^2(\Omega)} \\
&= h(u_{n+1}, v_{n+1})_{L^2(\Omega)} - h(\pi(\varphi_{n+1}), v_{n+1})_{L^2(\Omega)} \\
&\quad - h(a(\cdot)\varphi_n - J * \varphi_n, v_{n+1})_{L^2(\Omega)}.
\end{aligned} \quad (4.2)$$

Here the condition (A2) leads to the inequality

$$(\beta(\varphi_{n+1}), \varphi_{n+1} - \varphi_n)_{L^2(\Omega)} \ge \|\widehat{\beta}(\varphi_{n+1})\|_{L^1(\Omega)} - \|\widehat{\beta}(\varphi_n)\|_{L^1(\Omega)}. \quad (4.3)$$

Thus we deduce from (4.1)-(4.3), the Young inequality, (A1), and (A3) that there exists a constant $C_1 > 0$ such that

$$\begin{aligned}
&\frac{1}{2}\|\varphi_{n+1}\|^2_{L^2(\Omega)} - \frac{1}{2}\|\varphi_n\|^2_{L^2(\Omega)} + \frac{1}{2}\|\varphi_{n+1} - \varphi_n\|^2_{L^2(\Omega)} \\
&+ \frac{1}{2}\|v_{n+1}\|^2_{L^2(\Omega)} - \frac{1}{2}\|v_n\|^2_{L^2(\Omega)} + \frac{1}{2}\|v_{n+1} - v_n\|^2_{L^2(\Omega)} + h\|v_{n+1}\|^2_{L^2(\Omega)} \\
&+ \|\widehat{\beta}(\varphi_{n+1})\|_{L^1(\Omega)} - \|\widehat{\beta}(\varphi_n)\|_{L^1(\Omega)} \\
&\le h(u_{n+1}, v_{n+1})_{L^2(\Omega)} + C_1 h + C_1\|\varphi_{n+1}\|^2_{L^2(\Omega)} + C_1\|\varphi_n\|^2_{L^2(\Omega)} \\
&\quad + C_1\|v_{n+1}\|^2_{L^2(\Omega)}
\end{aligned} \tag{4.4}$$

for all $h \in (0, h_0)$. Next we multiply the first equation in (2.1) by $h(1 + u_{n+1})$ to obtain that

$$\begin{aligned}
&(\theta_{n+1} - \theta_n, 1 + u_{n+1})_{L^2(\Omega)} + h\int_\Omega |\nabla u_{n+1}|^2 + h\int_{\partial\Omega} |u_{n+1}|^2 \\
&= h(f_{n+1}, 1 + u_{n+1})_{L^2(\Omega)} - h(u_{n+1}, v_{n+1})_{L^2(\Omega)} \\
&\quad - h(v_{n+1}, 1)_{L^2(\Omega)} - h\int_{\partial\Omega} u_{n+1} + h\int_{\partial\Omega} g_{n+1}(1 + u_{n+1}).
\end{aligned} \tag{4.5}$$

Here, noting that $u_{n+1} = -\frac{1}{\theta_{n+1}}$ and $r - 1 \ge \ln r$ for all $r > 0$, we have that

$$\begin{aligned}
&(\theta_{n+1} - \theta_n, 1 + u_{n+1})_{L^2(\Omega)} \\
&= \|\theta_{n+1}\|_{L^1(\Omega)} - \|\theta_n\|_{L^1(\Omega)} + (\theta_{n+1} - \theta_n, u_{n+1})_{L^2(\Omega)} \\
&= \|\theta_{n+1}\|_{L^1(\Omega)} - \|\theta_n\|_{L^1(\Omega)} + \int_\Omega (\frac{\theta_n}{\theta_{n+1}} - 1) \\
&\ge \|\theta_{n+1}\|_{L^1(\Omega)} - \|\theta_n\|_{L^1(\Omega)} + \int_\Omega \ln\frac{\theta_n}{\theta_{n+1}} \\
&= \|\theta_{n+1}\|_{L^1(\Omega)} - \|\theta_n\|_{L^1(\Omega)} + \int_\Omega (-\ln\theta_{n+1} + \ln\theta_n).
\end{aligned} \tag{4.6}$$

There exist constants $C_*, C^* > 0$ such that

$$C_*(\|\nabla w\|^2_{L^2(\Omega)} + \|w\|^2_{L^2(\partial\Omega)}) \le \|w\|^2_{H^1(\Omega)} \le C^*(\|\nabla w\|^2_{L^2(\Omega)} + \|w\|^2_{L^2(\partial\Omega)}) \tag{4.7}$$

for all $w \in H^1(\Omega)$. Therefore we see from (4.5)-(4.7) and the Young inequality that there exists a constant $C_2 > 0$ such that

$$\begin{aligned}
&\|\theta_{n+1}\|_{L^1(\Omega)} - \|\theta_n\|_{L^1(\Omega)} + \int_\Omega (-\ln\theta_{n+1} + \ln\theta_n) + \frac{1}{2C^*}h\|u_{n+1}\|^2_{H^1(\Omega)} \\
&\le -h(u_{n+1}, v_{n+1})_{L^2(\Omega)} + C_2 h + C_2 h\|f_{n+1}\|^2_{L^2(\Omega)} + C_2 h\|g_{n+1}\|^2_{L^2(\partial\Omega)} \\
&\quad + C_2 h\|v_{n+1}\|^2_{L^2(\Omega)}
\end{aligned} \tag{4.8}$$

for all $h \in (0, h_0)$. Therefore we add (4.4) to (4.8) and sum over $n = 0, \dots, m-1$ with $1 \le m \le N$ to derive that

$$\begin{aligned} &\frac{1}{2}\|\varphi_m\|^2_{L^2(\Omega)} + \frac{1}{2}\|v_m\|^2_{L^2(\Omega)} + \|\widehat{\beta}(\varphi_m)\|_{L^1(\Omega)} \\ &+ \|\theta_m\|_{L^1(\Omega)} - \int_\Omega \ln\theta_m + \frac{1}{2C^*} h \sum_{n=0}^{m-1} \|u_{n+1}\|^2_{H^1(\Omega)} \\ &\le \frac{1}{2}\|\varphi_0\|^2_{L^2(\Omega)} + \frac{1}{2}\|v_0\|^2_{L^2(\Omega)} + \|\widehat{\beta}(\varphi_0)\|_{L^1(\Omega)} + \|\theta_0\|_{L^1(\Omega)} - \int_\Omega \ln\theta_0 \\ &\quad + (C_1 + C_2)T + C_2 h \sum_{n=0}^{m-1} \|f_{n+1}\|^2_{L^2(\Omega)} + C_2 h \sum_{n=0}^{m-1} \|g_{n+1}\|^2_{L^2(\partial\Omega)} \\ &\quad + 2C_1 h \sum_{n=0}^{m-1} \|\varphi_{n+1}\|^2_{L^2(\Omega)} + (C_1 + C_2) h \sum_{n=0}^{m-1} \|v_{n+1}\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.9}$$

