Electronic Journal of Differential Equations, Vol. 2023 (2023), No. 47, pp. 1–16.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: https://doi.org/10.58997/ejde.2023.47

EXISTENCE OF AT LEAST FOUR SOLUTIONS FOR SCHRODINGER EQUATIONS WITH MAGNETIC POTENTIAL INVOLVING AND SIGN-CHANGING WEIGHT FUNCTION

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In memory of Prof. Alan C. Lazer with admiration

Abstract. We consider the elliptic problem

$$-\Delta_A u + u = a_\lambda(x)|u|^{q-2}u + b_\mu(x)|u|^{p-2}u,$$

for $x \in \mathbb{R}^N$, $1 < q < 2 < p < 2^* = 2N/(N-2)$, $a_\lambda(x)$ is a sign-changing weight function, $b_\mu(x)$ satisfies some additional conditions, $u \in H^1_A(\mathbb{R}^N)$ and $A : \mathbb{R}^N \to \mathbb{R}^N$ is a magnetic potential. Exploring the Bahri-Li argument and some preliminary results we will discuss the existence of a four nontrivial solutions to the problem in question.

1. Introduction

In this work we are interested in studying the existence of a fourth solution for the concave-convex elliptic problem

$$\begin{gathered} -\Delta_A u + u = a_\lambda(x)|u|^{q-2}u + b_\mu(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \\ u \in H^1_A(\mathbb{R}^N), \end{gathered} \tag{1.1}$$

where $N \geq 3$, $-\Delta_A = (-i\nabla + A)^2$, $1 < q < 2 < p < 2^* = \frac{2N}{N-2}$, $a_\lambda(x)$ is a family of functions that can change signs, $b_\mu(x)$ is continuous and satisfies some additional conditions, $u : \mathbb{R}^N \to \mathbb{C}$ with $u \in H^1_A(\mathbb{R}^N)$ (such space will be defined later), $\lambda > 0$ and $\mu > 0$ are real parameters, and $A : \mathbb{R}^N \to \mathbb{R}^N$ is a magnetic potential in $L^2_{\rm loc}(\mathbb{R}^N, \mathbb{R}^N)$. For the relevance of this equation to the magnetic Laplacian in Physics, the reader is referred to Alves and Figueiredo [1] and Arioli and Szulkin [3].

In [12] the authors showed the existence of three solutions for this problem and proved their regularity. In this paper we show the existence of a fourth solution.

There are many works on problems similar problem to (1.1), but with $A = 0$. Ambrosetti, Brezis, and Cerami [2] considered the problem

$$-\Delta u + u = \lambda u^{q-1} + u^{p-1} \quad \text{in } \Omega,$$

2020 *Mathematics Subject Classification.* 35J20, 35Q60, 35Q55, 35B38.
Key words and phrases. Magnetic potential; Nehari method; sign-changing function; variational method.

Submitted January 13, 2022. Published July 11, 2023.

$$
\begin{gathered}
u>0 \quad \text{in } \Omega, \\
u=0 \quad \text{on } \partial\Omega,
\end{gathered}
$$

where Ω is a bounded regular domain of $\mathbb{R}^N$ ($N \geq 3$), with smooth boundary and $1<q<2<p\leq 2^*$. Combining the method of sub and super-solutions with the variational method, the authors proved the existence of $\lambda_0>0$ such that there are two solutions when $\lambda \in (0,\lambda_0)$, one solution when $\lambda=\lambda_0$, and no solution when $\lambda>\lambda_0$.

Wu [26] studied the concave-convex problem

$$
\begin{gathered}
-\Delta u+u=\lambda f(x)u^{q-1}+u^{p-1} \quad \text{in } \Omega, \\
u>0 \quad \text{in } \Omega, \\
u=0 \quad \text{on } \partial\Omega,
\end{gathered}
$$

with $f \in C(\overline{\Omega})$ a sign changing function and $1<q<2<p<2^*$. It was proved that the problem has at least two positive solutions for λ small enough. For p,q as above, many studies have been devoted to the existence and multiplicity of solutions to concave-convex elliptic problems in bounded domains; see for example Brown [8], Brown and Wu [6], Brown and Zhang [7], Hsu [18], Hsu and Lin [17], and their references.

For an unbounded domains, we can cite Chen [10], Huang, Wu and Wu [19], who studied a similar problems in $\mathbb{R}^N$. Wu [25] studied the problem

$$
\begin{gathered}
-\Delta u+u=f_\lambda(x)u^{q-1}+g_\mu u^{p-1} \quad \text{in } \mathbb{R}^N, \\
u\geq 0 \quad \text{in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N),
\end{gathered}
$$

with $1<q<2<p<2^*$, $g_\mu \geq 0$, and f_λ being able to change sign. Wu [25] showed the existence of at least four solutions to the problem when λ and μ small enough. This result was extend in [12], investigating if it would be possible to obtain similar consequences when we use the magnetic Laplacian in place of the usual Laplacian. In this work we will show the existence of a fourth solution for this problem.

The first results for non-linear Schrödinger equations with $A \neq 0$ can be attributed to Esteban and Lions [14]. They obtained the existence of stationary solutions for the equation

$$
-\Delta_A u+Vu=|u|^{p-2}u, u\neq 0, u \in L^2(\mathbb{R}^N),
$$

with $V=1$ and $p \in (2,\infty)$ using a minimization method with constant magnetic field. This was done for the general case.

Chabrowski and Szulkin [9] worked with this operator in the critical case, that is $p=2^*$, and with the electric potential V being able to change signs. Already Cingolani, Jeanjean and Secchi [11] considered the existence of multi-peak solutions in the subcritical case.

Alves and Figueiredo [1] considered the problem

$$
-\Delta_A u=\mu|u|^{q-2}u+|u|^{2^*-2}u, \quad u\neq 0, \Omega \subset \mathbb{R}^N,
$$

with $\mu>0$ and $2\leq q<2^*$. They related number of solutions with the topology of Ω.

The authors in [12] studied non-zero A case with a weight function that changes signs in the concave-convex case, just like the problem stated in this work. They proved the existence of three solutions for the problem. Now we would like to show the existence of a fourth solution. There the authors used category theory, the Nehari manifold, and the fibering map.

In what follows, we will present a set of preliminary results. Observe that

$$J_{\lambda,\mu}(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla_A u|^2 + |u|^2)\,dx - \frac{1}{q}\int_{\mathbb{R}^N} a_\lambda(x)|u|^q\,dx - \frac{1}{p}\int_{\mathbb{R}^N} b_\mu(x)|u|^p\,dx, \quad (1.2)$$

is the functional associated with problem (1.1) and is of class C^1 in $H^1_A(\mathbb{R}^N)$ as can shown in [22]. Also, the critical points of $J_{\lambda,\mu}(u)$ are weak solutions of problem (1.1). We will use the following hypotheses: We assume $a(x) \in L^{q'}(\mathbb{R}^N)$, $q' = \frac{p}{p-q}$, and $a_\pm = \pm\max\{\pm a(x), 0\} \neq 0$. Let

$$a_\lambda(x) = \lambda a_+(x) + a_-(x).$$

and assume that

- (A1) $a(x) \in L^{q'}(\mathbb{R}^N)$, $q' = \frac{p}{p-q}$, and there exists $\hat{c} > 0$ and $r_{a_-} > 0$, such that
$$a_-(x) > -\hat{c}\exp(-r_{a_-}|x|) \quad \text{for all } x \in \mathbb{R}^N.$$

