Electronic Journal of Differential Equations, Vol. 2023 (2023), No. 53, pp. 1–17.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2023.53

OPTIMAL ENERGY DECAY RATES FOR VISCOELASTIC WAVE EQUATIONS WITH NONLINEARITY OF VARIABLE EXPONENT

MUHAMMAD I. MUSTAFA

Abstract. In this article, we consider the viscoelastic wave equation
$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + a|u_t|^{m(\cdot)-2}u_t = 0$$
with a nonlinear feedback having a variable exponent $m(x)$. We investigate the interaction between the two types of damping and establish an optimal decay result under very general assumptions on the relaxation function g. We construct explicit formulae which provide faster energy decay rates than the ones already existing in the literature.

1. Introduction

In this article we consider the problem
$$\begin{gathered} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + a|u_t|^{m(\cdot)-2}u_t = 0, \quad \text{in } \Omega \times (0,T) \\ u = 0, \quad \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \end{gathered} \tag{1.1}$$
a wave equation subject to the effect of an internal frictional damping and a viscoelastic damping. Here Ω is a bounded domain of $\mathbb{R}^n$ ($n \geq 1$) with a smooth boundary $\partial\Omega$, a is a positive constant, g is a non-increasing positive function and $m \in C(\overline{\Omega})$ is satisfying
$$1 < m_1 \leq m(x) \leq m_2 < 2^*, \tag{1.2}$$
with
$$m_1 = \inf_{x\in\Omega} m(x), \quad m_2 = \sup_{x\in\Omega} m(x), \quad 2^* = \begin{cases} \frac{2n}{n-2}, & \text{if } n > 2 \\ \infty, & \text{if } n = 1, 2, \end{cases}$$
and satisfying the log-Hölder continuity condition
$$|m(x) - m(y)| \leq -\frac{A}{\log|x-y|}, \tag{1.3}$$
for a.e. $x, y \in \Omega$ with $|x-y| < \delta$, $0 < \delta < 1$, $A > 0$.

2020 *Mathematics Subject Classification.* 35B40, 74D99, 93D15, 93D20.
Key words and phrases. Viscoelasticity; frictional damping; variable exponent; energy decay.

Submitted March 22, 2023. Published August 28, 2023.

In the absence of the viscoelastic term, when m is a constant satisfying $1 < m < 2^*$, there have been many results about the existence and energy decay rates of the solutions, we refer the readers to [8, 15, 17, 29, 33] and the references therein. In recent years, more attention has been paid to the study of mathematical nonlinear models of hyperbolic, parabolic and elliptic equations with variable exponents of nonlinearity. Some models from physical phenomena like flows of electro-rheological fluids or fluids with temperature-dependent viscosity, filtration processes in a porous media, nonlinear viscoelasticity, and image processing, give rise to such problems. More details on the subject can be found in [4, 5]. Regarding hyperbolic problems with nonlinearities of variable-exponent type, only few works have appeared. The issues of existence and blow up of solutions were treated in [2, 3, 13, 21, 23, 32]. For the stability, Messaoudi and Talahmeh [22] looked at

$$u_{tt} - \Delta u + \alpha |u_t|^{m(x)-1} u_t = 0, \tag{1.4}$$

with $\alpha \equiv 1$ and $2 \leq m(x) < 2^*$, and proved decay estimates for the solution under suitable assumptions on the variable exponent m. Mustafa et al. [25, 28] obtained decay rate estimates for (1.4) in bounded and unbounded domains with $1 < m(x) < 2^*$ and a nonconstant time-dependent coefficient $\alpha(t)$.

On the other hand, when the unique damping mechanism is given by the memory term, many stability results have been established with different types of relaxation function g. When g decays exponentially (resp. polynomially), we refer to [16, 24, 31] for subsequent results showing that the energy decays at the same rate of g. For more general types of g, Messaoudi [19] used the condition

$$g'(t) \leq -\xi(t) g(t) \tag{1.5}$$

where ξ is a non-increasing differentiable function, and established a more general decay result. Also, a condition of the form

$$g'(t) \leq -H\big(g(t)\big) \tag{1.6}$$

where H is a convex function satisfying some smoothness properties, was introduced by Alabau-Boussouira and Cannarsa [1] to obtain decay results in terms of H. Mustafa [25] studied viscoelastic wave equations with relaxation functions of more general type than the ones in (1.5) and (1.6), namely

$$g'(t) \leq -\xi(t) H\left(g(t)\right) \tag{1.7}$$

where H is increasing and convex without any additional constraints, and established energy decay results that address both the optimality and generality.

The interaction between viscoelastic and frictional dampings was given a great deal of attention. We first refer to the work of Fabrizio and Polidoro [10] who studied the following equation

$$u_{tt} - \Delta u + \int_0^t g(t-\tau) \Delta u(\tau) d\tau + h(u_t) = 0, \tag{1.8}$$

with linear function h, and showed that the the viscoelasticity with poorly behaving relaxation kernel destroys the exponential decay rates generated by linear frictional dissipation. Cavalcanti and Oquendo [9] treated (1.8) and established exponential (resp. polynomial) stability for g decaying exponentially (resp. polynomially) and h has linear (resp. polynomial) growth near 0. Recently, Mustafa [27] obtained energy decay rates for (1.8) with general h and general g satisfying (1.7).

System (1.1), with constant m satisfying $1 < m < 2^*$ and g satisfying (1.5), was investigated by Messaoudi [20] and stability results depending on m and ξ were obtained. Later on, Belhannache et al. [7] extended the result of [20] to the case when g satisfies (1.7). For variable exponent $m(x)$, we refer to Gao and Gao [12] and Park and Kang [30] who studied (1.1), with nonlinear source term, and proved existence and blow up results. Hassan et al. [14] treated (1.1), used condition (1.7) and provided general energy estimates, but their results lack optimality in some cases.

Our aim in this work is to investigate (1.1), with $m(x)$ satisfying (1.2) and (1.3) and g satisfying (1.7). We study both cases when $m_1 \geq 2$ and $m_1 < 2$ and establish explicit formulae depending on both m and g which combine the generality and optimality and provide faster energy decay rates than the ones obtained in [14].

2. Preliminaries

In this section, we present some preliminary facts about Lebesgue and Sobolev spaces with variable exponents (see [11, 17]) and introduce our assumptions. Let $p : \Omega \to [1, \infty)$ be a measurable function, where Ω is a domain of $\mathbb{R}^n$. We define the Lebesgue space with a variable exponent $p(\cdot)$ by

$$L^{p(\cdot)}(\Omega) = \big\{u : \Omega \to \mathbb{R}, \text{measurable in } \Omega \text{ and } \varrho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}\,dx < \infty\big\}.$$

Equipped with the Luxembourg-type norm

$$\|u\|_{p(\cdot)} := \inf\{\lambda > 0 : \varrho_{p(\cdot)}\big(\frac{u}{\lambda}\big) \leq 1\},$$

$L^{p(\cdot)}(\Omega)$ is a Banach space. If $1 < p_1 \leq p(x) \leq p_2 < \infty$ holds, then, for any $u \in L^{p(\cdot)}(\Omega)$,

$$\min\{\|u\|_{p(\cdot)}^{p_1}, \|u\|_{p(\cdot)}^{p_2}\} \leq \varrho_{p(\cdot)}(u) \leq \max\{\|u\|_{p(\cdot)}^{p_1}, \|u\|_{p(\cdot)}^{p_2}\}.$$

We, next, define the variable-exponent Sobolev space

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : \nabla u \text{ exists and } |\nabla u| \in L^{p(\cdot)}(\Omega)\}.$$

This space is a Banach space with respect to the norm $\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}$. Furthermore, we set $W_0^{1,p(\cdot)}(\Omega)$ to be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. Here we note that the space $W_0^{1,p(\cdot)}(\Omega)$ is usually defined in a different way for the variable exponent case. However, both definitions are equivalent under (1.3).