On the other hand,

$$\|\theta_m\|_{L^1(\Omega)} - \int_\Omega \ln\theta_m = \int_\Omega (\theta_m - \ln\theta_m) \ge \frac{1}{3}\int_\Omega (\theta_m + |\ln\theta_m|). \tag{4.10}$$

Thus it follows from (4.9) and (4.10) that

$$\begin{aligned} &(\frac{1}{2} - 2C_1 h)\|\varphi_m\|^2_{L^2(\Omega)} + (\frac{1}{2} - (C_1 + C_2)h)\|v_m\|^2_{L^2(\Omega)} + \|\widehat{\beta}(\varphi_m)\|_{L^1(\Omega)} \\ &+ \frac{1}{3}\|\theta_m\|_{L^1(\Omega)} + \frac{1}{3}\|\ln\theta_m\|_{L^1(\Omega)} + \frac{1}{2C^*} h \sum_{n=0}^{m-1} \|u_{n+1}\|^2_{H^1(\Omega)} \\ &\le \frac{1}{2}\|\varphi_0\|^2_{L^2(\Omega)} + \frac{1}{2}\|v_0\|^2_{L^2(\Omega)} + \|\widehat{\beta}(\varphi_0)\|_{L^1(\Omega)} + \|\theta_0\|_{L^1(\Omega)} + \|\ln\theta_0\|_{L^1(\Omega)} \\ &\quad + (C_1 + C_2)T + C_2 h \sum_{n=0}^{m-1} \|f_{n+1}\|^2_{L^2(\Omega)} + C_2 h \sum_{n=0}^{m-1} \|g_{n+1}\|^2_{L^2(\partial\Omega)} \\ &\quad + 2C_1 h \sum_{j=0}^{m-1} \|\varphi_j\|^2_{L^2(\Omega)} + (C_1 + C_2) h \sum_{j=0}^{m-1} \|v_j\|^2_{L^2(\Omega)} \end{aligned}$$

and then there exist constants $C_3 > 0$ and $h_1 \in (0, h_0)$ such that

$$\begin{aligned} &\|\varphi_m\|^2_{L^2(\Omega)} + \|v_m\|^2_{L^2(\Omega)} + \|\widehat{\beta}(\varphi_m)\|_{L^1(\Omega)} \\ &+ \|\theta_m\|_{L^1(\Omega)} + \|\ln\theta_m\|_{L^1(\Omega)} + h \sum_{n=0}^{m-1} \|u_{n+1}\|^2_{H^1(\Omega)} \\ &\le C_3 + C_3 h \sum_{j=0}^{m-1} \|\varphi_j\|^2_{L^2(\Omega)} + C_3 h \sum_{j=0}^{m-1} \|v_j\|^2_{L^2(\Omega)} \end{aligned}$$

for all $h \in (0, h_1)$ and $m = 1, \dots, N$. Therefore, owing to the discrete Gronwall lemma (see e.g., [5, Prop. 2.2.1]), there exists a constant $C_4 > 0$ such that

$$\begin{aligned} &\|\varphi_m\|^2_{L^2(\Omega)} + \|v_m\|^2_{L^2(\Omega)} + \|\widehat{\beta}(\varphi_m)\|_{L^1(\Omega)} \\ &+ \|\theta_m\|_{L^1(\Omega)} + \|\ln\theta_m\|_{L^1(\Omega)} + h \sum_{n=0}^{m-1} \|u_{n+1}\|^2_{H^1(\Omega)} \le C_4 \end{aligned}$$

for all $h \in (0, h_1)$ and $m = 1, \dots, N$. □

Lemma 4.2. *Let h_1 be as in Lemma 4.1. Then there exist constants $h_2 \in (0, h_1)$ and $C > 0$ depending on the data such that*

$$\|\overline{\theta}_h\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\ln \overline{\theta}_h\|^2_{L^2(0,T;H^1(\Omega))} \le C$$

for all $h \in (0, h_2)$.

Proof. Testing the first equation in (2.1) by $h\theta_{n+1}$ leads to the identity

$$\begin{aligned} &\frac{1}{2}\|\theta_{n+1}\|^2_{L^2(\Omega)} - \frac{1}{2}\|\theta_n\|^2_{L^2(\Omega)} + \frac{1}{2}\|\theta_{n+1} - \theta_n\|^2_{L^2(\Omega)} + h(-\Delta u_{n+1}, \theta_{n+1})_{L^2(\Omega)} \\ &= h(f_{n+1}, \theta_{n+1})_{L^2(\Omega)} - h(v_{n+1}, \theta_{n+1})_{L^2(\Omega)}. \end{aligned} \tag{4.11}$$

Here, since $u_{n+1} = -\frac{1}{\theta_{n+1}}$, $\theta_{n+1} > 0$, and $g_{n+1} \le 0$, we have that

$$\begin{aligned} &h(-\Delta u_{n+1}, \theta_{n+1})_{L^2(\Omega)} \\ &= h\int_\Omega \nabla u_{n+1} \cdot \nabla\theta_{n+1} + h\int_{\partial\Omega} u_{n+1}\theta_{n+1} - h\int_{\partial\Omega} g_{n+1}\theta_{n+1} \\ &\ge h\int_\Omega |\nabla \ln\theta_{n+1}|^2 - h|\partial\Omega|. \end{aligned} \tag{4.12}$$

Therefore we can verify that Lemma 4.2 holds by combining (4.11), (4.12), by summing over $n = 0, \dots, m-1$ with $1 \le m \le N$, by applying the discrete Gronwall lemma, Lemma 4.1, the Poincaré-Wirtinger inequality. □

Lemma 4.3. *Let h_2 be as in Lemma 4.2. Then there exists a constant $C > 0$ depending on the data such that*

$$\|(\widehat{\theta}_h)_t\|_{L^2(0,T;(H^1(\Omega))^*)} \le C$$

for all $h \in (0, h_2)$.

