Electronic Journal of Differential Equations, Vol. 2023 (2023), No. 46, pp. 1–15.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: https://doi.org/10.58997/ejde.2023.46

EXISTENCE AND MULTIPLICITY OF SOLUTIONS TO A FRACTIONAL p-LAPLACIAN ELLIPTIC DIRICHLET PROBLEM

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Abstract. In this article, the authors consider a fractional p-Laplacian elliptic Dirichlet problem. Using critical point theory and the variational method, they investigate the existence of at least one, two, and three solutions to the problem. Examples illustrating the results are interspaced in the paper.

1. Introduction

In this article, we examine the nonlinear elliptic equation involving the fractional p-Laplacian and depending on a real parameter $\lambda > 0$,

$$\begin{gathered} (-\Delta)_p^s u = \lambda f(x,u) + h(u), \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \mathbb{R}^N \backslash \Omega, \end{gathered} \tag{1.1}$$

where $sp < N$, Ω is a bounded open subset of $\mathbb{R}^N$ with a Lipschitz boundary, the fractional p-Laplacian operator $(-\Delta)_p^s$ is defined by

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \backslash B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} dy, \quad x \in \mathbb{R}^N.$$

Here, $0 < s < 1 < p < +\infty$, $B_\varepsilon(x) = \{y \in \mathbb{R}^N : |x-y| < \varepsilon\}$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies a Carathédory condition, and $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function of order $p-1$ with a Lipschitz constant $L > 0$, i.e.,

$$|h(\xi_1) - h(\xi_2)| \le L|\xi_1 - \xi_2|^{p-1} \quad \text{for all } \xi_1, \xi_2 \in \mathbb{R}, \tag{1.2}$$

and such that $h(0) = 0$.

In recent years, a great deal of attention has been focused on the study of fractional and nonlocal operators of elliptic type, for both pure mathematical research and concrete real-world applications. Fractional and nonlocal operators appear in many fields such as optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, multiple scattering, minimal surfaces, materials science, water waves, and Lévy processes; see, e.g., [2, 8, 14, 17] and the references therein.

2020 *Mathematics Subject Classification.* 35R11, 35A15, 35J60, 35B38.
Key words and phrases. Fractional p-Laplacian; weak solution; critical points; variational method.

Submitted April 13, 2023. Published July 3, 2023.

This is one of the reasons why nonlocal fractional problems are widely studied in the literature in many different contexts.

The application of a mountain pass theorem to Dirichlet problems involving nonlocal integro-differential operators of fractional Laplacian type are given in [19, 20]. Wei and Su [21] showed that the fractional Laplacian problem possesses infinitely many weak solutions. Lehrer et al. [15] investigated the existence of nonnegative solutions to problem (1.1) in the case $h \equiv 0$. Their problem is set on a unbounded domain and compactness issues have to be handled. Iannizzotto et al. [12] studied existence and multiplicity results for fractional p-Laplacian type problems via Morse theory. Kim [13] applied abstract critical point results to establish an estimate of a positive interval for the parameter λ within which the problem (1.1) with $h \equiv 0$ admits at least one or two nontrivial weak solutions provided the nonlinearity f satisfies a subcritical growth condition. In addition, under certain conditions, he established an *a priori* estimate in $L^\infty(\Omega)$ for any possible weak solution by applying a bootstrap argument.

In this paper we obtain three different results about the existence of weak solutions to the problem (1.1) by using critical point theorems established in [4, 5, 7].

The first aim of this paper is to provide an estimate of the positive interval for the parameter λ in which the problem (1.1) possesses at least one nontrivial weak solution in the case where the nonlinear term f satisfies a subcritical growth condition. We also wish to consider the existence of two solutions to our problem by using a result of Bonanno [5, Theorem 3.2]. In a recent paper, Bonanno and Chinnì [6] studied the existence of at least two distinct weak solutions to a problem involving a $p(x)$-Laplacian by applying critical point theory. Our first main result will require the $(\text{P.S.})^{[r]}$ condition, while in our second one, we will ask that the (AR)-condition holds and use it to ensure that the (usual) (PS)-condition is satisfied. We refer the reader to the papers [3, 6, 11] where this approach was applied successfully.

Finally, our third goal is to obtain the existence of three solutions to (1.1); this problem is less studied by researchers. In this case, we consider problem (1.1) where the nonlinearity f has subcritical growth, and we apply variational methods and critical point theory. The main tool used is the critical point theorem of Bonanno and Marano [7, Theorem 3.6].

The remainder of this paper is organized as follows. First, in Section 2, we recall briefly some basic results for fractional Sobolev spaces. In Section 3, we obtain the existence of at least one, two, or three nontrivial weak solutions to the problem (1.1) provided the parameter λ belongs to a positive interval to be determined.

2. Preliminaries

This section is devoted to the definition of the fractional Sobolev spaces and related properties that will be used in the next section.

For $s \in (0,1)$ and $p \in (1,+\infty)$, the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is defined as

$$W^{s,p}(\mathbb{R}^N) := \Big\{u \in L^p(\mathbb{R}^N) : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy < +\infty\Big\},$$

which is an interpolation Banach space between $L^p(\mathbb{R}^N)$ and $W^{1,p}(\mathbb{R}^N)$. The norm for this space is

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} := (\|u\|^p_{L^p(\mathbb{R}^N)} + |u|^p_{W^{s,p}(\mathbb{R}^N)})^{1/p},$$

where

$$\|u\|_{L^p(\mathbb{R}^N)}^p := \int_{\mathbb{R}^N} |u|^p \, dx \quad \text{and} \quad |u|_{W^{s,p}(\mathbb{R}^N)}^p := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \, dx \, dy.$$

It is known (see [1]) that $W^{s,p}(\mathbb{R}^N)$ is a separable and reflexive Banach space and that $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{s,p}(\mathbb{R}^N)$, i.e., $W_0^{s,p}(\mathbb{R}^N) = W^{s,p}(\mathbb{R}^N)$.