Hölder's Inequality: Let $p, q, s \geq 1$ be measurable functions defined on Ω such that

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)}, \quad \text{for a.e. } y \in \Omega.$$

If $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{q(\cdot)}(\Omega)$, then $fg \in L^{s(\cdot)}(\Omega)$ and

$$\|fg\|_{s(\cdot)} \leq 2\,\|f\|_{p(\cdot)}\|g\|_{q(\cdot)}.$$

Poincaré's Inequality: Let Ω be a bounded domain of $\mathbb{R}^n$ and $p(\cdot)$ satisfies (1.3), then

$$\|u\|_{p(\cdot)} \leq C\|\nabla u\|_{p(\cdot)}, \quad \text{for all } u \in W_0^{1,p(\cdot)}(\Omega),$$

where the positive constant C depends on p_1, p_2 and Ω only. In particular, the space $W_0^{1,p(\cdot)}(\Omega)$ has an equivalent norm given by $\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{p(\cdot)}$.

Embedding Property: Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Assume that $p, q \in C(\overline{\Omega})$ such that

$$1 < p_1 \le p(x) \le p_2 < +\infty, \quad 1 < q_1 \le q(x) \le q_2 < +\infty, \quad \text{for all } x \in \overline{\Omega},$$

and $q(x) < p^*(x)$ in $\overline{\Omega}$ with

$$p^*(x) = \begin{cases} \frac{np(x)}{n-p(x)}, & \text{if } p_2 < n \\ +\infty, & \text{if } p_2 \ge n, \end{cases}$$

then there is a continuous and compact embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$.

On the relaxation function g and the variable exponent $m(x)$, we consider the following assumption

(A1) $m \in C(\overline{\Omega})$ is satisfying (1.2) and (1.3) and $g : [0, \infty) \to (0, \infty)$ is a C^1 function satisfying

$$1 - \int_0^{+\infty} g(s) ds = l > 0, \tag{2.1}$$

$$g'(t) \le -\xi(t) H_1(g(t)), \quad \forall t \ge 0, \tag{2.2}$$

where $\xi : [0, \infty) \to (0, \infty)$ is a non-increasing differentiable function and $H_1 : (0, \infty) \to (0, \infty)$ is a C^1 function which is linear or strictly increasing and strictly convex ^{2} function on $(0, r_1]$, with $H_1(0) = H_1'(0) = 0$.

Remarks. (1) The function $H_2(t) = t^{\frac{m_1}{2m_1-2}}$ is strictly increasing and strictly convex when $1 < m_1 < 2$. We will be using $H(t) = \min\{H_1(t), H_2(t)\}$ and $r \le r_1$ is small enough so that either $H(t) = H_1(t)$ or $H(t) = H_2(t)$ on the interval $(0, r]$.
(2) The well-known Jensen inequality will be of essential use in establishing our result. If Y is a convex function on $[d_1, d_2]$, $y : \Omega \to [d_1, d_2]$ and w are integrable functions on Ω, $w(x) \ge 0$, and $\int_\Omega w(x)\, dx = d_3 > 0$, then Jensen's inequality states

$$Y\Big[\frac{1}{d_3} \int_\Omega y(x) w(x)\, dx\Big] \le \frac{1}{d_3} \int_\Omega Y[y(x)] w(x)\, dx.$$

(3) If H^* be the convex conjugate of H in the sense of Young [6, p. 61-64], then

$$H^*(s) = s(H')^{-1}(s) - H[(H')^{-1}(s)]$$

and H^* satisfies the generalized Young inequality

$$AB \le H^*(A) + H(B). \tag{2.3}$$

(4) If H is a strictly increasing and strictly convex ^{2} function on $(0, r]$, with $H(0) = H'(0) = 0$, then it has an extension $\overline{H}$ which is strictly increasing and strictly convex ^{2} function on $(0, \infty)$. For instance, if $H(r) = a, H'(r) = b, H''(r) = c$, we can define $\overline{H}$, for $t > r$, by

$$\overline{H}(t) = \frac{c}{2} t^2 + (b - cr)t + (a + \frac{c}{2} r^2 - br). \tag{2.4}$$

At the end of this section, we state, without proof, the following existence and regularity result.

Proposition 2.1 ([11, 30]). *Let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ be given. If* (A1) *holds, then problem* (1.1) *has a unique global (weak) solution*

$$u \in L^\infty((0, T); H_0^1(\Omega)), u_t \in L^\infty((0, T); L^2(\Omega)) \cap L^{m(\cdot)}((0, T) \times \Omega).$$

Moreover, if $(u_0, u_1) \in \big(H^2(\Omega) \cap H_0^1(\Omega)\big) \times H_0^1(\Omega)$*, then the solution satisfies*

$$u \in L^\infty((0,T); H^2(\Omega) \cap H_0^1(\Omega)) \cap W^{1,\infty}((0,T); H_0^1(\Omega)) \cap W^{2,\infty}((0,T); L^2(\Omega)).$$

3. Technical Lemmas

We introduce the energy functional

$$E(t) := \frac{1}{2}\int_\Omega \Big(u_t^2 + [1 - \int_0^t g(s)\,ds]|\nabla u|^2\Big)\,dx + \frac{1}{2}(g \circ \nabla u)(t),$$

where

$$(g \circ v)(t) = \int_\Omega \int_0^t g(t-s)|v(t) - v(s)|^2\,ds\,dx.$$

In this section, we establish several lemmas and construct a Lyapunov functional $\mathcal{L}$ equivalent to E. We will use c, in this paper, to denote a generic positive constant.

Lemma 3.1. *Let u be the solution of* (1.1). *Then the energy functional satisfies*

$$E'(t) = \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)\int_\Omega |\nabla u|^2\,dx - a\int_\Omega |u_t|^{m(x)}\,dx \le 0. \tag{3.1}$$

Proof. By multiplying equation (1.1) by u_t and integrating over Ω, using integration by parts, hypothesis (A) and some manipulations, we obtain (3.1). $\square$

We consider the following partition of Ω,

$$\Omega_* = \{x \in \Omega : m(x) < 2\} \quad \text{and} \quad \Omega_{**} = \{x \in \Omega : m(x) \ge 2\}.$$

Lemma 3.2. *The functional K_1 defined by*

$$K_1(t) := \int_\Omega u u_t\,dx \tag{3.2}$$

satisfies, along the solution of (1.1), *the estimate*

$$\begin{aligned} K_1'(t) \le &-\frac{l}{4}\int_\Omega |\nabla u|^2\,dx + \int_\Omega u_t^2\,dx + \frac{C_\alpha}{l}(h \circ \nabla u)(t) + c\int_{\Omega_*} |u_t|^{2m(x)-2}\,dx \\ &+ c\int_\Omega |u_t|^{m(x)}\,dx \end{aligned} \tag{3.3}$$

for any $0 < \alpha < 1$, where

$$C_\alpha = \int_0^\infty \frac{g^2(s)}{\alpha g(s) - g'(s)}ds \quad \textit{and} \quad h(t) = \alpha g(t) - g'(t). \tag{3.4}$$