We assume that $b_\mu(x) = b_1(x) + \mu b_2(x)$, where

- (A2) $b_1(x) > 0$ in continuous in $\mathbb{R}^N$, with $b_1(x) \to 1$ as $|x| \to \infty$ and there exists $r_{b_1} > 0$, such that
$$1 \geq b_1(x) \geq 1 - c_0\exp(-r_{b_1}|x|) \quad \text{for some } c_0 < 1 \text{ and all } x \in \mathbb{R}^N.$$
- (A3) $b_2(x) > 0$ is continuous in $\mathbb{R}^N$, $b_2(x) \to 0$ as $|x| \to \infty$ and exists $r_{b_2} > 0$, with $r_{b_2} < \min\{r_{a_-}, r_{b_1}, q\}$ such that
$$b_2(x) \geq d_0\exp(-r_{b_2}|x|) \quad \text{for some } d_0 < 1 \text{ and all } x \in \mathbb{R}^N.$$

The above hypotheses were used in [12]. We define

$$\Upsilon_0 = (2-q)^{2-q}\Big(\frac{p-2}{\|a_+\|_{q'}}\Big)^{p-2}\Big(\frac{S_p}{p-q}\Big)^{p-q},$$

where

$$S_p = \inf_{u\in H^1_A(\mathbb{R}^N\setminus\{0\})} \frac{\big(\int_{\mathbb{R}^N} |\nabla_A u|^2 + u^2 dx\big)^{1/2}}{\big(\int_{\mathbb{R}^N} |u|^p dx\big)^{2/p}} > 0. \quad (1.3)$$

In [12], under assumptions (A1)–(A3), it was proved that (1.1) has at least one solution, provided that

$$\lambda^{p-2}(1+\mu\|b_2\|_\infty)^{2-q} < \big(\frac{q}{2}\big)^{p-2}\Upsilon_0 \quad (1.4)$$

holds for each $\lambda > 0$ and $\mu > 0$. Then, assuming that the potential is asymptotic to a constant at infinity, they prove the existence of at least two solutions u^+ and u^- with $J_{\lambda,\mu}(u^+) < 0 < J_{\lambda,\mu}(u^-)$.

In the previous result, the existence is valid for all λ and μ satisfying inequality (1.4). So, if we additionally set values of λ and μ conveniently small we obtain the multiplicity result, that is, there exist at least three solutions. Actually they showed the existence of $\lambda_0 > 0$ and $\mu_0 > 0$ with

$$\lambda_0^{p-2}(1+\mu_0\|b_2\|_\infty)^{2-q} < \big(\frac{q}{2}\big)^{p-2}\Upsilon_0,$$

such that for all $\lambda \in (0,\lambda_0)$ and $\mu \in (0,\mu_0)$, problem (1.1) has at least three solutions.

In this work, we observe that for the problem in question, the numbers λ_0 and μ_0 as previously mentioned are independent of the value of a_-. However, considering some additional hypotheses and taking values of $\|a_-\|_{q'}$ sufficiently small we obtain another solution. Before stating this result we present the following hypotheses:

(A4) $b_1(x) < 1$ in $\mathbb{R}^N$ in a positive measure set;

(A5) $r_{b_1} > 2$.

Theorem 1.1. *Suppose that the potential $A(x)$ converges to some constant $d \in \mathbb{N}^N$ as $|x| \to \infty$. Assuming* (A1)–(A5)*, there are positive values $\tilde{\lambda_0} \le \lambda_0$, $\tilde{\mu_0} \le \mu_0$, and ν_0 such that for $\lambda \in (0, \tilde{\lambda_0})$, $\mu \in (0, \tilde{\mu_0})$, and $\|a_-\|_{q'} < \nu_0$, problem* (1.1) *has at least four solutions.*

For the first three solutions of this problem, the Nehari method was used together with category theory. We will use variational methods to prove the above theorem. We will work under a few more assumptions to estimate different energy levels and will use the Bahri-Li min-max argument to show that for very small values of $\|a_-\|_{q'}$, the problem has at least four distinct solutions.

2. Initial considerations

According to Tang [23], we denote by $H_A(\mathbb{R}^N)$ the Hilbert space obtained by the closing of $C_0^\infty(\mathbb{R}^N, \mathbb{C})$ with the inner product

$$\langle u, v\rangle_A = \operatorname{Re}\int_{\mathbb{R}} (\nabla_A u \overline{\nabla_A v} + u\bar{v} dx),$$

where $\nabla_A u := (D_1 u, D_2 u, \dots, D_N u)$ and $D_j := -i\partial_j - A_j(x)$, with $j = 1, 2, \dots, N$, and with $A(x) = (A_1(x), \dots, A_N(x))$. The norm induced by this product is

$$\|u\|_A^2 := \int_{\mathbb{R}} (|\nabla_A u|^2 + u^2 dx).$$

Esteban and Lions [14, Section II] proved that that for all $u \in H_A^1(\mathbb{R}^N)$ the diamagnetic inequality holds, i.e.

$$|\nabla|u|(x)| = \big|\operatorname{Re}\big(\nabla u \frac{\overline{u}}{|u|}\big)\big| = \big|\operatorname{Re}\big((\nabla u - iAu)\frac{\overline{u}}{|u|}\big)\big| \le |\nabla_A u(x)|.$$

So, if $u \in H_A^1(\mathbb{R}^N)$ we have that $|u|$ belongs to the usual Sobolev space $H_0^1(\mathbb{R}^N)$.

2.1. **Preliminary results.** To obtain the existence, we introduced the Nehari manifold

$$M_{\lambda,\mu} = \{u \in H_A^1(\mathbb{R}^N) \setminus \{0\} : \langle J'_{\lambda,\mu}(u), u\rangle = 0\},$$

where $\langle \cdot, \cdot\rangle$ denotes the usual duality between $H_A^1(\mathbb{R}^N)$ and its dual $H_A^1(\mathbb{R}^N)^*$. The Nehari manifold is linked to the functions $F_u : t \to J_{\lambda,\mu}(tu)$, $(t > 0)$, called fibering maps. Note that the fabering map it was defined and depends on u, λ, and μ; so that proper notation should be $F_{u,\lambda,\mu}$, but to simplify the notation, we write F_u. If $u \in H_A^1(\mathbb{R}^N)$, we have

$$F_u(t) = \frac{t^2}{2}\|u\|_A^2 - \frac{t^q}{q}\int_{\mathbb{R}^N} a_\lambda(x)|u|^q\,dx - \frac{t^p}{p}\int_{\mathbb{R}^N} b_\mu(x)|u|^p\,dx, \tag{2.1}$$

$$F'_u(t) = t\|u\|_A^2 - t^{q-1}\int_{\mathbb{R}^N} a_\lambda(x)|u|^q\,dx - t^{p-1}\int_{\mathbb{R}^N} b_\mu(x)|u|^p\,dx, \tag{2.2}$$

$$F''_u(t) = \|u\|_A^2 - (q-1)t^{q-2}\int_{\mathbb{R}^N} a_\lambda(x)|u|^q\,dx - (p-1)t^{p-2}\int_{\mathbb{R}^N} b_\mu(x)|u|^p\,dx. \tag{2.3}$$

The following remark relates the Nehari manifold and the Fibering map.