For our problem we consider the subspace of $W^{s,p}(\mathbb{R}^N)$ given by

$$X_s^p(\Omega) = \{u \in W^{s,p}(\mathbb{R}^N) : u(x) = 0 \;\text{ a.e. in }\; \mathbb{R}^N \backslash \Omega\}$$

with the norm

$$\|u\|_{X_s^p(\Omega)} := (\|u\|_{L^p(\Omega)}^p + |u|_{W^{s,p}(\mathbb{R}^N)}^p)^{1/p}, \tag{2.1}$$

which is known to be a uniformly convex Banach space (see [22, Lemma 2.4]). We will need the following lemmas to prove our main theorems.

Lemma 2.1 ([13, Lemma 2.1]). *Let Ω be a bounded open set in $\mathbb{R}^N$, $s \in (0,1)$, and $p \in [1,+\infty)$. Then*

$$\|u\|_{L^p(\Omega)}^p \le \frac{sp|\Omega|^{sp/N}}{2\omega_N^{\frac{sp}{N}+1}} |u|_{W^{s,p}(\mathbb{R}^N)}^p$$

for any $u \in \tilde{W}^{s,p}(\mathbb{R}^N)$. Here, $|\Omega|$ is the Lebesgue measure of Ω, ω_N denotes the volume of the N-dimensional unit ball, and $\tilde{W}^{s,p}(\mathbb{R}^N)$ is the space of all $u \in X_s^p(\Omega)$ such that $\tilde{u} \in W^{s,p}(\mathbb{R}^N)$, where $\tilde{u}$ is the extension by zero of u.

Remark 2.2. In view of Lemma 2.1, it is clear from (2.1) that there is an equivalence between the norms in $W^{s,p}(\mathbb{R}^N)$ and $X_s^p(\Omega)$.

Lemma 2.3 ([16]). *Let $s \in (0,1)$ and $p \in [1,+\infty)$ be such that $sp < N$. Then, for any $u \in W^{s,p}(\mathbb{R}^N)$,*

$$\|u\|_{L^{p_s^*}(\Omega)}^p \le C_{p_s^*} |u|_{W^{s,p}(\mathbb{R}^N)}^p,$$

where

$$C_{p_s^*} = \frac{(N+2p)^{3p} p^{p+2} 2^{(N+1)(N+2)} s(1-s)}{N^{\frac{p}{p_s^*}} |S^{N-1}|^{\frac{sp}{N}+1} (N-sp)^{p-1}}.$$

Here, $|S^{N-1}|$ denotes the surface area of the $(N-1)$-dimensional unit sphere and p_s^* is the fractional critical Sobolev exponent, that is, $p_s^* = \frac{pN}{N-sp}$.

Remark 2.4. Recall that for each $s \in (0,1)$ and $p \in [1,+\infty)$ such that $sp < N$, from [9, Theorem 4.54], we have the continuous embedding

$$X_s^p(\Omega) \hookrightarrow L^q(\Omega) \quad \text{for all } q \in [1, p_s^*].$$

In particular, the space $X_s^p(\Omega)$ is compactly embedded in $L^q(\Omega)$ for any $q \in [1, p_s^*)$. In fact, according to Lemma 2.3, for each $u \in X_s^p(\Omega)$, there exists $C_q > 0$ such that

$$\|u\|_{L^q(\Omega)} \le C_q^{1/p} |u|_{W^{s,p}(\mathbb{R}^N)}.$$

The constant C_q is important in obtaining an interval on λ in which (1.1) has one or more nontrivial weak solutions.

Definition 2.5 ([4, p. 2993], [5, p. 210]). Let Φ and Ψ be two continuously Gâteaux differentiable functionals defined on a real Banach space X and fix $r \in \mathbb{R}$. The functional $I = \Phi - \Psi$ is said to satisfy the Palais-Smale condition cut off upper at r, denoted by (P.S.)$^{[r]}$ if any sequence $\{u_n\}_{n\in\mathbb{N}}$ in X such that

(1) $\{I(u_n)\}$ is bounded,
(2) $\lim_{n\to\infty} \|I'(u_n)\|_{X^*} = 0$, and
(3) $\Phi(u_n) < r$ for each $n \in \mathbb{N}$,

has a convergent subsequence.

If only conditions (1) and (2) hold, then $I = \Phi - \Psi$ is said to satisfy the (usual) Palais-Smale (P.S.) condition.

We next wish to define what is meant by a weak solution of our problem.

Definition 2.6. Let $0 < s < 1 < p < +\infty$. We say that $u \in X_s^p(\Omega)$ is a weak solution of problem (1.1) if

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}}\,dx\,dy\\&= \lambda\int_\Omega f(x,u)v\,dx + \int_\Omega h(u)v\,dx\end{aligned}$$

for all $v \in X_s^p(\Omega)$.

We define $\Phi : X_s^p(\Omega) \to \mathbb{R}$ by

$$\Phi(u) := \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy - \int_\Omega H(u)\,dx \quad \text{for all } u \in X_s^p(\Omega), \tag{2.2}$$

where $H(t) = \int_0^t h(\xi)d\xi$ for $t \in \mathbb{R}$. The functional Φ is Fréchet differentiable and its Fréchet derivative is given by

$$\begin{aligned}\langle \Phi'(u), v\rangle &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}}\,dx\,dy\\&\quad - \int_\Omega h(u)v\,dx\end{aligned}$$

for any $v \in X_s^p(\Omega)$.

We will need the condition

(H1) there exist nonnegative functions α, $\beta \in L^\infty(\Omega)$ such that
$$|f(x,t)| \le \alpha(x) + \beta(x)|t|^{q-1} \quad \text{for all } (x,t) \in \Omega\times\mathbb{R},$$
where $1 < q < p_s^*$.