Proof. Direct computations, using (1.1), (2.1), and Young's inequality, yield

$$\begin{aligned} &K_1'(t) \\ &= \int_\Omega u_t^2\,dx + \int_\Omega u\Delta u\,dx - \int_\Omega u\int_0^t g(t-s)\Delta u(s)\,ds\,dx - a\int_\Omega u|u_t|^{m(x)-2}u_t\,dx \\ &= \int_\Omega u_t^2\,dx - \Big(1 - \int_0^t g(s)ds\Big)\int_\Omega |\nabla u|^2\,dx \\ &\quad + \int_\Omega \nabla u \cdot \int_0^t g(t-s)(\nabla u(s) - \nabla u(t))\,ds\,dx - a\int_\Omega |u_t|^{m(x)-2}u_t u\,dx \\ &\le \int_\Omega u_t^2\,dx - l\int_\Omega |\nabla u|^2\,dx + \frac{1}{l}\int_\Omega \Big(\int_0^t g(t-s)|\nabla u(s) - \nabla u(t)|ds\Big)^2\,dx \end{aligned}$$

$$+\frac{l}{4}\int_\Omega |\nabla u|^2\,dx + a\int_{\Omega_*} |u_t|^{m(x)-1}|u|\,dx + a\int_{\Omega_{**}} |u_t|^{m(x)-1}|u|\,dx.$$

Using Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\int_\Omega \Big(\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)|\,ds\Big)^2 dx\\
&= \int_\Omega \Big(\int_0^t \frac{g(t-s)}{\sqrt{\alpha g(t-s)-g'(t-s)}}\sqrt{\alpha g(t-s)-g'(t-s)}|\nabla u(s)-\nabla u(t)|ds\Big)^2 dx\\
&\le \Big(\int_0^t \frac{g^2(s)}{\alpha g(s)-g'(s)}ds\Big)\int_\Omega\int_0^t \big[\alpha g(t-s)-g'(t-s)\big]|\nabla u(s)-\nabla u(t)|^2\,ds\,dx\\
&\le C_\alpha (h\circ\nabla u)(t).
\end{aligned} \tag{3.5}$$

Now, using Young's and Poincaré's inequalities,

$$\begin{aligned}
a\int_{\Omega_*} |u_t|^{m(x)-1}|u|\,dx &\le \int_{\Omega_*}\big[\delta_0|u|^2 + \frac{a^2}{4\delta_0}|u_t|^{2m(x)-2}\big]\,dx\\
&\le c\delta_0\|\nabla u\|_2^2 + \frac{a^2}{4\delta_0}\int_{\Omega_*}|u_t|^{2m(x)-2}\,dx.
\end{aligned}$$

On the other hand, If meas $(\Omega_{**})\neq 0$ so $m_2\ge 2$, we use Young's inequality with $p(x)=\frac{m(x)}{m(x)-1}$ and $p'(x)=m(x)$ for $x\in\Omega_{**}$ and the Sobolev embedding $H_0^1(\Omega)\hookrightarrow L^{m_2}(\Omega)$ to obtain

$$\begin{aligned}
a\int_{\Omega_{**}} |u_t|^{m(x)-1}|u|\,dx &\le a\int_{\Omega_{**}}\big[\varepsilon|u|^{m(x)} + C_\varepsilon(x)|u_t|^{m(x)}\big]\,dx\\
&\le a\varepsilon\int_\Omega \Big(|u|^2+|u|^{m_2}\Big)\,dx + a\int_{\Omega_{**}} C_\varepsilon(x)|u_t|^{m(x)}\,dx\\
&\le c\varepsilon\left(\|\nabla u\|_2^2+\|\nabla u\|_2^{m_2}\right) + a\int_{\Omega_{**}} C_\varepsilon(x)|u_t|^{m(x)}\,dx\\
&\le c\varepsilon\left(1+\|\nabla u\|_2^{m_2-2}\right)\|\nabla u\|_2^2 + a\int_{\Omega_{**}} C_\varepsilon(x)|u_t|^{m(x)}\,dx\\
&\le c\varepsilon\Big(1+E(0)^{\frac{m_2-2}{2}}\Big)\|\nabla u\|_2^2 + a\int_{\Omega_{**}} C_\varepsilon(x)|u_t|^{m(x)}\,dx.
\end{aligned}$$

If we fix $\varepsilon=\frac{l}{4c\big(1+E(0)^{\frac{m_2-2}{2}}\big)}$, then

$$C_\varepsilon(x)=\frac{m(x)-1}{[m(x)]^{\frac{m(x)}{m(x)-1}}\varepsilon^{\frac{1}{m(x)-1}}}$$

is bounded since $m(x)$ is bounded, and we obtain

$$a\int_{\Omega_{**}} |u_t|^{m(x)-1}|u|\,dx \le \frac{l}{4}\|\nabla u\|_2^2 + c\int_\Omega |u_t|^{m(x)}\,dx.$$

Combining all the above estimates, with δ_0 small enough, gives (3.3). $\square$

Lemma 3.3. *The functional K_2 defined by*

$$K_2(t) := -\int_\Omega u_t\int_0^t g(t-s)(u(t)-u(s))\,ds\,dx \tag{3.6}$$

satisfies, for any $0 < \delta < 1$, the estimate

$$
\begin{aligned}
K_2'(t) \le -\Big(\int_0^t g(s)ds - \delta\Big)\int_\Omega u_t^2\,dx + \delta\int_\Omega |\nabla u|^2\,dx + \frac{c[C_\alpha+1]}{\delta}(h\circ\nabla u)(t) \\
+ c\delta(g\circ\nabla u)(t) + \frac{a^2}{4}\int_{\Omega_*}|u_t|^{2m(x)-2}\,dx + a\int_\Omega C_\delta(x)|u_t|^{m(x)}\,dx.
\end{aligned}
\tag{3.7}
$$

Proof. By using (1.1) and integrating by parts, we have

$$
\begin{aligned}
K_2'(t) &= \int_\Omega \nabla u\cdot\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx \\
&\quad - \int_\Omega\Big(\int_0^t g(t-s)\nabla u(s)ds\Big)\Big(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\Big)\,dx \\
&\quad - \int_\Omega u_t\int_0^t g'(t-s)(u(t)-u(s))\,ds\,dx - \Big(\int_0^t g(s)ds\Big)\int_\Omega u_t^2\,dx \\
&\quad + a\int_\Omega |u_t|^{m(x)-2}u_t\int_0^t g(t-s)(u(t)-u(s))\,ds\,dx \\
&= \Big(1-\int_0^t g(s)ds\Big)\int_\Omega\nabla u\cdot\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx \\
&\quad + \int_\Omega\Big(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\Big)^2\,dx \\
&\quad - \int_\Omega u_t\int_0^t g'(t-\tau)(u(t)-u(\tau))d\tau\,dx - \Big(\int_0^t g(s)\,ds\Big)\int_\Omega u_t^2\,dx \\
&\quad + a\int_{\Omega_*}|u_t|^{m(x)-2}u_t\int_0^t g(t-s)(u(t)-u(s))\,ds\,dx \\
&\quad + a\int_{\Omega_{**}}|u_t|^{m(x)-2}u_t\int_0^t g(t-s)(u(t)-u(s))\,ds\,dx
\end{aligned}
$$

