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SPACE-TIME DECAY RATES OF A TWO-PHASE FLOW MODEL WITH MAGNETIC FIELD IN $\mathbb{R}^3$

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Abstract. We investigate the space-time decay rates of strong solution to a two-phase flow model with magnetic field in the whole space $\mathbb{R}^3$. Based on the temporal decay results by Xiao [24] we show that for any integer $\ell \geq 3$, the space-time decay rate of $k(0 \leq k \leq \ell)$-order spatial derivative of the strong solution in the weighted Lebesgue space L^2_γ is $t^{-\frac{3}{4}-\frac{k}{2}+\gamma}$. Moreover, we prove that the space-time decay rate of $k(0 \leq k \leq \ell-2)$-order spatial derivative of the difference between two velocities of the fluid in the weighted Lebesgue space L^2_γ is $t^{-\frac{5}{4}-\frac{k}{2}+\gamma}$, which is faster than ones of the two velocities themselves.

1. Introduction and main results

In this article, we study the space-time decay rates of strong solutions to the compressible isothermal Euler equations coupled with compressible magnetohydrodynamic (MHD) system through a drag forcing term in the whole space $\mathbb{R}^3$. The coupled system models the motions of particles immersed in the electrically conducting fluid with the effect of magnetic field. The system takes the following form

$$
\begin{aligned}
&\rho_t + \operatorname{div}(\rho u) = 0,\\
&(\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla \rho = -\rho(u-v),\\
&n_t + \operatorname{div}(nv) = 0,\\
&(nv)_t + \operatorname{div}(nv \otimes v) + \nabla P(n) - \mu\Delta v - (\mu+\lambda)\nabla \operatorname{div} v\\
&= \rho(u-v) + (\nabla \times B) \times B,\\
&B_t - \nabla \times (v \times B) = -\nu \nabla \times (\nabla \times B),\\
&\operatorname{div} B = 0,
\end{aligned}
\tag{1.1}
$$

where $(x,t) \in \mathbb{R}^3 \times \mathbb{R}_+$ is the spatial coordinate and time. Let $\rho(x,t)$ and $n(x,t)$ denote the densities of fluid and let $u(x,t)$ and $v(x,t)$ be the corresponding velocities of $\rho(x,t)$ and $n(x,t)$ respectively. $P = P(n) = An^a (A > 0, a \geq 1)$ represents the pressure. μ and λ are the shear viscosity and the bulk viscosity coefficients of the

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fluid satisfying the following physical restrictions

$$\mu > 0, \quad \frac{2}{3}\mu + \lambda \geq 0.$$

Here B represents magnetic field and ν denotes the coefficient of magnetic diffusivity acting as a magnetic diffusion. We consider the coupled system (1.1) with initial data

$$(\rho, u, n, v, B)|_{t=0} = (\rho_0(x), u_0(x), n_0(x), v_0(x), B_0(x)), \quad x \in \mathbb{R}^3, \tag{1.2}$$

satisfying

$$(\rho_0(x), u_0(x), n_0(x), v_0(x), B_0(x)) \to (\bar{\rho}, 0, \bar{n}, 0, 0) \quad \text{as } \|x| \to \infty,$$

where the positive constants $\bar{\rho}$ and $\bar{n}$ are the reference densities.

1.1. **History of the problem.** When the magnetic field is not taken into account ($B = 0$) in (1.1), system (1.1) reduces to the two-phase fluid model

$$\begin{gathered} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla \rho = -\rho(u - v), \\ n_t + \operatorname{div}(nv) = 0, \\ (nv)_t + \operatorname{div}(nv \otimes v) + \nabla P(n) - \mu \Delta v - (\mu + \lambda)\nabla \operatorname{div} v = \rho(u - v). \end{gathered} \tag{1.3}$$

We notice that Choi [4] firstly addressed the formal derivation of the coupled hydrodynamic system (1.3) from kinetic-fluid equations, which is a type of Vlasov-Fokker-Planck/compressible Navier-Stokes equations. When the magnetic field is taken into account ($B \neq 0$) as in (1.1), its derivation is slightly different from (1.3). Denote the distribution of particles at the position-velocity $(x, \omega) \in \mathbb{R}^3 \times \mathbb{R}^3$ and at time $t \in \mathbb{R}_+$ by $f(x, \omega, t)$, the isentropic compressible fluid density and velocity by $n(x, t)$ and $v(x, t)$ respectively. We intend to study the following kinetic-fluid equations with local alignment and noise forces for the particles to model the motions of particles immersed in the compressible and electrically conducting fluid with the effect of magnetic field:

$$\begin{gathered} f_t + \omega \cdot \nabla_x f + \nabla_\omega \cdot ((v - \omega) f) = -\alpha \nabla_\omega \cdot ((u_f - \omega) f) + \sigma \Delta_\omega f, \\ n_t + \nabla_x \cdot (nv) = 0, \\ (nv)_t + \nabla_x \cdot (nv \otimes v) + \nabla_x P(n) - \mu \Delta_x v - (\mu + \lambda) \nabla_x \nabla_x \cdot v \\ = (\nabla \times B) \times B + \int_{\mathbb{R}^3} (\omega - v) f \mathrm{d}\omega, \\ B_t - \nabla \times (v \times B) = -\nu \nabla \times (\nabla \times B), \end{gathered} \tag{1.4}$$

where u_f is the averaged local velocity defined by

$$u_f(x, t) := \frac{\int_{\mathbb{R}^3} \omega f(x, \omega, t) \mathrm{d}\omega}{\int_{\mathbb{R}^3} f(x, \omega, t) \mathrm{d}\omega}. \tag{1.5}$$

We consider a regime where the local alignment and noise forces are strong, i.e., $\alpha = \sigma = \varepsilon^{-1}$. Let $(f^\varepsilon, n^\varepsilon, v^\varepsilon, B^\varepsilon)$ be the solution to the system (1.4) with $\alpha = \sigma = \varepsilon^{-1}$. It follows from $(1.4)_1$ that

$$-\nabla_\omega \cdot ((u_{f^\varepsilon} - \omega) f^\varepsilon) + \Delta_\omega f^\varepsilon = \varepsilon (f_t^\varepsilon + \omega \cdot \nabla_x f^\varepsilon + \nabla_\omega \cdot ((v^\varepsilon - \omega) f^\varepsilon)) \to 0$$

as $\varepsilon \to 0$. So $f^\varepsilon(x,\omega,t)$ is expected to converge to

$$f(x,\omega,t) = \frac{\rho_f(x,t)}{(2\pi)^{3/2}} \mathrm{e}^{-\frac{|u_f(x,t)-\omega|^2}{2}}, \tag{1.6}$$

where

$$\rho_f(x,t) = \int_{\mathbb{R}^3} f(x,\omega,t)\mathrm{d}\omega. \tag{1.7}$$

Integrating $(1.4)_1$ with respect to ω over $\mathbb{R}^3$, if $\rho_{f^\varepsilon} \to \rho_f$ and $u_{f^\varepsilon} \to u_f$ as $\varepsilon \to 0$ then we obtain the continuity equation $(1.1)_1$. According to the (1.5)-(1.7), we have

$$\int_{\mathbb{R}^3} (u_{f^\varepsilon} - \omega) f^\varepsilon \mathrm{d}\omega = 0. \tag{1.8}$$

Multiplying $(1.4)_1$ by ω, integrating the resulting equation with respect to ω over $\mathbb{R}^3$ and combining (1.8), we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \omega f^\varepsilon \mathrm{d}\omega &= \int_{\mathbb{R}^3} \omega(-\nabla_x \cdot (\omega f^\varepsilon) - \nabla_\omega \cdot ((v^\varepsilon - \omega) f^\varepsilon))\mathrm{d}\omega \\ &\quad - \frac{1}{\varepsilon} \int_{\mathbb{R}^3} \omega(\nabla_\omega \cdot ((u_{f^\varepsilon} - \omega) f^\varepsilon) - \Delta_\omega f^\varepsilon)\mathrm{d}\omega \\ &= -\nabla_x \cdot \int_{\mathbb{R}^3} \omega \otimes \omega f^\varepsilon \mathrm{d}\omega + \int_{\mathbb{R}^3} (v^\varepsilon - \omega) f^\varepsilon \mathrm{d}\omega \\ &:= I_1^\varepsilon + I_2^\varepsilon. \end{aligned} \tag{1.9}$$

Substituting (1.5) and (1.7) into (1.9), one has

$$\begin{aligned} I_1^\varepsilon = -\nabla_x \cdot \Big(&\int_{\mathbb{R}^3} (\omega - u_{f^\varepsilon}) \otimes (\omega - u_{f^\varepsilon}) f^\varepsilon \mathrm{d}\omega - \int_{\mathbb{R}^3} u_{f^\varepsilon} \otimes u_{f^\varepsilon} f^\varepsilon \mathrm{d}\omega \\ &+ \int_{\mathbb{R}^3} \omega \otimes u_{f^\varepsilon} f^\varepsilon \mathrm{d}\omega + \int_{\mathbb{R}^3} u_{f^\varepsilon} \otimes \omega f^\varepsilon \mathrm{d}\omega \Big) \end{aligned}$$

and

$$I_2^\varepsilon = \rho_{f^\varepsilon}(v^\varepsilon - u_{f^\varepsilon}).$$

Using (1.5)-(1.7) and assuming the appropriate convergence of solutions, we can deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \omega f^\varepsilon \mathrm{d}\omega \to \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \omega f \mathrm{d}\omega = \frac{\mathrm{d}}{\mathrm{d}t}(\rho_f u_f),$$

and

$$\begin{aligned} I_1^\varepsilon \to I_1 &= -\nabla_x \cdot \Big(\int_{\mathbb{R}^3} (\omega - u_f) \otimes (\omega - u_f) \frac{\rho_f}{(2\pi)^{3/2}} \mathrm{e}^{-\frac{|u_f - \omega|^2}{2}} \mathrm{d}\omega \\ &\quad - \int_{\mathbb{R}^3} u_f \otimes u_f \frac{\rho_f}{(2\pi)^{3/2}} \mathrm{e}^{-\frac{|u_f - \omega|^2}{2}} \mathrm{d}\omega + \int_{\mathbb{R}^3} \omega \otimes u_f f \mathrm{d}\omega + \int_{\mathbb{R}^3} u_f \otimes \omega f \mathrm{d}\omega \Big) \\ &= -\nabla_x \cdot \Big(\rho_f \mathbb{I}_3 - \rho_f u_f \otimes u_f + \rho_f u_f \otimes u_f + \rho_f u_f \otimes u_f \Big) \\ &= -\nabla_x \rho_f - \nabla_x \cdot (\rho_f u_f \otimes u_f), \end{aligned}$$

where

$$\mathbb{I}_3 = \int_{\mathbb{R}^3} (\omega - u_f) \otimes (\omega - u_f) \frac{1}{(2\pi)^{3/2}} \mathrm{e}^{-\frac{|u_f - \omega|^2}{2}} \mathrm{d}\omega,$$

$$\int_{\mathbb{R}^3} \frac{1}{(2\pi)^{3/2}} \mathrm{e}^{-\frac{|u_f - \omega|^2}{2}} \mathrm{d}\omega = 1.$$

By (1.5) and (1.6), we have

$$I_2^\varepsilon \to I_2 = \rho_f(v - u_f),$$

which implies $(1.1)_2$. Finally, as $\varepsilon \to 0$, we can obtain $(1.1)_3$-$(1.1)_5$.

System (1.4) models the interactions between particles and a fluid. This type of the kinetic-fluid system has attracted a lot of attention because of applications in biotechnology, medicine, sedimentation phenomena, compressibility of droplets of a spray, diesel engines, etc. We can refer [2, 16] for more physical background.

Kinetic theory in the mathematical study of nonlinear partial differential equations has attracted considerable attention in the last few decades. There is much progress on the topics of the kinetic-fluid equations and related models. For system (1.4) without magnetic field and local alignment force (i.e., $B = 0$ and $\alpha = 0$), the existence of weak global solution was established by Mellet and Vasseur [15]. Baranger and Desvillettes [1] studied the local-in-time existence of classical solution for the Vlasov/compressible Euler equations. For the existence of global classical solution near equilibrium to Vlasov-Fokker-Planck/Euler equation, Duan and Liu discussed it in [6]. We also refer the readers to [5, 10] for the study of Vlasov-Fokker-Planck equations with the local alignment force.

For the three-dimensional compressible MHD system, if the initial density has a uniform positive lower bound, Vol'pert and Hudjaev investigated the local well-posedness of the Cauchy problem in [20]. The result was extended by Fan and Yu in [8], where the initial density does not need to be positive and may vanish in an open set. When the initial data are discontinuous and have large oscillations, Wu, Zhang and Zou [23] showed the optimal time-decay rates of the weak solutions in $L^r(2 \le r \le \infty)$-norm and the first-order derivative of the velocity and magnetic field in L^2-norm. We also refer the interested readers to study the multidimensional case in [25], when the initial data are close to a stable equilibrium state in Besov spaces, the authors established the existence and uniqueness of a global strong solution. When the magnetic field is not involved ($B = 0$) in (1.1), system (1.1) is reduced to (1.3). When the initial data are small in H^ℓ-norm, Choi established the existence of global strong solutions for both the periodic domain $\mathbb{T}^3$ and the whole space $\mathbb{R}^3$ in [4]. The author also obtained the large-time behavior of strong solutions for the periodic domain case, but the strategy in [4] can not be applied to the whole space case, as the Poincaré's inequality is not available for the whole space case. Recently, Wu, Zhang, and Zou [22] solved this problem in a perturbation framework. They proved that the perturbation and its k-order derivative decay in L^2-norm are $t^{-3/4}$ and $t^{-\frac{3}{4}-\frac{k}{2}}$ in the whole space $\mathbb{R}^3$ respectively. They also showed that the time decay rate is optimal, which coincides with that of the heat equation. We also refer [26] for the existence and large-time behavior of the system (1.3). For the two-fluid system with magnetic field (1.1), Xiao obtained the existence and large time behavior of global strong solution for the 3D Cauchy problem in [24]. His main results are as follows. Let $(\rho - \bar{\rho}, u, n - \bar{n}, v, B)$ be the strong solution to equations (1.1)-(1.2) and assume that $(\rho_0 - \bar{\rho}, u_0, n_0 - \bar{n}, v_0, B_0) \in H^\ell(\mathbb{R}^3)$ for an integer $\ell \ge 3$. Then there exists a constant $\delta_0 > 0$ such that if

$$\|(\rho_0 - \bar{\rho}, u_0, n_0 - \bar{n}, v_0, B_0)\|_{H^\ell} \le \delta_0, \tag{1.10}$$

then the Cauchy problem (1.1)-(1.2) admits a unique globally classical solution (ρ, u, n, v, B) such that

$$\begin{aligned}
&\|(\rho-\bar{\rho}, u, n-\bar{n}, v, B)(t)\|_{H^{\ell}}^{2} \\
&+\int_{0}^{t}\left(\|\nabla(\rho-\bar{\rho}, u, n-\bar{n}, B)(\tau)\|_{H^{\ell-1}}^{2}+\|(u-v, \nabla v)(\tau)\|_{H^{\ell}}^{2}\right) \mathrm{d} \tau \\
&\leq C\left\|\left(\rho_{0}-\bar{\rho}, u_{0}, n_{0}-\bar{n}, v_{0}, B_{0}\right)\right\|_{H^{\ell}}^{2}, \quad t \geq 0 .
\end{aligned}$$

Upper bounds: If additionally, (1.10) holds for a small constant $\delta_0 > 0$ and

$$\left\|\left(\rho_{0}-\bar{\rho}, u_{0}, n_{0}-\bar{n}, v_{0}, B_{0}\right)\right\|_{L^{1}}<\infty,$$

then

$$\|\nabla^{k}(\rho-\bar{\rho}, u, n-\bar{n}, v, B)(t)\|_{L^{2}} \leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \tag{1.11}$$

$$\|(\rho-\bar{\rho}, u, n-\bar{n}, v, B)(t)\|_{L^{p}} \leq C(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)}, \tag{1.12}$$

for t is large enough, $0 \leq k \leq \ell$ and $2 \leq p \leq \infty$, where C is a positive constant independent of t. The author also proved the lower bound optimal decay rates.