Define the function $F : \Omega\times\mathbb{R} \to \Omega\times\mathbb{R}$ by

$$F(x,t) = \int_0^t f(x,\xi)d\xi \quad \text{for } (x,\xi) \in \Omega\times\mathbb{R}$$

and the functionals Ψ, $I_\lambda : X_s^p(\Omega) \to \mathbb{R}$ by

$$\begin{gathered}\Psi(u) := \int_\Omega F(x,u)\,dx,\\ I_\lambda(u) = \Phi(u) - \lambda\Psi(u)\end{gathered} \tag{2.3}$$

for all $u \in X_s^p(\Omega)$. In what follows, we will assume that the Lipschitz constant $L > 0$ belonging to the function h in (1.2) satisfies

$$L < \frac{2^{1-p}\omega_N^{\frac{sp}{N}+1}}{ps|\Omega|^{ps/N}}, \tag{2.4}$$

from which we see that

$$0 < L < \frac{2^{1-p}\omega_N^{\frac{sp}{N}+1}}{ps|\Omega|^{ps/N}} < \frac{2\omega_N^{\frac{sp}{N}+1}}{ps|\Omega|^{ps/N}}. \tag{2.5}$$

3. Main results

We begin by presenting a result that guarantees the existence of at least one solution to problem (1.1). We will need the constant

$$\mu := \Big[\frac{2^{2p+N-sp}}{(p-sp)(N+p-sp)} + \frac{2^{1+sp}}{sp(p-sp+1)} + \frac{1}{sp(N-sp)}\Big]\omega_N^2 N^2.$$

Theorem 3.1. *Let $p \geq 2$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying* (H1)*, and assume that there exist three real positive constants τ, ρ, and δ such that:*

(H2)
$$2\omega_N^2 \rho^{N-sp}\delta^p \mu \frac{2\omega_N^{\frac{sp}{N}+1} + Lsp|\Omega|^{sp/N}}{2\omega_N^{\frac{sp}{N}+1} - Lsp|\Omega|^{sp/N}} < \tau^p;$$

(H3)
$$\begin{aligned}&\frac{\omega_N\Big(\|\alpha\|_\infty|\Omega|^{1/p'}\Big(\frac{ps|\Omega|^{sp/N}}{2\omega_N^{\frac{sp}{N}+1}}\Big)^{1/p}\tau + q^{-1}C_q^{q/p}\|\beta\|_\infty\tau^q\Big)}{\Big(2\omega_N^{\frac{sp}{N}+1} - Lsp|\Omega|^{sp/N}\Big)\tau^p}\\ &< \frac{\rho^{sp}\operatorname{ess\,inf}_{x\in\Omega} F(x,\delta)}{2^{N+1}\delta^p\mu\Big(2\omega_N^{\frac{sp}{N}+1} + Lsp|\Omega|^{sp/N}\Big)},\end{aligned}$$

where $1/p + 1/p' = 1$;

(H4) *$F(x,t) \geq 0$ for each $(x,t) \in \Omega \times \mathbb{R}^+$.*

Then, for each

$$\begin{aligned}\lambda \in \Lambda_w := \Big(&\frac{2^N(2\omega_N^{\frac{sp}{N}+1} + Lsp|\Omega|^{sp/N})\delta^p\mu}{p\rho^{sp}\omega_N^{sp/N}\operatorname{ess\,inf}_{x\in\Omega} F(x,\delta)},\\ &\frac{(2\omega_N^{\frac{sp}{N}+1} - Lsp|\Omega|^{sp/N})\tau^p}{2p\omega_N^{\frac{sp}{N}+1}(\|\alpha\|_\infty|\Omega|^{1/p'}(\frac{ps|\Omega|^{sp/N}}{2\omega_N^{\frac{sp}{N}+1}})^{1/p}\tau + q^{-1}C_q^{q/p}\|\beta\|_\infty\tau^q)}\Big),\end{aligned} \tag{3.1}$$

problem (1.1) *admits at least one nontrivial solution $u_\lambda \in X_s^p(\Omega)$.*

Proof. Our goal is to apply [5, Theorem 2.3] to problem (1.1). To this end, we take the real Banach space $X_s^p(\Omega)$ with the norm as defined in Section 2, and Φ and Ψ to be the functionals defined in (2.2) and (2.3). Taking into account that h is a Lipschitz continuous function of order $p-1$ with Lipschitz constant (see (2.4))

$$0 < L < \frac{2^{1-p}\omega_N^{\frac{sp}{N}+1}}{ps|\Omega|^{ps/N}}$$

and $h(0) = 0$, we have

$$\frac{1}{p}|u|^p_{W^{s,p}(\mathbb{R}^N)} - \frac{L}{p}\|u\|^p_{L^p(\Omega)} \leq \Phi(u) \leq \frac{1}{p}|u|^p_{W^{s,p}(\mathbb{R}^N)} + \frac{L}{p}\|u\|^p_{L^p(\Omega)},$$

namely,

$$\begin{aligned} \frac{2\omega_N^{\frac{sp}{N}+1} - Lps|\Omega|^{sp/N}}{2p\omega_N^{\frac{sp}{N}+1}} |u|^p_{W^{s,p}(\mathbb{R}^N)} &\le \Phi(u) \\ &\le \frac{2\omega_N^{\frac{sp}{N}+1} + Lps|\Omega|^{sp/N}}{2p\omega_N^{\frac{sp}{N}+1}} |u|^p_{W^{s,p}(\mathbb{R}^N)}. \end{aligned} \tag{3.2}$$

From the first inequality in (3.2), it follows that Φ is coercive. It is also clear that $\Phi \in C^1(X_s^p(\Omega), \mathbb{R})$. To show that Φ' admits a continuous inverse, in view of [23, Theorem 26.A(d)], it suffices to show that Φ' is coercive, hemicontinuous, and uniformly monotone.

