

EXISTENCE OF SOLUTIONS FOR SINGULAR ELLIPTIC PROBLEMS WITH SINGULAR NONLINEARITIES AND CRITICAL CAFFARELLI-KOHN-NIRENBERG EXPONENT

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ABSTRACT. In this article, we consider a singular elliptic problem with singular nonlinearities and critical Caffarelli-Kohn-Nirenberg exponent. By using variational methods and Palais-Smale condition, we show the existence of at least two nontrivial solutions. The result depends crucially on the parameters $a, b, N, \beta, \gamma, \lambda, \mu$.

1. INTRODUCTION

In this article, we consider the existence of multiple nontrivial nonnegative solutions of the problem

$$\begin{aligned} -\operatorname{div}\left(\frac{\nabla u}{|x|^{2a}}\right) - \mu \frac{u}{|x|^{2(a+1)}} &= h(x) \frac{|u|^{2^*-2}u}{|x|^{2^*b}} + \frac{\lambda}{|x|^\beta |u|^\gamma} \quad \text{in } \Omega, x \neq 0 \\ u &= 0 \quad x \in \partial\Omega \end{aligned} \quad (1.1)$$

where Ω is a smooth bounded domain in \mathbb{R}^N , $N \geq 3$, $-\infty < a < \frac{N-2}{2}$, $a \leq b < a+1$, $0 \leq \beta < \frac{N}{2^*+1}(2^* + \gamma - 1)$, $0 < \gamma < 1$, $2^* = \frac{2N}{N-2+2(b-a)}$ is the critical Caffarelli-Kohn-Nirenberg exponent, $-\infty < \mu < \bar{\mu}_a = [\frac{N-2(a+1)}{2}]^2$, λ is a real parameter and h is a bounded positive function on \mathbb{R}^N .

In recent years, people have paid much attention to the singular elliptic problem

$$\begin{aligned} -\Delta u - \mu|x|^{-2}u &= h(x)|u|^{p-2}u + \lambda u \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \quad (1.2)$$

where Ω is a smooth bounded domain in \mathbb{R}^N ($N \geq 3$), $0 \in \Omega$, $\lambda > 0$, $0 \leq \mu < \bar{\mu}_0 := (N-2)^2/4$ and $2^* = 2N/(N-2)$ is the critical Sobolev exponent, see [6, 7] and references therein. Ali and Iaiia [1] studied the existence and nonexistence for singular sublinear problems on exterior domains when $\mu = 0$. Some results are already available for (1.1). Wang and Zhou [13] proved that there exist at least two solutions for (1.1) with $a = 0$, $0 < \mu \leq \bar{\mu}_0 = (N-2)^2/4$. Boucekif and Matallah [4] showed the existence of two solutions of (1.1) under certain conditions

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on a weighted function h , when $0 < \mu \leq \bar{\mu}_a$, $\lambda \in (0, \Lambda_*)$, $-\infty < a < (N-2)/2$ and $a \leq b < a+1$, with Λ_* a positive constant.

The regular problem corresponding to $a = b = \mu = 0$ was considered on a regular bounded domain Ω by Tarantello [10]. She proved that, with a nonhomogeneous term $f \in H^{-1}(\Omega)$, the dual of $H_0^1(\Omega)$, not identically zero and satisfying a suitable condition, the problem considered admits two distinct solutions.

Before formulating our results, we give some definitions and notation. We denote by $\mathcal{H}_\mu = \mathcal{H}_\mu(\Omega)$, the closure of $C_0^\infty(\Omega \setminus \{0\})$ with respect to the norms

$$\|u\|_0 = \left(\int_{\Omega} |x|^{-2a} |\nabla u|^2 dx \right)^{1/2}$$

and

$$\|u\|_\mu = \left(\int_{\Omega} (|x|^{-2a} |\nabla u|^2 - \mu |x|^{-2(a+1)} |u|^2) dx \right)^{1/2} \text{ for } -\infty < \mu < \bar{\mu}_a.$$

From the weighted Hardy inequality [6] that is

$$\int_{\Omega} |x|^{-2(a+1)} u^2 dx \leq \frac{1}{\bar{\mu}_a} \int_{\Omega} |x|^{-2a} |\nabla u|^2 dx \quad (1.3)$$

it is easy to see that the norm $\|u\|_\mu$ is equivalent to $\|u\|_0$. More explicitly, we have

$$\left(1 - (\sqrt{\bar{\mu}_a} - a)^{-2} \mu^+\right)^{1/2} \|u\|_0 \leq \|u\|_\mu \leq \left(1 - (\sqrt{\bar{\mu}_a} - a)^{-2} \mu^-\right)^{1/2} \|u\|_0,$$

with $\mu^+ = \max(\mu, 0)$ and $\mu^- = \min(\mu, 0)$ for all $u \in \mathcal{H}_\mu$.

Next we list here a few integral inequalities. It is clear that degeneracy and singularity occur in problem (p1). In these situations, the classical methods do not directly apply so that the existence results may become a delicate matter that is closely related to some phenomena due to the degenerate (or singular) character of the differential equation. The starting point of the variational approach is the following Caffarelli-Kohn-Nirenberg inequality in [5] which states there is a positive constant $C_{a,b}$ such that

$$\left(\int_{\Omega} |x|^{-2_* b} |u|^{2_*} dx \right)^{1/2_*} \leq C_{a,b} \left(\int_{\Omega} |x|^{-2a} |\nabla u|^2 dx \right)^{1/2} \quad (1.4)$$

for any $u \in \mathcal{H}_\mu$ where $-\infty < a < \frac{N-2}{2}$, $a \leq b < a+1$, $2_* = \frac{2N}{N-2+2(b-a)}$.