The space-time decay rate of the strong solution has attracted more and more attention. In the following, we will state the progress on the topic about the space-time decay in the weighted Sobolev space H^{ℓ}_{γ}. Takahashi first established the space-time decay of strong solutions to the Navier-Stokes equations in [18]. In [11, 13], Kukavica et al. used the parabolic interpolation inequality to obtain the sharp decay rates of the higher-order derivatives for the solutions in the weighted Lebesgue space L^2_{γ}. In [12, 14], Kukavica et al. also established the strong solution's space-time decay rate in $L^p_{\gamma}(2 \leq p \leq \infty)$ and extended the result to $n(n \geq 2)$ dimensions. Using the Fourier splitting method, Gao, Lyu and Yao obtained space-time decay rate for the compressible Hall-MHD equations in [9].

However, to the best of our knowledge, up to now, there is no result on the space-time decay rate of the two-fluid system with the magnetic field (1.1). The main motivation of this paper is to give a definite answer to this issue. More precisely, we establish the space-time decay rates of the $k(0 \leq k \leq \ell)$-order derivative of strong solution to the Cauchy problem (1.1)-(1.2) in the weighted Lebesgue space L^2_{γ} and the sharp space-time decay rate of $k(0 \leq \ell - 2)$-order derivative of the difference between two velocities of the fluid in the weighted Lebesgue space L^2_{γ}.

1.2. **Notation.** We use L^p and H^{ℓ} to denote the usual Lebesgue space $L^p\left(\mathbb{R}^3\right)$ and Sobolev spaces $H^{\ell}\left(\mathbb{R}^3\right) = W^{\ell,2}\left(\mathbb{R}^3\right)$ with norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_{H^{\ell}}$ respectively. We denote $\|(f,g)\|_X := \|f\|_X + \|g\|_X$ for simplicity. The notation $f \lesssim g$ means that $f \leq Cg$ for a generic positive constant $C > 0$ that only depends on the parameters coming from the problem. We often drop x-dependence of differential operators, that is $\nabla f = \nabla_x f = (\partial_{x_1} f, \partial_{x_2} f, \partial_{x_3} f)$ and ∇^k denotes any partial derivative ∂^{α} with multi-index $\alpha, |\alpha| = k$. For any $\gamma \in \mathbb{R}$, denote the weighted Lebesgue space by $L^p_{\gamma}\left(\mathbb{R}^3\right)(2 \leq p < +\infty)$ with respect to the spatial variables:

$$L_{\gamma}^{p}(\mathbb{R}^{3}):=\left\{f(x): \mathbb{R}^{3} \rightarrow \mathbb{R},\|f\|_{L_{\gamma}^{p}(\mathbb{R}^{3})}^{p}:=\int_{\mathbb{R}^{3}}|x|^{p \gamma}|f(x)|^{p} d x<+\infty\right\}.$$

Then, we can define the weighted Sobolev space:

$$H_{\gamma}^{s}\left(\mathbb{R}^{3}\right) \triangleq\left\{f \in L_{\gamma}^{2}\left(\mathbb{R}^{3}\right) \mid\|f\|_{H_{\gamma}^{s}(\mathbb{R}^{3})}^{2}:=\sum_{k \leq s}\|\nabla^{k} u\|_{L_{\gamma}^{2}(\mathbb{R}^{3})}^{2}<+\infty\right\}.$$

Let Λ^s be the pseudo differential operator defined by

$$\Lambda^s f = \mathfrak{F}^{-1}\left(|\xi|^s \widehat{f}\right), \quad \text{for } s \in \mathbb{R},$$

where $\widehat{f}$ and $\mathfrak{F}(f)$ are the Fourier transform of f. The homogenous Sobolev space $\dot{H}^s\left(\mathbb{R}^3\right)$ with norm given by

$$\|f\|_{\dot{H}^s} \triangleq \|\Lambda^s f\|_{L^2}.$$

The function f is in Schwartz class $\mathcal{S}$, if it is infinitely differentiable and if all of its derivatives decrease rapidly at infinity. That is,

$$\sup_x |x^\alpha D^\beta f(x)| < \infty,$$

for all $\alpha, \beta \in \mathbb{N}^3$.

1.3. **Main results.** Inspired by the work of [21], we investigate the space-time decay rates of strong solution in the weighted Lebesgue space L^2_γ as follows.

Theorem 1.1. *Let $(\rho-\bar{\rho}, u, n-\bar{n}, v, B)$ be the strong solution to the equations (1.1)-(1.2) with initial data $(\rho_0-\bar{\rho}, u_0, n_0-\bar{n}, v_0, B_0)$ belonging to the Schwartz class $\mathcal{S}$. In addition, for any integer $\ell \geq 3$, assume that $(\rho_0-\bar{\rho}, u_0, n_0-\bar{n}, v_0, B_0) \in H^\ell\left(\mathbb{R}^3\right) \cap H^\ell_\gamma\left(\mathbb{R}^3\right) \cap L^1\left(\mathbb{R}^3\right)$. If there exists a small constant $\delta_0 > 0$ such that*

$$\|\left(\rho_0-\bar{\rho}, u_0, n_0-\bar{n}, v_0, B_0\right)\|_{H^\ell} \leq \delta_0,$$

then there exists a large enough T such that

$$\|\nabla^k(\rho-\bar{\rho}, u, n-\bar{n}, v, B)(t)\|_{L^2_\gamma} \leq C t^{-\frac{3}{4}-\frac{k}{2}+\gamma}, \tag{1.13}$$

for all $t > T$, $0 \leq k \leq \ell$ and $\gamma \geq 0$, where C is a positive constant independent of t.

Remark 1.2. Applying Gagliardo-Nirenberg-Sobolev inequality, we can obtain the space-time decay rates of smooth solution in the weighted Lebesgue space L^p_γ as follows. For any $f \in L^2\left(\mathbb{R}^3\right) \cap \dot{H}^2\left(\mathbb{R}^3\right)$, we have $\|f\|_{L^\infty(\mathbb{R}^3)} \leq \|f\|^{1/4}_{L^2(\mathbb{R}^3)} \|f\|^{\frac{3}{4}}_{\dot{H}^2(\mathbb{R}^3)}$. So we can obtain the estimate $\||x|^\gamma \nabla^k(\rho-\bar{\rho}, u, n-\bar{n}, v, B)(t)\|_{L^\infty} (k \in [0, \ell-2])$ from the estimates $\||x|^\gamma \nabla^k(\rho-\bar{\rho}, u, n-\bar{n}, v, B)(t)\|_{L^2}$ and $\||x|^\gamma \nabla^k(\rho-\bar{\rho}, u, n-\bar{n}, v, B)(t)\|_{\dot{H}^2}$. Using the interpolation inequality, we can show that there exists a large enough T such that

$$\|\nabla^k(\rho-\bar{\rho}, u, n-\bar{n}, v, B)(t)\|_{L^p_\gamma} \leq C t^{-\frac{3}{2}\left(1-\frac{1}{p}\right)-\frac{k}{2}+\gamma},$$

for $t > T$, $2 \leq p \leq \infty$ and $0 \leq k \leq \ell-2$, where C is a positive constant independent of t.

Remark 1.3. Under the assumptions of Theorem 1.1, applying the similar method as [26], we can easily show that there exists a large enough T such that

$$\|\nabla^k(u-v)(t)\|_{L^2} \leq C(1+t)^{-\frac{5}{4}-\frac{k}{2}}, \tag{1.14}$$

for all $t > T$ and $0 \leq k \leq \ell-2$, where C is a positive constant independent of t.

Theorem 1.4. *Let $(\rho-\bar{\rho}, u, n-\bar{n}, v, B)$ be the strong solution to the system* (1.1)-(1.2) *with initial data $(\rho_0-\bar{\rho}, u_0, n_0-\bar{n}, v_0, B_0)$ belonging to the Schwartz class $\mathcal{S}$. Under the assumptions in Theorem 1.1, then there exists a large enough T such that*

$$\|\nabla^k(u-v)(t)\|_{L^2_\gamma} \leq C(1+t)^{-\frac{5}{4}-\frac{k}{2}+\gamma}, \tag{1.15}$$

for all $t > T$, $0 \le k \le \ell - 2$ and $\gamma \ge 0$, where C is a positive constant independent of t.

Now, let us outline the strategies for proving Theorem 1.1 and 1.4, and explain the main difficulties in the process.

Proof of Theorem 1.1. We employ delicate weighted energy estimates, the strategy of induction and interpolation trick. Firstly, using of several lemmas in Section 2 and (1.11), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}(t) \le C_0 t^{-5/4}\mathrm{E}(t) + C_1 t^{-\frac{3}{4\gamma}}\mathrm{E}(t)^{\frac{2\gamma-1}{2\gamma}} + C_2 t^{-\frac{3}{2\gamma}}\mathrm{E}(t)^{\frac{\gamma-1}{\gamma}}, \tag{1.16}$$

for t is large enough and $\gamma > \frac{3}{2}$, where $\mathrm{E}(t) := \|(m,u,\sigma,v,B)\|_{L^2_\gamma}^2$ and C_0, C_1, C_2 are positive constants independent of t. Applying Lemma 2.5 for (1.16) and the interpolation trick, we show that the Theorem 1.1 holds for case $k = 0$. Secondly, Using the similar method as $k = 0$ and Minkowski's inequality, we can show that the Theorem 1.1 holds for $k = 1$. Finally, according to the strategy of induction, we prove that Theorem 1.1 holds for $0 \le k \le \ell$. The main difficulties come from those terms like $\alpha_2\bar{n}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^k(u\cdot\nabla m)\cdot\nabla^k m\,\mathrm{d}x$ and $\bar{n}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^k\big((\alpha_1 - \frac{P'(n)}{n})\nabla\sigma\big)\cdot\nabla^k v\,\mathrm{d}x$, which contain three difficulties. The first difficulty is the absence of the dissipation $\|\nabla^{k+1}m\|_{L^2_\gamma}^2$. We use integration by parts to overcome this difficulty (see (3.38)). The second difficulty is that we can only obtain the decay rate of $\|\nabla^{\ell-1}u\|_{H^1}$ rather than the decay rate of $\|\nabla^\ell u\|_{H^1}$, we make different estimates for $\sum_{j=2}^{k-2}\||x|^{2\gamma}\nabla^j u\nabla^{k-j+1}m\nabla^k m\|_{L^1}$ and $\||x|^{2\gamma}\nabla^{k-1}u\nabla^2 m\nabla^k m\|_{L^1}$ (see (3.40) and (3.41)). The last difficulty is that the Lemma 2.2 does not hold in the weighted Lebesgue space L^2_γ, so we have to estimate $\nabla^j\big((\alpha_1 - \frac{P'(n)}{n})\nabla\sigma\big)$ in L^p-norm (see (3.49)-(3.52)). Overcoming all of these difficulties, applying some lemmas in Section 2 and (1.11), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}(t) \le{}& C_0 t^{-5/4}\mathrm{E}(t) + C_1 t^{-(\frac{3}{4}+\frac{k}{2})\frac{1}{\gamma}}\mathrm{E}(t)^{\frac{2\gamma-1}{2\gamma}} \\ &+ C_2 t^{-(\frac{3}{4}+\frac{k}{2})\frac{2}{\gamma}}\mathrm{E}(t)^{\frac{\gamma-1}{\gamma}} + C_3 t^{-\frac{5}{2}-k+2\gamma},\end{aligned} \tag{1.17}$$

where $\mathrm{E}(t) := \|\nabla^k(m,u,\sigma,v,B)\|_{L^2_\gamma}^2$ and C_0, C_1, C_2, C_3 are positive constants independent of t. Using the Gronwall-type Lemma 2.5 for (1.17), we prove that Theorem 1.1 is true for $\gamma > \frac{3+2k}{2}$. Applying the interpolation trick, we complete the proof of Theorem 1.1. □

Proof of Theorem 1.4. We make full use of energy estimates, the result of Theorem 1.1 and (1.11) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^k(u-v)\|_{L^2_\gamma}^2 + C'\|\nabla^k(u-v)\|_{L^2_\gamma}^2 \lesssim (1+t)^{-\frac{5}{2}-k+2\gamma}, \tag{1.18}$$

for t is large enough, where C' is a positive constant independent of t. We note that there exist terms like $-\bar{\mu}\langle\nabla^k\Delta v, |x|^{2\gamma}\nabla^k(u-v)\rangle$ in (4.2) so $0 \le k \le \ell - 2$ in Theorem 1.4. Using the Gronwall's inequality of differential form to (1.18), we complete the proof. □

The paper will be organized as follows. In section 2, we rewrite the Cauchy problem (1.1)-(1.2) and present some lemmas, which are used frequently throughout this paper. In section 3, using the strategy of induction, we prove the Theorem 1.1. In section 4, applying the energy estimate, we prove the Theorem 1.4.

2. Reformulation and preliminaries

2.1. **Reformulation.** In this section, we reformulate the Cauchy problem (1.1)-(1.2). We denote

$$m=\ln\rho-\ln\bar\rho,\quad \sigma=n-\bar n,\quad \alpha_1=\frac{P'(\bar n)}{\bar n},\quad \alpha_2=\frac{\bar\rho}{\bar n},\quad \bar\mu=\frac{\mu}{\bar n},\quad \bar\lambda=\frac{\lambda}{\bar n}.$$

Then system (1.1)-(1.2) can be rewritten as

$$\begin{gathered} m_t+\operatorname{div}u=F_1,\\ u_t+\nabla m+(u-v)=F_2,\\ \sigma_t+\bar n\operatorname{div}v=F_3,\\ v_t+\alpha_1\nabla\sigma-\bar\mu\Delta v-(\bar\mu+\bar\lambda)\nabla\operatorname{div}v-\alpha_2(u-v)=F_4,\\ B_t-\nu\Delta B=F_5,\\ \operatorname{div}B=0, \end{gathered}\tag{2.1}$$

and

$$(m,u,\sigma,v,B)\big|_{t=0}=\big(m_0(x),u_0(x),\sigma_0(x),v_0(x),B_0(x)\big),\tag{2.2}$$

where

$$\begin{gathered} F_1=-u\cdot\nabla m,\quad F_2=-u\cdot\nabla u,\quad F_3=-v\cdot\nabla\sigma-\sigma\operatorname{div}v,\\ F_4=-v\cdot\nabla v+\Big(\alpha_1-\frac{P'(n)}{n}\Big)\nabla\sigma+\big(\frac{\mu}{n}-\bar\mu\big)\Delta v+\Big(\frac{\mu+\lambda}{n}-(\bar\mu+\bar\lambda)\Big)\nabla\operatorname{div}v\\ +\big(\frac{\rho}{n}-\alpha_2\big)(u-v)+(\operatorname{div}B)B+B\cdot\nabla B-\frac12\nabla|B|^2,\\ F_5=-(\operatorname{div}v)B+B\cdot\nabla v-v\cdot\nabla B. \end{gathered}\tag{2.3}$$

It is noted that in the homogeneous case (i.e. $F_i=0$ for $i=1,2,\dots,5$, $(2.1)_5$ is decoupled from $(2.1)_1$-$(2.1)_4$. This key observation enables us to employ the sharp linear estimates for (m,u,σ,v) obtained in [26] and estimates for B obtained in [27]. We can refer [24] for more details.

Here are several useful tools, which will be frequently used in the whole article.