Proof. We can obtain this lemma by the first equation in (2.7) and Lemma 4.1. □

Lemma 4.4. *Let h_2 be as in Lemma 4.2. Then there exists a constant $C > 0$ depending on the data such that*

$$h \max_{1\le m\le N} \Big\| \sum_{n=0}^{m-1}(-u_{n+1})\Big\|_{H^2(\Omega)} \le C$$

for all $h \in (0, h_2)$.

Proof. We can prove this lemma by Lemmas 4.1, 4.2 and the elliptic regularity theory (cf. [7, Proof of Lemma 4.5]). □

Lemma 4.5. *Let h_2 be as in Lemma 4.2. Then there exist constants $h_3 \in (0, h_2)$ and $C > 0$ depending on the data such that*

$$\|\overline{\varphi}_h\|^2_{L^\infty(\Omega\times(0,T))} + \|\overline{v}_h\|^2_{L^\infty(\Omega\times(0,T))} \le C$$

for all $h \in (0, h_3)$.

Proof. From [7, Proof of Lemma 4.6], we can confirm that there exists a constant $C_1 > 0$ such that

$$\begin{aligned} \frac{1}{2}|\varphi_m(x)|^2 + \frac{1}{2}|v_m(x)|^2 \le h\sum_{n=0}^{m-1} u_{n+1}(x)v_{n+1}(x) + C_1 h \sum_{n=0}^{m-1} \|\varphi_{n+1}\|^2_{L^\infty(\Omega)} \\ + C_1 h \sum_{n=0}^{m-1} \|v_{n+1}\|^2_{L^\infty(\Omega)} + C_1 \end{aligned} \tag{4.13}$$

for all $h \in (0, h_2)$ and for a.a. $x \in \Omega$, $m = 1, \dots, N$. Here, noting that $-u_j > 0$ a.e. in Ω for $j = 0, 1, \dots, N$, we deduce from Lemma 4.4 and the continuity of the embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ that there exists a constant $C_2 > 0$ such that

$$\begin{aligned} h\sum_{n=0}^{m-1} u_{n+1}(x)v_{n+1}(x) &= h\sum_{n=0}^{m-1} (-u_{n+1}(x))(-v_{n+1}(x)) \\ &\le \Big(\max_{1\le m\le N} \|-v_m\|_{L^\infty(\Omega)}\Big) h \sum_{n=0}^{m-1} (-u_{n+1}(x)) \\ &\le \Big(\max_{1\le m\le N} \|v_m\|_{L^\infty(\Omega)}\Big) h \Big\| \sum_{n=0}^{m-1} (-u_{n+1}) \Big\|_{L^\infty(\Omega)} \\ &\le C_2 \max_{1\le m\le N} \|v_m\|_{L^\infty(\Omega)} \end{aligned} \tag{4.14}$$

for all $h \in (0, h_2)$ and for a.a. $x \in \Omega$, $m = 1, \dots, N$. Thus we see from (4.13) and (4.14) that

$$\begin{aligned} &\frac{1}{2}|\varphi_m(x)|^2 + \frac{1}{2}|v_m(x)|^2 \\ &\le C_2 \max_{1\le m\le N} \|v_m\|_{L^\infty(\Omega)} + C_1 h \sum_{n=0}^{m-1} \|\varphi_{n+1}\|^2_{L^\infty(\Omega)} + C_1 h \sum_{n=0}^{m-1} \|v_{n+1}\|^2_{L^\infty(\Omega)} + C_1 \end{aligned}$$

for a.a. $x \in \Omega$ and for all $h \in (0, h_2)$, $m = 1, \dots, N$, whence the inequality

$$\begin{aligned} &\frac{1}{2}\|\varphi_m\|^2_{L^\infty(\Omega)} + \frac{1}{2}\|v_m\|^2_{L^\infty(\Omega)} \\ &\le C_2 \max_{1\le m\le N} \|v_m\|_{L^\infty(\Omega)} + C_1 h \sum_{n=0}^{m-1} \|\varphi_{n+1}\|^2_{L^\infty(\Omega)} + C_1 h \sum_{n=0}^{m-1} \|v_{n+1}\|^2_{L^\infty(\Omega)} + C_1 \end{aligned}$$

holds. Then there exist constants $h_3 \in (0, h_2)$ and $C_3 > 0$ such that

$$\begin{aligned} &\|\varphi_m\|^2_{L^\infty(\Omega)} + \|v_m\|^2_{L^\infty(\Omega)} \\ &\le C_3 \max_{1\le m\le N} \|v_m\|_{L^\infty(\Omega)} + C_3 h \sum_{j=0}^{m-1} \|\varphi_j\|^2_{L^\infty(\Omega)} + C_3 h \sum_{j=0}^{m-1} \|v_j\|^2_{L^\infty(\Omega)} + C_3 \end{aligned}$$

for all $h \in (0, h_3)$ and $m = 1, \dots, N$. Hence by the discrete Gronwall lemma there exists a constant $C_4 > 0$ such that

$$\|\varphi_m\|^2_{L^\infty(\Omega)} + \|v_m\|^2_{L^\infty(\Omega)} \le C_4 + C_4 \max_{1\le m\le N} \|v_m\|_{L^\infty(\Omega)}$$

for all $h \in (0, h_3)$ and $m = 1, \dots, N$. Therefore it holds that

$$\max_{1\le m\le N} \|\varphi_m\|^2_{L^\infty(\Omega)} + \max_{1\le m\le N} \|v_m\|^2_{L^\infty(\Omega)} \le C_4 + C_4 \max_{1\le m\le N} \|v_m\|_{L^\infty(\Omega)}$$

$$\leq C_4 + \frac{1}{2}\max_{1\leq m\leq N}\|v_m\|^2_{L^\infty(\Omega)} + \frac{C_4^2}{2},$$

which leads to Lemma 4.5. □

Lemma 4.6. *Let h_3 be as in Lemma 4.5. Then there exists a constant $C > 0$ depending on the data such that*

$$\|\underline{\varphi}_h\|_{L^\infty(\Omega\times(0,T))} \leq C$$

for all $h \in (0, h_3)$.