Remark 2.1. Let F_u be the application defined above and $u \in H^1_A(\mathbb{R}^N)$. Then:

(i) $u \in M_{\lambda,\mu}$ if, and only if, $F'_u(1) = 0$;
(ii) more generally $tu \in M_{\lambda,\mu}$, and only if, $F'_u(t) = 0$.

From the previous remark we conclude that the elements in $M_{\lambda,\mu}$, correspond to the critical points of the Fibering map. Thus, as $F_u(t) \in C^2(\mathbb{R}^+, \mathbb{R})$, we can divide the Nehari manifold into three parts

$$\begin{aligned} M^+_{\lambda,\mu} &= \{u \in M_{\lambda,\mu} : F''_{\lambda,\mu}(1) > 0\}, \\ M^-_{\lambda,\mu} &= \{u \in M_{\lambda,\mu} : F''_{\lambda,\mu}(1) < 0\}, \\ M^0_{\lambda,\mu} &= \{u \in M_{\lambda,\mu} : F''_{\lambda,\mu}(1) = 0\}. \end{aligned}$$

The lemma below shows that under some conditions, $M^0_{\lambda,\mu}$ is empty, as shown in [6, Lemma 2.2].

Lemma 2.2. *Let $\mu \geq 0$ and $\lambda > 0$ such that*

$$\lambda^{p-2}(1 + \mu\|b_2\|_\infty)^{2-q} < \Upsilon_0. \tag{2.4}$$

Then $M^0_{\lambda,\mu} = \emptyset$.

As shown in [12], under certain conditions on λ and μ, we have a minimizer in $M^+_{\lambda,\mu}$ and another in $M^-_{\lambda,\mu}$. The minimum levels of energy will be denoted respectively by

$$\begin{aligned} m^+_{\lambda,\mu} &= \inf_{u \in M^+_{\lambda,\mu}} J_{\lambda,\mu}(u), \\ m^-_{\lambda,\mu} &= \inf_{u \in M^-_{\lambda,\mu}} J_{\lambda,\mu}(u). \end{aligned}$$

To establish the existence of the first two solutions and compare with the energy level of the fourth solution, we will need the following result that was shown in [12].

Lemma 2.3. *For each $u \in H^1_A(\mathbb{R}^N) \setminus \{0\}$ and $\mu > 0$ we have:*

(i) *If $\int_{\mathbb{R}^N} a_\lambda(x)|u|^q\,dx \leq 0$, there is a single $t^-(u) > t_{\max}(u)$ such that $t^-(u)u \in M^-_{\lambda,\mu}$. Also, $F_u(t)$ is increasing in $(0, t^-(u))$, decreasing in $(t^-(u), +\infty)$ and $F_u(t) \to -\infty$ as $t \to +\infty$.*
(ii) *If $\int_{\mathbb{R}^N} a_\lambda(x)|u|^q\,dx > 0$ and λ is such that $\lambda^{p-2}(1 + \mu\|b_2\|_\infty)^{2-q} < \Upsilon_0$, so there is $0 < t^+(u) < t_{\max}(u) < t^-(u)$ such that $t^\pm(u)u \in M^\pm_{\lambda,\mu}$. Also, $F_u(t)$ is decreasing in $(0, t^+(u))$, increasing in $(t^+(u), t^-(u))$ and decreasing in $(t^-(u), +\infty)$. Furthermore, $F_u(t) \to -\infty$ as $t \to +\infty$.*

Our next lemma shows that these points are well defined, and i prove can be found in [17, Lemma 2.1].

Lemma 2.4. *The functional $J_{\lambda,\mu}$ is coercive and bounded from below in $M_{\lambda,\mu}$.*

For the next results we need some estimates on $m^\pm_{\lambda,\mu}$. To do this, from (2.4) we have

$$\|u\|^2_A < \frac{p-q}{p-2} \int_{\mathbb{R}^N} a_\lambda(x)|u|^q\,dx \leq \Upsilon_0^{1/(p-2)} \frac{p-q}{p-2} S_p^{-q/2} \|a_+\|_{L^{q'}} \|u\|^q_A.$$

Therefore,

$$\|u\|_A \leq \Big(\Upsilon_0^{1/(p-2)} \frac{p-q}{p-2} S_p^{-q/2} \|a_+\|_{L^{q'}}\Big)^{1/(2-q)} \|u\|_A^q, \tag{2.5}$$

for all $u \in M^+_{\lambda,\mu}$. Also, if $\lambda = 0$, then (2.4) is satisfied, so that by Lemma 2.3(i), $M^+_{\lambda,\mu} = \emptyset$, and we have $M_{\lambda,\mu} = M^-_{\lambda,\mu}$ for all $\mu \geq 0$. By has been seen, we will show the following results on the values of $m^\pm_{\lambda,\mu}$.

Lemma 2.5. (i) *If $\lambda^{p-2}(1+\mu\|b_2\|_\infty)^{2-q} < (\frac{q}{2})^{p-2}\Upsilon_0$, then $m^-_{\lambda,\mu} > 0$;*

(ii) *For $\lambda > 0$ and $\mu \geq 0$ with $\lambda^{p-2}(1+\mu\|b_2\|_\infty)^{2-q} < \Upsilon_0$, then $m^+_{\lambda,\mu} < 0$. In particular, if $\lambda^{p-2}(1+\mu\|b_2\|_\infty)^{2-q} < (\frac{q}{2})^{p-2}\Upsilon_0$, then*

$$m^+_{\lambda,\mu} = \inf_{M_{\lambda,\mu}} J_{\lambda,\mu}(u).$$

The proof of the above lemma is similar to one in [25, Theorem 3.1]; we omit it. By Lemma 2.5, we can conclude that for every $u \in H^1_A(\mathbb{R}^N) \setminus \{0\}$

$$J_{\lambda,\mu}(t^-(u)u) = \max_{t\leq 0} J_{\lambda,\mu}(tu), \tag{2.6}$$

whenever $\lambda^{p-2}(1+\mu\|b_2\|_\infty)^{2-q} < \big(\frac{q}{2}\big)^{p-2}\Upsilon_0$, with $\lambda \geq 0$ and $\mu > 0$.

3. Existence of m_∞

In this section we define the energy level of the limit problem and make some estimates for energy levels of the solutions in the Nehari manifold. Then, we will have tools to show that the fourth solution has a different level than other solutions. For this, consider the semilinear elliptic problem

$$\begin{gathered} -\Delta_A u + u = |u|^{p-2}u \quad \text{in } \mathbb{R}^N, \\ u \in H^1_A(\mathbb{R}^N). \end{gathered} \tag{3.1}$$

We define $J_\infty(u) = \frac{1}{2}\|u\|_A^2 - \frac{1}{p}\|u\|_p^p$, as the functional associated with problem (3.1). Then J_∞ is a C^2 functional in $H^1_A(\mathbb{R}^N)$. The Nehari manifold associated with (3.1) is

$$M_\infty = \{u \in H^1_A(\mathbb{R}^N) \setminus \{0\} : J'_\infty(u)u = 0\}.$$

In this problem we can observe that if $u \in N_\infty$, then $\|u\|_A^2 = \|u\|_p^p$. Now consider the minimization problem

$$m_\infty = \inf_{M_\infty} J_\infty(u). \tag{3.2}$$

In [12] it was shown that there exists $\bar{u} \in H^1_A(\mathbb{R}^N)$ such that $m_\infty = \inf_{N_\infty} J_\infty(u) = J_\infty(\bar{u})$. From these considerations we will show the following result that gives us a description of a sequence (PS) of $J_{\lambda,\mu}$.