Lemma 2.1 (Gagliardo-Nirenberg inequality)**.** *Let $0\le i,j\le k$, then*

$$\|\nabla^i f\|_{L^p}\lesssim\|\nabla^j f\|_{L^q}^{1-a}\|\nabla^k f\|_{L^r}^a,$$

where a satisfies

$$\frac i3-\frac1p=\big(\frac j3-\frac1q\big)(1-a)+\big(\frac k3-\frac1r\big)a.$$

Especially, when $p=3$, $q=r=2$, $i=j=0$, $k=1$, combining Cauchy's inequality, we have

$$\|f\|_{L^3}\lesssim\|f\|_{L^2}^{1/2}\|\nabla f\|_{L^2}^{1/2}\lesssim\|f\|_{H^1},\tag{2.4}$$

when $p=\infty,q=r=2,i=0,j=1,k=2$, combining Cauchy's inequality, we have

$$\|f\|_{L^\infty}\lesssim\|\nabla f\|_{L^2}^{1/2}\|\nabla^2 f\|_{L^2}^{1/2}\lesssim\|\nabla f\|_{H^1},\tag{2.5}$$

while $i=j=0$, $k=1$, $a=1$, $p=q=r=2$ and using Minkowski's inequality, we have

$$\|f\|_{L^6_\gamma}\lesssim\|\nabla\left(|x|^\gamma f\right)\|_{L^2}\lesssim\Big(\|\nabla f\|_{L^2_\gamma}+\|f\|_{L^2_{\gamma-1}}\Big).\tag{2.6}$$

Proof. This is a special case of [17] and some inequalities based on our needs. □

Lemma 2.2. *Assume that the function $f(\varrho)$ satisfies*
$$f(\varrho) \sim \varrho \text{ and } \|f^{(k)}(\varrho)\| \le C_k \quad \text{for all } k \ge 1,$$
then for any integer $k \ge 0$ and $p \ge 2$, we have
$$\|\nabla^k f(\varrho)\|_{L^p} \le C_k \|\nabla^k \varrho\|_{L^p},$$
where C_k is a constant independent of t. Especially, in this paper,
$$\alpha_1 - \frac{P'(n)}{n} \sim \sigma, \frac{\mu}{n} - \bar{\mu} \sim \sigma, \frac{\mu+\lambda}{n} - (\bar{\mu}+\bar{\lambda}) \sim \sigma, \frac{\rho}{n} - \alpha_2 \sim (m,\sigma).$$

For a proof of the above lemma, we refer to [3, Lemma A.4] for $p = 2$, and to [19, Lemma 2.2] for $p \ge 2$.

Lemma 2.3. *For each vector function $f \in C_0^\infty(\mathbb{R}^3)$, and bounded scalar function g, we have*
$$\Big| \int_{\mathbb{R}^3} \left(\nabla |x|^{2\gamma}\right) \cdot f g \, \mathrm{d}x \Big| \lesssim \|g\|_{L^2_\gamma} \|f\|_{L^2_{\gamma-1}}.$$

Proof. The left side of the above inequality can be rewritten as
$$\Big| 2\gamma \int_{\mathbb{R}^3} |x|^{2\gamma-2} x_j \partial_i x_j g f_i \, dx \Big|.$$
Then using Hölder's inequality, we have the desired inequality □

Lemma 2.4 (Interpolation inequality with weights)**.** *If $p, r \geqslant 1$, $s + n/r, \alpha + n/p, \beta + n/q > 0$, and $0 \leqslant \theta \leqslant 1$ then*
$$\|f\|_{L^r_s} \le \|f\|^\theta_{L^p_\alpha} \|f\|^{1-\theta}_{L^q_\beta},$$
for $f \in C_0^\infty(\mathbb{R}^n)$ provided that
$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q},$$
and $s = \theta\alpha + (1-\theta)\beta$. In particular when $s = p = q = 2$, $\theta = \frac{\gamma-1}{\gamma}$, $s = \gamma - 1$, $\alpha = \gamma$, $\beta = 0$, we have
$$\|f\|_{L^2_{\gamma-1}} \le \|f\|^{\frac{\gamma-1}{\gamma}}_{L^2_\gamma} \|f\|^{1/\gamma}_{L^2}. \tag{2.7}$$

Proof. By computations, we have
$$\begin{aligned}
&\int_U |x|^{sr} |f|^r \, dx \\
&= \int_U |x|^{\alpha\theta r} |f|^{\theta r} |x|^{\beta(1-\theta)r} |f|^{(1-\theta)r} \, dx \\
&\le \Big(\int_U \left(|x|^{\alpha\theta r} |f|^{\theta r} \right)^{\frac{p}{\theta r}} dx \Big)^{\theta r/p} \Big(\int_U \left(|x|^{\beta(1-\theta)r} |f|^{(1-\theta)r} \right)^{\frac{q}{(1-\theta)r}} dx \Big)^{\frac{(1-\theta)r}{q}}.
\end{aligned}$$
This completes the proof. □

Lemma 2.5 (Gronwall-type Lemma). *Let $\alpha_0 > 1$, $\alpha_1 < 1$, $\alpha_2 < 1$, $\beta_1 < 1$, and $\beta_2 < 1$. Assume that a continuously differential function $F : [1,\infty) \to [0,\infty)$ satisfies*

$$\frac{d}{dt}F(t) \le C_0 t^{-\alpha_0}F(t) + C_1 t^{-\alpha_1}F(t)^{\beta_1} + C_2 t^{-\alpha_2}F(t)^{\beta_2} + C_3 t^{\gamma_1 - 1}, \quad t \ge 1$$
$$F(1) \le K_0,$$

where $C_0, C_1, C_2, C_3, K_0 \ge 0$ and $\gamma_i = \frac{1-\alpha_i}{1-\beta_i} > 0$ for $i = 1,2$. Assume that $\gamma_1 \ge \gamma_2$, then there exists a constant C^* depending on α_0, α_1, β_1, α_2, β_2, K_0, C_i, $i = 1,2,3$, such that

$$F(t) \le C^* t^{\gamma_1},$$

for all $t \ge 1$.

Fr a proof of the above lemma, see [21, Lemma 2.1].

Lemma 2.6 (Gronwall's inequality of differential form). *Let $\eta(\cdot)$ be a nonnegative, absolutely continuous function on $[0,T]$, which satisfies for a.e. t the differential inequality*

$$\eta'(t) \le \phi(t)\eta(t) + \psi(t),$$

where $\phi(t)$ and $\psi(t)$ are nonnegative and summable functions on $[0,T]$. Then

$$\eta(t) \le e^{\int_0^t \phi(s)ds}\Big[\eta(0) + \int_0^t \psi(s)ds\Big],$$

for $0 \le t \le T$.

For a proof of the above lemma, see [7, Appendix B.2.j].

3. The proof of Theorem 1.1

Inspired by the work in [21], we will address the space-time decay rate of the strong solution of the coupled system (1.1)-(1.2). Under the assumptions of Theorem 1.1, it is clear that there exists a large enough T, such that

$$\|\nabla^k(m,u,\sigma,v,B)(t)\|_{L^2} \le Ct^{-\frac{3}{4}-\frac{k}{2}}, \tag{3.1}$$

for all $t > T$ and $0 \le k \le \ell$, where C is a positive constant independent of t.

Lemma 3.1. *Under the assumptions of Theorem 1.1, there exists a large enough T such that the solution (m,u,σ,v,B) of system (2.1)-(2.2) has the estimate*

$$\|\left(m,u,\sigma,v,B\right)(t)\|_{L^2_\gamma} \le Ct^{-\frac{3}{4}+\gamma}, \tag{3.2}$$

for all $t > T$ and $\gamma \ge 0$, where C is a positive constant independent of t.

Proof. Multiplying $(2.1)_1$–$(2.1)_5$ by $\alpha_2\bar{n}|x|^{2\gamma}m$, $\alpha_2\bar{n}|x|^{2\gamma}u$, $\alpha_1|x|^{2\gamma}\sigma$, $\bar{n}|x|^{2\gamma}v$ and $|x|^{2\gamma}B$ respectively, and the summing them, integrating over $\mathbb{R}^3$, and then using integration by parts, one has

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\alpha_2\bar{n}\|m\|^2_{L^2_\gamma} + \alpha_2\bar{n}\|u\|^2_{L^2_\gamma} + \alpha_1\|\sigma\|^+_{L^2_\gamma}\bar{n}\|v\|^2_{L^2_\gamma} + \|B\|^2_{L^2_\gamma}\Big)\\
&+ \alpha_2\bar{n}\|u-v\|^2_{L^2_\gamma} + \bar{n}\bar{\mu}\|\nabla v\|^2_{L^2_\gamma} + \bar{n}(\bar{\lambda}+\bar{\mu})\|\operatorname{div} v\|^2_{L^2_\gamma} + \nu\|\nabla B\|^2_{L^2_\gamma}\\
&= \alpha_2\bar{n}\int_{\mathbb{R}^3}\nabla\left(|x|^{2\gamma}\right)\cdot mu\,\mathrm{d}x + \alpha_1\bar{n}\int_{\mathbb{R}^3}\nabla\left(|x|^{2\gamma}\right)\cdot\sigma v\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&- \bar{n}\bar{\mu}\int_{\mathbb{R}^3} \nabla\left(|x|^{2\gamma}\right)\cdot v\nabla v\,\mathrm{d}x - \bar{n}(\bar{\lambda}+\bar{\mu})\int_{\mathbb{R}^3}\nabla\left(|x|^{2\gamma}\right)\cdot v\,\mathrm{div}\,v\,\mathrm{d}x\\
&+ \int_{\mathbb{R}^3}\left(\alpha_2\bar{n}|x|^{2\gamma}m\cdot F_1\right)\,\mathrm{d}x + \int_{\mathbb{R}^3}\left(\alpha_2\bar{n}|x|^{2\gamma}u\cdot F_2\right)\,\mathrm{d}x\\
&+ \int_{\mathbb{R}^3}\left(\alpha_1|x|^{2\gamma}\sigma\cdot F_3\right)\,\mathrm{d}x + \int_{\mathbb{R}^3}\left(\bar{n}|x|^{2\gamma}v\cdot F_4\right)\mathrm{d}x\\
&+ \int_{\mathbb{R}^3}\left(|x|^{2\gamma}B\cdot F_5\right)\mathrm{d}x\\
=\;& \alpha_2\bar{n}\int_{\mathbb{R}^3}\nabla\left(|x|^{2\gamma}\right)\cdot mu\,\mathrm{d}x + \alpha_1\bar{n}\int_{\mathbb{R}^3}\nabla\left(|x|^{2\gamma}\right)\cdot\sigma v\,\mathrm{d}x\\
&- \bar{n}\bar{\mu}\int_{\mathbb{R}^3}\nabla\left(|x|^{2\gamma}\right)\cdot v\nabla v\,\mathrm{d}x - \bar{n}(\bar{\lambda}+\bar{\mu})\int_{\mathbb{R}^3}\nabla\left(|x|^{2\gamma}\right)\cdot v\,\mathrm{div}\,v\mathrm{d}x\\
&- \alpha_2\bar{n}\int_{\mathbb{R}^3}|x|^{2\gamma}(u\cdot\nabla m)m\,\mathrm{d}x - \alpha_2\bar{n}\int_{\mathbb{R}^3}|x|^{2\gamma}(u\cdot\nabla u)u\,\mathrm{d}x\\
&- \alpha_1\int_{\mathbb{R}^3}|x|^{2\gamma}(v\cdot\nabla\sigma)\sigma\mathrm{d}x - \alpha_1\int_{\mathbb{R}^3}|x|^{2\gamma}(\sigma\,\mathrm{div}\,v)\sigma\mathrm{d}x\\
&- \bar{n}\int_{\mathbb{R}^3}|x|^{2\gamma}(v\cdot\nabla v)v\,\mathrm{d}x + \bar{n}\int_{\mathbb{R}^3}|x|^{2\gamma}\Big((\alpha_1 - \frac{P'(n)}{n})\nabla\sigma\Big)\cdot v\,\mathrm{d}x\\
&+ \bar{n}\int_{\mathbb{R}^3}|x|^{2\gamma}\Big((\frac{\mu}{n}-\bar{\mu})\Delta v\Big)\cdot v\,\mathrm{d}x\\
&+ \bar{n}\int_{\mathbb{R}^3}|x|^{2\gamma}\Big((\frac{\mu+\lambda}{n}-(\bar{\mu}+\bar{\lambda}))\nabla\,\mathrm{div}\,v\Big)\cdot v\,\mathrm{d}x\\
&+ \bar{n}\int_{\mathbb{R}^3}|x|^{2\gamma}\Big((\frac{\rho}{n}-\alpha_2)(u-v)\Big)\cdot v\,\mathrm{d}x\\
&+ \bar{n}\int_{\mathbb{R}^3}|x|^{2\gamma}((\mathrm{div}\,B)B)\cdot v\,\mathrm{d}x + \bar{n}\int_{\mathbb{R}^3}|x|^{2\gamma}(B\cdot\nabla B)\cdot v\,\mathrm{d}x\\
&- \frac{1}{2}\bar{n}\int_{\mathbb{R}^3}|x|^{2\gamma}\left(\nabla|B|^2\right)\cdot v\,\mathrm{d}x - \int_{\mathbb{R}^3}|x|^{2\gamma}((\mathrm{div}\,v)B)\cdot B\,\mathrm{d}x\\
&+ \int_{\mathbb{R}^3}|x|^{2\gamma}(B\cdot\nabla v)\cdot B\,\mathrm{d}x - \int_{\mathbb{R}^3}|x|^{2\gamma}(v\cdot\nabla B)\cdot B\,\mathrm{d}x\\
:=\;& \sum_{j=1}^{19} I_{1,j}.
\end{aligned}
\tag{3.3}
$$

Applying Lemma 2.3, we have

$$
\begin{aligned}
|I_{1,1}| + |I_{1,2}| &\lesssim \|\nabla(|x|^{2\gamma})mu\|_{L^1} + \|\nabla(|x|^{2\gamma})\sigma v\|_{L^1}\\
&\lesssim \|(m,\sigma)\|_{L^2_\gamma}\|(u,v)\|_{L^2_{\gamma-1}}.
\end{aligned}
\tag{3.4}
$$

Using Lemma 2.3 and Cauchy's inequality, we have

$$
\begin{aligned}
|I_{1,3}| &\lesssim \|\nabla(|x|^{2\gamma})v\nabla v\|_{L^1}\\
&\lesssim \|\nabla v\|_{L^2_\gamma}\|v\|_{L^2_{\gamma-1}}\\
&\le \frac{\bar{n}\bar{\mu}}{4}C\|\nabla v\|_{L^2_\gamma}^2 + C(\bar{n}\bar{\mu})\|v\|_{L^2_{\gamma-1}}^2.
\end{aligned}
\tag{3.5}
$$

Applying a method similar to the one for (3.5), one has

$$|I_{1,4}| \le \frac{\bar{n}(\bar{\lambda}+\bar{\mu})}{4} C\|\operatorname{div} v\|_{L^2_\gamma}^2 + C(\bar{n}\bar{\lambda}\bar{\mu})\|v\|_{L^2_{\gamma-1}}^2. \tag{3.6}$$

Using Hölder's inequality, Lemma 2.1 (Gagliardo-Nirenberg inequality), Cauchy's inequality and (3.1), we have

$$\begin{aligned} |I_{1,5}| &\lesssim \|\nabla m\|_{L^\infty}\|u\|_{L^2_\gamma}\|m\|_{L^2_\gamma} \\ &\lesssim \|\nabla^2 m\|_{H^1}\|u\|_{L^2_\gamma}\|m\|_{L^2_\gamma} \\ &\lesssim t^{-\frac{7}{4}}\|(m,u)\|_{L^2_\gamma}^2. \end{aligned} \tag{3.7}$$

Applying a method similar to the one for (3.7), one has

$$\sum_{j=6}^{9}|I_{1,j}| + \sum_{j=14}^{19}|I_{1,j}| \lesssim t^{-\frac{7}{4}}\|(u,v,\sigma,B)\|_{L^2_\gamma}^2. \tag{3.8}$$

Applying integration by parts, Minkowski's inequality, Hölder's inequality, Lemma 2.3, Lemma 2.2, Lemma 2.1 (Gagliardo-Nirenberg inequality), Cauchy's inequality and (3.1), we have

$$\begin{aligned} |I_{1,10}| &\lesssim \|\nabla\Big(|x|^{2\gamma}\big(\alpha_1 - \frac{P'(n)}{n}\big)v\Big)\sigma\|_{L^1} \\ &\lesssim \|\Big(\alpha_1 - \frac{P'(n)}{n}\Big)\|_{L^\infty}\|v\|_{L^2_\gamma}\|\sigma\|_{L^2_{\gamma-1}} \\ &\quad + \|\nabla\Big(\alpha_1 - \frac{P'(n)}{n}\Big)\|_{L^\infty}\|v\|_{L^2_\gamma}\|\sigma\|_{L^2_\gamma} \\ &\quad + \|\Big(\alpha_1 - \frac{P'(n)}{n}\Big)\|_{L^\infty}\|\nabla v\|_{L^2_\gamma}\|\sigma\|_{L^2_\gamma} \\ &\lesssim \|\nabla\sigma\|_{H^1}\|\nabla v\|_{L_\gamma}^2 + \|(\nabla\sigma,\nabla^2\sigma)\|_{H^1}\|(\sigma,v)\|_{L_\gamma}^2 + \|\nabla\sigma\|_{H^1}\|\sigma\|_{L^2_{\gamma-1}}^2 \\ &\lesssim t^{-5/4}\|\nabla v\|_{L_\gamma}^2 + t^{-5/4}\|(\sigma,v)\|_{L_\gamma}^2 + \|\sigma\|_{L^2_{\gamma-1}}^2. \end{aligned} \tag{3.9}$$