By Lemma 2.1, it is clear that for any $u \in X_s^p(\Omega)$, we have

$$\begin{aligned} &\frac{\langle \Phi'(u), u\rangle}{\|u\|_{X_s^p(\Omega)}} \\ &\ge \frac{1}{\|u\|_{X_s^p(\Omega)}} \Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))^2}{|x-y|^{N+sp}}\,dx\,dy - \int_\Omega h(u)u\,dx \Big) \\ &\ge \frac{2\omega_N^{\frac{sp}{N}+1}}{(2\omega_N^{\frac{sp}{N}+1} + sp|\Omega|^{sp/N})|u|_{W^{s,p}(\mathbb{R}^N)}} \big(|u|^p_{W^{s,p}(\mathbb{R}^N)} - L\|u\|^p_{L^p(\Omega)} \big) \\ &\ge \frac{2\omega_N^{\frac{sp}{N}+1} - Lsp|\Omega|^{sp/N}}{2\omega_N^{\frac{sp}{N}+1} + sp|\Omega|^{sp/N}} |u|^{p-1}_{W^{s,p}(\mathbb{R}^N)}. \end{aligned}$$

Since $L < \frac{2\omega_N^{\frac{sp}{N}+1}}{ps|\Omega|^{ps/N}}$, this implies

$$\lim_{\|u\|_{X_s^p(\Omega)}\to\infty} \frac{\langle \Phi'(u), u\rangle}{\|u\|_{X_s^p(\Omega)}} = \infty,$$

i.e., Φ' is coercive. The fact that Φ' is hemicontinuous can be shown using standard arguments (see, for example, [18]).

Finally, we show that Φ' is uniformly monotone. First recall the inequality that for any $\xi, \psi \in \mathbb{R}$,

$$(|\xi|^{r-2}\xi - |\psi|^{r-2}\psi)(\xi-\psi) \ge 2^{-r}|\xi-\psi|^r, \quad \text{if } r \ge 2. \tag{3.3}$$

In view of (2.4) and Lemma 2.1, for every $u, v \in X_s^p(\Omega)$, there exists a positive constant k_1 such that

$$\begin{aligned} &\langle \Phi'(u) - \Phi'(v), u-v\rangle \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\Big(|u(x)-u(y)|^{p-2}(u(x)-u(y)) - |v(x)-v(y)|^{p-2}(v(x)-v(y))\Big)}{|x-y|^{N+sp}} \\ &\quad \times ((u-v)(x) - (u-v)(y))\,dx\,dy - \int_\Omega (h(u)-h(v))(u-v)\,dx \\ &\ge 2^{-p}|u-v|^p_{W^{s,p}(\mathbb{R}^N)} - L\|u-v\|^p_{L^p(\Omega)} \\ &\ge \Big(2^{-p} - \frac{Lps|\Omega|^{ps/N}}{2\omega_N^{\frac{sp}{N}+1}}\Big)|u-v|^p_{W^{s,p}(\mathbb{R}^N)} \\ &\ge k_1\|u-v\|^p_{X_s^p(\Omega)}. \end{aligned}$$

From condition (H1) and Remark 2.4, the functional Ψ belongs to $C^1(X_s^p(\Omega),\mathbb{R})$ and has a compact derivative. This ensures that the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies $(P.S.)^{[r]}$ for each $r > 0$ (see [4, Proposition 2.1]).

To apply [5, Theorem 2.] to the functional I_λ, first note that $\inf_{X_s^p(\Omega)} \Phi = \Phi(0) = \Psi(0) = 0$. We need to show that there is an $r > 0$ and $\bar{v} \in X_s^p(\Omega)$ with $0 < \Phi(\bar{v}) < r$ such that $\frac{\sup_{\Phi(u)\le r}\Psi(u)}{r} < \frac{\Psi(\bar{v})}{\Phi(\bar{v})}$. To this end, set

$$r := \frac{2\omega_N^{\frac{sp}{N}+1} - Lps|\Omega|^{sp/N}}{2p\omega_N^{\frac{sp}{N}+1}}\tau^p$$

with L satisfying (2.5), and define w by

$$w(x) = \begin{cases} 0, & \text{if } x \in \mathbb{R}^N \setminus B_N(0,\rho), \\ \delta, & \text{if } x \in B_N(0,\frac{\rho}{2}), \\ \frac{2\delta}{\rho}(\rho - |x|), & \text{if } x \in B_N(0,\rho)\setminus B_N(0,\frac{\rho}{2}). \end{cases} \tag{3.4}$$

We take $B_\rho = B_N(0,\rho)$; then

$$\begin{aligned}
\Phi(w) &\le \frac{2\omega_N^{\frac{sp}{N}+1} + Lps|\Omega|^{sp/N}}{2p\omega_N^{\frac{sp}{N}+1}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \\
&= \frac{2\omega_N^{\frac{sp}{N}+1} + Lps|\Omega|^{sp/N}}{2p\omega_N^{\frac{sp}{N}+1}} \Big(\int_{B_\rho\setminus B_{\rho/2}}\int_{B_\rho\setminus B_{\rho/2}} \frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \\
&\quad + 2\int_{B_\rho\setminus B_{\rho/2}}\int_{\mathbb{R}^N\setminus B_\rho} \frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \\
&\quad + 2\int_{B_{\rho/2}}\int_{B_\rho\setminus B_{\rho/2}} \frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \\
&\quad + 2\int_{\mathbb{R}^N\setminus B_\rho}\int_{B_{\rho/2}} \frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \Big) \\
&= \frac{2\omega_N^{\frac{sp}{N}+1} + Lps|\Omega|^{sp/N}}{2p\omega_N^{\frac{sp}{N}+1}} (R_1 + 2R_2 + 2R_3 + 2R_4).
\end{aligned}$$