Proof. This lemma can be obtained by (A4) and Lemma 4.5. □

Lemma 4.7. *Let h_3 be as in Lemma 4.5. Then there exists a constant $C > 0$ depending on the data such that*

$$\|\beta(\overline{\varphi}_h)\|_{L^\infty(\Omega\times(0,T))} \leq C$$

for all $h \in (0, h_3)$.

Proof. We can prove this lemma by the continuity of β and Lemma 4.5. □

Lemma 4.8. *Let h_3 be as in Lemma 4.5. Then there exists a constant $C > 0$ depending on the data such that*

$$\|\overline{z}_h\|_{L^2(0,T;L^2(\Omega))} \leq C$$

for all $h \in (0, h_3)$.

Proof. We can verify that this lemma holds by the second equation in (2.7), Lemmas 4.1, 4.5-4.7, and the conditions (A1), (A3). □

Lemma 4.9. *Let h_3 be as in Lemma 4.5. Then there exists a constant $C > 0$ depending on the data such that*

$$\begin{aligned}&\|\widehat{\theta}_h\|_{H^1(0,T;(H^1(\Omega))^*)\cap L^\infty(0,T;L^2(\Omega))} + \|\widehat{v}_h\|_{H^1(0,T;L^2(\Omega))\cap L^\infty(\Omega\times(0,T))}\\ &+ \|\widehat{\varphi}_h\|_{W^{1,\infty}(0,T;L^\infty(\Omega))} \leq C\end{aligned}$$

for all $h \in (0, h_3)$.

Proof. Lemmas 4.2, 4.3, 4.5, 4.8, along with (2.8)-(2.10), lead to Lemma 4.9. □

5. Existence for (1.3) and error estimate

In this section we will derive the existence and uniqueness of solutions to (1.3) by passing to the limit in (2.7) as $h \searrow 0$ and will establish an error estimate between the solution of (1.3) and the solution of (2.7).

Lemma 5.1. *Let h_3 be as in Lemma 4.5. Then there exists a constant $M_1 > 0$ depending on the data such that*

$$\begin{aligned}&\|(1\star(\overline{u}_h - \overline{u}_\tau))(t)\|^2_{H^1(\Omega)}\\ &\leq M_1(h+\tau) + M_1\int_0^t \|(1\star(\overline{u}_h - \overline{u}_\tau))(s)\|^2_{H^1(\Omega)}\,ds\\ &\quad + M_1\int_0^t \|\widehat{v}_h(s) - \widehat{v}_\tau(s)\|^2_{L^2(\Omega)}\,ds + M_1\|\overline{f}_h - \overline{f}_\tau\|^2_{L^2(0,T;L^2(\Omega))}\\ &\quad + M_1\|\overline{g}_h - \overline{g}_\tau\|^2_{L^2(0,T;L^2(\partial\Omega))}\end{aligned}\tag{5.1}$$

for all $h, \tau \in (0, h_3)$ and all $t \in [0, T]$.

Proof. We have that

$$
\begin{aligned}
&(\widehat{\theta}_h - \widehat{\theta}_\tau, w)_{L^2(\Omega)} + (\widehat{\varphi}_h - \widehat{\varphi}_\tau, w)_{L^2(\Omega)} + \int_\Omega \nabla(1 \star (\overline{u}_h - \overline{u}_\tau)) \cdot \nabla w \\
&+ \int_{\partial\Omega} (1 \star (\overline{u}_h - \overline{u}_\tau)) w \\
&= ((1 \star (\overline{f}_h - \overline{f}_\tau)), w)_{L^2(\Omega)} + \int_{\partial\Omega} (1 \star (\overline{g}_h - \overline{g}_\tau)) w
\end{aligned} \tag{5.2}
$$

a.e. in $(0, T)$ for all $w \in H^1(\Omega)$. Taking $w = \overline{u}_h - \overline{u}_\tau$ in (5.2) and integrating over $(0, t)$, where $t \in [0, T]$, we have

$$
\begin{aligned}
&\int_0^t (\widehat{\theta}_h(s) - \widehat{\theta}_\tau(s), \overline{u}_h(s) - \overline{u}_\tau(s))_{L^2(\Omega)}\, ds \\
&+ \int_0^t (\widehat{\varphi}_h(s) - \widehat{\varphi}_\tau(s), \overline{u}_h(s) - \overline{u}_\tau(s))_{L^2(\Omega)}\, ds + \frac{1}{2}\|\nabla(1 \star (\overline{u}_h - \overline{u}_\tau))(s)\|^2_{L^2(\Omega)} \\
&+ \frac{1}{2}\|(1 \star (\overline{u}_h - \overline{u}_\tau))(s)\|^2_{L^2(\partial\Omega)} \\
&= \int_0^t ((1 \star (\overline{f}_h - \overline{f}_\tau))(s), (1 \star (\overline{u}_h - \overline{u}_\tau))(s))_{L^2(\Omega)}\, ds \\
&\quad + \int_0^t \Big(\int_{\partial\Omega} (1 \star (\overline{g}_h - \overline{g}_\tau))(s)(1 \star (\overline{u}_h - \overline{u}_\tau))(s) \Big)\, ds.
\end{aligned} \tag{5.3}
$$

Here we see from the identity $\overline{\theta}_h = -\frac{1}{\overline{u}_h}$ that

$$
\begin{aligned}
&\int_0^t (\widehat{\theta}_h(s) - \widehat{\theta}_\tau(s), \overline{u}_h(s) - \overline{u}_\tau(s))_{L^2(\Omega)}\, ds \\
&= \int_0^t \Big\langle \widehat{\theta}_h(s) - \overline{\theta}_h(s), \overline{u}_h(s) - \overline{u}_\tau(s) \Big\rangle_{(H^1(\Omega))^*, H^1(\Omega)}\, ds \\
&\quad + \int_0^t \Big\langle \overline{\theta}_\tau(s) - \widehat{\theta}_\tau(s), \overline{u}_h(s) - \overline{u}_\tau(s) \Big\rangle_{(H^1(\Omega))^*, H^1(\Omega)}\, ds \\
&\quad + \int_0^t (\alpha(\overline{u}_h(s)) - \alpha(\overline{u}_\tau(s)), \overline{u}_h(s) - \overline{u}_\tau(s))_{L^2(\Omega)}\, ds \\
&\geq \int_0^t \Big\langle \widehat{\theta}_h(s) - \overline{\theta}_h(s), \overline{u}_h(s) - \overline{u}_\tau(s) \Big\rangle_{(H^1(\Omega))^*, H^1(\Omega)}\, ds \\
&\quad + \int_0^t \Big\langle \overline{\theta}_\tau(s) - \widehat{\theta}_\tau(s), \overline{u}_h(s) - \overline{u}_\tau(s) \Big\rangle_{(H^1(\Omega))^*, H^1(\Omega)}\, ds,
\end{aligned} \tag{5.4}
$$