Lemma 3.1. *Let $\{u_n\} \subset M^-_{\lambda,\mu}$ be a $(PS)_\beta$ sequence in $H^1_A(\mathbb{R}^N)$ of $J_{\lambda,\mu}$, this is, a sequence satisfying $J_{\lambda,\mu}(u_n) = \beta + o_n(1)$ and $J'_{\lambda,\mu}(u_n) = o_n(1)$ in H_A^{-1} as $n \to \infty$, where*

$$m^+_{\lambda,\mu} + m_\infty < \beta < m^-_{\lambda,\mu} + m_\infty,$$

then there is a subsequence $\{u_n\}$ and $u_0 \in H^1_A(\mathbb{R}^N)$, with a non zero u_0, such that $u_n = u_0 + o_n(1)$ strong in $H^1_A(\mathbb{R}^N)$ and $J_{\lambda,\mu}(u_0) = \beta$. Moreover, u_0 is a solution of (1.1).

Proof. From (A1)–(A3), we obtain by a standard argument that $\{u_n\}$ is a bounded sequence in $H^1_A(\mathbb{R}^N)$. Then there is a subsequence $\{u_n\}$ and $u_0 \in H^1_A(\mathbb{R}^N)$ such that $u_n \rightharpoonup u_0$ weakly in $H^1_A(\mathbb{R}^N)$ as $n \to \infty$. Taking $v_n = u_n - u_0$, we have $v_n \rightharpoonup 0$ weak in $H^1_A(\mathbb{R}^N)$ as $n \to \infty$. Denoting by $B(0,1)$ the ball centered on the origin of radius 1, we have in $B(0,1)$ the strong convergence

$$\int_{B(0,1)} |u_n|^q \to \int_{B(0,1)} |u_0|^q.$$

By the Dominated Convergence Theorem we obtain

$$\int_{B(0,1)} a_\lambda \big||u_n|^q - |u_0|^q\big| \to 0, \quad \text{as } n \to \infty.$$

Then, by Hölder and the integrability of a_λ it follows that

$$\begin{aligned}
&\Big|\int a_\lambda(x)(|u_n|^q - |u_0|^q)\Big| \\
&\le o_n(1) + \int_{B^c(0,1)} a_\lambda(x)\big||u_n|^q - |u_0|^q\big| \\
&\le o_n(1) + \Big(\int_{B^c(0,1)} a_\lambda(x)^{q^*}\Big)^{1/q^*} (\|u_n\|_p^q + \|u_0\|_p^q) \\
&\le o_n(1) + \epsilon C.
\end{aligned}$$

As $\epsilon > 0$ it is arbitrary, we have

$$\int a_\lambda(x)(|u_n|^q - |u_0|^q) = o_n(1).$$

On the other hand, by (A2) and (A3) and the Brezis-Lieb lemma (see [24]), we can conclude that $\mu \int b_2(x)|v_n|^p = o_n(1)$, $\int (1-b_1(x))|v_n|^p = o_n(1)$ and $\int b_\mu(x)(|u_n|^p - |v_n|^p - |u_0|^p) = o_n(1)$, which together with the above inequality gives us

$$J_{\lambda,\mu}(u_n) = J_\infty(v_n) + J_{\lambda,\mu}(u_0) + o_n(1).$$

In a similar way we obtain that $J'_\infty(v_n)v_n = J'_{\lambda,\mu}(u_n)u_n - J'_{\lambda,\mu}(u_0)u_0 + o_n(1)$. By hypothesis $J'_{\lambda,\mu}(u_n) \to 0$ strong in $H^1_A(\mathbb{R}^N)^{-1}$ and $u_n \rightharpoonup u_0$ weak in $H^1_A(\mathbb{R}^N)$ as $n \to \infty$ and so we have $J'_{\lambda,\mu}(u_0) = 0$. Now, define $\delta = \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,1)} |v_n|^p$. So we have two cases:

(i) $\delta > 0$, and
(ii) $\delta = 0$.

Suppose that (i) happens. Then there will be a sequence $\{y_n\} \subset \mathbb{R}^N$ such that $\int_{B(y_n,1)} |v_n|^p \ge \frac{\delta}{2}$ and for all $n \in \mathbb{N}$. Define $\tilde{v}_n(x) = v_n(x + y_n)$. We have that $\{\tilde{v}_n\}$ is bounded and $\tilde{v}_n \rightharpoonup v$ weak and almost everywhere. Making a change of variables we obtain

$$\int_{B(0,1)} |\tilde{v}_n|^p \ge \frac{\delta}{4}.$$

Then

$$\int_{B(0,1)} |v|^p \ge \frac{\delta}{4}, \tag{3.3}$$

giving us $v \ne 0$. But, $v_n \rightharpoonup 0$ weakly; then

$$\int_{\mathbb{R}^N} |v_n|^p \ge \int_{B(y_n,1)} |v_n|^p \ge \frac{\delta}{2} > 0. \tag{3.4}$$

We see that
$$J_\infty(v_n) = \frac{1}{2}\int (|\nabla_A v_n|^2 + v_n^2)dx - \frac{1}{p}\int |v_n|^p dx.$$
Likewise,
$$F_{v_n}(t) = J_\infty(tv_n) = \frac{t^2}{2}\|v_n\|_A^2 - \frac{t^p}{p}\|v_n\|^p.$$
For each $n \in \mathbb{N}$, we can get t_n such that $t_n v_n \in M_\infty$. So we build a sequence $\{t_n\} \subset \mathbb{R}^N$ with $t_n \to t_0$ as $n \to \infty$, such that $t_n v_n \in M_\infty$, that is, such that $J'_\infty(t_n v_n)t_n v_n = 0$. We see also that
$$J'_\infty(v_n)v_n = \|v_n\|_A^2 - \|v_n\|^p = o_n(1)$$
and
$$F'_{v_n}(t) = J'_\infty(tv_n)v_n = t\|v_n\|_A^2 - t^{p-1}\|v_n\|^p = o_n(1). \tag{3.5}$$
With this, we have
$$(t_n - t_n^{p-1})\|v_n\|_A^2 = t_n(1 - t_n^{p-2})\|v_n\|_A^2 = o_n(1). \tag{3.6}$$
From (3.3) we know that $\|v_n\|_A^2 \not\to 0$ (that is, v_n does not converge to zero). Also note that $t_n^{2-p} = \frac{\int |v_n|^p}{\|v_n\|_A^2} \geq \frac{\delta}{2c}$. With that and by (3.6) we obtain that $(1-t_n^{p-2}) \to 0$, giving us that $t_n \to 1$. Now, see that $v_n \rightharpoonup 0$ weak in $H_A^1(\mathbb{R}^N)$ as $n \to \infty$. With this and by the fact $t_n \to 1$, we can conclude that
$$J_{\lambda,\mu}(u_n) = J_\infty(t_n v_n) + J_{\lambda,\mu}(u_0) + o_n(1) \geq m_\infty + J_{\lambda,\mu}(u_0).$$
Note that by hypotheses $J_{\lambda,\mu}(u_n) = \beta + o_n(1)$ with $\beta < m_\infty + m_{\lambda,\mu}^+$. From there we obtain
$$\beta + o_n(1) = J_{\lambda,\mu}(u_n) = J_\infty(t_n v_n) + J_{\lambda,\mu}(u_0) + o_n(1) \geq m_\infty + J_{\lambda,\mu}(u_0),$$
giving us
$$m_\infty + J_{\lambda,\mu}(u_0) \leq \beta + o_n(1) < m_\infty + m_{\lambda,\mu}^+ + o_n(1);$$
therefore
$$J_{\lambda,\mu}(u_0) < m_{\lambda,\mu}^+ + o_n(1). \tag{3.7}$$
We have already seen that $J'_{\lambda,\mu}(u_n)$ converges strongly to zero, therefore we obtain $J'_{\lambda,\mu}(u_0) = 0$. Thus $u_0 \in M_{\lambda,\mu}$. Still, by Lemma 2.2, $M_{\lambda,\mu}^0 = \emptyset$ and by Lemma 2.5, we conclude that $m^+ > 0$ and $m^- < 0$. Then
$$J_{\lambda,\mu}(u_0) \geq \inf_{M_{\lambda,\mu}} J_{\lambda,\mu}(u) = \inf_{M_{\lambda,\mu}^+} J_{\lambda,\mu}(u) = m^+,$$
which contradicts what we have concluded in (3.7). We have proved that (ii) occurs. In this case, $\{v_n\}$ such that $\int |v_n|^p \to 0$ if $n \to \infty$.