Next, we estimate R_1, R_2, R_3, and R_4 by direct calculations. Recall that if g is a continuous radial function (i.e., $g(x) = \tilde{g}(|x|)$) on a closed ball B_γ of radius γ, then

$$\int_{B_\gamma} g(x)\,dx = N\omega_N \int_0^\gamma \tilde{g}(r) r^{N-1} dr.$$

We then have

$$\begin{aligned}
R_1 &= \int_{B_\rho\setminus B_{\rho/2}}\int_{B_\rho\setminus B_{\rho/2}} \frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \\
&\le \frac{2^p\delta^p}{\rho^p} \int_{B_\rho\setminus B_{\rho/2}}\int_{B_\rho\setminus B_{\rho/2}} \frac{|x-y|^p}{|x-y|^{N+sp}}\,dx\,dy \\
&\le \frac{2^p\delta^p\omega_N N}{\rho^p} \int_{B_\rho\setminus B_{\rho/2}}\int_0^{\rho+|y|} r^{p-sp-1} dr dy \\
&\le \frac{2^p\delta^p\omega_N N}{\rho^p} \int_{B_\rho\setminus B_{\rho/2}} \frac{(\rho+|y|)^{p-sp}}{p-sp} dy
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2^p\delta^p\omega_N^2N^2}{(p-sp)\rho^p}\int_{\frac{3}{2}\rho}^{2\rho} r^{p+N-sp-1}dr \\
&= \frac{2^p\delta^p\omega_N^2\rho^{N-sp}N^2}{(p-sp)(p+N-sp)}\Big(2^{p+N-sp}-(\frac{3}{2})^{p+N-sp}\Big).
\end{aligned}
$$

$$
\begin{aligned}
R_2 &= \int_{B_\rho\setminus B_{\rho/2}}\int_{\mathbb{R}^N\setminus B_\rho} \frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \\
&= \frac{2^p\delta^p}{\rho^p}\int_{B_\rho\setminus B_{\rho/2}}\int_{\mathbb{R}^N\setminus B_\rho} \frac{|\rho-|y||^p}{|x-y|^{N+sp}}\,dx\,dy \\
&= \frac{2^p\delta^p\omega_N N}{\rho^p}\int_{B_\rho\setminus B_{\rho/2}}\int_{\rho-|y|}^{+\infty} \frac{|\rho-|y||^p}{r^{sp+1}}dr dy \\
&= \frac{2^p\delta^p\omega_N N}{\rho^p sp}\int_{B_\rho\setminus B_{\rho/2}} |\rho-|y||^{p-sp}dy \\
&= \frac{2^p\delta^p\omega_N^2 N^2}{\rho^p sp}\int_0^{\frac{\rho}{2}} r^{p-sp}(\rho-r)^{N-1}dr \\
&\le \frac{\delta^p\rho^{N-sp}\omega_N^2N^2}{2^{1-sp}sp(p-sp+1)}.
\end{aligned}
$$

$$
\begin{aligned}
R_3 &= \int_{B_{\rho/2}}\int_{B_\rho\setminus B_{\rho/2}} \frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \\
&= \frac{2^p\delta^p}{\rho^2}\int_{B_{\rho/2}}\int_{B_\rho\setminus B_{\rho/2}} \frac{|-\frac{\rho}{2}+|x||^p}{|x-y|^{N+sp}}\,dx\,dy \\
&= \frac{2^p\delta^p}{\rho^p}\int_{B_\rho\setminus B_{\rho/2}}\int_{B_{\rho/2}} \frac{|-\frac{\rho}{2}+|x||^p}{|x-y|^{N+sp}}dy\,dx \\
&= \frac{2^p\delta^p\omega_N N}{\rho^p}\int_{B_\rho\setminus B_{\rho/2}} \Big|-\frac{\rho}{2}+|x|\Big|^p\int_{|x|-\frac{\rho}{2}}^{|x|+\frac{\rho}{2}} \frac{1}{r^{sp+1}}dr\,dx \\
&\le \frac{2^p\delta^p\omega_N N}{\rho^p sp}\int_{B_\rho\setminus B_{\rho/2}} \Big|-\frac{\rho}{2}+|x|\Big|^{p-sp}\,dx \\
&= \frac{2^p\delta^p\omega_N^2 N^2}{\rho^p sp}\int_0^{\frac{\rho}{2}} r^{p-sp}\Big(r+\frac{\rho}{2}\Big)^{N-1}\,dr \\
&\le \frac{\rho^{N-sp}\delta^p\omega_N^2N^2}{2^{1-sp}sp(p-sp+1)}.
\end{aligned}
$$

$$
\begin{aligned}
R_4 &= \int_{B_{\rho/2}}\int_{\mathbb{R}^N\setminus B_\rho} \frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \\
&= \delta^p\int_{B_{\rho/2}}\int_{\mathbb{R}^N\setminus B_\rho} \frac{1}{|x-y|^{N+sp}}\,dx\,dy \\
&= \delta^p\omega_N N\int_{B_{\rho/2}}\int_{\rho-|y|}^{\infty} r^{-sp-1}drdy \\
&= \delta^p\omega_N N\int_{B_{\rho/2}} \frac{1}{sp(\rho-|y|)^{sp}}dy
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\delta^p \omega_N^2 N^2}{sp} \int_{\rho/2}^{\rho} r^{N-sp-1} dr \\
&= \frac{\delta^p \omega_N^2 N^2 \rho^{N-sp}}{sp(N-sp)} \Big(1 - \frac{1}{2^{N-sp}}\Big) \\
&\le \frac{\delta^p \omega_N^2 N^2 \rho^{N-sp}}{sp(N-sp)}.
\end{aligned}
$$

Then, we have $w \in X_s^p(\Omega)$ and

$$
\Phi(w) \le \frac{2\omega_N^{\frac{sp}{N}+1} + Lsp|\Omega|^{sp/N}}{p\omega_N^{\frac{sp}{N}-1}} \delta^p \rho^{N-sp} \mu. \tag{3.5}
$$

Hence, it follows from (H2) that $0 < \Phi(w) < r$. From (H4), we have

$$
\Psi(w) \ge \int_{B_{\rho/2}} F(x, w)\, dx \ge \operatorname{ess\,inf}_{x\in\Omega} F(x, \delta) \frac{\omega_N \rho^N}{2^N}. \tag{3.6}
$$

By (3.2), the estimate $\Phi(u) \le r$ implies that $|u|^p_{W^{s,p}(\mathbb{R}^N)} \le \tau^p$. From Lemma 2.1, for every $u \in \Phi^{-1}(-\infty, r]$ we have

$$
\|u\|_{L^p(\Omega)} \le \Big(\frac{ps|\Omega|^{sp/N}}{2\omega_N^{\frac{sp}{N}+1}}\Big)^{1/p} \tau.
$$