where $\alpha(r) := -1/r$ for $r \in D(\alpha) := \{r \in \mathbb{R} \mid r < 0\}$ and the monotonicity of α was used. Integrating by parts with respect to time yields that

$$
\begin{aligned}
&\int_0^t (\widehat{\varphi}_h(s) - \widehat{\varphi}_\tau(s), \overline{u}_h(s) - \overline{u}_\tau(s))_{L^2(\Omega)}\, ds \\
&= \int_0^t ((1 \star (\overline{v}_h - \overline{v}_\tau))(s), (1 \star (\overline{u}_h - \overline{u}_\tau))'(s))_{L^2(\Omega)}\, ds \\
&= ((1 \star (\overline{v}_h - \overline{v}_\tau))(t), (1 \star (\overline{u}_h - \overline{u}_\tau))(t))_{L^2(\Omega)} \\
&\quad - \int_0^t (\overline{v}_h(s) - \overline{v}_\tau(s), (1 \star (\overline{u}_h - \overline{u}_\tau))(s))_{L^2(\Omega)}\, ds.
\end{aligned} \tag{5.5}
$$

Also, it holds that

$$
\begin{aligned}
&\int_0^t ((1\star(\overline{f}_h-\overline{f}_\tau))(s),\overline{u}_h(s)-\overline{u}_\tau(s))_{L^2(\Omega)}\,ds\\
&=\int_0^t ((1\star(\overline{f}_h-\overline{f}_\tau))(s),(1\star(\overline{u}_h-\overline{u}_\tau))'(s))_{L^2(\Omega)}\,ds\\
&=((1\star(\overline{f}_h-\overline{f}_\tau))(t),(1\star(\overline{u}_h-\overline{u}_\tau))(t))_{L^2(\Omega)}\\
&\quad-\int_0^t (\overline{f}_h(s)-\overline{f}_\tau(s),(1\star(\overline{u}_h-\overline{u}_\tau))(s))_{L^2(\Omega)}\,ds
\end{aligned}
\tag{5.6}
$$

and

$$
\begin{aligned}
&\int_0^t \Big(\int_{\partial\Omega}(1\star(\overline{g}_h-\overline{g}_\tau))(s)(\overline{u}_h(s)-\overline{u}_\tau(s))\Big)\,ds\\
&=\int_0^t \Big(\int_{\partial\Omega}(1\star(\overline{g}_h-\overline{g}_\tau))(s)(1\star(\overline{u}_h-\overline{u}_\tau))'(s)\Big)\,ds\\
&=\int_{\partial\Omega}(1\star(\overline{g}_h-\overline{g}_\tau))(t)(1\star(\overline{u}_h-\overline{u}_\tau))(t)\\
&\quad-\int_0^t \Big(\int_{\partial\Omega}(\overline{g}_h-\overline{g}_\tau)(s)(1\star(\overline{u}_h-\overline{u}_\tau))(s)\Big)\,ds.
\end{aligned}
\tag{5.7}
$$

Therefore, since $\overline{v}_h-\overline{v}_\tau=\overline{v}_h-\widehat{v}_h+\widehat{v}_\tau-\overline{v}_\tau+\widehat{v}_h-\widehat{v}_\tau$, we can prove Lemma 5.1 by (5.3)-(5.7), the Schwarz inequality, the Young inequality, (2.11), (2.13), Lemmas 4.1, 4.3, 4.8. □

Lemma 5.2. *Let h_3 be as in Lemma 4.5. Then there exists a constant $M_2>0$ depending on the data such that*

$$
\begin{aligned}
&\|\widehat{\varphi}_h(t)-\widehat{\varphi}_\tau(t)\|_{L^2(\Omega)}^2+\|\widehat{v}_h(t)-\widehat{v}_\tau(t)\|_{L^2(\Omega)}^2\\
&\le M_2(h+\tau)+M_2\int_0^t \|\widehat{\varphi}_h(s)-\widehat{\varphi}_\tau(s)\|_{L^2(\Omega)}^2\,ds\\
&\quad+M_2\int_0^t \|\widehat{v}_h(s)-\widehat{v}_\tau(s)\|_{L^2(\Omega)}^2\,ds+M_2\|(1\star(\overline{u}_h-\overline{u}_\tau))(t)\|_{H^1(\Omega)}^2
\end{aligned}
\tag{5.8}
$$

for all $h,\tau\in(0,h_3)$ and all $t\in[0,T]$.

Proof. We see from (2.14) and Lemma 4.1 that there exists a constant $C_1>0$ such that

$$
\begin{aligned}
&\int_0^t \|\underline{\varphi}_h(s)-\underline{\varphi}_\tau(s)\|_{L^2(\Omega)}^2\,ds\\
&\le 3\int_0^t \|\overline{\varphi}_h(s)-\overline{\varphi}_\tau(s)\|_{L^2(\Omega)}^2\,ds+3h^2\int_0^t \|(\widehat{\varphi}_h)_s(s)\|_{L^2(\Omega)}^2\,ds\\
&\quad+3\tau^2\int_0^t \|(\widehat{\varphi}_\tau)_s(s)\|_{L^2(\Omega)}^2\,ds\\
&\le 3\int_0^t \|\overline{\varphi}_h(s)-\overline{\varphi}_\tau(s)\|_{L^2(\Omega)}^2\,ds+C_1h^2+C_1\tau^2
\end{aligned}
\tag{5.9}
$$

for all $h, \tau \in (0, h_3)$ and all $t \in [0, T]$. Here, owing to (2.12) and Lemma 4.5, it holds that there exists a constant $C_2 > 0$ such that