As we already have $J'_\infty(v_n)v_n = o_n(1)$ with $J'_\infty(v_n)v_n = \|v_n\|_A^2 - \|v_n\|_p^p$ and $\int |v_n|^p \to 0$, we conclude that $\|v_n\|^2 \to 0$ giving us $u_n \to u_0$ strong in $H_A^1(\mathbb{R}^N)$. See also that $u_0 \neq 0$. In fact, note that if $u_0 = 0$ so $\tilde{v}_n = v_n = u_n$ and $\int_{B(0,1)} |u_n|^p \geq \frac{\delta}{4}$, which we have already seen to be an absurd. □

To address the existence of a second solution to (1.1), certain considerations need to be made. Note that equation
$$-\Delta_A u + u = a_\lambda(x)|u|^{q-2}u + b_\mu(x)|u|^{p-2}u \tag{3.8}$$

is such that $a_\lambda(x) \to 0$ and $b_\mu(x) \to 1$ as $|x| \to \infty$. Adding the hypothesis of $A \to d$ with d constant as $|x| \to \infty$, problem (3.8) converges to the problem

$$-\Delta_d u + u = |u|^{p-2}u., \tag{3.9}$$

where $-\Delta_d = (-i\nabla + d)^2$. Now, by a result of Ding and Liu [13, Lemma 2.5], u is a least energy solutions of Problem (3.9) if and only if $v(x) := |u(x)| \in H^1$ it is a least energy solution to the problem

$$-\Delta v + v = v^{p-1}; \quad v > 0. \tag{3.10}$$

Furthermore, the equations (3.9) and (3.10) share the same least energy. Specifically, we have

$$J_\infty(u) = I_\infty(v) = m_\infty,$$

where J_∞ and I_∞ represent the corresponding functionals associated with the aforementioned problems. According to Berestick, Lions [5] or Kwong [21], equation (3.10) has a unique solution z_0 symmetric, positive, and radial. By [15, Theorem 2], for all $\epsilon > 0$, exists A_ϵ, B_0 and C_ϵ positive such that

$$A_\epsilon \exp(-(1+\epsilon)|x|) \le z_0(x) \le B_0 \exp(-|x|), \tag{3.11}$$

$$|\nabla z_0(x)| \le C_\epsilon \exp(-(1-\epsilon)|x|). \tag{3.12}$$

According to Kurata [20, Lemma 4], defining $w_0 = z_0 e^{-idx}$, we have that w_0 is the unique, symmetrical, positive and radial solution of (3.9). So we will have $J_\infty(w_0) = m_\infty$. We see also that $z_0 = |w_0|$, which together with (3.11) gives us the inequalities

$$A_\epsilon \exp(-(1+\epsilon)|x|) \le |w_0(x)| \le B_0 \exp(-|x|), \tag{3.13}$$

$$|\nabla w_0(x)| \le C_\epsilon \exp(-(1-\epsilon)|x|). \tag{3.14}$$

Next, To prove the existence of a second solution, we make some estimates on the minimum energy levels in the Nehari Manifold. Not to overload the notation, we write $u^+ := u^+_{\lambda,\mu}$. Considering $J(u^+) = m^+$, $m^- = \inf_{u \in M^-_{\lambda,\mu}} J_{\lambda,\mu}(u)$, and $m_\infty = \inf_{u \in M_\infty} J_\infty(u) = J_\infty(w_0)$, we will make the following estimate for such energy levels.

Proposition 3.2. *For all $\lambda > 0$ and $\mu > 0$ satisfying $\lambda^{p-2}(1+\mu\|b_2\|_\infty)^{2-q} < \Upsilon_0$, we have $m^- < m^+ + m^\infty$.*

The proof of the above proposition is similar to that of [12, Proposition 6.1]; we omit it.

4. Third solution

To obtain the third solution of problem (1.1), we need some additional results. For $\lambda = 0$ and $\mu = 0$ we define the sets

$$M^-_{a_0,b_0} = \{u \in H^1_A(\mathbb{R}^N) \setminus \{0\} : \langle J'_{a_0,b_0}(u), u\rangle = 0\}$$

where

$$\begin{aligned} J_{a_0,b_0} &= \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla_A u|^2 + |u|^2)\,dx - \frac{1}{q}\int a_0(x)|u|^q dx - \frac{1}{p}\int b_0(x)|u|^p dx \\ &= \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla_A u|^2 + |u|^2)\,dx - \frac{1}{q}\int a_-(x)|u|^q dx - \frac{1}{p}\int b_1(x)|u|^p dx. \end{aligned}$$

Lemma 4.1. *With the above notation we have*

$$\inf_{u\in M^-_{a_0,b_0}} J_{a_0,b_0}(u) = \inf_{u\in M^\infty} J_\infty(u) = m^\infty.$$

Proof. Let w_k be as defined above. Because $\lambda = 0$, we have $a(x) = \lambda a_+(x) + a_-(x) = a_-(x) < 0$ from where $\int_{\mathbb{R}^N} a_-|t^-(w_k)w_k|^q dx \le 0$, hence by Lemma 2.3(i) there is only one $t^-(w_k) > \big(\frac{2-q}{p-q}\big)^{\frac{1}{p-2}}$ such that $t^-(w_k)w_k \in M^-_{a_0,b_0}$ for all $k > 0$; that is, $J'_{a_0,b_0}(t^-(w_k)w_k) = 0$, giving us

$$\|t^-(w_k)w_k\|^2_A = \int_{\mathbb{R}^N} a_-|t^-(w_k)w_k|^q dx + \int_{\mathbb{R}^N} b_-|t^-(w_k)w_k|^p dx. \tag{4.1}$$