Hence, condition (H1), Hölder's inequality, and the content of Remark 2.4 imply that, for each $u \in \Phi^{-1}(-\infty, r]$,

$$
\begin{aligned}
\Psi(u) &= \int_\Omega F(x, u) \le \int_\Omega |\alpha(x)||u(x)|\, dx + q^{-1} \int_\Omega |\beta(x)||u(x)|^q\, dx \quad (3.7)\\
&\le \|\alpha\|_\infty |\Omega|^{1/p'} \|u\|_{L^p(\Omega)} + q^{-1}\|\beta\|_\infty \|u\|^q_{L^q(\Omega)} \quad (3.8)\\
&\le \|\alpha\|_\infty |\Omega|^{1/p'} \Big(\frac{ps|\Omega|^{sp/N}}{2\omega_N^{\frac{sp}{N}+1}}\Big)^{1/p} \tau + q^{-1} C_q^{q/p} \|\beta\|_\infty \tau^q.
\end{aligned}
$$

In view of (3.5), (3.6), the above inequality, and (H3), we obtain

$$
\begin{aligned}
\frac{\sup_{u\in\Phi^{-1}(-\infty,r]} \Psi(u)}{r} &\le \frac{\|\alpha\|_\infty |\Omega|^{1/p'} \Big(\frac{ps|\Omega|^{sp/N}}{2\omega_N^{\frac{sp}{N}+1}}\Big)^{1/p} \tau + q^{-1} C_q^{q/p} \|\beta\|_\infty \tau^q}{r} \\
&< \frac{\operatorname{ess\,inf}_{x\in\Omega} F(x, \delta) \frac{\omega_N \rho^N}{2^N}}{r} \le \frac{\Psi(w)}{\Phi(w)},
\end{aligned} \tag{3.9}
$$

which means that $\frac{\sup_{\Phi(u)\le r} \Psi(u)}{r} < \frac{\Psi(\bar v)}{\Phi(\bar v)}$ holds for some $\bar v \in X_s^p(\Omega)$. Hence, for each $\lambda \in \Big(\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{\Phi(u)\le r} \Psi(u)}\Big)$ the functional I_λ admits at least one critical point u_λ with

$$
0 < \Phi(u_\lambda) < r,
$$

which in turn is a nontrivial solution of problem (1.1). $\square$

Remark 3.2. Condition (H3) in Theorem 3.1 can be replaced by the less general but more easily verifiable condition

$$
\|\alpha\|_\infty |\Omega|^{1/p'} \Big(\frac{ps|\Omega|^{sp/N}}{2\omega_N^{\frac{sp}{N}+1}}\Big)^{1/p} \tau + q^{-1} C_q^{q/p} \|\beta\|_\infty \tau^q < \frac{\omega_N \rho^N}{2^N} \rho^{sp} \operatorname{ess\,inf}_{x\in\Omega} F(x, \delta).
$$

As an illustration of Theorem 3.1, we have the following example.

Example 3.3. On the domain $\Omega = \{(x_1, x_2) : x_1^2 + x_2^2 < 1\} \subset \mathbb{R}^2$, consider the problem

$$(-\Delta)_2^{1/4} u = \lambda f(x,u) + \sin(u), \quad \text{in } \Omega,$$
$$u = 0, \quad \text{on } \mathbb{R}^N \backslash \Omega.$$

Here we have $N = 2$, $p = 2$, $p' = 2$, and $s = 1/4$. For $t \in \mathbb{R}$, let

$$f(t) = \begin{cases} e^t, & t \le 1, \\ e, & t > 1. \end{cases}$$

From the definition of f, we have

$$F(t) = \begin{cases} e^t - 1, & t \le 1, \\ et - 1, & t > 1. \end{cases}$$

By choosing $\alpha(x) = e$, $\beta(x) = 10^{-13}$, and $q = 2$, we see that the function f satisfies condition (H1). Choosing $\delta = 1$, $\rho = 4$, and $\tau = 280$, simple calculations show that the remaining conditions in Theorem 3.1 also hold. Hence, for every

$$\lambda \in \Big(\frac{12(4\pi + 1)}{e - 1}, \frac{70(4\pi - 1)}{\pi e + 0.169 \times 10^{-2}}\Big),$$

the above problem admits at least one nontrivial weak solution.

Our second aim in this paper is to obtain a result on the existence of two distinct solutions to problem (1.1). The following theorem is obtained by applying [5, Theorem 3.2].

Theorem 3.4. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying* (H1). *Moreover, assume that*

(H5) *(Ambrosetti-Rabinowitz Condition) there exist $\nu > \frac{2^p+1}{2^p-1}p$ and $R > 0$ such that*

$$0 < \nu F(x,t) < t f(x,t) \quad \text{for all } x \in \Omega \text{ and } |t| \ge R.$$

Then, for each

$$\lambda \in \Lambda_r := \Big(0, \frac{(2\omega_N^{\frac{sp}{N}+1} - Lsp|\Omega|^{sp/N})\tau^p}{2p\omega_N^{\frac{sp}{N}+1}(\alpha\|_\infty |\Omega|^{1/p'} (\frac{ps|\Omega|^{sp/N}}{2\omega_N^{\frac{sp}{N}+1}})^{1/p}\tau + q^{-1}C_q^q\|\beta\|_\infty \tau^q)}\Big),$$

problem (1.1) *admits at least two nontrivial solutions.*

Proof. Let Φ and Ψ be the functionals defined in (2.2) and (2.3). Notice that they satisfy all regularity assumptions required in [5, Theorem 3.2]). Arguing as in the proof of Theorem 3.1, choosing

$$r := \frac{2\omega_N^{\frac{sp}{N}+1} - Lps|\Omega|^{sp/N}}{2p\omega_N^{\frac{sp}{N}+1}}\tau^p$$

with L as in (2.5), for each $\lambda \in \Lambda_r$ we obtain

$$\frac{\sup_{u \in \Phi^{-1}(-\infty, r]} \Psi(u)}{r} \le \frac{\|\alpha\|_\infty |\Omega|^{1/p'} (\frac{ps|\Omega|^{sp/N}}{2\omega_N^{\frac{sp}{N}+1}})^{1/p}\tau + q^{-1}C_q^q\|\beta\|_\infty \tau^q}{r} < \frac{1}{\lambda}$$