$$
\begin{aligned}
&3\int_0^t \|\overline{\varphi}_h(s) - \overline{\varphi}_\tau(s)\|_{L^2(\Omega)}^2\,ds \\
&= 3\int_0^t \|\overline{\varphi}_h(s) - \widehat{\varphi}_h(s) + \widehat{\varphi}_\tau(s) - \overline{\varphi}_\tau(s) + \widehat{\varphi}_h(s) - \widehat{\varphi}_\tau(s)\|_{L^2(\Omega)}^2\,ds \\
&\le 9\int_0^t \|\overline{\varphi}_h(s) - \widehat{\varphi}_h(s)\|_{L^2(\Omega)}^2\,ds + 9\int_0^t \|\widehat{\varphi}_\tau(s) - \overline{\varphi}_\tau(s)\|_{L^2(\Omega)}^2\,ds \\
&\quad + 9\int_0^t \|\widehat{\varphi}_h(s) - \widehat{\varphi}_\tau(s)\|_{L^2(\Omega)}^2\,ds \\
&\le C_2h^2 + C_2\tau^2 + 9\int_0^t \|\widehat{\varphi}_h(s) - \widehat{\varphi}_\tau(s)\|_{L^2(\Omega)}^2\,ds
\end{aligned} \tag{5.10}
$$

for all $h, \tau \in (0, h_3)$ and all $t \in [0, T]$. We derive from the identity $\overline{v}_h(s) = (\widehat{\varphi}_h)_s(s)$, (2.13) and Lemma 4.8 that there exists a constant $C_3 > 0$ such that

$$
\begin{aligned}
&\|\widehat{\varphi}_h(t) - \widehat{\varphi}_\tau(t)\|_{L^2(\Omega)}^2 \\
&= \Big\|\int_0^t (\overline{v}_h(s) - \overline{v}_\tau(s))\,ds\Big\|_{L^2(\Omega)}^2 \\
&= \Big\|\int_0^t (\overline{v}_h(s) - \widehat{v}_h(s) + \widehat{v}_\tau(s) - \overline{v}_\tau(s) + \widehat{v}_h(s) - \widehat{v}_\tau(s))\,ds\Big\|_{L^2(\Omega)}^2 \\
&\le C_3h^2 + C_3\tau^2 + C_3\int_0^t \|\widehat{v}_h(s) - \widehat{v}_\tau(s)\|_{L^2(\Omega)}^2\,ds
\end{aligned} \tag{5.11}
$$

for all $h, \tau \in (0, h_3)$ and all $t \in [0, T]$. Thus, since

$$
\begin{aligned}
&\widehat{v}_h - \widehat{v}_\tau + \widehat{\varphi}_h - \widehat{\varphi}_\tau + a(\cdot)(1 \star (\underline{\varphi}_h - \underline{\varphi}_\tau)) - J * (1 \star (\underline{\varphi}_h - \underline{\varphi}_\tau)) \\
&+ 1 \star (\beta(\overline{\varphi}_h) - \beta(\overline{\varphi}_\tau)) + 1 \star (\pi(\overline{\varphi}_h) - \pi(\overline{\varphi}_\tau)) \\
&= 1 \star (\overline{u}_h - \overline{u}_\tau),
\end{aligned}
$$

we deduce from (A1), Lemma 4.5, the local Lipschitz continuity of β, (A3), and (5.9)-(5.11) that there exists a constant $C_4 > 0$ such that

$$
\begin{aligned}
&\|\widehat{v}_h(t) - \widehat{v}_\tau(t)\|_{L^2(\Omega)}^2 \\
&\le C_4(h^2 + \tau^2) + C_4\int_0^t \|\widehat{\varphi}_h(s) - \widehat{\varphi}_\tau(s)\|_{L^2(\Omega)}^2\,ds \\
&\quad + C_4\int_0^t \|\widehat{v}_h(s) - \widehat{v}_\tau(s)\|_{L^2(\Omega)}^2\,ds + C_4\|(1 \star (\overline{u}_h - \overline{u}_\tau))(t)\|_{H^1(\Omega)}^2
\end{aligned} \tag{5.12}
$$

for all $h, \tau \in (0, h_3)$ and all $t \in [0, T]$. On the other hand, it follows from the identity $\overline{v}_h(s) = (\widehat{\varphi}_h)_s(s)$, the Schwarz inequality, the Young inequality, (2.13), Lemmas 4.8

and 4.9 that there exists a constant $C_5 > 0$ such that

$$
\begin{aligned}
\frac{1}{2}\|\widehat{\varphi}_h(t) - \widehat{\varphi}_\tau(t)\|^2_{L^2(\Omega)} &= \int_0^t (\overline{v}_h(s) - \overline{v}_\tau(s), \widehat{\varphi}_h(s) - \widehat{\varphi}_\tau(s))_{L^2(\Omega)}\, ds \\
&= \int_0^t (\overline{v}_h(s) - \widehat{v}_h(s), \widehat{\varphi}_h(s) - \widehat{\varphi}_\tau(s))_{L^2(\Omega)}\, ds \\
&\quad + \int_0^t (\widehat{v}_\tau(s) - \overline{v}_\tau(s), \widehat{\varphi}_h(s) - \widehat{\varphi}_\tau(s))_{L^2(\Omega)}\, ds \\
&\quad + \int_0^t (\widehat{v}_h(s) - \widehat{v}_\tau(s), \widehat{\varphi}_h(s) - \widehat{\varphi}_\tau(s))_{L^2(\Omega)}\, ds \\
&\le C_5 h + C_5\tau + \frac{1}{2}\int_0^t \|\widehat{v}_h(s) - \widehat{v}_\tau(s)\|^2_{L^2(\Omega)}\, ds \\
&\quad + \frac{1}{2}\int_0^t \|\widehat{\varphi}_h(s) - \widehat{\varphi}_\tau(s)\|^2_{L^2(\Omega)}\, ds
\end{aligned} \tag{5.13}
$$

for all $h, \tau \in (0, h_3)$ and all $t \in [0, T]$. Therefore we can show Lemma 5.2 by (5.12) and (5.13). ☐

Lemma 5.3. *Let h_3 be as in Lemma 4.5. Then there exists a constant $M > 0$ depending on the data such that*

$$
\begin{aligned}
&\|1 \star (\overline{u}_h - \overline{u}_\tau)\|_{C([0,T];H^1(\Omega))} + \|\widehat{\varphi}_h - \widehat{\varphi}_\tau\|_{C([0,T];L^2(\Omega))} + \|\widehat{v}_h - \widehat{v}_\tau\|_{C([0,T];L^2(\Omega))} \\
&\le M(h^{1/2} + \tau^{1/2}) + M\|\overline{f}_h - \overline{f}_\tau\|_{L^2(0,T;L^2(\Omega))} + M\|\overline{g}_h - \overline{g}_\tau\|_{L^2(0,T;L^2(\partial\Omega))}
\end{aligned}
$$

for all $h, \tau \in (0, h_3)$.