As w_0 is a solution of problem (3.10) and remembering that the functional associated with (3.10) is $I(u) = \frac{1}{2}\|u\|^2_A - \frac{1}{p}\|u\|^p_p$, and $I'(u) = \|u\|^2_A - \|u\|^p_p$ we have

$$I'(w_0)w_0 = \|w_0\|^2_A - \|w_0\|^p_p = 0.$$

Therefore,

$$\begin{aligned} m_\infty = I(w_0) &= \frac{1}{2}\|w_0\|^2_A - \frac{1}{p}\|w_0\|^p_p \\ &= \frac{1}{2}\|w_0\|^2_A - \frac{1}{p}\|w_0\|^2_A = \frac{p-2}{2p}\|w_0\|^{\cdot}_A \end{aligned}$$

Being w_0 solution of problem (3.10) follows that $w_k(x) = w_0(x + ke)$. With this and $I'(w_0)w_0 = 0$, we have $I'(w_k)w_k = 0$. So that

$$\|w_k\|^2_A = \int_{\mathbb{R}^N} |w_k|^q dx = \frac{2p}{p-2} m^\infty \quad \text{for all } k \ge 0. \tag{4.2}$$

It is known that w_n is bounded in $L^{r'}$ and $w_n \to 0$ a.e., by Theorem [16, Theorem 13.44] that $w_n \rightharpoonup 0$ weakly in $L^{r'}$. By the condition (A1), $a_- \in (L^{r'})' = L^r$ we obtain

$$\int_{\mathbb{R}^N} a_-|w_k|^q dx \to 0 \quad \text{as } k \to \infty. \tag{4.3}$$

In addition, by (A2) and (A3) we have

$$\int_{\mathbb{R}^N} (1-b_1)|w_k|^q dx = \int_{B(0,R)} (1-b_1)|w_k|^q dx + \int_{B^c(0,R)} (1-b_1)|w_k|^q dx \to 0, \tag{4.4}$$

as $|w_k| \to \infty$. By (4.1), (4.3), and (4.4) we have that $t^-(w_k) \to 1$ as $k \to \infty$. Likewise

$$\lim_{k\to\infty} J_{a_0,b_0}(t^-(w_k)w_k) = \lim_{k\to\infty} J_\infty(t^-(w_k)w_k) = m_\infty.$$

Thus

$$m_\infty = \inf_{u\in M^\infty} J_\infty(u) = \lim_{k\to\infty} J_\infty(t^-(w_k)w_k) \ge \inf_{u\in M^-_{a_0,b_0}} J_{a_0,b_0}(u). \tag{4.5}$$

We also have $u \in M_{a_0,b_0}$, by Lemma 2.3(i), $J_{a_0,b_0}(u) = \sup_{t\ge 0} J_{a_0,b_0}(tu)$, and furthermore, there is a single $t^\infty > 0$ such that $t^\infty u \in M^\infty$. So

$$\begin{aligned} J_{a_0,b_0}(t^\infty u) &= \frac{1}{2}\|t^\infty u\|^2_A - \frac{(t^\infty)^q}{q}\int_{\mathbb{R}^N} a_-|u|^q dx - \frac{(t^\infty)^p}{p}\int_{\mathbb{R}^N} b_1|u|^p dx \\ &\ge \frac{1}{2}\|t^\infty u\|^2_A - \frac{(t^\infty)^p}{p}\int_{\mathbb{R}^N} |u|^p dx \\ &= J_\infty(t^\infty u) \ge m_\infty; \end{aligned}$$

therefore

$$\inf_{u\in M_{a_0,b_0}} J_{a_0,b_0}(t^\infty u) \geq m_\infty. \tag{4.6}$$

By (4.5) and (4.6), we have

$$\inf_{u\in M_{a_0,b_0}} J_{a_0,b_0}(u) = \inf_{u\in M^\infty} J_\infty(u) = m^\infty. \qquad \square$$

To obtain the fourth solution of the problem, we need a lemma that establishes suitable values of λ and μ.

Lemma 4.2. *Exist $\lambda_0 > 0$ and $\mu_0 > 0$ with*

$$\lambda_0^{p-2}(1+\mu_0\|b_1\|_\infty)^{2-q} < \big(\frac{q}{2}\big)^{p-2}\Upsilon_0,$$

such that for all $\lambda \in (0,\lambda_0)$ and all $\mu \in (0,\mu_0)$, we have

$$\int_{\mathbb{R}^N} \frac{x}{|x|}(|\nabla u|^2 + u^2)dx \neq 0$$

for all $u \in M^-_{a_\lambda,b_\mu}$ with $J_{\lambda,\mu}(u) < m^+_{a_\lambda,b_\mu} + m^\infty$.

The above lemma can be obtained arguing as in [12, Lemma 7.6]; we omit its proof.

5. Fourth solution

In this section we will work to estimate of the energy levels of the functional associated with the main problem, to prove the existence of a solution whose energy level satisfies the conditions of Proposition 3.1(ii); that is, to find a distinct solution from the three solutions in the previous sections. For $\alpha > 0$, we define

$$J_{0,\alpha b_0}(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla_A u|^2 + u^2 dx - \frac{1}{p}\int_{\mathbb{R}^N} \alpha b_0|u|^p dx,$$
$$M_{0,\alpha b_0} = \{u \in H^1_A(\mathbb{R}^N)\setminus\{0\} : \langle J'_{0,\alpha b_0}(u), u\rangle = 0\}.$$

We now define the following subset of the unitary ball

$$\mathcal{B} = \{u \in H^1_A(\mathbb{R}^N)\setminus\{0\} : u \geq 0 \text{ and } \|u\|_A = 1\}.$$

Let us recall that for every $u \in H^1_A(\mathbb{R}^N)\setminus\{0\}$ there exists a unique $t^-(u) > 0$ and $t_0(u) > 0$ such that $t^-(u) \in M^-_{a_\lambda,b_\mu}$ and $t_0(u) \in M_{0,b_0}$. To apply the minimax argument of Bahri-Li we present the following result.

Lemma 5.1. *For each $u \in \mathcal{B}$ we will have*

(i) *There is a unique $t_0^\alpha = t_0^\alpha(u) > 0$ such that $t_0^\alpha u \in M_{0,\alpha b_0}$ and*

$$\sup_{t\geq 0} J_{0,\alpha b_0}(tu) = J_{0,\alpha b_0}(t_0^\alpha u) = \frac{p-2}{2p}\Big(\int_{\mathbb{R}^N}\alpha b_0|u|^p dx\Big)^{-2/(p-2)}.$$

(ii) *For $\rho \in (0,1)$,*

$$J_{a_\lambda,b_\mu}(t^-(u)u) \geq \frac{(1-\rho)^{\frac{p}{p-2}}}{(1+\mu\|b_2/b_1\|_\infty)^{\frac{2}{p-2}}} J_{0,b_0}(t_0(u)u) - \frac{2-q}{2q}(\rho S_p)^{\frac{q}{q-2}}(\lambda\|a_+\|_{q^*})^{\frac{2}{2-q}}$$

and

$$J_{a_\lambda,b_\mu}(t^-(u)u) \leq \frac{(1+\rho)^{\frac{p}{p-2}}}{2} J_{0,b_0}(t_0(u)u) + \frac{2-q}{2q}(\rho S_p)^{\frac{q}{q-2}}(\lambda\|a_+\|_{q^*} + \|a_-\|_{q^*})^{\frac{2}{2-q}}.$$