(see (3.9)). Now, from condition (H5), a straight forward calculation shows that there are positive constants m and C such that

$$F(x,t) \geq m|t|^\nu - C \quad \text{for all } x \in \Omega \text{ and } t \in \mathbb{R}. \tag{3.10}$$

Hence, for every $\lambda \in \Lambda_r$, $u \in X_s^p(\Omega) \setminus \{0\}$ and $t > 1$, we obtain

$$\begin{aligned} I_\lambda(tu(x)) &= \Phi(tu(x)) - \lambda \int_\Omega F(x,tu)\,dx \\ &\leq \frac{2\omega_N^{\frac{sp}{N}+1} + Lps|\Omega|^{sp/N}}{2p\omega_N^{\frac{sp}{N}+1}} t^p \|u\|^p_{X_s^p(\Omega)} - m\lambda t^\nu \int_\Omega |u|^\nu\,dx + \lambda C|\Omega|. \end{aligned}$$

Since $\nu > p$, this condition guarantees that I_λ is unbounded from below. We recall that I_λ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in X_s^p(\Omega)$ is the functional $I'_\lambda(u) \in (X_s^p(\Omega))^*$ given by

$$\begin{aligned} \langle I'_\lambda(u), v\rangle &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}}\,dx\,dy \\ &\quad - \lambda\int_\Omega f(x,u)v\,dx - \int_\Omega h(u)v\,dx, \end{aligned}$$

for every $v \in X_s^p(\Omega)$.

To show that I_λ satisfies the (PS)-condition, let $\{u_n\}_{n\in\mathbb{N}} \subset X_s^p(\Omega)$ be a sequence such that $\{I_\lambda(u_n)\}_{n\in\mathbb{N}}$ is bounded and $I'_\lambda(u_n) \to 0$ in $(X_s^p(\Omega))^*$ as $n \to +\infty$. Then, there exists a positive constant s_0 such that

$$|I_\lambda(u_n)| \leq s_0 \quad \text{and} \quad \|I'_\lambda(u_n)\| \leq s_0 quad \text{for all } n \in \mathbb{N}.$$

Using condition (H5), Lemma 2.1, (2.5), and the definition of I'_λ, we see that for all $n \in \mathbb{N}$, there exists $D > 0$ such that

$$\begin{aligned} \nu s_0 + s_0\|u_n\|_{X_s^p(\Omega)} &\geq \nu I_\lambda(u_n) - \langle I'_\lambda(u_n), u_n\rangle \\ &\geq \frac{\nu}{p}|u_n|^p_{W^{s,p}(\mathbb{R}^N)} - \frac{\nu L}{p}\|u_n\|^p_{L^p(\Omega)} - |u_n|^p_{W^{s,p}(\mathbb{R}^N)} - L\|u_n\|^p_{L^p(\Omega)} \\ &\quad + \lambda\int_\Omega (f(x,u_n)u_n - \nu F(x,u_n))\,dx \\ &\geq \Big(\frac{\nu}{p} - 1\Big)|u_n|^p_{W^{s,p}(\mathbb{R}^N)} - L\Big(\frac{\nu}{p}+1\Big)\|u_n\|^p_{L^p(\Omega)} - D \\ &\geq \Big(\frac{\nu}{p} - 1\Big)|u_n|^p_{W^{s,p}(\mathbb{R}^N)} - L\Big(\frac{\nu}{p}+1\Big)\frac{ps|\Omega|^{sp/N}}{2\omega_N^{\frac{sp}{N}+1}}|u_n|^p_{W^{s,p}(\mathbb{R}^N)} - D \\ &\geq \Big(\Big(\frac{\nu}{p} - 1\Big) - 2^{-p}\Big(\frac{\nu}{p}+1\Big)\Big)|u_n|^p_{W^{s,p}(\mathbb{R}^N)} - D. \end{aligned}$$

Since $\nu > \frac{2^p+1}{2^p-1}p$, the equivalence in Remark 2.2 shows that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded.

Since $X_s^p(\Omega)$ is a reflexive Banach space, we have, up to taking a subsequence if necessary,

$$u_n \rightharpoonup u \;\text{ in } X_s^p(\Omega).$$

By the fact that $I'_\lambda(u_n) \to 0$ and $u_n \rightharpoonup u$ in $X_s^p(\Omega)$, we obtain

$$(I'_\lambda(u_n) - I'_\lambda(u))(u_n - u) \to 0.$$

Furthermore,

$$\int_{\Omega}(f(x,u_n)-f(x,u))(u_n-u)\,dx \to 0 \text{ as } n\to+\infty,$$

$$\int_{\Omega}(h(u_n)-h(u))(u_n-u)\,dx \to 0 \text{ as } n\to+\infty.$$

An easy computation shows that

$$\begin{aligned}
&\langle I'_\lambda(u_n)-I'_\lambda(u),u_n-u\rangle\\
&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|(u_n-u)(x)-(u_n-u)(y)|^{p-2}((u_n-u)(x)-(u_n-u)(y))^2}{|x-y|^{N-sp}}\,dx\,dy\\
&\quad-\int_{\Omega}(h(u_n)-h(u))(u_n-u)\,dx-\lambda\int_{\Omega}(f(x,u_n)-f(x,u))(u_n-u)\,dx\\
&\geq k_3\|u_n-u\|^p_{X^p_s(\Omega)}-\int_{\Omega}(h(u_n)-h(u))(u_n-u)\,dx\\
&\quad-\lambda\int_{\Omega}(f(x,u_n)-f(x,u))(u_n-u)\,dx,
\end{aligned}$$

where k_3 is a positive constant. This implies that the sequence $\{u_n\}_{n\in\mathbb{N}}$ converges strongly to u in $X^p_s(\Omega)$. Therefore, I_λ satisfies the (PS)-condition and so all conditions of [5, Theorem 3.2]) are satisfied. Hence, for each $\lambda\in\Lambda_r$, the function I_λ admits at least two distinct critical points that are solutions of the problem (1.1). □

In our final result, we discuss the existence of at least three solutions to problem (1.1).