Applying a method similar to the one for (3.9), one has

$$\sum_{j=11}^{12}|I_{1,j}| \lesssim t^{-5/4}\|(\nabla v,\operatorname{div} v)\|_{L_\gamma}^2 + t^{-5/4}\|v\|_{L_\gamma}^2 + \|(\sigma,v)\|_{L^2_{\gamma-1}}^2. \tag{3.10}$$

Using Hölder's inequality, Lemma 2.2, Lemma 2.1 (Gagliardo-Nirenberg inequality), Triangle inequality, Cauchy's inequality and (3.1), we obtain

$$\begin{aligned} |I_{1,13}| &\lesssim \|(\frac{\rho}{n} - \alpha_2)\|_{L^\infty}\|(u-v)\|_{L^2_\gamma}\|v\|_{L^2_\gamma} \\ &\lesssim \|\nabla(m,\sigma)\|_{H^1}\|(u,v)\|_{L^2_\gamma}^2 \\ &\lesssim t^{-5/4}\|(u,v)\|_{L^2_\gamma}^2. \end{aligned} \tag{3.11}$$

Substituting (3.4)-(3.11) into (3.3), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\alpha_2\bar{n}\|m\|_{L^2_\gamma}^2+\alpha_2\bar{n}\|u\|_{L^2_\gamma}^2+\alpha_1\|\sigma\|_{L^2_\gamma}^2+\bar{n}\|v\|_{L^2_\gamma}^2+\|B\|_{L^2_\gamma}^2\Big)\\
&+\alpha_2\bar{n}\|u-v\|_{L^2_\gamma}^2+\bar{n}\bar{\mu}\|\nabla v\|_{L^2_\gamma}^2+\bar{n}(\bar{\lambda}+\bar{\mu})\|\operatorname{div} v\|_{L^2_\gamma}^2+\nu\|\nabla B\|_{L^2_\gamma}^2\\
&\le \frac{\bar{n}\bar{\mu}}{4}\|\nabla v\|_{L^2_\gamma}^2+\frac{\bar{n}(\bar{\lambda}+\bar{\mu})}{4}\|\operatorname{div} v\|_{L^2_\gamma}^2+Ct^{-5/4}\|(\nabla v,\operatorname{div} v)\|_{L_\gamma}^2\\
&\quad+Ct^{-5/4}\|(m,u,\sigma,v,B)\|_{L^2_\gamma}^2+C\|(m,\sigma)\|_{L^2_\gamma}\|(u,v)\|_{L^2_{\gamma-1}}\\
&\quad+C\|(\sigma,v)\|_{L^2_{\gamma-1}}^2,
\end{aligned}
\tag{3.12}
$$

For t is large enough, we have

$$
t^{-5/4}\le \min\big\{\frac{\bar{n}\bar{\mu}}{4},\frac{\bar{n}(\bar{\lambda}+\bar{\mu})}{4}\big\}. \tag{3.13}
$$

Substituting (3.13) into (3.12), there exists a large enough T such that

$$
\begin{aligned}
&\frac{d}{dt}\Big(\alpha_2\bar{n}\|m\|_{L^2_\gamma}^2+\alpha_2\bar{n}\|u\|_{L^2_\gamma}^2+\alpha_1\|\sigma\|_{L^2_\gamma}^2+\bar{n}\|v\|_{L^2_\gamma}^2+\|B\|_{L^2_\gamma}^2\Big)\\
&+\alpha_2\bar{n}\|u-v\|_{L^2_\gamma}^2+\bar{n}\bar{\mu}\|\nabla v\|_{L^2_\gamma}^2+\bar{n}(\bar{\lambda}+\bar{\mu})\|\operatorname{div} v\|_{L^2_\gamma}^2+\nu\|\nabla B\|_{L^2_\gamma}^2\\
&\lesssim t^{-5/4}\|(m,u,\sigma,v,B)\|_{L^2_\gamma}^2+\|(m,\sigma)\|_{L^2_\gamma}\|(u,v)\|_{L^2_{\gamma-1}}+\|(\sigma,v)\|_{L^2_{\gamma-1}}^2,
\end{aligned}
\tag{3.14}
$$

for all $t>T$. Substituting (2.7) and (3.1) into (3.14), we have

$$
\begin{aligned}
&\frac{d}{dt}\Big(\alpha_2\bar{n}\|m\|_{L^2_\gamma}^2+\alpha_2\bar{n}\|u\|_{L^2_\gamma}^2+\alpha_1\|\sigma\|_{L^2_\gamma}^2+\bar{n}\|v\|_{L^2_\gamma}^2+\|B\|_{L^2_\gamma}^2\Big)\\
&+\alpha_2\bar{n}\|u-v\|_{L^2_\gamma}^2+\bar{n}\bar{\mu}\|\nabla v\|_{L^2_\gamma}^2+\bar{n}(\bar{\lambda}+\bar{\mu})\|\operatorname{div} v\|_{L^2_\gamma}^2+\nu\|\nabla B\|_{L^2_\gamma}^2\\
&\lesssim t^{-5/4}\|(m,u,\sigma,v,B)\|_{L^2_\gamma}^2+\|(m,\sigma)\|_{L^2_\gamma}\|(u,v)\|_{L^2_\gamma}^{\frac{\gamma-1}{\gamma}}\|(u,v)\|_{L^2}^{1/\gamma}\\
&\quad+\|(\sigma,v)|_{L^2_\gamma}^{\frac{2(\gamma-1)}{\gamma}}\|(\sigma,v)\|_{L^2}^{2/\gamma}\\
&\lesssim t^{-5/4}\|(m,u,\sigma,v,B)\|_{L^2_\gamma}^2+t^{-\frac{3}{4\gamma}}\|(m,\sigma)\|_{L^2_\gamma}^{\frac{2\gamma-1}{\gamma}}+t^{-\frac{3}{2\gamma}}\|(\sigma,v)\|_{L^2_\gamma}^{\frac{2(\gamma-1)}{\gamma}}.
\end{aligned}
$$

Denoting $\mathrm{E}(t):=\|(m,u,\sigma,v,B)\|_{L^2_\gamma}^2$, we obtain

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}(t)\le C_0t^{-5/4}\mathrm{E}(t)+C_1t^{-\frac{3}{4\gamma}}\mathrm{E}(t)^{\frac{2\gamma-1}{2\gamma}}+C_2t^{-\frac{3}{2\gamma}}\mathrm{E}(t)^{\frac{\gamma-1}{\gamma}},
$$

where C_0,C_1,C_2 are positive constants independent of t. If $\gamma>\frac{3}{2}$, then we can apply Lemma 2.5 with $\alpha_0=\frac{5}{4}>1,\alpha_1=\frac{3}{4\gamma}<1,\beta_1=\frac{2\gamma-1}{2\gamma}<1,\alpha_2=\frac{3}{2\gamma}<1,\beta_2=\frac{\gamma-1}{\gamma}<1$,$\gamma_1=\frac{1-\alpha_1}{1-\beta_1}=-\frac{3}{2}+2\gamma>\gamma_2=\frac{1-\alpha_2}{1-\beta_2}=-\frac{3}{2}+\gamma$ to obtain

$$
\mathrm{E}(t)\le Ct^{-\frac{3}{2}+2\gamma}, \tag{3.15}
$$

for all $t>T$. Lemma 3.1 is proved for all $\gamma>\frac{3}{2}$ and the conclusion for the case of $[0,3/2]$ is proved by Lemma 2.4(interpolation inequality with weights). More precisely, combining (3.1) and (3.15), we have

$$
\|(m,u,\sigma,v,B)(t)\|_{L^2_{\gamma_0}}\le C\|(m,u,\sigma,v,B)(t)\|_{L^2}^{1-\frac{\gamma_0}{\gamma}}\|(m,u,\sigma,v,B)(t)\|_{L^2_\gamma}^{\frac{\gamma_0}{\gamma}}\le Ct^{-\frac{3}{4}+\gamma_0},
$$

for all $t>T$ and $\gamma_0\in[0,\gamma](\gamma>\frac{3}{2})$. Thus, the proof is complete. □

Lemma 3.2. *Under the assumptions of Theorem 1.1 and* (3.1)*, there exists a large enough* T *such that the solution* (m,u,σ,v,B) *of coupled system* (2.1)-(2.2) *satisfies*

$$\|\nabla\left(m,u,\sigma,v,B\right)(t)\|_{L^2_\gamma}\le Ct^{-\frac{5}{4}+\gamma}, \tag{3.16}$$

for all $t>T$ *and* $\gamma\ge 0$*, where* C *is a positive constant independent of* t*.*

Proof. Applying ∇ to each equation of $(2.1)_1$–$(2.1)_5$, then multiplying equations $(2.1)_1$–$(2.1)_5$ by $\alpha_2\bar{n}|x|^{2\gamma}\nabla m$, $\alpha_2\bar{n}|x|^{2\gamma}\nabla u$, $\alpha_1|x|^{2\gamma}\nabla\sigma$, $\bar{n}|x|^{2\gamma}\nabla v$, and $|x|^{2\gamma}\nabla B$ respectively, summing them and then integrating over $\mathbb{R}^3$, and using integration by parts, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\alpha_2\bar{n}\|\nabla m\|^2_{L^2_\gamma}+\alpha_2\bar{n}\|\nabla u\|^2_{L^2_\gamma}+\alpha_1\|\nabla\sigma\|^2_{L^2_\gamma}+\bar{n}\|\nabla v\|^2_{L^2_\gamma}+\|\nabla B\|^2_{L^2_\gamma}\Big)\\
&+\alpha_2\bar{n}\|\nabla(u-v)\|^2_{L^2_\gamma}+\bar{n}\bar{\mu}\|\nabla^2 v\|^2_{L^2_\gamma}+\bar{n}(\bar{\lambda}+\bar{\mu})\|\nabla\operatorname{div}v\|^2_{L^2_\gamma}+\nu\|\nabla^2 B\|^2_{L^2_\gamma}\\
&=\alpha_2\bar{n}\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\cdot\nabla m\nabla u\,\mathrm{d}x+\alpha_1\bar{n}\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\cdot\nabla\sigma\nabla v\,\mathrm{d}x\\
&\quad-\bar{n}\bar{\mu}\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\cdot\nabla v\nabla^2 v\,\mathrm{d}x-\bar{n}(\bar{\lambda}+\bar{\mu})\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\cdot\nabla v\nabla\operatorname{div}v\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^3}\Big(\alpha_2\bar{n}|x|^{2\gamma}m\cdot\nabla F_1\Big)\mathrm{d}x+\int_{\mathbb{R}^3}\left(\alpha_2\bar{n}|x|^{2\gamma}u\cdot\nabla F_2\right)\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^3}\left(\alpha_1|x|^{2\gamma}\sigma\cdot\nabla F_3\right)\mathrm{d}x+\int_{\mathbb{R}^3}\left(\bar{n}|x|^{2\gamma}v\cdot\nabla F_4\right)\mathrm{d}x+\int_{\mathbb{R}^3}\left(|x|^{2\gamma}B\cdot\nabla F_5\right)\mathrm{d}x\\
&=\alpha_2\bar{n}\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\cdot\nabla m\nabla u\,\mathrm{d}x+\alpha_1\bar{n}\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\cdot\nabla\sigma\nabla v\,\mathrm{d}x\\
&\quad-\bar{n}\bar{\mu}\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\cdot\nabla v\nabla^2 v\,\mathrm{d}x-\bar{n}(\bar{\lambda}+\bar{\mu})\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\cdot\nabla v\nabla\operatorname{div}v\,\mathrm{d}x\\
&\quad-\alpha_2\bar{n}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla(u\cdot\nabla m)\nabla m\,\mathrm{d}x-\alpha_2\bar{n}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla(u\cdot\nabla u)\nabla u\,\mathrm{d}x\\
&\quad-\alpha_1\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla(v\cdot\nabla\sigma)\cdot\nabla\sigma\mathrm{d}x-\alpha_1\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla(\sigma\operatorname{div}v)\cdot\nabla\sigma\mathrm{d}x\\
&\quad-\bar{n}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla(v\cdot\nabla v)\cdot\nabla v\,\mathrm{d}x+\bar{n}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla\Big((\alpha_1-\frac{P'(n)}{n})\nabla\sigma\Big)\cdot\nabla v\,\mathrm{d}x\\
&\quad+\bar{n}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla\left((\frac{\mu}{n}-\bar{\mu})\Delta v\right)\cdot\nabla v\,\mathrm{d}x\\
&\quad+\bar{n}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla\Big((\frac{\mu+\lambda}{n}-(\bar{\mu}+\bar{\lambda}))\nabla\mathrm{div}v\Big)\cdot\nabla v\,\mathrm{d}x\\
&\quad+\bar{n}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla\left((\frac{\rho}{n}-\alpha_2)(u-v)\right)\cdot\nabla v\,\mathrm{d}x+\bar{n}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla\left((\operatorname{div}B)B\right)\cdot\nabla v\,\mathrm{d}x\\
&\quad+\bar{n}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla\left(B\cdot\nabla B\right)\cdot\nabla v\,\mathrm{d}x-\frac{1}{2}\bar{n}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla\left(\nabla|B|^2\right)\cdot\nabla v\,\mathrm{d}x\\
&\quad-\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla\left((\operatorname{div}v)B\right)\cdot\nabla B\,\mathrm{d}x+\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla\left(B\cdot\nabla v\right)\cdot\nabla B\,\mathrm{d}x\\
&\quad-\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla\left(v\cdot\nabla B\right)\cdot\nabla B\,\mathrm{d}x
\end{aligned}$$

$$:= \sum_{j=1}^{19} I_{2,j}. \tag{3.17}$$

Applying Lemma 2.3, we have

$$\sum_{i=1}^{2} |I_{2,j}| \lesssim \|\nabla(m,\sigma)\|_{L^2_\gamma} \|\nabla(u,v)\|_{L^2_{\gamma-1}}. \tag{3.18}$$

Using Lemma 2.3 and Cauchy's inequality, we have

$$\begin{aligned} |I_{2,3}| &\lesssim \|\nabla(|x|^{2\gamma})\nabla v\nabla^2 v\|_{L^1} \\ &\lesssim \|\nabla^2 v\|_{L^2_\gamma} \|\nabla v\|_{L^2_{\gamma-1}} \\ &\le \frac{\bar{n}\bar{\mu}}{4} C\|\nabla^2 v\|^2_{L^2_\gamma} + (\bar{n}\bar{\mu})C\|\nabla v\|^2_{L^2_{\gamma-1}}. \end{aligned} \tag{3.19}$$

Applying a method similar to the one for (3.19), one has

$$|I_{2,4}| \le \frac{\bar{n}(\bar{\lambda}+\bar{\mu})}{4} C\|\nabla \operatorname{div} v\|^2_{L^2_\gamma} + C(\bar{n}\bar{\lambda}\bar{\mu})\|\nabla v\|^2_{L^2_{\gamma-1}}. \tag{3.20}$$

Using Minkowski's inequality, integration by parts, Hölder's inequality, Lemma 2.3, Lemma 2.1 (Gagliardo-Nirenberg inequality), Cauchy's inequality and (3.1), we have

$$\begin{aligned} |I_{2,5}| &\lesssim \||x|^{2\gamma}\nabla u|\nabla m|^2\|_{L^1} + \||x|^{2\gamma} u\nabla^2 m\nabla m\|_{L^1} \\ &\lesssim \||x|^{2\gamma}\nabla u|\nabla m|^2\|_{L^1} + \|\nabla(|x|^{2\gamma} u)|\nabla m|^2\|_{L^1} \\ &\lesssim \||x|^{2\gamma}\nabla u|\nabla m|^2\|_{L^1} + \|\nabla(|x|^{2\gamma}) u|\nabla m|^2\|_{L^1} \\ &\lesssim \|\nabla u\|_{L^\infty} \|\nabla m\|^2_{L^2_\gamma} + \|u\|_{L^\infty} \|\nabla m\|_{L^2_\gamma} \|\nabla m\|_{L^2_{\gamma-1}} \\ &\lesssim \|\nabla^2\|_{H^1} \|\nabla m\|^2_{L^2_\gamma} + \|\nabla u\|_{H^1} \big[\|\nabla m\|^2_{L^2_\gamma} + \|\nabla m\|^2_{L^2_{\gamma-1}}\big] \\ &\lesssim t^{-5/4} \|\nabla m\|^2_{L^2_\gamma} + \|\nabla m\|^2_{L^2_{\gamma-1}}. \end{aligned} \tag{3.21}$$