Proof. (i) For each $u \in \mathcal{B}$, we consider

$$K_u(t) = J_{0,\alpha b_0}(tu) = \frac{1}{2}t^2 - \frac{1}{2}t^p \int_{\mathbb{R}^N} \alpha b_0 |u|^p dx,$$

so $K_u(t) \to -\infty$ as $t \to \infty$, and

$$K_u'(t) = t - t^{p-1} \int_{\mathbb{R}^N} \alpha b_0 |u|^p dx.$$

Thus, $K_u'(t_0^\alpha) = 0$, and $t_0^\alpha u \in M_{0,\alpha b_0}$ as

$$t_0^\alpha = t_0^\alpha(u) = \Big(\int_{\mathbb{R}^N} \alpha b_0 |u|^p dx \Big)^{\frac{1}{2-p}} > 0.$$

Moreover, $K_u''(t) = 1 - (p-1)t^{p-2} \int_{\mathbb{R}^N} \alpha b_0 |u|^p dx$. So, for $t_0^\alpha(u)$ we have

$$K_u''(t_0^\alpha) = 2 - p < 0,$$

that is, t_0^α is a maximum point of K_u. Then, there exists a unique $t_0^\alpha = t_0^\alpha(u) > 0$ such that $t_0^\alpha u \in M_{0,\alpha b_0}$ and also by definition $K_u(t) = J(tu)$ we obtain

$$\sup_{t \geq 0} J_{0,\alpha b_0}(tu) = J_{0,\alpha b_0}(t_0^\alpha u) = \frac{p-2}{2p} \Big(\int_{\mathbb{R}^N} \alpha b_0 |u|^p dx \Big)^{\frac{-2}{2-p}}.$$

(ii) We consider $\alpha = (1+\mu\|b_2/b_1\|_\infty)/(1-\rho)$. Then, for each $u \in \mathcal{B}$ and $\rho \in (0,1)$, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} a_\lambda |t_0^\alpha u|^q dx &\leq \lambda S_p^{-q/2} \|a_+\|_{q^*} \|t_0^\alpha u\|_A^q \\
&\leq \frac{2-q}{2} \Big((\rho S_p)^{\frac{-q}{2}} \lambda \|a_+\|_{q^*} \Big)^{\frac{2}{2-q}} + \frac{q}{2} \Big((\rho)^{\frac{q}{2}} \|t_0^\alpha u\|_A \Big)^{2/q} \quad (5.1)\\
&= \frac{2-q}{2} (\rho S_p)^{\frac{q}{q-2}} (\lambda \|a_+\|_{q^*})^{\frac{2}{2-q}} + \frac{q\rho}{2} \|t_0^\alpha u\|_A^2.
\end{aligned}$$

Then, from part (i) and by (5.1), we have

$$\begin{aligned}
&\sup_{t \geq 0} J_{a_\lambda, b_\mu}(tu) \\
&\geq J_{a_\lambda, b_\mu}(t_0^\alpha u) \\
&\geq \frac{1-\rho}{2} \|t_0^\alpha u\|_A^2 - \frac{2-q}{2q} (\rho S_p)^{\frac{q}{q-2}} (\lambda \|a_+\|_{q^*})^{\frac{2}{2-q}} \\
&\quad - \frac{(1+\mu\|b_2/b_1\|_\infty)}{p} \int_{\mathbb{R}^N} b_0 |t_0^\alpha u|^p dx \\
&= (1-\rho) J_{0,\alpha b_0}(t_0^\alpha u) - \frac{2-q}{2q} (\rho S_p)^{\frac{q}{q-2}} (\lambda \|a_+\|_{q^*})^{\frac{2}{2-q}} \\
&= \frac{(p-2)(1-\rho)^{\frac{p}{p-2}}}{2p((1+\mu\|b_2/b_1\|_\infty) \int_{\mathbb{R}^N} b_0 |u|^p dx)^{\frac{2}{p-2}}} - \frac{2-q}{2q} (\rho S_p)^{\frac{q}{q-2}} (\lambda \|a_+\|_{q^*})^{\frac{2}{2-q}} \\
&= \frac{(1-\rho)^{\frac{p}{p-2}}}{(1+\mu\|b_2/b_1\|_\infty)^{\frac{2}{p-2}}} J_{0,\alpha b_0}(t_0(u)u) - \frac{2-q}{2q} (\rho S_p)^{\frac{q}{q-2}} (\lambda \|a_+\|_{q^*})^{\frac{2}{2-q}}.
\end{aligned}$$

By Lemma 2.3 and by Theorem 2.5, we have

$$\sup_{t \geq 0} J_{a_\lambda, b_\mu}(tu) = J_{a_\lambda, b_\mu}(t^-(u)u).$$

Thus,

$$J_{a_\lambda,b_\mu}(t^-(u)u) \geq \frac{(1-\rho)^{\frac{p}{p-2}}}{(1+\mu\|b_2/b_1\|_\infty)^{\frac{2}{p-2}}} J_{0,\alpha b_0}(t_0(u)u) - \frac{2-q}{2q}(\rho S_p)^{\frac{q}{q-2}}(\lambda\|a_+\|_{q^*})^{\frac{2}{2-q}}.$$

Furthermore, by Hölder, Sobolev, and Young's inequalities,

$$\begin{aligned}
\Big|\int_{\mathbb{R}^N} a_\lambda |tu|^q dx\Big| &\leq \int_{\mathbb{R}^N} a_\lambda |tu|^q dx \leq (\lambda\|a_+\|_{q^*} + \|a_-\|_{q^*}) S_p^{-q/2}\|tu\|_A^q \\
&\leq \frac{2-q}{2}(\rho S_p)^{\frac{q}{q-2}}(\lambda\|a_+\|_{q^*} + \|a_-\|_{q^*})^{\frac{2}{2-q}} + \frac{q\rho}{2}\|tu\|_A^2.
\end{aligned}$$

Also

$$\begin{aligned}
J_{a_\lambda,b_\mu}(tu) &\leq \frac{(1+\rho)}{2}t^2 + \frac{2-q}{2q}(\rho S_p)^{\frac{q}{q-2}}(\lambda\|a_+\|_{q^*} + \|a_-\|_{q^*})^{\frac{2}{2-q}} - \frac{1}{p}\int_{\mathbb{R}^N} b_0|tu|^p dx \\
&\leq \frac{(1+\rho)^{\frac{p}{p-2}}}{2} J_{0,b_0}(t_0(u)u) + \frac{2-q}{2q}(\rho S_p)^{\frac{q}{q-2}}(\lambda\|a_+\|_{q^*} + \|a_-\|_{q^*})^{\frac{2}{2-q}}.
\end{aligned}$$

Then

$$J_{a_\lambda,b_\mu}(t^-(u)u) \leq \frac{(1+\rho)^{\frac{p}{p-2}}}{2} J_{0,b_0}(t_0(u)u) + \frac{2-q}{2q}(\rho S_p)^{\frac{q}{q-2}}(\lambda\|a_+\|_{q^*} + \|a_-\|_{q^*})^{\frac{2}{2-q}}.$$