Using Young's and Poincaré's inequalities and similar calculations as in (3.5), we obtain

$$
\begin{aligned}
&\Big(1-\int_0^t g(s)ds\Big)\int_\Omega\nabla u\cdot\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx \\
&\le \delta\int_\Omega|\nabla u|^2\,dx + \frac{c}{\delta}C_\alpha(h\circ\nabla u)(t),
\end{aligned}
$$

$$
\begin{aligned}
&-\int_\Omega u_t\int_0^t g'(t-s)(u(t)-u(s))\,ds\,dx \\
&= \int_\Omega u_t\int_0^t h(t-s)(u(t)-u(s))\,ds\,dx - \int_\Omega u_t\int_0^t \alpha g(t-s)(u(t)-u(s))\,ds\,dx \\
&\le \delta\int_\Omega u_t^2\,dx + \frac{1}{2\delta}\int_\Omega\Big(\int_0^t\sqrt{h(t-s)}\sqrt{h(t-s)}|u(s)-u(t)|ds\Big)^2\,dx \\
&\quad + \frac{\alpha^2}{2\delta}\int_\Omega\Big(\int_0^t g(t-s)|u(s)-u(t)|ds\Big)^2\,dx \\
&\le \delta\int_\Omega u_t^2\,dx + \frac{\big(\int_0^t h(s)\,ds\big)}{2\delta}(h\circ u)(t) + \frac{\alpha^2C_\alpha}{2\delta}(h\circ u)(t)
\end{aligned}
$$

$$\le \delta \int_\Omega u_t^2\,dx + \frac{c}{\delta}(h\circ\nabla u)(t) + \frac{cC_\alpha}{\delta}(h\circ\nabla u)(t),$$

and

$$\begin{aligned}
&a\int_{\Omega_*}|u_t|^{m(x)-2}u_t\int_0^t g(t-s)(u(t)-u(s))\,ds\,dx\\
&\le \int_{\Omega_*}\Big(\int_0^t g(t-s)(u(t)-u(s))\,ds\Big)^2\,dx + \frac{a^2}{4}\int_{\Omega_*}|u_t|^{2m(x)-2}\,dx\\
&\le cC_\alpha(h\circ\nabla u)(t) + \frac{a^2}{4}\int_{\Omega_*}|u_t|^{2m(x)-2}\,dx\,.
\end{aligned}$$

On Ω_{**}, we use a similar argument as in Lemma 3.2 to obtain

$$\begin{aligned}
&a\int_{\Omega_{**}}|u_t|^{m(x)-2}u_t\int_0^t g(t-s)(u(t)-u(s))\,ds\,dx\\
&\le a\delta\int_{\Omega_{**}}\Big|\int_0^t g^{\frac{m(x)-1}{m(x)}}(t-s)g^{\frac{1}{m(x)}}(t-s)(u(t)-u(s))ds\Big|^{m(x)}\,dx\\
&\quad + a\int_{\Omega_{**}}C_\delta(x)|u_t|^{m(x)}\,dx\\
&\le a\delta\int_{\Omega_{**}}\Big(\int_0^t g(s)ds\Big)^{m(x)-1}\Big(\int_0^t g(t-s)|u(t)-u(s)|^{m(x)}ds\Big)\,dx\\
&\quad + a\int_{\Omega_{**}}C_\delta(x)|u_t|^{m(x)}\,dx\\
&\le c\delta\int_\Omega\int_0^t g(t-s)\left(|u(t)-u(s)|^2+|u(t)-u(s)|^{m_2}\right)\,ds\,dx\\
&\quad + a\int_{\Omega_{**}}C_\delta(x)|u_t|^{m(x)}\,dx\\
&\le c\delta\int_0^t g(t-s)\left(\|\nabla u(t)-\nabla u(s)\|_2^2+\|\nabla u(t)-\nabla u(s)\|_2^{m_2}\right)\,ds\\
&\quad + a\int_{\Omega_{**}}C_\delta(x)|u_t|^{m(x)}\,dx\\
&\le c\delta\int_0^t g(t-s)\left(1+\|\nabla u(t)-\nabla u(s)\|_2^{m_2-2}\right)\|\nabla u(t)-\nabla u(s)\|_2^2\,ds\\
&\quad + a\int_{\Omega_{**}}C_\delta(x)|u_t|^{m(x)}\,dx\\
&\le c\delta\Big(1+E(0)^{\frac{m_2-2}{2}}\Big)(g\circ\nabla u)(t) + a\int_{\Omega_{**}}C_\delta(x)|u_t|^{m(x)}\,dx.
\end{aligned}$$

Combining the above estimates, (3.7) is established. □

Next, we use the functional

$$K_3(t)=\int_\Omega\int_0^t f(t-s)|\nabla u(s)|^2\,ds\,dx \tag{3.8}$$

where $f(t)=\int_t^\infty g(s)ds$.

Lemma 3.4. *The functional K_3 satisfies, along the solution of* (1.1), *the estimate*
$$K_3'(t) \le -\frac{1}{2}(g \circ \nabla u)(t) + 3(1-l)\int_\Omega |\nabla u(t)|^2\,dx. \tag{3.9}$$

Proof. By Young's inequality and the fact that $f'(t) = -g(t)$, we see that
$$\begin{aligned}
K_3'(t) &= f(0)\int_\Omega |\nabla u(t)|^2\,dx - \int_\Omega\int_0^t g(t-s)|\nabla u(s)|^2\,ds\,dx\\
&= -\int_\Omega\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)|^2\,ds\,dx\\
&\quad - 2\int_\Omega \nabla u(t)\cdot\int_0^t g(t-s)(\nabla u(s)-\nabla u(t))\,ds\,dx + f(t)\int_\Omega |\nabla u(t)|^2\,dx.
\end{aligned}$$
But
$$\begin{aligned}
&- 2\int_\Omega \nabla u(t)\cdot\int_0^t g(t-s)(\nabla u(s)-\nabla u(t))\,ds\,dx\\
&\le 2(1-l)\int_\Omega |\nabla u(t)|^2\,dx + \frac{\int_0^t g(s)ds}{2(1-l)}\int_\Omega\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)|^2\,ds\,dx.
\end{aligned}$$
Then, as $f(t) \le f(0) = (1-l)$ and $\int_0^t g(s)ds \le (1-l)$, we obtain (3.9). □

Lemma 3.5. *The functional $\mathcal{L}$ defined by*
$$\mathcal{L}(t) := NE(t) + N_1K_1(t) + N_2K_2(t)$$
for suitable choice of $N, N_1, N_2 > 0$ and for all $t \ge t_1$, satisfies
$$\mathcal{L}'(t) \le -4(1-l)\int_\Omega |\nabla u|^2\,dx - \int_\Omega u_t^2\,dx + \frac{1}{4}(g\circ\nabla u)(t) + c\int_{\Omega_*} |u_t|^{2m(x)-2}\,dx \tag{3.10}$$
and
$$\mathcal{L}(t) \sim E(t) \tag{3.11}$$
which means that, for some constants $a_1, a_2 > 0$,
$$a_1E(t) \le \mathcal{L}(t) \le a_2E(t).$$

Proof. Let $g_1 = \int_0^{t_1} g(s)ds > 0$ for some fixed $t_1 > 0$. By combining (3.1), (3.3), (3.7), recalling that $g' = (\alpha g - h)$, and taking $\delta = 1/(8cN_2)$, we obtain that for all $t \ge t_1$,
$$\begin{aligned}
&\mathcal{L}'(t)\\
&\le -\big(\frac{l}{4}N_1 - \frac{1}{8c}\big)\int_\Omega |\nabla u|^2\,dx - \big(g_1N_2 - \frac{1}{8c} - N_1\big)\int_\Omega u_t^2\,dx\\
&\quad + \big(\frac{\alpha}{2}N + \frac{1}{8}\big)(g\circ\nabla u)(t) - \big(\frac{1}{2}N - 8c^2N_2^2 - C_\alpha[\frac{1}{l}N_1 + 8c^2N_2^2]\big)(h\circ\nabla u)(t)\\
&\quad - \int_\Omega (aN - cN_1 - aC_\delta(x)N_2)\,|u_t|^{m(x)}\,dx + \big(cN_1 + \frac{a^2}{4}N_2\big)\int_{\Omega_*} |u_t|^{2m(x)-2}\,dx
\end{aligned}$$
Now we choose N_1 large enough so that
$$\frac{l}{4}N_1 - \frac{1}{8c} > 4(1-l),$$

and N_2 large enough so that

$$g_1 N_2 - \frac{1}{8c} - N_1 > 1.$$

As δ now is fixed and $C_\delta(x)$ is bounded, we have

$$-\left(aN - cN_1 - aC_\delta(x)N_2\right) \leq -\left(aN - cN_1 - cN_2\right).$$