Applying a method similar to the one for (3.21), we have

$$\sum_{j=6}^{7} |I_{2,j}| + |I_{2,9}| \lesssim t^{-5/4} \|\nabla(m,u,\sigma,v)\|^2_{L^2_\gamma} + \|\nabla(m,u,\sigma,v)\|^2_{L^2_{\gamma-1}}. \tag{3.22}$$

Applying Minkowski's inequality, Hölder's inequality, Lemma 2.1, Cauchy's inequality and (3.1), we have

$$\begin{aligned} |I_{2,8}| &\lesssim \||x|^{2\gamma} \operatorname{div} v|\nabla\sigma|^2\|_{L^1} + \||x|^{2\gamma}\nabla \operatorname{div} v\sigma\nabla\sigma\|_{L^1} \\ &\lesssim \|\nabla v\|_{L^\infty} \|\nabla\sigma\|^2_{L^2_\gamma} + \|\sigma\|_{L^\infty} \|\nabla \operatorname{div} v\|_{L^2_\gamma} \|\nabla\sigma\|_{L^2_\gamma} \\ &\lesssim \|\nabla\sigma\|_{H^1} \|\nabla \operatorname{div} v\|^2_{L^2_\gamma} + \|(\nabla\sigma,\nabla^2 v)\|_{H^1} \|\nabla\sigma\|^2_{L^2_\gamma} \\ &\lesssim t^{-5/4} \|\nabla \operatorname{div} v\|^2_{L^2_\gamma} + t^{-5/4} \|\nabla\sigma\|^2_{L^2_\gamma}. \end{aligned} \tag{3.23}$$

Using Minkowski's inequality, integration by parts, Hölder's inequality, Lemma 2.3, Lemma 2.1 (Gagliardo-Nirenberg inequality), Lemma 2.2, Cauchy's inequality and (3.1), we have

$$\begin{aligned} |I_{2,10}| &\lesssim \Big\||x|^{2\gamma}\Big(\alpha_1 - \frac{P'(n)}{n}\Big)\nabla^2\sigma\nabla v\Big\|_{L^1} + \Big\||x|^{2\gamma}\nabla\Big(\alpha_1 - \frac{P'(n)}{n}\Big)\nabla\sigma\nabla v\Big\|_{L^1} \\ &\lesssim \Big\|\nabla(|x|^{2\gamma}\Big(\alpha_1 - \frac{P'(n)}{n}\Big)\nabla v)\nabla\sigma\Big\|_{L^1} + \Big\||x|^{2\gamma}\nabla\Big(\alpha_1 - \frac{P'(n)}{n}\Big)\nabla\sigma\nabla v\Big\|_{L^1} \end{aligned}$$

$$
\begin{aligned}
&\lesssim \|\Big(\alpha_1-\frac{P'(n)}{n}\Big)\|_{L^\infty}\|\nabla v\|_{L^2_\gamma}\|\nabla\sigma\|_{L^2_{\gamma-1}}\\
&\quad+\|\nabla\Big(\alpha_1-\frac{P'(n)}{n}\Big)\|_{L^\infty}\|\nabla v\|_{L^2_\gamma}\|\nabla\sigma\|_{L^2_\gamma}\\
&\quad+\|\Big(\alpha_1-\frac{P'(n)}{n}\Big)\|_{L^\infty}\|\nabla^2 v\|_{L^2_\gamma}\|\nabla\sigma\|_{L^2_\gamma}\\
&\lesssim \|\nabla\sigma\|_{H^1}\|\nabla^2 v\|^2_{L^2_\gamma}+\|(\nabla\sigma,\nabla^2\sigma)\|_{H^1}\|\nabla(\sigma,v)\|^2_{L^2_\gamma}+\|\nabla\sigma\|_{H^1}\|\nabla v\|^2_{L^2_{\gamma-1}}\\
&\lesssim t^{-5/4}\|\nabla^2 v\|^2_{L^2_\gamma}+t^{-5/4}\|\nabla(\sigma,v)\|^2_{L^2_\gamma}+\|\nabla v\|^2_{L^2_{\gamma-1}}. \qquad (3.24)
\end{aligned}
$$

Applying method similar to the one for (3.24), we have

$$
\sum_{j=11}^{12}|I_{2,j}|\lesssim t^{-5/4}\|(\nabla^2 v,\nabla\operatorname{div} v)\|^2_{L_\gamma}+t^{-5/4}\|\nabla(\sigma,v)\|^2_{L_\gamma}+\|\nabla v\|^2_{L^2_{\gamma-1}}. \qquad (3.25)
$$

Using Hölder's inequality, Lemma 2.1 (Gagliardo-Nirenberg inequality), Lemma 2.2, Triangle inequality, Cauchy's inequality and (3.1), we have

$$
\begin{aligned}
|I_{2,13}|&\lesssim \||x|^{2\gamma}(\frac{\rho}{n}-\alpha_2)\nabla(u-v)\nabla v\|_{L^1}+\||x|^{2\gamma}\nabla(\frac{\rho}{n}-\alpha_2)(u-v)\nabla v\|_{L^1}\\
&\lesssim \|(\frac{\rho}{n}-\alpha_2)\|_{L^\infty}\|\nabla(u-v)\|_{L^2_\gamma}\|\nabla v\|_{L^2_\gamma}\\
&\quad+\|\nabla(\frac{\rho}{n}-\alpha_2)\|_{L^\infty}\|(u-v)\|_{L^2_\gamma}\|\nabla v\|_{L^2_\gamma}\\
&\lesssim \nabla\|(m,\sigma)\|_{H^1}\Big[\|\nabla u\|_{L^2_\gamma}\|\nabla v\|_{L^2_\gamma}+\|\nabla v\|^2_{L^2_\gamma}\Big]\\
&\quad+\|\nabla^2(m,\sigma)\|_{H^1}\|(u,v)\|_{L^2_\gamma}\|\nabla v\|_{L^2_\gamma}\\
&\lesssim t^{-5/4}\|\nabla(u,v)\|_{L^2_\gamma}+t^{-\frac74}\times t^{-\frac34+\gamma}\|\nabla(u,v)\|_{L^2_\gamma}\\
&\lesssim t^{-5/4}\|\nabla(u,v)\|^2_{L^2_\gamma}+t^{-\frac72+2\gamma}.
\end{aligned} \qquad (3.26)
$$

Using the same method as in (3.23), we have

$$
\sum_{j=14}^{19}|I_{2,j}|\lesssim t^{-5/4}\|\nabla^2(v,B)\|^2_{L^2_\gamma}+t^{-5/4}\|\nabla B\|^2_{L^2_\gamma}. \qquad (3.27)
$$

Substituting estimates (3.18)-(3.27) into (3.17), we have

$$
\begin{aligned}
&\frac12\frac{d}{dt}\Big(\alpha_2\bar n\|\nabla m\|^2_{L^2_\gamma}+\alpha_2\bar n\|\nabla u\|^2_{L^2_\gamma}+\alpha_1\|\nabla\sigma\|^2_{L^2_\gamma}+\bar n\|\nabla v\|^2_{L^2_\gamma}+\|\nabla B\|^2_{L^2_\gamma}\Big)\\
&\quad+\alpha_2\bar n\|\nabla(u-v)\|^2_{L^2_\gamma}+\bar n\bar\mu\|\nabla^2 v\|^2_{L^2_\gamma}+\bar n(\bar\lambda+\bar\mu)\|\nabla\operatorname{div} v\|^2_{L^2_\gamma}+\nu\|\nabla^2 B\|^2_{L^2_\gamma}\\
&\le Ct^{-5/4}\|(\nabla^2 v,\nabla\operatorname{div} v,\nabla^2 B)\|^2_{L^2_\gamma}+\frac{\bar n\bar\mu}{4}\|\nabla^2 v\|^2_{L^2_\gamma}+\frac{\bar n(\bar\lambda+\bar\mu)}{4}\|\operatorname{div} v\|^2_{L^2_\gamma}\\
&\quad+Ct^{-5/4}\|\nabla(m,u,\sigma,v,B)\|^2_{L^2_\gamma}+C\|\nabla(m,\sigma)\|_{L^2_\gamma}\|\nabla(u,v)\|_{L^2_{\gamma-1}}\\
&\quad+\|\nabla(m,u,\sigma,v)\|^2_{L^2_{\gamma-1}}+Ct^{-\frac72+2\gamma},
\end{aligned} \qquad (3.28)
$$

where C is a positive constant independent of t. For t is large enough, we have

$$
t^{-5/4}\le\frac12\min\big\{\frac{\bar n\bar\mu}{2},\frac{\bar n(\bar\lambda+\bar\mu)}{2},\nu\big\}. \qquad (3.29)
$$

Substituting (3.29) into (3.28), there exists a large enough T such that

$$\begin{aligned}&\frac{d}{dt}\Big(\alpha_2\bar{n}\|\nabla m\|_{L^2_\gamma}^2+\alpha_2\bar{n}\|\nabla u\|_{L^2_\gamma}^2+\alpha_1\|\nabla\sigma\|_{L^2_\gamma}^2+\bar{n}\|\nabla v\|_{L^2_\gamma}^2+\|\nabla B\|_{L^2_\gamma}^2\Big)\\&+\alpha_2\bar{n}\|\nabla(u-v)\|_{L^2_\gamma}^2+\bar{n}\bar{\mu}\|\nabla^2 v\|_{L^2_\gamma}^2+\bar{n}(\bar{\lambda}+\bar{\mu})\|\nabla\operatorname{div} v\|_{L^2_\gamma}^2+\nu\|\nabla^2 B\|_{L^2_\gamma}^2\\&\lesssim t^{-5/4}\|\nabla(m,u,\sigma,v,B)\|_{L^2_\gamma}^2+\|\nabla(m,\sigma)\|_{L^2_\gamma}\|\nabla(u,v)\|_{L^2_{\gamma-1}}\\&\quad+\|\nabla(m,u,\sigma,v)\|_{L^2_{\gamma-1}}^2+t^{-\frac{7}{2}+2\gamma},\end{aligned}\tag{3.30}$$

for all $t>T$. Substituting (2.7) and (3.1) into (3.30), we have

$$\begin{aligned}&\frac{d}{dt}\Big(\alpha_2\bar{n}\|\nabla m\|_{L^2_\gamma}^2+\alpha_2\bar{n}\|\nabla u\|_{L^2_\gamma}^2+\alpha_1\|\nabla\sigma\|_{L^2_\gamma}^2+\bar{n}\|\nabla v\|_{L^2_\gamma}^2+\|\nabla B\|_{L^2_\gamma}^2\Big)\\&+\alpha_2\bar{n}\|\nabla(u-v)\|_{L^2_\gamma}^2+\bar{n}\bar{\mu}\|\nabla^2 v\|_{L^2_\gamma}^2+\bar{n}(\bar{\lambda}+\bar{\mu})\|\nabla\operatorname{div} v\|_{L^2_\gamma}^2+\nu\|\nabla^2 B\|_{L^2_\gamma}^2\\&\lesssim t^{-5/4}\|\nabla(m,u,\sigma,v,B)\|_{L^2_\gamma}^2+\|\nabla(m,\sigma)\|_{L^2_\gamma}\|\nabla(u,v)\|_{L^2_\gamma}^{\frac{\gamma-1}{\gamma}}\|\nabla(u,v)\|_{L^2}^{1/\gamma}\\&\quad+\|\nabla(m,u,\sigma,v)\|_{L^2_\gamma}^{\frac{2(\gamma-1)}{\gamma}}\|\nabla(m,u,\sigma,v)\|_{L^2}^{2/\gamma}+t^{-\frac{7}{2}+2\gamma}\\&\lesssim t^{-5/4}\|\nabla(m,u,\sigma,v,B)\|_{L^2_\gamma}^2+t^{-\frac{5}{4\gamma}}\|\nabla(m,u,\sigma,v)\|_{L^2_\gamma}^{\frac{2\gamma-1}{\gamma}}\\&\quad+t^{-\frac{5}{2\gamma}}\|\nabla(m,u,\sigma,v)\|_{L^2_\gamma}^{\frac{2(\gamma-1)}{\gamma}}+t^{-\frac{7}{2}+2\gamma}.\end{aligned}$$

Denoting $\mathrm{E}(t):=\|\nabla(m,u,\sigma,v,B)\|_{L^2_\gamma}^2$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}(t)\le C_0t^{-5/4}\mathrm{E}(t)+C_1t^{-\frac{5}{4\gamma}}\mathrm{E}(t)^{\frac{2\gamma-1}{2\gamma}}+C_2t^{-\frac{5}{2\gamma}}\mathrm{E}(t)^{\frac{\gamma-1}{\gamma}}+C_3t^{-\frac{7}{2}+2\gamma},$$

for all $t>T$, where C_0,C_1,C_2,C_3 are positive constants independent of t. If $\gamma>\frac{5}{2}$, then we can apply Lemma 2.5 with $\alpha_0=\frac{5}{4}>1$, $\alpha_1=\frac{5}{4\gamma}<1,\beta_1=\frac{2\gamma-1}{2\gamma}<1$, $\alpha_2=\frac{5}{2\gamma}<1,\beta_2=\frac{\gamma-1}{\gamma}<1$, $\gamma_1=\frac{1-\alpha_1}{1-\beta_1}=-\frac{5}{2}+2\gamma>\gamma_2=\frac{1-\alpha_2}{1-\beta_2}=-\frac{5}{2}+\gamma$ to obtain

$$\mathrm{E}(t)\le Ct^{-\frac{5}{2}+2\gamma},\tag{3.31}$$

for all $t>T$. The Lemma 3.2 is proved for all $\gamma>5/2$ and the conclusion for the case of $[0,\frac{5}{2}]$ is proved by Lemma 2.4(interpolation inequality with weights). Thus, the proof is complete. □

Lemma 3.3. *Under the assumptions of Theorem 1.1 and* (3.1)*, there exists a large enough T such that the solution (m,u,σ,v,B) of the coupled system* (2.1)-(2.2) *satisfies*

$$\|\nabla^k(m,u,\sigma,v,B)(t)\|_{L^2_\gamma}\le Ct^{-\frac{3}{4}-\frac{k}{2}+\gamma},\tag{3.32}$$

for all $t>T$, $0\le k\le\ell$ and $\gamma\ge 0$, where C is a positive constant independent of t.