Proof. Combining (5.1) and (5.8) leads to the inequality

$$
\begin{aligned}
&\frac{1}{2}\|(1 \star (\overline{u}_h - \overline{u}_\tau))(t)\|^2_{H^1(\Omega)} + \frac{1}{2M_2}\|\widehat{\varphi}_h(t) - \widehat{\varphi}_\tau(t)\|^2_{L^2(\Omega)} \\
&+ \frac{1}{2M_2}\|\widehat{v}_h(t) - \widehat{v}_\tau(t)\|^2_{L^2(\Omega)} \\
&\le \big(M_1 + \frac{1}{2}\big)(h + \tau) + M_1\int_0^t \|(1 \star (\overline{u}_h - \overline{u}_\tau))(s)\|^2_{H^1(\Omega)}\, ds \\
&\quad + \frac{1}{2}\int_0^t \|\widehat{\varphi}_h(s) - \widehat{\varphi}_\tau(s)\|^2_{L^2(\Omega)}\, ds + \big(M_1 + \frac{1}{2}\big)\int_0^t \|\widehat{v}_h(s) - \widehat{v}_\tau(s)\|^2_{L^2(\Omega)}\, ds \\
&\quad + M_1\|\overline{f}_h - \overline{f}_\tau\|^2_{L^2(0,T;L^2(\Omega))} + M_1\|\overline{g}_h - \overline{g}_\tau\|^2_{L^2(0,T;L^2(\partial\Omega))}.
\end{aligned}
$$

Thus by the Gronwall lemma we can obtain Lemma 5.3. ☐

Proof of Theorem 1.2. From Lemmas 4.1-4.3, 4.5-4.9, 5.3, the Aubin-Lions lemma for the compact embedding $L^2(\Omega) \hookrightarrow (H^1(\Omega))^*$, and properties (2.11)-(2.14), there exist some functions u, θ, φ, ξ such that

$$
\begin{gathered}
u \in L^2(0,T;H^1(\Omega)),\ \theta \in H^1(0,T;(H^1(\Omega))^*) \cap L^\infty(0,T;L^2(\Omega)), \\
\varphi \in W^{2,2}(0,T;L^2(\Omega)) \cap W^{1,\infty}(0,T;L^\infty(\Omega)),\ \xi \in L^\infty(\Omega \times (0,T))
\end{gathered}
$$

and

$$
\widehat{\theta}_h \to \theta \quad \text{weakly}^* \text{ in } H^1(0,T;(H^1(\Omega))^*) \cap L^\infty(0,T;L^2(\Omega)), \tag{5.14}
$$

$$
\widehat{\theta}_h \to \theta \quad \text{strongly in } C([0,T];(H^1(\Omega))^*), \tag{5.15}
$$

$$\alpha(\overline{u}_h) = \overline{\theta}_h \to \theta \quad \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)), \tag{5.16}$$
$$\overline{u}_h \to u \quad \text{weakly in } L^2(0,T;H^1(\Omega)), \tag{5.17}$$
$$\overline{z}_h \to \varphi_{tt} \quad \text{weakly in } L^2(0,T;L^2(\Omega)), \tag{5.18}$$
$$\widehat{v}_h \to \varphi_t \quad \text{strongly in } C([0,T];L^2(\Omega)), \tag{5.19}$$
$$\overline{v}_h \to \varphi_t \quad \text{weakly}^* \text{ in } L^\infty(\Omega\times(0,T)), \tag{5.20}$$
$$\widehat{\varphi}_h \to \varphi \quad \text{weakly}^* \text{ in } W^{1,\infty}(0,T;L^\infty(\Omega)), \tag{5.21}$$
$$\widehat{\varphi}_h \to \varphi \quad \text{strongly in } C([0,T];L^2(\Omega)), \tag{5.22}$$
$$\overline{\varphi}_h \to \varphi \quad \text{weakly}^* \text{ in } L^\infty(\Omega\times(0,T)), \tag{5.23}$$
$$\underline{\varphi}_h \to \varphi \quad \text{weakly}^* \text{ in } L^\infty(\Omega\times(0,T)), \tag{5.24}$$
$$\beta(\overline{\varphi}_h) \to \xi \quad \text{weakly}^* \text{ in } L^\infty(\Omega\times(0,T)) \tag{5.25}$$

as $h = h_j \searrow 0$, where $\alpha(r) := -\frac{1}{r}$ for $r \in D(\alpha) := \{r \in \mathbb{R} \mid r < 0\}$. We see from (2.11), Lemmas 4.1, 4.3, (5.15), and (5.17) that

$$\begin{aligned}
&\int_0^T (\alpha(\overline{u}_h(t)), \overline{u}_h(t))_{L^2(\Omega)}\,dt \\
&= \int_0^T (\overline{\theta}_h(t), \overline{u}_h(t))_{L^2(\Omega)}\,dt \\
&= \int_0^T \langle \overline{\theta}_h(t) - \widehat{\theta}_h(t), \overline{u}_h(t)\rangle_{(H^1(\Omega))^*,H^1(\Omega)}\,dt + \int_0^T \langle \widehat{\theta}_h(t), \overline{u}_h(t)\rangle_{(H^1(\Omega))^*,H^1(\Omega)}\,dt \\
&\to \int_0^T \langle \theta(t), u(t)\rangle_{(H^1(\Omega))^*,H^1(\Omega)}\,dt = \int_0^T (\theta(t), u(t))_{L^2(\Omega)}\,dt
\end{aligned}$$

as $h = h_j \searrow 0$. Thus, noting that $\alpha : D(\alpha) \subset \mathbb{R} \to \mathbb{R}$ is maximal monotone, we can obtain that

$$\theta = \alpha(u) = -\frac{1}{u} \quad \text{a.e. in } \Omega\times(0,T) \tag{5.26}$$