Next, as $\frac{\alpha g^2(s)}{\alpha g(s)-g'(s)} < g(s)$, it is easy to show, using the Lebesgue dominated convergence theorem, that

$$\alpha C_\alpha = \int_0^\infty \frac{\alpha g^2(s)}{\alpha g(s) - g'(s)} ds \longrightarrow 0 \quad \text{as } \alpha \longrightarrow 0.$$

Hence, there is $0 < \alpha_0 < 1$ such that if $\alpha < \alpha_0$, then

$$\alpha C_\alpha < \frac{1}{16[\frac{1}{l}N_1 + 8c^2N_2^2]}.$$

Let us choose N large enough and choose α satisfying

$$aN - cN_1 - cN_2 > 0, \quad \frac{1}{4}N - 8c^2N_2^2 > 0, \quad \alpha = \frac{1}{4N} < \alpha_0$$

which implies

$$\frac{1}{2}N - 8c^2N_2^2 - C_\alpha[\frac{1}{2l}N_1 + 8c^2N_2^2] > 0.$$

So, we arrive at

$$\mathcal{L}'(t) \leq -4(1-l)\int_\Omega |\nabla u|^2\,dx - \int_\Omega u_t^2\,dx + \frac{1}{4}(g \circ \nabla u)(t) + c\int_{\Omega_*} |u_t|^{2m(x)-2}\,dx.$$

On the other hand, we find that

$$\begin{aligned}
&|\mathcal{L}(t) - NE(t)| \\
&\leq N_1|K_1(t)| + N_2|K_2(t)| \\
&\leq N_1\int_\Omega |uu_t|\,dx + N_2\int_\Omega \Big|u_t\int_0^t g(t-s)(u(t)-u(s))ds\Big|\,dx \\
&\leq \frac{N_1}{2}\int_\Omega u^2\,dx + \frac{N_1+N_2}{2}\int_\Omega u_t^2\,dx + \frac{N_2}{2}\int_\Omega \Big|\int_0^t g(t-s)(u(t)-u(s))ds\Big|^2\,dx \\
&\leq c\Big[\int_\Omega |\nabla u|^2\,dx + \int_\Omega u_t^2\,dx + (g\circ\nabla u)(t)\Big] \\
&\leq cE(t).
\end{aligned}$$

Therefore, we can choose N even larger (if needed) so that (3.11) is satisfied. $\square$

4. Main result

Theorem 4.1. *Assume that* (A1) *holds. Then there exist positive constants $\varepsilon_0 \leq r$, $k_1 \leq 1$, and k_2 such that the energy functional satisfies*

$$E(t) \leq k_2 H_0^{-1}\Big(k_1\int_0^t \xi(s)ds\Big) \tag{4.1}$$

where

$$H_0(t)=\begin{cases}\int_t^1 \frac{1}{sH_1'(\varepsilon_0 s)}ds, & \text{if } m_1\ge 2\\ \int_t^1 \frac{1}{sH_2'(\varepsilon_0 s)}ds, & \text{if } 1<m_1<2 \text{ and } H_1 \text{ is linear}\\ \int_t^1 \frac{1}{sH'(\varepsilon_0 s)}ds, & \text{if } \frac{4}{3}<m_1<2 \text{ and } H_1 \text{ is nonlinear}\\ \int_t^1 \frac{1}{sH'(\varepsilon_0 s^{\frac{m_1}{2m_1-2}})}ds, & \text{if } 1<m_1\le\frac{4}{3} \text{ and } H_1 \text{ is nonlinear.}\end{cases}$$

Here, H_0 is strictly decreasing and convex on $(0,r]$, with $\lim_{t\to 0}H_0(t)=+\infty$.

Proof. We start by estimating the last term in (3.10). When $m_1\ge 2$, we have that

$$\operatorname{meas}(\Omega_*)=0 \implies \int_{\Omega_*}|u_t|^{2m(x)-2}\,dx=0.$$

But, if $1<m_1<2$, with

$$\Omega_1=\{x\in\Omega_*:|u_t|\le 1\},\quad \Omega_2=\Omega_*\backslash\Omega_1$$

and $\frac{2m(x)-2}{m(x)}\ge\frac{2m_1-2}{m_1}$, we use Jensen's inequality to obtain

$$\begin{aligned}\int_{\Omega_*}|u_t|^{2m(x)-2}\,dx&=\int_{\Omega_1}|u_t|^{2m(x)-2}\,dx+\int_{\Omega_2}|u_t|^{2m(x)-2}\,dx\\&=\int_{\Omega_1}\big[|u_t|^{m(x)}\big]^{\frac{2m(x)-2}{m(x)}}\,dx+\int_{\Omega_2}|u_t|^{m(x)+m(x)-2}\,dx\\&\le\int_{\Omega_1}\big[|u_t|^{m(x)}\big]^{\frac{2m_1-2}{m_1}}\,dx+\int_{\Omega_2}|u_t|^{m(x)}\,dx\\&\le\Big[\int_{\Omega_1}|u_t|^{m(x)}\,dx\Big]^{\frac{2m_1-2}{m_1}}+\int_{\Omega_2}|u_t|^{m(x)}\,dx\\&\le c[-E'(t)]^{\frac{2m_1-2}{m_1}}-cE'(t)\\&=cH_2^{-1}(-E'(t))-cE'(t)\end{aligned}\tag{4.2}$$

and use Young's inequality to obtain

$$\frac{E(t)^{\frac{2-m_1}{2m_1-2}}[-E'(t)]^{\frac{2m_1-2}{m_1}}}{E(t)^{\frac{2-m_1}{2m_1-2}}}\le\frac{\varepsilon E(t)^{\frac{m_1}{2m_1-2}}-C_\varepsilon E'(t)}{E(t)^{\frac{2-m_1}{2m_1-2}}}=\varepsilon E(t)-\frac{C_\varepsilon E'(t)}{E(t)^{\frac{2-m_1}{2m_1-2}}}.$$

With $\varepsilon=\frac{b_0}{2c}$ and $b_0=\min\{2(1-l),1/2\}$, this yields

$$c\int_{\Omega_*}|u_t|^{2m(x)-2}\,dx\le\frac{b_0}{2}E(t)-\frac{cE'(t)}{E(t)^{\frac{2-m_1}{2m_1-2}}}-cE'(t).\tag{4.3}$$

Next, we prove that

$$\begin{gathered}\int_0^\infty E(s)ds<\infty,\quad \text{if } m_1>\frac{4}{3},\\ \int_0^\infty E(s)^{\frac{m_1}{2m_1-2}}ds<\infty,\quad \text{if } 1<m_1\le\frac{4}{3}\end{gathered}\tag{4.4}$$

For this purpose, we use Lemmas 3.4 and 3.5 to deduce that $L(t) = \mathcal{L}(t) + K_3(t)$ is nonnegative and it satisfies, for all $t \geq t_1$,

$$\begin{aligned} L'(t) &\leq -\Big[(1-l)\int_\Omega |\nabla u|^2\,dx + \int_\Omega u_t^2\,dx + \frac{1}{4}(g \circ \nabla u)\Big] + c\int_{\Omega_*} |u_t|^{2m(x)-2}\,dx \\ &\leq -b_0 E(t) + c\int_{\Omega_*} |u_t|^{2m(x)-2}\,dx. \end{aligned} \tag{4.5}$$

If $m_1 \geq 2$ then $\mathrm{meas}(\Omega_*) = 0$, so $L'(t) \leq -b_0 E(t)$ implies

$$b_0 \int_0^t E(s)ds \leq -\int_0^t L'(s)ds \leq L(0) - L(t) \leq L(0).$$

If $1 < m_1 < 2$, then, using (4.3), (4.5) becomes

$$L'(t) \leq -\frac{b_0}{2}E(t) - \frac{cE'(t)}{E(t)^{\frac{2-m_1}{2m_1-2}}} - cE'(t).$$