Proof. We use induction to prove the for estimate (3.32). In fact, inequalities (3.2) and (3.16) imply (3.32) when $k=0$ and $k=1$. By the general step of induction, assume that the estimate (3.32) holds for $0\le j\le k-1(2\le k\le\ell)$, i.e.,

$$\|\nabla^j(m,u,\sigma,v,B)(t)\|_{L^2_\gamma}\le Ct^{-\frac{3}{4}-\frac{j}{2}+\gamma},\tag{3.33}$$

for $0\le j\le k-1$. Then, we need to verify that (3.33) holds for $j=k$. Applying ∇^k to each equation of $(2.1)_1$–$(2.1)_5$, multiplying the equations $(2.1)_1$–$(2.1)_5$ by $\alpha_2\bar{n}|x|^{2\gamma}\nabla^k m$, $\alpha_2\bar{n}|x|^{2\gamma}\nabla^k u$, $\alpha_1|x|^{2\gamma}\nabla^k\sigma$, $\bar{n}|x|^{2\gamma}\nabla^k v$, and $|x|^{2\gamma}\nabla^k B$ respectively, summing them and then integrating over $\mathbb{R}^3$, using integration by parts the terms of the left side in (2.1), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\alpha_2\bar{n}\|\nabla^k m\|_{L^2_\gamma}^2+\alpha_2\bar{n}\|\nabla^k u\|_{L^2_\gamma}^2+\alpha_1\|\nabla^k\sigma\|_{L^2_\gamma}^2+\bar{n}\|\nabla^k v\|_{L^2_\gamma}^2+\|\nabla^k B\|_{L^2_\gamma}^2\Big)\\
&+\alpha_2\bar{n}\|\nabla^k(u-v)\|_{L^2_\gamma}^2+\bar{n}\bar{\mu}\|\nabla^{k+1}v\|_{L^2_\gamma}^2+\bar{n}(\bar{\lambda}+\bar{\mu})\|\nabla^k \operatorname{div} v\|_{L^2_\gamma}^2+\nu\|\nabla^{k+1}B\|_{L^2_\gamma}^2\\
&=\alpha_2\bar{n}\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\cdot\nabla^k m\nabla^k u\,\mathrm{d}x+\alpha_1\bar{n}\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\cdot\nabla^k\sigma\nabla^k v\,\mathrm{d}x\\
&\quad-\bar{n}\bar{\mu}\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\cdot\nabla^k v\nabla^{k+1}v\,\mathrm{d}x-\bar{n}(\bar{\lambda}+\bar{\mu})\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\cdot\nabla^k v\nabla^k\operatorname{div} v\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^3}\big(\alpha_2\bar{n}|x|^{2\gamma}m\cdot\nabla^k F_1\big)\,\mathrm{d}x+\int_{\mathbb{R}^3}\big(\alpha_2\bar{n}|x|^{2\gamma}u\cdot\nabla^k F_2\big)\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^3}\big(\alpha_1|x|^{2\gamma}\sigma\cdot\nabla^k F_3\big)\,\mathrm{d}x+\int_{\mathbb{R}^3}\big(\bar{n}|x|^{2\gamma}v\cdot\nabla^k F_4\big)\,\mathrm{d}x+\int_{\mathbb{R}^3}\big(|x|^{2\gamma}B\cdot\nabla^k F_5\big)\,\mathrm{d}x\\
&=\alpha_2\bar{n}\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\cdot\nabla^k m\nabla^k u\,\mathrm{d}x+\alpha_1\bar{n}\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\cdot\nabla^k\sigma\nabla^k v\,\mathrm{d}x\\
&\quad-\bar{n}\bar{\mu}\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\cdot\nabla^k v\nabla^{k+1}v\,\mathrm{d}x-\bar{n}(\bar{\lambda}+\bar{\mu})\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\cdot\nabla^k v\nabla^k\operatorname{div} v\,\mathrm{d}x\\
&\quad-\alpha_2\bar{n}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^k(u\cdot\nabla m)\cdot\nabla^k m\,\mathrm{d}x-\alpha_2\bar{n}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^k(u\cdot\nabla u)\cdot\nabla^k u\,\mathrm{d}x\\
&\quad-\alpha_1\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^k(v\cdot\nabla\sigma)\cdot\nabla^k\sigma\mathrm{d}x-\alpha_1\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^k(\sigma\operatorname{div} v)\cdot\nabla^k\sigma\mathrm{d}x\\
&\quad-\bar{n}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^k(v\cdot\nabla v)\cdot\nabla^k v\,\mathrm{d}x+\bar{n}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^k\Big((\alpha_1-\frac{P'(n)}{n})\nabla\sigma\Big)\cdot\nabla^k v\,\mathrm{d}x\\
&\quad+\bar{n}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^k\Big((\frac{\mu}{n}-\bar{\mu})\Delta v\Big)\cdot\nabla^k v\,\mathrm{d}x\\
&\quad+\bar{n}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^k\Big((\frac{\mu+\lambda}{n}-(\bar{\mu}+\bar{\lambda}))\nabla\operatorname{div}v\Big)\cdot\nabla^k v\,\mathrm{d}x\\
&\quad+\bar{n}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^k\Big((\frac{\rho}{n}-\alpha_2)(u-v)\Big)\cdot\nabla^k v\,\mathrm{d}x\\
&\quad+\bar{n}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^k\left((\operatorname{div} B)B\right)\cdot\nabla^k v\,\mathrm{d}x+\bar{n}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^k\left(B\cdot\nabla B\right)\cdot\nabla^k v\,\mathrm{d}x\\
&\quad-\frac{1}{2}\bar{n}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^k\left(\nabla|B|^2\right)\cdot\nabla^k v\,\mathrm{d}x-\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^k\left((\operatorname{div} v)B\right)\cdot\nabla^k B\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^k\left(B\cdot\nabla v\right)\cdot\nabla^k B\,\mathrm{d}x-\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^k\left(v\cdot\nabla B\right)\cdot\nabla^k B\,\mathrm{d}x\\
&:=\sum_{j=1}^{19}I_{3,j}.
\end{aligned}\tag{3.34}
$$

Applying Lemma 2.3, we have

$$
\sum_{i=1}^{2}|I_{3,j}|\lesssim\|\nabla^k(m,\sigma)\|_{L^2_\gamma}\|\nabla^k(u,v)\|_{L^2_{\gamma-1}}.\tag{3.35}
$$

Using Lemma 2.3 and Cauchy's inequality, we have

$$\begin{aligned}|I_{3,3}| &\lesssim \|\nabla\left(|x|^{2\gamma}\right)\nabla^k v\nabla^{k+1}v\|_{L^1}\\ &\lesssim \|\nabla^{k+1}v\|_{L^2_\gamma}\|\nabla^k v\|_{L^2_{\gamma-1}}\\ &\le \frac{\bar n\bar\mu}{4}C\|\nabla^{k+1}v\|^2_{L^2_\gamma}+C(\bar n\bar\mu)\|\nabla^k v\|^2_{L^2_{\gamma-1}}.\end{aligned} \tag{3.36}$$

Applying a similar method to the one for (3.36), we have

$$|I_{3,4}| \le \frac{\bar n(\bar\lambda+\bar\mu)}{4}C\|\nabla^k \operatorname{div} v\|^2_{L^2_\gamma}+C(\bar n\bar\lambda\bar\mu)\|\nabla^k v\|^2_{L^2_{\gamma-1}}. \tag{3.37}$$

Using integration by parts, we have

$$\begin{aligned}&-\frac{1}{\alpha_2\bar n}I_{3,5}\\ &=\int_{\mathbb{R}^3}|x|^{2\gamma}u\nabla^{k+1}m\cdot\nabla^k m\,\mathrm{d}x+\sum_{j=1}^{k}C_k^j\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^j u\nabla^{k-j+1}m\nabla^k m\,\mathrm{d}x\\ &=-\frac12\int_{\mathbb{R}^3}\nabla\left(|x|^{2\gamma}\right)u|\nabla^k m|^2\,\mathrm{d}x+(k-\frac12)\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla u|\nabla^k m|^2\,\mathrm{d}x\\ &\quad+\sum_{j=2}^{k-2}C_k^j\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^j u\nabla^{k-j+1}m\nabla^k m\,\mathrm{d}x\\ &\quad+\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^k u\nabla m\nabla^k m\,\mathrm{d}x+k\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^{k-1}u\nabla^2 m\nabla^k m\,\mathrm{d}x\\ &:=\sum_{j=1}^{5}I_{3,5,j}.\end{aligned} \tag{3.38}$$

Applying Hölder's inequality, Lemma 2.3, Lemma 2.1 (Gagliardo-Nirenberg inequality), (3.33), Cauchy's inequality, and (3.1), we have

$$\begin{aligned}|I_{3,5,1}| &\lesssim \|u\|_{L^\infty}\|\nabla^k m\|_{L^2_\gamma}\|\nabla^k m\|_{L^2_{\gamma-1}}\\ &\lesssim \|\nabla u\|_{H^1}\|\nabla^k m\|_{L^2_\gamma}\|\nabla^k m\|_{L^2_{\gamma-1}}\\ &\lesssim t^{-5/4}\|\nabla^k m\|^2_{L^2_\gamma}+\|\nabla^k m\|^2_{L^2_{\gamma-1}}.\end{aligned} \tag{3.39}$$

$$\begin{aligned}\sum_{j=2}^{j=4}|I_{3,5,j}| &\lesssim \|\nabla u\|_{L^\infty}\|\nabla^k m\|^2_{L^2_\gamma}+\sum_{j=2}^{k-2}\|\nabla^j u\|_{L^\infty}\|\nabla^{k-j+1}m\|_{L^2_\gamma}\|\nabla^k m\|_{L^2_\gamma}\\ &\quad+\|\nabla^k u\|_{L^2_\gamma}\|\nabla m\|_{L^\infty}\|\nabla^k m\|_{L^2_\gamma}\\ &\lesssim \|\nabla^2 u\|_{H^1}\|\nabla^k m\|^2_{L^2_\gamma}+\sum_{j=2}^{k-2}\|\nabla^{j+1}u\|_{H^1}\|\nabla^{k-j+1}m\|_{L^2_\gamma}\|\nabla^k m\|_{L^2_\gamma}\\ &\quad+\|\nabla^2 m\|_{H^1}\|\nabla^k m\|_{L^2_\gamma}\|\nabla^k u\|_{L^2_\gamma}\\ &\lesssim t^{-\frac74}\|\nabla^k(m,u)\|^2_{L^2_\gamma}+t^{-\frac54-\frac k2+\gamma}\times t^{-5/4}\|\nabla^k(m,u)\|_{L^2_\gamma}\\ &\lesssim t^{-5/4}\|\nabla^k m\|^2_{L^2_\gamma}+t^{-\frac52-k+2\gamma}.\end{aligned} \tag{3.40}$$

$$\begin{aligned}|I_{3,5,5}| &\lesssim \|\nabla^{k-1}u\|_{L^3}\|\nabla^2 m\|_{L^6_\gamma}\|\nabla^k m\|_{L^2_\gamma}\\ &\lesssim \|\nabla^{k-1}u\|_{H^1}(\|\nabla^3 m\|_{L^2_\gamma}+\|\nabla^2 m\|_{L^2_{\gamma-1}})\|\nabla^k m\|_{L^2_\gamma}\\ &\lesssim t^{-\frac{5}{4}-\frac{k}{2}+\gamma}\times t^{-\frac{9}{4}}\|\nabla^k m\|_{L^2_\gamma}\\ &\lesssim t^{-5/4}\|\nabla^k m\|^2_{L^2_\gamma}+t^{-\frac{5}{2}-k+2\gamma}.\end{aligned} \tag{3.41}$$

Combining (3.38)-(3.41), we have

$$|I_{3,5}| \lesssim t^{-5/4}\|\nabla^k(m,u)\|^2_{L^2_\gamma}+\|\nabla^k m\|^2_{L^2_{\gamma-1}}+t^{-\frac{5}{2}-k+2\gamma}. \tag{3.42}$$

Applying a method similar to the one in $I_{3,5}$, we have

$$\sum_{j=6}^{7}|I_{3,j}|+|I_{3,9}| \lesssim t^{-5/4}\|\nabla^k(u,\sigma,v)\|^2_{L^2_\gamma}+\|\nabla^k(u,v,\sigma)\|^2_{L^2_{\gamma-1}}+t^{-\frac{5}{2}-k+2\gamma}. \tag{3.43}$$

According to (3.34), we have

$$\begin{aligned}&-\frac{1}{\alpha_1}I_{3,8}\\ &=\int_{\mathbb{R}^3}|x|^{2\gamma}\sigma\nabla^k \operatorname{div} v\cdot\nabla^k\sigma\,\mathrm{d}x+\sum_{j=1}^{k}C_k^j\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^j\sigma\nabla^{k-j}\operatorname{div} v\nabla^k\sigma\,\mathrm{d}x\\ &:= I_{3,5,1}+I_{3,5,2}.\end{aligned} \tag{3.44}$$

Using Minkowski's inequality, Hölder's inequality, Lemma 2.1 (Gagliardo-Nirenberg inequality), Cauchy's inequality and (3.1), we have

$$\begin{aligned}|I_{3,8,1}| &\lesssim \||x|^{2\gamma}\sigma\nabla^k\operatorname{div} v\nabla^k\sigma\|_{L^1}\\ &\lesssim \|\sigma\|_{L^\infty}\|\nabla^k\operatorname{div} v\|_{L^2_\gamma}\|\nabla^k\sigma\|_{L^2_\gamma}\\ &\lesssim \|\nabla\sigma\|_{H^1}\|\nabla^k\operatorname{div} v\|_{L^2_\gamma}\|\nabla^k\sigma\|_{L^2_\gamma}\\ &\lesssim t^{-5/4}\|\nabla^k\operatorname{div} v\|^2_{L^2_\gamma}+t^{-5/4}\|\nabla^k\sigma\|^2_{L^2_\gamma}.\end{aligned} \tag{3.45}$$

Applying the similar method to (3.40)-(3.41), one has

$$|I_{3,8,2}| \lesssim t^{-5/4}\|\nabla^k(\sigma,v)\|^2_{L^2_\gamma}+t^{-\frac{5}{2}-k+2\gamma}. \tag{3.46}$$

Combining (3.45) and (3.46), we have

$$|I_{3,8}| \lesssim t^{-5/4}\|\nabla^k\operatorname{div} v\|^2_{L^2_\gamma}+t^{-5/4}\|\nabla^k(\sigma,v)\|^2_{L^2_\gamma}+t^{-\frac{5}{2}-k+2\gamma}. \tag{3.47}$$

Applying integration by parts, one has

$$\begin{aligned}\frac{1}{\bar n}I_{3,10} &= \int_{\mathbb{R}^3}|x|^{2\gamma}\Big(\alpha_1-\frac{P'(n)}{n}\Big)\nabla^{k+1}\sigma\cdot\nabla^k v\,\mathrm{d}x\\ &\quad+\sum_{j=1}^{k}C_k^j\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^j\Big(\alpha_1-\frac{P'(n)}{n}\Big)\nabla^{k-j+1}\sigma\nabla^k v\,\mathrm{d}x\\ &= -\int_{\mathbb{R}^3}\nabla\left(|x|^{2\gamma}\right)\Big(\alpha_1-\frac{P'(n)}{n}\Big)\nabla^k v\nabla^k\sigma\,\mathrm{d}x\\ &\quad-\int_{\mathbb{R}^3}|x|^{2\gamma}\Big(\alpha_1-\frac{P'(n)}{n}\Big)\nabla^{k+1}v\nabla^k\sigma\,\mathrm{d}x\\ &\quad+(k-1)\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla\Big(\alpha_1-\frac{P'(n)}{n}\Big)\nabla^k v\nabla^k\sigma\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=2}^{k-1} C_k^j \int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^j\Big(\alpha_1-\frac{P'(n)}{n}\Big)\nabla^{k-j+1}\sigma\nabla^k v\,\mathrm{d}x\\
&+\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^k\Big(\alpha_1-\frac{P'(n)}{n}\Big)\nabla\sigma\nabla^k v\,\mathrm{d}x\\
:=&\sum_{j=1}^{5} I_{3,10,j}.
\end{aligned}\tag{3.48}
$$

Using Hölder's inequality, Lemma 2.3, Lemma 2.1 (Gagliardo–Nirenberg inequality), Lemma 2.2, (3.1), and Cauchy inequality, we have

$$
\begin{aligned}
|I_{3,10,1}| &\lesssim \|\nabla\left(|x|^{2\gamma}\right)\Big(\alpha_1-\frac{P'(n)}{n}\Big)\nabla^k v\nabla^k\sigma\|_{L^1}\\
&\lesssim \|\Big(\alpha_1-\frac{P'(n)}{n}\Big)\|_{L^\infty}\|\nabla^k v\|_{L^2_\gamma}\|\nabla^k\sigma\|_{L^2_{\gamma-1}}\\
&\lesssim \|\nabla\sigma\|_{H^1}\|\nabla^k v\|_{L^2_\gamma}\|\nabla^k\sigma\|_{L^2_{\gamma-1}}\\
&\lesssim t^{-5/4}\|\nabla^k v\|^2_{L^2_\gamma}+\|\nabla^k\sigma\|^2_{L^2_{\gamma-1}}.
\end{aligned}\tag{3.49}
$$

Applying a method similar to the on in (3.49), we have

$$
|I_{3,10,2}|+|I_{3,10,3}|\lesssim t^{-5/4}\|\nabla^{k+1}v\|^2_{L^2_\gamma}+t^{-5/4}\ \|\nabla^k(\sigma,v)\|^2_{L^2_\gamma}.\tag{3.50}
$$