Theorem 3.5. *Let $p>q$ and $f:\Omega\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function satisfying* (H1) *and let* (H3), (H4) *hold. In addition, assume that there exist three positive constants τ, ρ, and δ, such that*

(H6) $\frac{\omega_N^2\delta^p\rho^{N-sp}N^2}{2^{N-sp-1}sp(N+p-sp)}>\tau^p$.

Then, if (3.1) *holds, problem* (1.1) *admits at least three distinct weak solutions.*

Proof. Here we will apply [7, Theorem 3.6]. We consider the functionals Φ and Ψ defined in (2.2) and (2.3). Once again, they satisfy the regularity assumptions needed in [7, Theorem 3.6]. Now, we argue as in the proof of Theorem 3.1 with $w(k)$ defined in (3.4),

$$r:=\frac{2\omega_N^{\frac{sp}{N}+1}-Lps|\Omega|^{sp/N}}{2p\omega_N^{\frac{sp}{N}+1}}\tau^p,$$

and $0<L<\frac{2^{1-p}\omega_N^{\frac{sp}{N}+1}}{ps|\Omega|^{ps/N}}$. Given that lower bounds for R_1, R_3, and R_4 are greater than zero, we have

$$\Phi(w)\geq\frac{2\omega_N^{\frac{sp}{N}+1}-Lps|\Omega|^{sp/N}}{2p\omega_N^{\frac{sp}{N}+1}}(0+2R_2+2\times0+2\times0).$$

In view of (H6), we have $\Phi(w)>r>0$. Therefore, from (H3), inequality (3.9) holds, and so

$$\frac{\sup_{\Phi(u)\leq r}\Psi(u)}{r}<\frac{\Psi(\overline{v})}{\Phi(\overline{v})}$$

holds for some $\bar{v} \in X_s^p(\Omega)$.

Now, we prove that for each $\lambda \in \Lambda_w$, the functional I_λ is coercive. Using condition (H1), Hölder's inequality, and Remark 2.4, we easily obtain that for all $u \in X_s^p(\Omega)$,

$$\begin{aligned} I_\lambda(u) &\geq \frac{2\omega_N^{\frac{sp}{N}+1} - Lps|\Omega|^{sp/N}}{2p\omega_N^{\frac{sp}{N}+1}} |u|^p_{W^{s,p}(\mathbb{R}^N)} - \lambda \int_\Omega F(x,u)\,dx \\ &\geq \frac{2\omega_N^{\frac{sp}{N}+1} - Lps|\Omega|^{sp/N}}{2p\omega_N^{\frac{sp}{N}+1}} |u|^p_{W^{s,p}(\mathbb{R}^N)} \\ &\quad - \|\alpha\|_\infty |\Omega|^{1/p'} \|u\|_{L^p(\Omega)} - q^{-1}\|\beta\|_\infty \|u\|^q_{L^q(\Omega)} \end{aligned}$$

by (3.7).

Since

$$L < \frac{2\omega_N^{\frac{sp}{N}+1}}{ps|\Omega|^{ps/N}}$$

and $p > q$, we see that $I_\lambda \to +\infty$ as $\|u\| \to +\infty$, so the functional I_λ is coercive. Thus, for each $\lambda \in \Lambda_w$, [7, Theorem 3.6] implies that the functional I_λ admits at least three critical points in $X_s^p(\Omega)$ that are solutions of the problem (1.1). $\square$

We conclude this article with an example of Theorem 3.5.

Example 3.6. Let $\Omega = \{(x_1, x_2) : x_1^2 + x_2^2 < 1\} \subset \mathbb{R}^2$, and consider the problem

$$\begin{gathered} (-\Delta)_2^{1/4} u = \lambda f(x,u) + \tan(u), \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \mathbb{R}^N \backslash \Omega. \end{gathered}$$

We have $N = 2$, $p = 2$, $p' = 2$, and $s = 1/4$. For $t \in \mathbb{R}$, let

$$f(t) = \begin{cases} t/2, & t \leq 1, \\ 1/2, & t > 1. \end{cases}$$

From f, we have

$$F(t) = \begin{cases} t^2/4, & t \leq 1, \\ \frac{t}{2} - \frac{1}{4}, & t > 1. \end{cases}$$

By choosing $\alpha(x) = 1/2$, $\beta(x) = 10^{-10}$, and $q = 3/2$, we see that condition (H1) holds. If we take $\delta = 1$, $\rho = 90$, and $\tau = 64$, simple calculations show that all the conditions in Theorem 3.5 are satisfied. Hence, for every

$$\lambda \in \Big(12(4\pi + 1), \frac{45(4\pi - 1)}{\pi + .19 \times 10^{-2}}\Big),$$

the above problem admits at least three nontrivial weak solutions.

Remark 3.7. It would be possible to replace the requirement that $\alpha \in L^\infty(\Omega)$ in condition (H1) by the less restrictive condition that this function belong to the space $L^{\frac{p}{p-1}}(\Omega)$ and modifying our calculations. The conclusions we have obtained would remain true.

Conclusions

We considered a nonlinear elliptic fractional Dirichlet boundary value problem involving a p-Laplacian and containing a positive parameter. Our interest was in obtaining the existence of at least one, two, and three solutions to the problem. In doing this we estimated an interval for the parameter λ in which problem (1.1) possesses at least one nontrivial weak solution provided the nonlinear term satisfied a subcritical growth condition.

To obtain the existence of two solutions, we used a result of Bonanno [5] and required that the (P.S.)$^{[r]}$ condition or the (AR)-condition holds. In order to obtain the existence of three solutions, we asked that the nonlinear term has subcritical growth and used variational methods and a critical point theorem of Bonanno and Marano [7].

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