Here, when $4/3 < m_1 < 2$, we notice that

$$\begin{aligned} \int_0^t \frac{-cE'(s)}{E(t)^{\frac{2-m_1}{2m_1-2}}}ds &= \frac{c(2m_1-2)}{3m_1-4}\Big[E(0)^{\frac{3m_1-4}{2m_1-2}} - E(t)^{\frac{3m_1-4}{2m_1-2}}\Big] \\ &\leq \frac{c(2m_1-2)}{3m_1-4}E(0)^{\frac{3m_1-4}{2m_1-2}} = d_0 \end{aligned}$$

implies

$$\frac{b_0}{2}\int_0^t E(s)ds \leq -\int_0^t \Big[L'(s) + cE'(t) + \frac{cE'(s)}{E(t)^{\frac{2-m_1}{2m_1-2}}}\Big]ds \leq L(0) + cE(0) + d_0$$

which gives $(4.4)_1$. Otherwise,

$$E(t)^{\frac{2-m_1}{2m_1-2}}\left[L'(t) + cE'(t)\right] \leq -\frac{b_0}{2}E(t)^{\frac{m_1}{2m_1-2}} - cE'(t).$$

This means, as $E(t)$ is decreasing, that

$$L_0(t) = E(t)^{\frac{2-m_1}{2m_1-2}}\left[L(t) + cE(t)\right] + cE(t)$$

is nonnegative and

$$L_0'(t) \leq -\frac{b_0}{2}E(t)^{\frac{m_1}{2m_1-2}}, \quad \text{for all } t \geq t_1$$

which now gives $(4.4)_2$.

To this end, we multiply (3.10) by $\xi(t)$ and use (4.2), the fact that ξ is non-increasing and the functional $F := \xi\mathcal{L} + cE$ satisfies $F \sim E$ and deduce, for some constant $m > 0$ and for all $t \geq t_1$,

$$F'(t) \leq \begin{cases} -m\xi(t)E(t) + c\xi(t)(g \circ \nabla u)(t), & \text{if } m_1 \geq 2 \\ -m\xi(t)E(t) + c\xi(t)(g \circ \nabla u)(t) + c\xi(t)H^{-1}(-E'(t)), & \text{if } 1 < m_1 < 2. \end{cases} \tag{4.6}$$

Case 1: H_1 is nonlinear on $[0, r]$ and $1 < m_1 \leq 4/3$. First, we define

$$I(t) := \int_0^t \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2\,dx\,ds$$

and consider $\overline{H}$ to be an extension of H such that $\overline{H}$ is a strictly increasing and strictly convex [2] function on $(0,\infty)$ [see (2.4)], then the use of hypothesis (2.2), (3.1), and Jensen's inequality leads to

$$\begin{aligned}
&\int_0^t g(s)\int_\Omega |\nabla u(t)-\nabla u(t-s)|^2\,dx\,ds\\
&\le \frac{I(t)}{I(t)}\int_0^t \overline{H}^{-1}\Big(\frac{-g'(s)}{\xi(s)}\Big)\int_\Omega |\nabla u(t)-\nabla u(t-s)|^2\,dx\,ds\\
&\le I(t)\overline{H}^{-1}\Big(\frac{1}{I(t)}\int_0^t\Big(\frac{-g'(s)}{\xi(s)}\Big)\int_\Omega |\nabla u(t)-\nabla u(t-s)|^2\,dx\,ds\Big)\\
&\le I(t)\overline{H}^{-1}\Big(\frac{1}{I(t)\xi(t)}\int_0^t\Big(-g'(s)\Big)\int_\Omega |\nabla u(t)-\nabla u(t-s)|^2\,dx\,ds\Big)\\
&\le I(t)\overline{H}^{-1}\Big(\frac{-2E'(t)}{I(t)\xi(t)}\Big).
\end{aligned}\tag{4.7}$$

Inserting (4.7) into $(4.6)_2$, defining

$$F_0(t) := \overline{H}'\Big(\varepsilon_0\big[\frac{E(t)}{E(0)}\big]^{\frac{m_1}{2m_1-2}}\Big)F(t),$$

with $\varepsilon_0 < r$, and using the fact that $E' \le 0$, $\overline{H}' > 0$, and $\overline{H}'' > 0$, give

$$\begin{aligned}
F_0'(t) &\le \overline{H}'\Big(\varepsilon_0\big[\frac{E(t)}{E(0)}\big]^{\frac{m_1}{2m_1-2}}\Big)F'(t)\\
&\le -m\xi(t)E(t)\overline{H}'\Big(\varepsilon_0\big[\frac{E(t)}{E(0)}\big]^{\frac{m_1}{2m_1-2}}\Big)\\
&\quad + c\xi(t)I(t)\overline{H}^{-1}\Big(\frac{-2E'(t)}{I(t)\xi(t)}\Big)\overline{H}'\Big(\varepsilon_0\big[\frac{E(t)}{E(0)}\big]^{\frac{m_1}{2m_1-2}}\Big)\\
&\quad + c\xi(t)H^{-1}(-E'(t))\overline{H}'\Big(\varepsilon_0\big[\frac{E(t)}{E(0)}\big]^{\frac{m_1}{2m_1-2}}\Big).
\end{aligned}\tag{4.8}$$

If $A = \overline{H}'\big(\varepsilon_0[\frac{E(t)}{E(0)}]^{\frac{m_1}{2m_1-2}}\big)$ and one time $B = \overline{H}^{-1}\big(\frac{-2E'(t)}{I(t)\xi(t)}\big)$ and another time $B = \overline{H}^{-1}(-E'(t))$ are used in the generalized Young inequality (2.3), we obtain

$$\begin{aligned}
F_0'(t) &\le -m\xi(t)E(t)\overline{H}'\Big(\varepsilon_0\big[\frac{E(t)}{E(0)}\big]^{\frac{m_1}{2m_1-2}}\Big)\\
&\quad + c\varepsilon_0\xi(t)I(t)\big[\frac{E(t)}{E(0)}\big]^{\frac{m_1}{2m_1-2}}\overline{H}'\Big(\varepsilon_0\big[\frac{E(t)}{E(0)}\big]^{\frac{m_1}{2m_1-2}}\Big) - cE'(t)\\
&\quad + c\varepsilon_0\xi(t)\big[\frac{E(t)}{E(0)}\big]^{\frac{m_1}{2m_1-2}}\overline{H}'\Big(\varepsilon_0\big[\frac{E(t)}{E(0)}\big]^{\frac{m_1}{2m_1-2}}\Big) - c\xi(t)E'(t).
\end{aligned}$$

Using that $E(t)^{\frac{2-m_1}{2m_1-2}}I(t)$ is uniformly bounded by some constant C because of $(4.4)_2$ and $\overline{H}'\Big(\varepsilon_0\big[\frac{E(t)}{E(0)}\big]^{\frac{m_1}{2m_1-2}}\Big) = H'\Big(\varepsilon_0\big[\frac{E(t)}{E(0)}\big]^{\frac{m_1}{2m_1-2}}\Big)$, the choice of ε_0, yields

$$\begin{aligned}
F_0'(t) &\le -m\xi(t)E(t)H'\Big(\varepsilon_0\big[\frac{E(t)}{E(0)}\big]^{\frac{m_1}{2m_1-2}}\Big) + c\varepsilon_0\xi(t)\frac{E(t)}{E(0)}H'\Big(\varepsilon_0\big[\frac{E(t)}{E(0)}\big]^{\frac{m_1}{2m_1-2}}\Big)\\
&\quad - cE'(t).
\end{aligned}$$