We note that Lemma 2.2 can not be applied in the weighted space. For $I_{3,10,4}$ and $I_{3,10,5}$, applying Hölder's inequality, Lemma 2.1 (Gagliardo–Nirenberg inequality) and (3.33) skillfully, we have

$$
\begin{aligned}
|I_{3,10,4}| &\lesssim \sum_{j=2}^{k-1}\||x|^{2\gamma}\nabla^j\Big(\alpha_1-\frac{P'(n)}{n}\Big)\nabla^{k-j+1}\sigma\nabla^k v\|_{L^1}\\
&\lesssim \sum_{j=2}^{k-1}\|\nabla^j\Big(\alpha_1-\frac{P'(n)}{n}\Big)\|_{L^3}\|\nabla^{k-j+1}\sigma\|_{L^2_\gamma}\|\nabla^k v\|_{L^6_\gamma}\\
&\lesssim \sum_{j=2}^{k-1}\|\nabla^j\sigma\|_{H^1}\|\nabla^{k-j+1}\sigma\|_{L^2_\gamma}\left(\|\nabla^{k+1}v\|_{L^2_\gamma}+\|\nabla^k v\|_{L^2_{\gamma-1}}\right)\\
&\lesssim t^{-\frac54-\frac{k}{2}+\gamma}\times t^{-3/4}\left(\|\nabla^{k+1}v\|_{L^2_\gamma}+\|\nabla^k v\|_{L^2_{\gamma-1}}\right)\\
&\lesssim t^{-5/4}\|\nabla^{k+1}v\|^2_{L^2_\gamma}+\|\nabla^k v\|^2_{L^2_{\gamma-1}}+t^{-\frac52-k+2\gamma}.
\end{aligned}\tag{3.51}
$$

$$
\begin{aligned}
|I_{3,10,5}| &\lesssim \||x|^{2\gamma}\nabla^k\Big(\alpha_1-\frac{P'(n)}{n}\Big)\nabla\sigma\nabla^k v\|_{L^1}\\
&\lesssim \|\nabla^k\Big(\alpha_1-\frac{P'(n)}{n}\Big)\|_{L^2}\||x|^{\gamma}\nabla\sigma\|_{L^\infty}\|\nabla^k v\|_{L^2_\gamma}\\
&\lesssim \|\nabla^k\sigma\|_{L^2}\Big(\|\nabla\left(|x|^{\gamma}\nabla\sigma\right)\|_{L^2}^{1/2}\|\nabla^2\left(|x|^{\gamma}\nabla\sigma\right)\|_{L^2}^{1/2}\Big)\|\nabla^k v\|_{L^2_\gamma}\\
&\lesssim \|\nabla^k\sigma\|_{L^2}\left(\|\nabla^2\left(|x|^{\gamma}\nabla\sigma\right)\|_{L^2}+\|\nabla\left(|x|^{\gamma}\nabla\sigma\right)\|_{L^2}\right)\|\nabla^k v\|_{L^2_\gamma}\\
&\lesssim \|\nabla^k\sigma\|_{L^2}\Big(\|\nabla^3\sigma\|_{L^2_\gamma}+\|\nabla^2\sigma\|_{L^2_{\gamma-1}}+\|\nabla\sigma\|_{L^2_{\gamma-2}}\\
&\quad+\|\nabla^2\sigma\|_{L^2_\gamma}+\|\nabla\sigma\|_{L^2_{\gamma-1}}\Big)\|\nabla^k v\|_{L^2_\gamma}\\
&\lesssim t^{-\frac{5}{4}-\frac{k}{2}+\gamma}\times t^{-5/4}\|\nabla^k v\|_{L^2_\gamma}\\
&\lesssim t^{-5/4}\|\nabla^k v\|_{L^2_\gamma}^2+t^{-\frac{5}{2}-k+2\gamma}.
\end{aligned}
\tag{3.52}
$$

Combining (3.48)-(3.52), we have

$$
|I_{3,10}|\lesssim t^{-5/4}\|\nabla^{k+1}v\|_{L^2_\gamma}^2+t^{-5/4}\|\nabla^k(\sigma,v)\|_{L^2_\gamma}^2+\|\nabla^k(\sigma,v)\|_{L^2_{\gamma-1}}^2+t^{-\frac{5}{2}-k+2\gamma}.
\tag{3.53}
$$

Using the same method as in $|I_{3,10}|$, one has

$$
\begin{aligned}
|I_{3,11}|+|I_{3,12}| &\lesssim t^{-5/4}\|\nabla^{k+1}v\|_{L^2_\gamma}^2+t^{-5/4}\|\nabla^k\operatorname{div}v\|_{L^2_\gamma}^2\\
&\quad+t^{-5/4}\|\nabla^k v\|_{L^2_\gamma}^2+\|\nabla^k v\|_{L^2_{\gamma-1}}^2+t^{-\frac{5}{2}-k+2\gamma}.
\end{aligned}
\tag{3.54}
$$

According to (3.3), we have

$$
\begin{aligned}
\frac{1}{\bar n}I_{3,13} &= \int_{\mathbb{R}^3}|x|^{2\gamma}(\frac{\rho}{n}-\alpha_2)\nabla^k(u-v)\nabla^k v\,\mathrm{d}x\\
&\quad+\sum_{j=1}^{k-1}C_k^j\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^j(\frac{\rho}{n}-\alpha_2)\nabla^{k-j}(u-v)\nabla^k v\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^k(\frac{\rho}{n}-\alpha_2)(u-v)\nabla^k v\,\mathrm{d}x\\
&:=\sum_{j=1}^{3}I_{3,13,j}.
\end{aligned}
\tag{3.55}
$$

Applying Minkowski's inequality, Hölder's inequality, Lemma 2.1, Lemma 2.2, Triangle inequality, (3.1), and (3.33), we have

$$
\begin{aligned}
|I_{3,13,1}| &\lesssim \|(\frac{\rho}{n}-\alpha_2)\|_{L^\infty}\|\nabla^k(u-v)\|_{L^2_\gamma}\|\nabla^k v\|_{L^2_\gamma}\\
&\lesssim \|\nabla(m,\sigma)\|_{H^1}\|\nabla^k(u,v)\|_{L^2_\gamma}\|\nabla^k v\|_{L^2_\gamma}\\
&\lesssim t^{-5/4}\|\nabla^k(u,v)\|_{L^2_\gamma}^2.
\end{aligned}
\tag{3.56}
$$

$$
\begin{aligned}
|I_{3,13,2}| &\lesssim \sum_{j=1}^{k-1} \|\nabla^j(\frac{\rho}{n}-\alpha_2)\|_{L^3}\|\nabla^{k-j}(u-v)\|_{L^2_\gamma}\|\nabla^k v\|_{L^6_\gamma} \\
&\lesssim \sum_{j=2}^{k-1} \|\nabla^j(m,\sigma)\|_{H^1}\|\nabla^{k-j}(u,v)\|_{L^2_\gamma}\Big(\|\nabla^{k+1}v\|_{L^2_\gamma}+\|\nabla^k v\|_{L^2_{\gamma-1}}\Big) \\
&\lesssim t^{-\frac{5}{4}-\frac{k}{2}+\gamma}\times t^{-\frac{1}{4}}\Big(\|\nabla^{k+1}v\|_{L^2_\gamma}+\|\nabla^k v\|_{L^2_{\gamma-1}}\Big) \\
&\lesssim t^{-1/2}\|\nabla^{k+1}v\|^2_{L^2_\gamma}+\|\nabla^k v\|^2_{L^2_{\gamma-1}}+t^{-\frac{5}{2}-k+2\gamma}.
\end{aligned} \tag{3.57}
$$

$$
\begin{aligned}
|I_{3,13,3}| &\lesssim \|\nabla^k(\frac{\rho}{n}-\alpha_2)\|_{L^2}\||x|^{2\gamma}(u-v)\|_{L^\infty}\|\nabla^k v\|_{L^2_\gamma} \\
&\lesssim \|\nabla^k(m,\sigma)\|_{L^2}\Big(\|\nabla\left(|x|^\gamma(u+v)\right)\|^{1/2}_{L^2}\|\nabla^2\left(|x|^\gamma(u+v)\right)\|^{1/2}_{L^2}\Big)\|\nabla^k v\|_{L^2_\gamma} \\
&\lesssim \|\nabla^k(m,\sigma)\|_{L^2}\left(\|\nabla^2\left(|x|^\gamma(u+v)\right)\|_{L^2}+\|\nabla\left(|x|^\gamma(u+v)\right)\|_{L^2}\right)\|\nabla^k v\|_{L^2_\gamma} \\
&\lesssim \|\nabla^k(m,\sigma)\|_{L^2}\Big(\|\nabla^2(u,v)\|_{L^2_\gamma}+\|\nabla(u,v)\|_{L^2_{\gamma-1}}+\|(u,v)\|_{L^2_{\gamma-2}} \\
&\quad+\|\nabla(u,v)\|_{L^2_\gamma}+\|(u,v)\|_{L^2_{\gamma-1}}\Big)\|\nabla^k v\|_{L^2_\gamma} \\
&\lesssim t^{-\frac{5}{4}-\frac{k}{2}+\gamma}\times t^{-3/4}\|\nabla^k v\|_{L^2_\gamma} \\
&\lesssim t^{-5/4}\|\nabla^k v\|^2_{L^2_\gamma}+t^{-\frac{5}{2}-k+2\gamma}.
\end{aligned} \tag{3.58}
$$

Combining (3.55)–(3.58), one has

$$
|I_{3,13}| \lesssim t^{-1/2}\|\nabla^{k+1}v\|^2_{L^2_\gamma}+t^{-5/4}\|\nabla^k(u,v)\|^2_{L^2_\gamma}+\|\nabla^k v\|^2_{L^2_{\gamma-1}}+t^{-\frac{5}{2}-k+2\gamma}. \tag{3.59}
$$

According to (3.3), we have

$$
\begin{aligned}
\frac{1}{\bar n}I_{3,14} &= \int_{\mathbb{R}^3}|x|^{2\gamma}B\nabla^k \operatorname{div} B\nabla^k v\,\mathrm{d}x + k\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla B\nabla^{k-1}\operatorname{div} B\nabla^k v\,\mathrm{d}x \\
&\quad+\sum_{j=2}^{k-1}C_k^j\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^j B\nabla^{k-j}\operatorname{div} B\nabla^k v\,\mathrm{d}x \\
&\quad+\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^k B\operatorname{div} B\nabla^k v\,\mathrm{d}x \\
&:= \sum_{j=1}^{4} I_{3,14,j}.
\end{aligned} \tag{3.60}
$$

Using Minkowski's inequality, Hölder's inequality, Lemma 2.1 (Gagliardo–Nirenberg inequality) (3.1), (3.33), and Cauchy's inequality, we have

$$
\begin{aligned}
|I_{3,14,1}| &\lesssim \|B\|_{L^\infty}\|\nabla^{k+1}B\|_{L^2_\gamma}\|\nabla^k v\|_{L^2_\gamma} \\
&\lesssim \|\nabla B\|_{H^1}\|\nabla^{k+1}B\|_{L^2_\gamma}\|\nabla^k v\|_{L^2_\gamma} \\
&\lesssim t^{-5/4}\|\nabla^{k+1}B\|^2_{L^2_\gamma}+t^{-5/4}\|\nabla^k v\|^2_{L^2_\gamma}.
\end{aligned} \tag{3.61}
$$

Applying a method similar to the one for (3.61), we have

$$
|I_{3,14,2}|+|I_{3,14,4}| \lesssim t^{-5/4}\|\nabla^k(v,B)\|^2_{L^2_\gamma}. \tag{3.62}
$$

For $I_{3,14,3}$, we have

$$\begin{aligned}
|I_{3,14,3}| &\lesssim \sum_{j=2}^{k-1} \|\nabla^j B\|_{L^3} \|\nabla^{k-j+1} B\|_{L^2_\gamma} \|\nabla^k v\|_{L^6_\gamma} \\
&\lesssim \sum_{j=2}^{k-1} \|\nabla^j B\|_{H^1} \|\nabla^{k-j+1} B\|_{L^2_\gamma} \Big(\|\nabla^{k+1} v\|_{L^2_\gamma} + \|\nabla^k v\|_{L^2_{\gamma-1}} \Big) \\
&\lesssim t^{-\frac{5}{4}-\frac{k}{2}+\gamma} \times t^{-3/4} \Big(\|\nabla^{k+1} v\|_{L^2_\gamma} + \|\nabla^k v\|_{L^2_{\gamma-1}} \Big) \\
&\lesssim t^{-5/4} \|\nabla^{k+1} v\|^2_{L^2_\gamma} + \|\nabla^k v\|^2_{L^2_{\gamma-1}} + t^{-\frac{5}{2}-k+2\gamma}.
\end{aligned} \tag{3.63}$$

Combining (3.60)–(3.63), one has

$$\begin{aligned}
|I_{3,14}| &\lesssim t^{-5/4} \|\nabla^{k+1}(v,B)\|^2_{L^2_\gamma} + t^{-5/4} \|\nabla^k (v,B)\|^2_{L^2_\gamma} + \|\nabla^k v\|^2_{L^2_{\gamma-1}} \\
&\quad + t^{-\frac{5}{2}-k+2\gamma}.
\end{aligned} \tag{3.64}$$

Using the same method as $I_{3,14}$, we have

$$\begin{aligned}
\sum_{j=15}^{19} |I_{3,j}| &\lesssim t^{-5/4} \|\nabla^{k+1}(v,B)\|^2_{L^2_\gamma} + t^{-5/4} \|\nabla^k (v,B)\|^2_{L^2_\gamma} + \|\nabla^k (v,B)\|^2_{L^2_{\gamma-1}} \\
&\quad + t^{-\frac{5}{2}-k+2\gamma}.
\end{aligned} \tag{3.65}$$

Substituting the estimates (3.35)–(3.65) into (3.34), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\alpha_2 \bar n \|\nabla^k m\|^2_{L^2_\gamma} + \alpha_2 \bar n \|\nabla^k u\|^2_{L^2_\gamma} + \alpha_1 \|\nabla^k \sigma\|^2_{L^2_\gamma} + \bar n \|\nabla^k v\|^2_{L^2_\gamma} \\
&+ \|\nabla^k B\|^2_{L^2_\gamma} \Big) + \alpha_2 \bar n \|\nabla^k (u-v)\|^2_{L^2_\gamma} + \bar n \bar\mu \|\nabla^{k+1} v\|^2_{L^2_\gamma} \\
&+ \bar n(\bar\lambda + \bar\mu) \|\nabla^k \operatorname{div} v\|^2_{L^2_\gamma} + \nu \|\nabla^{k+1} B\|^2_{L^2_\gamma} \\
&\le \frac{\bar n \bar\mu}{4} \|\nabla^{k+1} v\|^2_{L^2_\gamma} + \frac{\bar n(\bar\lambda+\bar\mu)}{4} \|\nabla^k \operatorname{div} v\|^2_{L^2_\gamma} \\
&\quad + C t^{-5/4} \|(\nabla^k \operatorname{div} v, \nabla^{k+1} B)\|^2_{L^2_\gamma} + C t^{-1/2} \|\nabla^{k+1} v\|^2_{L^2_\gamma} \\
&\quad + C t^{-5/4} \|\nabla^k (m,u,\sigma,v,B)\|^2_{L^2_\gamma} + C \|\nabla^k (m,u,\sigma,v,B)\|^2_{L^2_{\gamma-1}} \\
&\quad + C \|\nabla^k (m,\sigma)\|_{L^2_\gamma} \|\nabla^k (u,v)\|_{L^2_{\gamma-1}} + C t^{-\frac{5}{2}-k+2\gamma}.
\end{aligned} \tag{3.66}$$

For t large enough, we have

$$t^{-1/2} \le \frac{1}{2} \min\{ \frac{\bar n \bar\mu}{2}, \frac{\bar n(\bar\lambda+\bar\mu)}{2}, \nu \}. \tag{3.67}$$