As we wanted to prove. □

Note that as $m^-_{a_\lambda,b_\mu} > 0$ for all $\lambda \in (0,\lambda_0)$ and $\mu \in (0,\mu_0)$, we can define

$$I_{a_\lambda,b_\mu}(u) = \sup_{t\geq 0} J_{a_\lambda,b_\mu}(tu) = J_{a_\lambda,b_\mu}(t^-(u)u) > 0,$$

where $t^-(u)u \in M^-_{a_\lambda,b_\mu}$. We can see that if λ,μ and $\|a_-\|_{q^*}$ are sufficiently small, we can use the minimax Bahri-Li's argument [4] for our functional J_{a_λ,b_μ}. Let

$$\Gamma_{a_\lambda,b_\mu} = \{\gamma \in C(\overline{B^N(0,k)},\mathbb{B}) : \gamma|_{\partial B^N(0,k)} = w_k/\|w_k\|_A\}$$

be for values of l large enough. We define

$$\begin{gathered}
n_{a_\lambda,b_\mu} = \inf_{\gamma\in\Gamma_{a_\lambda,b_\mu}} \sup_{x\in\mathbb{R}^N} I_{a_\lambda,b_\mu}(\gamma(x)), \\
n_{0,b_0} = \inf_{\gamma\in\Gamma_{0,b_0}} \sup_{x\in\mathbb{R}^N} I_{0,b_0}(\gamma(x))
\end{gathered}$$

By Lemma 5.1(ii), for $0<\rho<1$, we have

$$n_{a_\lambda,b_\mu} \geq \frac{(1-\rho)^{\frac{p}{p-2}}}{(1+\mu\|b_2/b_1\|_\infty)^{\frac{2}{p-2}}} n_{0,b_0} - \frac{2-q}{2q}(\rho S_p)^{\frac{q}{q-2}}(\lambda\|a_+\|_{q^*})^{\frac{2}{2-q}}, \tag{5.2}$$

$$n_{a_\lambda,b_\mu} \leq (1+\rho)^{\frac{p}{p-2}} n_{0,b_0} + \frac{2-q}{2q}(\rho S_p)^{\frac{q}{q-2}}(\lambda\|a_+\|_{q^*} + \|a_-\|_{q^*})^{\frac{2}{2-q}}. \tag{5.3}$$

We will use the following estimates of the energy levels.

Lemma 5.2. *$m^\infty < n_{0,b_0} < 2m^\infty$.*

Proof. From the results by Bahri and Li [4] we have that (1.1), with $a_\lambda = 0$ and $b_\lambda = b_0$, admits at least one solution u_0 with $J_{0,b_0}(u_0) = n_{0,b_0} < 2m^\infty$. In addition, by (A4), problem (1.1), with $a_\lambda = 0$ and $b_\lambda = b_0$, does not have a minimum energy solution; this implies the lower estimates. □

Theorem 5.3. *Let λ_0 and μ_0 be as in Lemma 4.2. Then there will be positive values $\tilde{\lambda_0} \le \lambda_0$, $\tilde{\mu_0} \le \mu_0$ and $\tilde{\nu_0} \le \nu_0$ such that for $\lambda \in (0, \tilde{\lambda_0})$, $\mu \in (0, \tilde{\mu_0})$ and $\|a_-\|_{q^*} < \nu_0$, we have*

$$m^+_{a_\lambda,b_\mu} + m^\infty < n_{a_\lambda,b_\mu} < m^-_{a_\lambda,b_\mu} + m^\infty.$$

In addition, (1.1) *admits a solution v_{a_λ,b_μ} with*

$$J_{a_\lambda,b_\mu}(v_{a_\lambda,b_\mu}) = n_{a_\lambda,b_\mu}.$$

Proof. By Lemma 5.1(ii), for $0 < \rho < 1$ we have

$$\begin{aligned} m^-_{a_\lambda,b_\mu} &\ge \frac{(1-\rho)^{\frac{p}{p-2}}}{(1+\mu\|b_2/b_1\|_\infty)^{\frac{2}{p-2}}} m^\infty - \frac{2-q}{2q}(\rho S_p)^{\frac{q}{q-2}}(\lambda\|a_+\|_{q^*})^{\frac{2}{2-q}}, \\ m^-_{a_\lambda,b_\mu} &\le (1+\rho)^{\frac{p}{p-2}} m^\infty + \frac{2-q}{2q}(\rho S_p)^{\frac{q}{q-2}}(\lambda\|a_+\|_{q^*} + \|a_-\|_{q^*})^{\frac{2}{2-q}}. \end{aligned}$$

For each $\epsilon > 0$ there are positive values $\tilde{\lambda_1} \le \lambda_0$, $\tilde{\mu_1} \le \mu_0$ and ν_1 such that $\lambda \in (0, \tilde{\lambda_1})$, $\mu \in (0, \tilde{\mu_1})$, and $\|a_-\|_{q^*} < \nu_1$, we have

$$m^\infty - \epsilon < n_{a_\lambda,b_\mu} < m^\infty + \epsilon.$$

Then

$$2m^\infty - \epsilon < n_{a_\lambda,b_\mu} + m^\infty < 2m^\infty + \epsilon.$$

Using 5.2 and 5.3, for all $\delta > 0$ there will be positive values $\tilde{\lambda_2} \le \lambda_0$, $\tilde{\mu_2} \le \mu_0$, and ν_2, such that for $\lambda \in (0, \tilde{\lambda_2})$, $\mu \in (0, \tilde{\mu_2})$ and $\|a_-\|_{q^*} < \nu_2$, we have

$$n_{0,b_0} - \delta < n_{a_\lambda,b_\mu} < n_{0,b_0} + \delta.$$

Fixing small values of $0 < \epsilon < (2m^\infty - n_{0,b_0})/2$, and being $m^\infty < n_{0,b_0} < 2m^\infty$, and choosing $\delta > 0$ so that for $\lambda < \tilde{\lambda_0} = \min\{\tilde{\lambda_1}, \tilde{\lambda_2}\}$, $\mu < \tilde{\mu_0} = \min\{\tilde{\mu_1}, \tilde{\mu_2}\}$ and $\|a_-\|_{q^*} < \nu_0 = \min\{\nu_1, \nu_2\}$, we will have

$$m^+_{a_\lambda,b_\mu} + m^\infty < m^\infty < n_{a_\lambda,b_\mu} < 2m^\infty - \epsilon < m^-_{a_\lambda,b_\mu} + m^\infty.$$

Thus, by Proposition 3.1(ii), we obtain that problem (1.1) has a solution v_{a_λ,b_μ} with

$$J_{a_\lambda,b_\mu}(v_{a_\lambda,b_\mu}) = n_{a_\lambda,b_\mu}.$$ □

Proof of Theorem 1.1. With the result of Theorem 5.3 we can complete the proof of Theorem 1.1. For $\lambda \in (0, \tilde{\lambda_0})$, $\mu \in (0, \tilde{\mu_0})$ and $\|a_-\|_{q^*} < \nu_0$, also using the results presented in the introduction about the existence of the first three solutions and Theorem 5.3, we obtain that the equation (1.1) admits at least four solutions. □

Acknowledgments. F. O. Paiva received research grants from FAPESP 17/ 16108-6. S. M. S. Lima was supported by CAPES/Brazil. This article was completed while she was visiting the Departament of Mathematics of UFJF, whose hospitality she gratefully acknowledges. O. H. Miyagaki received research grants from CNPq/Brazil 303256/2022-2 and FAPESP/Brazil grant 2019/24901-3.

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