Consequently, with $F_1 = F_0 + cE$, which for some $\alpha_1, \alpha_2 > 0$ satisfies

$$\alpha_1 F_1(t) \le E(t) \le \alpha_2 F_1(t), \tag{4.9}$$

and with a suitable choice of ε_0, we obtain, for some constant $k > 0$ and for all $t \ge t_1$,

$$F_1'(t) \le -k\xi(t)\Big(\frac{E(t)}{E(0)}\Big)H'\Big(\varepsilon_0\big[\frac{E(t)}{E(0)}\big]^{\frac{m_1}{2m_1-2}}\Big) = -k\xi(t)H_3\Big(\frac{E(t)}{E(0)}\Big), \tag{4.10}$$

where $H_3(t) = tH'\big(\varepsilon_0 t^{\frac{m_1}{2m_1-2}}\big)$. Using the strict convexity of H on $(0, r]$, we find that $H_3(t)$, $H_3'(t) > 0$, and $H_3(t) \le tH'(r)$ on $(0, 1]$. Thus, with

$$R(t) = \frac{\alpha_1 F_1(t)}{E(0)},$$

taking in account (4.9) and (4.10), we have

$$R(t) \sim E(t) \tag{4.11}$$

and, for some $k_1 > 0$,

$$R'(t) \le -k_1\xi(t)H_3(R(t)), \quad \forall t \ge t_1.$$

Then, the integration over (t_1, t) yields, for some $k_2 > 0$,

$$\begin{gathered}\int_{t_1}^t \frac{-R'(s)}{H_3(R(s))}ds \ge k_1\int_{t_1}^t \xi(s)ds \implies \int_{R(t)}^{R(t_1)} \frac{1}{H_3(s)}ds \ge k_1\int_{t_1}^t \xi(s)ds \\ \implies R(t) \le H_0^{-1}(k_1\int_{t_1}^t \xi(s)ds) \underset{\text{by (4.11)}}{\implies} E(t) \le k_2 H_0^{-1}(k_1\int_{t_1}^t \xi(s)ds),\end{gathered} \tag{4.12}$$

where $H_0(t) = \int_t^1 \frac{1}{H_3(s)}ds$. Here, we have used, based on the properties of H_3, the fact that H_0 is a strictly decreasing function on $(0, 1]$ and $\lim_{t\to 0} H_0(t) = +\infty$. Also, it is easy to notice that we can start the integration inside at zero where if $\overline{k}_1 < k_1$ is chosen so that $\overline{k}_1 \int_0^{2t_1} \xi(s)ds = k_1 \int_{t_1}^{2t_1} \xi(s)ds$, then, as H_0^{-1} is decreasing,

$$E(t) \le k_2 H_0^{-1}(k_1\int_{t_1}^t \xi(s)ds) \le k_2 H_0^{-1}(\overline{k}_1\int_0^t \xi(s)ds), \quad \forall t \ge 2t_1 \tag{4.13}$$

and so $(4.1)_4$ is established.
Case 2: H_1 is linear on $[0, r]$ or $m_1 > 4/3$. If H_1 is linear then

$$\xi(t)\int_0^t g(s)\int_\Omega |\nabla u(t) - \nabla u(t-s)|^2\,dx\,ds \le -cE'(t)$$

or if $m_1 > 4/3$ then $I(t)$, used above, is itself uniformly bounded by some constant C because of $(4.4)_1$. This enables us to repeat exactly the same steps starting at (4.8), but with $\overline{H}'\big(\varepsilon_0 \frac{E(t)}{E(0)}\big)$ instead of $\overline{H}'\big(\varepsilon_0\big[\frac{E(t)}{E(0)}\big]^{\frac{m_1}{2m_1-2}}\big)$, and with $H_3(t) = tH'(\varepsilon_0 t)$, and similarly obtain (4.13), and so $(4.1)_{1,2,3}$. □

Applications. Here, we give applications of our result to some concrete examples. If assumption (A1) is satisfied with $H_1(t) = t^p, 1 \leq p < 2$ and $H_2(t) = t^{\frac{m_1}{2m_1-2}}$, then $H(t) = \min\{H_1(t), H_2(t)\} = t^q$ on the interval $(0,1]$ where $q = \max\{p, \frac{m_1}{2m_1-2}\}$. Hence, (4.1) and simple calculations lead to

$$E(t) \leq \begin{cases} ke^{-k_1 \int_0^t \xi(s)ds}, & \text{if } m_1 \geq 2 \text{ and } p = 1 \\ k_2\Big(1 + \int_0^t \xi(s)ds\Big)^{-\frac{2m_1-2}{2-m_1}}, & \text{if } 1 < m_1 < 2 \text{ and } p = 1 \\ k_3\Big(1 + \int_0^t \xi(s)ds\Big)^{\frac{-1}{q-1}}, & \text{if } m_1 > \frac{4}{3} \text{ and } 1 < p < 2 \\ k_4\Big(1 + \int_0^t \xi(s)ds\Big)^{-\frac{2m_1-2}{m_1(q-1)}}, & \text{if } 1 < m_1 \leq \frac{4}{3} \text{ and } 1 < p < 2. \end{cases} \tag{4.14}$$

- If $g(t) = \frac{a}{(1+t)^v}$, for $v > 1$, then $g'(t) = -\xi(t)H_1(g(t))$ where $H_1(t) = t^p$, with $p = \frac{v+1}{v}$, and $\xi(t) \equiv constant$. By $(4.14)_{3,4}$,

$$E(t) \leq \begin{cases} k_3(1+t)^{\frac{-1}{q-1}}, & \text{if } m_1 > \frac{4}{3} \\ k_4(1+t)^{-\frac{2m_1-2}{m_1(q-1)}}, & \text{if } 1 < m_1 \leq \frac{4}{3}. \end{cases}$$

- If $g(t) = \frac{a}{(t+e)[\ln(t+e)]^v}$, for $v > 1$, then $g'(t) = -\xi(t)H_1\big(g(t)\big)$ where $H_1(t) = t^p$, with $p = \frac{v+1}{v}$, and $\xi(t) = \frac{[\ln(t+e)+v]}{a^{\frac{1}{v}}(t+e)^{1-\frac{1}{v}}}$. By $(4.14)_{3,4}$,

$$E(t) \leq \overline{k}\Big(1 + \int_0^t \frac{\ln(s+e)+v}{a^{\frac{1}{v}}(s+e)^{1-\frac{1}{v}}} ds\Big)^{-q_0} \underset{\text{for large } t}{\leq} \overline{k}\,\Big((t+e)^{\frac{1}{v}}\ln(t+e)\Big)^{-q_0}$$

where

$$q_0 = \begin{cases} \frac{1}{q-1}, & \text{if } m_1 > \frac{4}{3} \\ \frac{2m_1-2}{m_1(q-1)}, & \text{if } 1 < m_1 \leq \frac{4}{3}. \end{cases}$$

- If $g(t) = a\exp(-t^v)$, for $0 < v \leq 1$, then $g'(t) = -\xi(t)H_1(g(t))$ where $H_1(t) = t$ and $\xi(t) = vt^{v-1}$. By $(4.14)_{1,2}$,

$$E(t) \leq \begin{cases} ke^{-k_1 t^v}, & \text{if } m_1 \geq 2 \\ k_2(1+t^v)^{-\frac{2m_1-2}{2-m_1}}, & \text{if } 1 < m_1 < 2. \end{cases} \tag{4.15}$$

But, for $0 < v < 1$, g' can also be written as $g'(t) = -H_1(g(t))$ where $H_1(t) = \frac{vt}{[\ln(a/t)]^{\frac{1}{v}-1}}$ satisfies (A1) on the interval $(0, r_1]$ for any $0 < r_1 < a$. If $1 < m_1 < 2$, then $\frac{m_1}{2m_1-2} > 1$ and $H_2(t) = t^{\frac{m_1}{2m_1-2}}$ is strictly convex and one can easily discover that $H(t) = \min\{H_1(t), H_2(t)\} = H_2(t)$ near the origin. By Theorem 4.1, we obtain

$$E(t) \leq \begin{cases} ke^{-k_1 t^v}, & \text{if } m_1 \geq 2 \\ k_2(1+t)^{-\frac{2m_1-2}{2-m_1}}, & \text{if } \frac{4}{3} < m_1 < 2 \\ k_3(1+t)^{-\frac{(2m_1-2)^2}{m_1(2-m_1)}}, & \text{if } 1 < m_1 \leq \frac{4}{3} \end{cases}$$

which gives better rates than (4.15) in the case $4/3 < m_1 < 2$.