Substituting (3.67) into (3.66), there exists a large enough T such that

$$\begin{aligned}
&\frac{d}{dt}\Big(\alpha_2 \bar n \|\nabla^k m\|^2_{L^2_\gamma} + \alpha_2 \bar n \|\nabla^k u\|^2_{L^2_\gamma} + \alpha_1 \|\nabla^k \sigma\|^2_{L^2_\gamma} \\
&+ \bar n \|\nabla^k v\|^2_{L^2_\gamma} + \|\nabla^k B\|^2_{L^2_\gamma} \Big) + \alpha_2 \bar n \|\nabla^k (u-v)\|^2_{L^2_\gamma} + \bar n \bar\mu \|\nabla^{k+1} v\|^2_{L^2_\gamma} \\
&+ \bar n(\bar\lambda + \bar\mu) \|\nabla^k \operatorname{div} v\|^2_{L^2_\gamma} + \nu \|\nabla^{k+1} B\|^2_{L^2_\gamma} \\
&\lesssim t^{-5/4} \|\nabla^k (m,u,\sigma,v,B)\|^2_{L^2_\gamma} + \|\nabla^k (m,\sigma)\|_{L^2_\gamma} \|\nabla^k (u,v)\|_{L^2_{\gamma-1}} \\
&\quad + \|\nabla^k (m,u,\sigma,v,B)\|^2_{L^2_{\gamma-1}} + t^{-\frac{5}{2}-k+2\gamma},
\end{aligned} \tag{3.68}$$

for all $t > T$. Substituting (2.7) and (3.33) into (3.68), we have

$$\begin{aligned}
&\frac{d}{dt}\Big(\alpha_2\bar{n}\|\nabla^k m\|_{L^2_\gamma}^2+\alpha_2\bar{n}\|\nabla^k u\|_{L^2_\gamma}^2+\alpha_1\|\nabla^k\sigma\|_{L^2_\gamma}^2+\bar{n}\|\nabla^k v\|_{L^2_\gamma}^2+\|\nabla^k B\|_{L^2_\gamma}^2\Big)\\
&+\alpha_2\bar{n}\|\nabla^k(u-v)\|_{L^2_\gamma}^2+\bar{n}\bar{\mu}\|\nabla^{k+1}v\|_{L^2_\gamma}^2+\bar{n}(\bar{\lambda}+\bar{\mu})\|\nabla^k\operatorname{div} v\|_{L^2_\gamma}^2+\nu\|\nabla^{k+1}B\|_{L^2_\gamma}^2\\
&\lesssim t^{-5/4}\|\nabla^k(m,u,\sigma,v,B)\|_{L^2_\gamma}^2+\|\nabla^k(m,\sigma)\|_{L^2_\gamma}\|\nabla^k(u,v)\|_{L^2_\gamma}^{\frac{\gamma-1}{\gamma}}\|\nabla^k(u,v)\|_{L^2}^{1/\gamma}\\
&\quad+\|\nabla^k(m,u,\sigma,v,B)\|_{L^2_\gamma}^{\frac{2(\gamma-1)}{\gamma}}\|\nabla^k(m,u,\sigma,v,B)\|_{L^2}^{2/\gamma}+t^{-\frac{5}{2}-k+2\gamma}\\
&\lesssim t^{-5/4}\|\nabla^k(m,u,\sigma,v,B)\|_{L^2_\gamma}^2+t^{(-\frac{3}{4}-\frac{k}{2})\frac{1}{\gamma}}\|\nabla^k(m,u,\sigma,v)\|_{L^2_\gamma}^{\frac{2\gamma-1}{\gamma}}\\
&\quad+t^{(-\frac{3}{4}-\frac{k}{2})\frac{2}{\gamma}}\|\nabla^k(m,u,\sigma,v,B)\|_{L^2_\gamma}^{\frac{2(\gamma-1)}{\gamma}}+t^{-\frac{5}{2}-k+2\gamma}.
\end{aligned}$$

Denoting $\mathrm{E}(t) := \|\nabla^k(m,u,\sigma,v,B)\|_{L^2_\gamma}^2$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}(t)\le C_0t^{-5/4}\mathrm{E}(t)+C_1t^{(-\frac{3}{4}-\frac{k}{2})\frac{1}{\gamma}}\mathrm{E}(t)^{\frac{2\gamma-1}{2\gamma}}+C_2t^{(-\frac{3}{4}-\frac{k}{2})\frac{2}{\gamma}}\mathrm{E}(t)^{\frac{\gamma-1}{\gamma}}+C_3t^{-\frac{5}{2}-k+2\gamma},$$

for all $t > T$, where C_0, C_1, C_2, C_3 are positive constants independent of t. If $\gamma > \frac{3+2k}{2}$, then we can apply Lemma 2.5 with $\alpha_0 = \frac{5}{4} > 1$, $\alpha_1 = (\frac{3}{4}+\frac{k}{2})\frac{1}{\gamma} < 1$, $\beta_1 = \frac{2\gamma-1}{2\gamma} < 1, \alpha_2 = (\frac{3}{4}+\frac{k}{2})\frac{2}{\gamma} < 1$, $\beta_2 = \frac{\gamma-1}{\gamma} < 1$, $\gamma_1 = \frac{1-\alpha_1}{1-\beta_1} = -\frac{3}{2}-k+2\gamma > \gamma_2 = \frac{1-\alpha_2}{1-\beta_2} = -\frac{3}{2}-k+\gamma$ to obtain

$$\mathrm{E}(t)\le Ct^{-\frac{3}{2}-k+2\gamma}, \tag{3.69}$$

for all $t > T$ and $\gamma > \frac{3+2k}{2}$. Lemma 3.3 is proved for all $\gamma > \frac{3+2k}{2}$ and the conclusion for the case of $[0, \frac{3+2k}{2}]$ is proved by Lemma 2.4. Thus, the proof of Theorem 1.1 is complete. □

4. Proof of Theorem 1.4

Combining (1.11) and (1.13) and using direct energy estimate, we prove Theorem 1.4 as follows.

Proof. Taking $(2.1)_2$–$(2.1)_4$, we have

$$(u-v)_t+(1+\alpha_2)(u-v)=G, \tag{4.1}$$

where G is defined by

$$G=F_2-F_4-\nabla m+\alpha_1\nabla\sigma-\bar{\mu}\Delta v-(\bar{\mu}+\bar{\lambda})\nabla\operatorname{div} v.$$

For $0 \leq k \leq \ell-2$, applying ∇^k to (4.1) and then multiplying (4.1) by $|x|^{2\gamma}\nabla^k(u-v)$, integrating them over $\mathbb{R}^3$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^k(u-v)\|_{L^2_\gamma}^2 + (1+\alpha_2)\|\nabla^k(u-v)\|_{L^2_\gamma}^2 \\
&= -\langle\nabla^k(u\cdot\nabla u), |x|^{2\gamma}\nabla^k(u-v)\rangle + \langle\nabla^k(v\cdot\nabla v), |x|^{2\gamma}\nabla^k(u-v)\rangle \\
&\quad + \langle\nabla^k[(\frac{\mu}{n}-\bar{\mu})\Delta v], |x|^{2\gamma}\nabla^k(u-v)\rangle \\
&\quad - \langle\nabla^k\Big[\Big(\alpha_1 - \frac{P'(n)}{n}\Big)\nabla\sigma\Big], |x|^{2\gamma}\nabla^k(u-v)\rangle \\
&\quad - \langle\nabla^k\big[\big(\frac{\mu+\lambda}{n} - (\bar{\mu}+\bar{\lambda})\nabla \operatorname{div} v\big)\big], |x|^{2\gamma}\nabla^k(u-v)\rangle \\
&\quad - \langle\nabla^k\big[\big(\frac{\rho}{n}-\alpha_2\big)(u-v)\big], |x|^{2\gamma}\nabla^k(u-v)\rangle \qquad (4.2)\\
&\quad - \langle\nabla^k[(\operatorname{div} B)B], |x|^{2\gamma}\nabla^k(u-v)\rangle - \langle\nabla^k(B.\nabla B), |x|^{2\gamma}\nabla^k(u-v)\rangle \\
&\quad + \langle\nabla^k\big(\frac{1}{2}\nabla|B|^2\big), |x|^{2\gamma}\nabla^k(u-v)\rangle \\
&\quad - \langle\nabla^k(\nabla m), |x|^{2\gamma}\nabla^k(u-v)\rangle + \alpha_1\langle\nabla^k(\nabla\sigma), |x|^{2\gamma}\nabla^k(u-v)\rangle \\
&\quad - \bar{\mu}\langle\nabla^k\Delta v, |x|^{2\gamma}\nabla^k(u-v)\rangle - (\bar{\mu}+\bar{\lambda})\langle\nabla^k\nabla \operatorname{div} v, |x|^{2\gamma}\nabla^k(u-v)\rangle \\
&= \sum_{j=1}^{13} I_{4,j}.
\end{aligned}
$$

Applying Minkowski's inequality, Hölder's inequality, (1.11), (1.13) and Cauchy's inequality, we have

$$
\begin{aligned}
|I_{4,1}| &\lesssim \sum_{j=0}^{k}\|\nabla^j u\nabla^{k-j+1}u\nabla^k(u-v)\|_{L^1} \\
&\lesssim \sum_{j=0}^{k}\|\nabla^j u\|_{L^\infty}\|\nabla^{k-j+1}u\|_{L^2_\gamma}\|\nabla^k(u-v)\|_{L^2_\gamma} \\
&\lesssim \sum_{j=0}^{k}\|\nabla^{j+1} u\|_{H^1}\|\nabla^{k-j+1}u\|_{L^2_\gamma}\|\nabla^k(u-v)\|_{L^2_\gamma} \qquad (4.3)\\
&\lesssim (1+t)^{-\frac{5}{4}-\frac{k}{2}+\gamma-\frac{5}{4}}\|\nabla^k(u-v)\|_{L^2_\gamma} \\
&\lesssim (1+t)^{-\frac{5}{2}-k+2\gamma} + (1+t)^{-\frac{5}{2}}\|\nabla^k(u-v)\|_{L^2_\gamma}^2.
\end{aligned}
$$

Applying a method similar to the one for (4.3), we have

$$
|I_{4,2}| \lesssim (1+t)^{-\frac{5}{2}-k+2\gamma} + (1+t)^{-\frac{5}{2}}\|\nabla^k(u-v)\|_{L^2_\gamma}^2. \qquad (4.4)
$$

Using Minkowski's inequality, Hölder's inequality, Lemma 2.2, (1.11), (1.13), and Cauchy's inequality, we have

$$
\begin{aligned}
|I_{4,3}| &\lesssim \sum_{j=0}^{k} \|\nabla^j(\frac{\mu}{n}-\bar{\mu})\nabla^{k-j+2}u\nabla^k(u-v)\|_{L^1} \\
&\lesssim \sum_{j=0}^{k} \|\nabla^j(\frac{\mu}{n}-\bar{\mu})\|_{L^\infty}\|\nabla^{k-j+2}u\|_{L^2_\gamma}\|\nabla^k(u-v)\|_{L^2_\gamma} \\
&\lesssim \sum_{j=0}^{k} \|\nabla^j\sigma\|_{L^\infty}\|\nabla^{k-j+2}u\|_{L^2_\gamma}\|\nabla^k(u-v)\|_{L^2_\gamma} \\
&\lesssim \sum_{j=0}^{k} \|\nabla^{j+1}\sigma\|_{H^1}\|\nabla^{k-j+2}u\|_{L^2_\gamma}\|\nabla^k(u-v)\|_{L^2_\gamma} \\
&\lesssim (1+t)^{-\frac{5}{4}-\frac{k}{2}+\gamma-\frac{7}{4}}\|\nabla^k(u-v)\|_{L^2_\gamma} \\
&\lesssim (1+t)^{-\frac{5}{2}-k+2\gamma}+(1+t)^{-\frac{7}{2}}\|\nabla^k(u-v)\|^2_{L^2_\gamma}.
\end{aligned}
\tag{4.5}
$$

Using the same method as (4.5), we obtain

$$
|I_{4,4}| \lesssim (1+t)^{-\frac{5}{2}-k+2\gamma}+(1+t)^{-\frac{5}{2}}\|\nabla^k(u-v)\|^2_{L^2_\gamma}, \tag{4.6}
$$

$$
|I_{4,5}| \lesssim (1+t)^{-\frac{5}{2}-k+2\gamma}+(1+t)^{-\frac{7}{2}}\|\nabla^k(u-v)\|^2_{L^2_\gamma}. \tag{4.7}
$$

Applying Minkowski's inequality, Hölder's inequality, Lemma 2.2, Triangle inequality, (1.11), (1.13), and Cauchy's inequality, we have

$$
\begin{aligned}
|I_{4,6}| &\lesssim \sum_{j=0}^{k} \|\nabla^j(\frac{\rho}{n}-\alpha_2)\nabla^{k-j}(u-v)\nabla^k(u-v)\|_{L^1} \\
&\lesssim \sum_{j=0}^{k} \|\nabla^j(\frac{\rho}{n}-\alpha_2)\|_{L^\infty}\|\nabla^{k-j}(u-v)\|_{L^2_\gamma}\|\nabla^k(u-v)\|_{L^2_\gamma} \\
&\lesssim \sum_{j=0}^{k} \|\nabla^j(m,\sigma)\|_{L^\infty}\|\nabla^{k-j}(u,v)\|_{L^2_\gamma}\|\nabla^k(u-v)\|_{L^2_\gamma} \\
&\lesssim \sum_{j=0}^{k} \|\nabla^{j+1}(m,\sigma)\|_{H^1}\|\nabla^{k-j}(u,v)\|_{L^2_\gamma}\|\nabla^k(u-v)\|_{L^2_\gamma} \\
&\lesssim (1+t)^{-\frac{5}{4}-\frac{k}{2}+\gamma-\frac{3}{4}}\|\nabla^k(u-v)\|_{L^2_\gamma} \\
&\lesssim (1+t)^{-\frac{5}{2}-k+2\gamma}+(1+t)^{-\frac{3}{2}}\|\nabla^k(u-v)\|^2_{L^2_\gamma}.
\end{aligned}
$$

Applying a method similar to the one for (4.3), we have

$$
\sum_{j=7}^{9}|I_{4,j}| \lesssim (1+t)^{-\frac{5}{2}-k+2\gamma}+(1+t)^{-\frac{5}{2}}\|\nabla^k(u-v)\|^2_{L^2_\gamma}. \tag{4.8}
$$

Using Hölder's inequality, (1.13) and Cauchy's inequality, we have

$$\begin{aligned}|I_{4.10}| &\lesssim \|\nabla^{k+1} m\|_{L^2_\gamma} \|\nabla^k (u-v)\|_{L^2_\gamma} \\ &\lesssim (1+t)^{-\frac{5}{4}-\frac{k}{2}+\gamma} \|\nabla^k (u-v)\|_{L^2_\gamma} \\ &\le C(\epsilon)(1+t)^{-\frac{5}{2}-k+2\gamma} + \epsilon \|\nabla^k (u-v)\|^2_{L^2_\gamma}.\end{aligned} \tag{4.9}$$

Applying a method similar to one for (4.9), one has

$$|I_{4.11}| \le C(\epsilon)(1+t)^{-\frac{5}{2}-k+2\gamma} + \epsilon \|\nabla^k (u-v)\|^2_{L^2_\gamma}. \tag{4.10}$$

Using Hölder's inequality, (1.11) and (1.13), we have

$$\begin{aligned}|I_{4,12}| + |I_{4,13}| &\lesssim \|\nabla^{k+2} v\|_{L^2_\gamma} \|\nabla^k (u-v)\|_{L^2_\gamma} \\ &\lesssim t^{-\frac{7}{4}-\frac{k}{2}+\gamma} \|\nabla^k (u-v)\|_{L^2_\gamma} \\ &\lesssim t^{-\frac{5}{2}-k+2\gamma} + t^{-1} \|\nabla^k (u-v)\|^2_{L^2_\gamma}.\end{aligned} \tag{4.11}$$

Substituting (4.3)–(4.11) into (4.2), and noting that ϵ is small enough, then there exists large enough T such that

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\nabla^k (u-v)\|^2_{L^2_\gamma} + C' \|\nabla^k (u-v)\|^2_{L^2_\gamma} \lesssim (1+t)^{-\frac{5}{2}-k+2\gamma},$$

for all $t > T$, where C' is a positive constant independent of t. Using Lemma 2.6 (Gronwall's inequality of differential form), we have

$$\|\nabla^k (u-v)\|_{L^2_\gamma} \lesssim (1+t)^{-\frac{5}{4}-\frac{k}{2}+\gamma},$$

for all $t > T$, $0 \le k \le \ell - 2$ and $\gamma \ge 0$. Thus, the proof is complete. □

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