(see, e.g., [1, Lemma 1.3, p. 42]). On the other hand, it follows from (2.12), Lemma 4.5 and (5.22) that

$$\begin{aligned}
\|\overline{\varphi}_h - \varphi\|_{L^\infty(0,T;L^2(\Omega))} &\le \|\overline{\varphi}_h - \widehat{\varphi}_h\|_{L^\infty(0,T;L^2(\Omega))} + \|\widehat{\varphi}_h - \varphi\|_{L^\infty(0,T;L^2(\Omega))} \\
&\le |\Omega|^{1/2} h \|\overline{v}_h\|_{L^\infty(\Omega\times(0,T))} + \|\widehat{\varphi}_h - \varphi\|_{C([0,T];L^2(\Omega))} \\
&\to 0
\end{aligned} \tag{5.27}$$

as $h = h_j \searrow 0$. Then combining (5.25) and (5.27) yields that

$$\int_0^T (\beta(\overline{\varphi}_h(t)), \overline{\varphi}_h(t))_{L^2(\Omega)}\,dt \to \int_0^T (\xi(t), \varphi(t))_{L^2(\Omega)}\,dt$$

as $h = h_j \searrow 0$, and hence it holds that

$$\xi = \beta(\varphi) \quad \text{a.e. in } \Omega\times(0,T). \tag{5.28}$$

Therefore by (5.14), (5.15), (5.17)-(5.28), (A1), and (A3), and by observing that $\overline{f}_h \to f$ strongly in $L^2(0,T;L^2(\Omega))$ and $\overline{g}_h \to g$ strongly in $L^2(0,T;L^2(\partial\Omega))$ as $h \searrow 0$ (see e.g., [4, Section 5]), we can derive the existence of weak solutions to (1.3). Moreover, we can show the uniqueness of weak solutions to (1.3) in a similar way to the proofs of Lemmas 5.1, 5.2 and 5.3. □

Proof of Theorem 2.2. Since we have from $f \in L^2(0,T;L^2(\Omega)) \cap W^{1,1}(0,T;L^2(\Omega))$ and $g \in L^2(0,T;L^2(\partial\Omega)) \cap W^{1,1}(0,T;L^2(\partial\Omega))$ that there exists a constant $C_1 > 0$ such that

$$\begin{aligned}
&\|\overline{f}_h - f\|_{L^2(0,T;L^2(\Omega))} \le C_1 h^{1/2}, \\
&\|\overline{g}_h - g\|_{L^2(0,T;L^2(\partial\Omega))} \le C_1 h^{1/2}
\end{aligned}$$

for all $h > 0$ (see e.g., [4, Section 5]), we can prove Theorem 2.2 by Lemma 5.3. □

Remark 5.4. Even if in [3] we consider the approximation

$$\begin{gathered}
(\mu_M u_M + \rho_M(u_M))_t + (\varphi_M)_t - \Delta u_M = f \quad \text{in } \Omega \times (0,T), \\
(\varphi_M)_{tt} + (\varphi_M)_t + a(\cdot)\varphi_M - J * \varphi_M + \beta_M(\varphi_M) + \pi(\varphi_M) \\
= -(\rho_M(u_M))^{-1} \quad \text{in } \Omega \times (0,T), \\
\partial_\nu u_M + u_M = g \quad \text{on } \partial\Omega \times (0,T), \\
(u_M)(0) = -(\rho_M(u_0))^{-1}, \quad \varphi_M(0) = \varphi_0, \quad (\varphi_M)_t(0) = v_0 \quad \text{in } \Omega,
\end{gathered} \tag{5.29}$$

we do not know whether we can establish a priori estimates for (5.29) or not. Here $M \in \mathbb{N}$, $\mu_M := \frac{1}{1+M^2}$, the function $\rho_M : \mathbb{R} \to \mathbb{R}$ is defined by

$$\rho_M(r) := \begin{cases} \frac{1}{M+1} & \text{if } r < -(M+1), \\ -\frac{1}{r} & \text{if } -(M+1) \le r \le -\frac{1}{M+1}, \\ M+1 & \text{if } -\frac{1}{M+1} < r, \end{cases}$$

and the function $\beta_M : \mathbb{R} \to \mathbb{R}$ is defined by

$$\beta_M(r) := \begin{cases} -M & \text{if } \beta(r) \le -M, \\ \beta(r) & \text{if } -M < \beta(r) < M, \\ M & \text{if } M \le \beta(r). \end{cases}$$

Although we can obtain that

$$\frac{1}{2}|\varphi_M(x,t)|^2 = \frac{1}{2}|\varphi_0(x)|^2 + \int_0^t (\varphi_M)_t(x,s)\varphi_M(x,s)\,ds$$

and

$$\begin{aligned}
&\frac{1}{2}|(\varphi_M)_t(x,t)|^2 + \int_0^t |(\varphi_M)_t(x,s)|^2\,ds + \widehat{\beta}_M(\varphi_M(x,t)) \\
&= \int_0^t (\rho_M(u_M(x,s)))^{-1}(-(\varphi_M)_t(x,s))\,ds + \cdots,
\end{aligned}$$

where $\widehat{\beta}_M(r) = \int_0^r \beta_M(s)\,ds$, we do not know whether the $L^\infty(\Omega \times (0,T))$-estimate for $\big\{\int_0^t (\rho_M(u_M(x,s)))^{-1}\,ds\big\}_M$ can be derived or not, and then we do not know whether the $L^\infty(\Omega \times (0,T))$-estimates for $\{(\varphi_M)_t\}_M$, $\{\varphi_M\}_M$ and $\{\beta(\varphi_M)\}_M$ can be obtained or not. Even if we replace $-(\rho_M(u_M))^{-1}$ with u_M in (5.29), since the inequality $-u_M \ge 0$ does not hold, we see that

$$\int_0^t (-u_M(x,s))(-(\varphi_M)_t(x,s))\,ds \not\le \|-(\varphi_M)_t\|_{L^\infty(\Omega\times(0,T))} \int_0^t (-u_M(x,s))\,ds,$$

whence we do not know whether the $L^\infty(\Omega \times (0,T))$-estimates for $\{(\varphi_M)_t\}_M$, $\{\varphi_M\}_M$ and $\{\beta(\varphi_M)\}_M$ can be established or not. In this paper, we can prove the existence of solutions to (1.3) by introducing the time discrete problem (2.1) and obtain an error estimate between the solution of (1.3) and the solution of (2.7).

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