Conclusions. In this paper, the important issue of stabilization of the wave equation was addressed. The main contribution of this work is studying the competition between two different types of dissipative mechanisms and establishing, by carefully tailored techniques, explicit formulae for the energy decay rates with very general assumptions on the relaxation function and with variable exponent of the feedback, which is more useful from the physical point of view and needed in several applications. Our results combine the generality and optimality and improve earlier related results in the literature. We provided some numerical examples of exponential, polynomial or logarithmic energy decay estimates and all the rates in these examples are faster than the rates obtained in [14]. Our paper opens the door for further research and new suggestions that may be addressed in the future, for instance the control of the system by such types of damping but located on the boundary.

Acknowledgments. This work was supported bythe MASEP Research Group in the Research Institute of Sciences and Engineering at University of Sharjah.

References

[1] F. Alabau-Boussouira, P. Cannarsa; *A general method for proving sharp energy decay rates for memory-dissipative evolution equations*, C. R. Acad. Sci. Paris, Ser. I 347(2009), 867–872.

[2] S. Antontsev; *Wave equation with p(x, t)-Laplacian and damping term: blow-up of solutions*, C. R.Mec. 339(12) (2011), 751–755.

[3] S. Antontsev; *Wave equation with p(x, t)-Laplacian and damping term: existence and blow-up*, Differ. Equ. Appl. 3(4) (2011), 503–525.

[4] S. Antontsev, S. Shmarev; *Blow-up of solutions to parabolic equations with nonstandard growth conditions*, J. Comput. Appl. Math. 234(9) (2010), 2633–2645.

[5] S. Antontsev, V. Zhikov; *Higher integrability for parabolic equations of p(x, t)-Laplacian type*, Adv. Differ. Equ. 10(9) (2005), 1053–1080.

[6] V. I. Arnold; *Mathematical methods of classical mechanics*, Springer-Verlag, New York, 1989.

[7] F. Belhannache, M. M. Algharabli, S. A. Messaoudi; *Asymptotic stability for a viscoelastic equation with nonlinear damping and very general type of relaxation functions*, J. Dyn. Control Syst. (2019), 1-23.

[8] A. Benaissa, S. Mimouni; *Energy decay of solutions of a wave equation of p-Laplacian type with a weakly nonlinear dissipation*, JIPM.J. Inequal. Pure Appl. Math. 7(1) (2006), Article 15, pp 8.

[9] M. M. Cavalcanti, H. P. Oquendo; *Frictional versus viscoelastic damping in a semilinear wave equation*, SIAM J. Control Optim. 42(4) (2003), 1310-1324.

[10] M. Fabrizio, S. Polidoro; *Asymptotic decay for some differential systems with fading memory*, Appl. Anal. 81(6) (2002), 1245-1264.

[11] X. Fan, D. Zhao; *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl. 263(2) (2001), 424–446.

[12] Y. Gao, W. Gao; *Existence of weak solutions for viscoelastic hyperbolic equations with variable exponents*, Boundary Value Problems 2013 no. 1 (2013), 1-8.

[13] B. Guo, W. Gao; *Blow-up of solutions to quasilinear hyperbolic equations with p(x, t)-Laplacian and positive initial energy*, C. R. Mec. 342(9) (2014), 513–519.

[14] J. H. Hassan, S. A. Messaoudi; *General decay results for a viscoelastic wave equation with a variable exponent nonlinearity*, Asymptot. Anal. 125 (2021), 365–388.

[15] A. Haraux, E. Zuazua; *Decay estimates for some semilinear damped hyperbolic problems*, Arch. Rational Mech. Anal. (1988), 191-206.

[16] W. J. Hrusa; *Global existence and asymptotic stability for a semilinear Volterra equation with large initial data*, SIAM Journal of Math. Anal. 16(1) (1985), 110-134.

[17] V. Komornik; *Decay estimates for the wave equation with internal damping*, International Series of Numerical Mathematics 118 (1994), 253-266.

[18] D. Lars, P. Harjulehto, P. Hasto, M. Ruzicka; *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, Vol. 2017, 2017.

[19] S. A. Messaoudi; *General decay of solutions of a viscoelastic equation*, J. Math. Anal. Appl. 341 (2008), 1457-1467.

[20] S. A. Messaoudi; *General stability in viscoelasticity, in: Viscoelastic Viscoplastic Materials*, InTech (2016), DOI: 10.5772/64217.

[21] S. A. Messaoudi, A. A. Talahmeh; *A blow-up result for a quasilinear wave equation with variable-exponent nonlinearities*, Math. Methods Appl. Sci. 40 (2017), 6976-6986.

[22] S. A. Messaoudi, A. A. Talahmeh; *On wave equation: review and recent results*, Arab. J. Math. 7(2) (2018), 113–145.
[23] S. A. Messaoudi, A. A. Talahmeh, J. H. Al-Smail; *Nonlinear damped wave equation: existence and blow-up*, Comput.Math. Appl. 74 (2017), 3024-3041.
[24] J. E. Munoz Rivera; *Asymptotic behavior in linear viscoelasticity*, Quart. Appl. Math. 52(4) (1994), 628-648.
[25] M. I. Mustafa; *Energy estimates to the Cauchy problem of a weakly damped Klein-Gordon equation with variable-exponent nonlinearity*, Math. Methods Appl. Sci. 44 (11) (2021), 8999-9011.
[26] M. I. Mustafa; *Optimal decay rates for the viscoelastic wave equation*, Math. Methods Appl. Sci. 41 (1) (2018), 192–204.
[27] M. I. Mustafa; *Optimal energy decay result for nonlinear abstract viscoelastic dissipative systems*, Z. Angew. Math. Phys. 72 (2), 67 (2021), 1-15.
[28] M. I. Mustafa, S. A. Messaoudi, M. Zahri; *Theoretical and computational results of a wave equation with variable exponent and time dependent nonlinear damping*, Arab. J. Math. 10 (2) (2021), 443–458.
[29] M. Nakao; *Decay of solutions of the wave equation with a local nonlinear dissipation*, Math. Ann. 305 (1996), 403-417.
[30] S.-H. Park, J.-R. Kang; *Blow-up of solutions for a viscoelastic wave equation with variable exponents*, Math. Methods Appl. Sci. 42 (2019), 2083–2097.
[31] M. L. Santos; *Asymptotic behavior of solutions to wave equations with a memory conditions at the boundary*, Electron. J. Differ. Equ. 2001(73) (2001), 1-11.
[32] L. Sun, Y. Ren, W. Gao; *Lower and upper bounds for the blow-up time for nonlinear wave equation with variable sources*, Comput. Math. Appl. 71(1) (2016), 267–277.
[33] E. Zuazua; *Exponential decay for the semilinear wave equation with locally distributed damping*, Comm. Partial Differential Equations 15 (1990), 205-235.

Muhammad I. Mustafa
Department of Mathematics, University of Sharjah, P.O. Box 27272, Sharjah, United Arab Emirates
Email address: mmustafa@sharjah.ac.ae