We consider the approximation equation

$$\begin{aligned} -\operatorname{div}\left(\frac{\nabla u}{|x|^{2a}}\right) - \mu \frac{u}{|x|^{2(a+1)}} &= h(x) \frac{|u|^{2_*-2} u}{|x|^{2_* b}} + \frac{\lambda}{|x|^\beta (u+\theta)^\gamma} \quad \text{in } \Omega \setminus \{0\} \\ u &= 0 \quad x \in \partial\Omega, \end{aligned} \quad (1.5)$$

for any $\theta > 0$.

We look for solutions of problem (1.5) by finding critical points of C^1 -energy functional defined in [10],

$$\begin{aligned} J_\lambda(u) &:= (1/2) \|u\|_\mu^2 - (1/2_*) \int_{\Omega} h(x) |x|^{-2_* b} |u|^{2_*} dx \\ &\quad - \frac{\lambda}{1-\gamma} \int_{\Omega} \frac{(u^+ + \theta)^{1-\gamma} - \theta^{1-\gamma}}{|x|^\beta} dx. \end{aligned}$$

A point $u \in \mathcal{H}_\mu$ is a weak solution of (1.5) if it satisfies

$$\begin{aligned} \langle J'_\lambda(u), \varphi \rangle := & \int_\Omega \left(\frac{\nabla u \nabla \varphi}{|x|^{2a}} - \mu \frac{u \varphi}{|x|^{2(a+1)}} \right) dx - \int_\Omega h(x) \frac{|u|^{2^*-1} u \varphi}{|x|^{2^*b}} dx \\ & - \lambda \int_\Omega \frac{\varphi}{(u^+ + \theta)^\gamma |x|^\beta} dx = 0, \quad \text{for all } \varphi \in \mathcal{H}_\mu. \end{aligned}$$

Here $\langle \cdot, \cdot \rangle$ denotes the product in the duality \mathcal{H}'_μ , of \mathcal{H}_μ . We consider the following assumptions:

- (A1) $h \in L^\infty(\Omega)$, $\text{ess lim}_{|x| \rightarrow 0} h(x) = h_0 \in (0, \infty)$ and $h(x) \geq h_0$ a.e. in Ω ;
- (A2) $(a, \mu) \in (-1, 0) \times (0, \bar{\mu}_a - b) \cup [0, \frac{N-2}{2}) \times (a(a - N + 2), \bar{\mu}_a - b)$;
- (A3) $(a, \mu) \in [0, \frac{N-2}{2}) \times [0, \bar{\mu}_a)$;
- (A4) $N > 2(|b| + 1)$ and (δ_1) ,
- (A5) $N \geq 3$ and (A2) holds.

Xuan et al. [12] proved that when (A2) holds for each $\varepsilon > 0$, the function

$$y_\varepsilon = C_0 \varepsilon^{\frac{2}{2^*-2}} \left[\varepsilon^{\frac{2\sqrt{\bar{\mu}_a - \mu}}{\sqrt{\bar{\mu}_a - \mu} - b}} |x|^{\frac{2^*-2}{2}(\sqrt{\bar{\mu}_a} - \sqrt{\bar{\mu}_a - \mu})} + |x|^{\frac{2^*-2}{2}(\sqrt{\bar{\mu}_a} + \sqrt{\bar{\mu}_a - \mu})} \right]^{\frac{-2}{2^*-2}}, \quad (1.6)$$

with a suitable positive constant C_0 , is a weak solution of

$$-\text{div} \div \left(\frac{\nabla u}{|x|^{2a}} \right) - \mu \frac{u}{|x|^{2(a+1)}} = \frac{|u|^{2^*-2} u}{|x|^{2^*b}} \quad \text{in } \Omega \setminus \{0\}.$$

Furthermore,

$$\int_\Omega \left(\frac{|\nabla y_\varepsilon|^2}{|x|^{2a}} - \mu \frac{y_\varepsilon^2}{|x|^{2(a+1)}} \right) dx = \int_\Omega \frac{|y_\varepsilon|^{2^*}}{|x|^{2^*b}} dx.$$

In addition, we have that

$$D_\mu = \inf_{u \in \mathcal{H}_\mu \setminus \{0\}} E(u) = E(y_\varepsilon),$$

is the best constant with

$$E(u) := \frac{\int_\Omega (|x|^{-2a} |\nabla u|^2 - \mu |x|^{-2(a+1)} |u|^2) dx}{\left(\int_\Omega |x|^{-2^*b} |u|^{2^*} dx \right)^{2/2^*}}.$$

Kang et al. [9] obtained that, when (A3) holds for each $\varepsilon > 0$, the function

$$v_\varepsilon = [2_* 2 \varepsilon (\bar{\mu}_a - \mu)]^{\frac{1}{2^*-2}} \left[|x|^{(-\sqrt{\bar{\mu}_a} + \sqrt{\bar{\mu}_a - \mu})} (\varepsilon + |x|^{(2^*-2)\sqrt{\bar{\mu}_a - \mu}}) \right]^{\frac{-2}{2^*-2}}, \quad (1.7)$$

with a suitable positive constant C_0 , is a weak solution of

$$-\text{div} \left(\frac{\nabla u}{|x|^{2a}} \right) - \mu \frac{u}{|x|^{2(a+1)}} = \frac{|u|^{2^*-2} u}{|x|^{2^*b}}, \quad \text{in } \Omega \setminus \{0\},$$

and satisfies

$$\int_\Omega \left(\frac{|\nabla v_\varepsilon|^2}{|x|^{2a}} - \mu \frac{v_\varepsilon^2}{|x|^{2(a+1)}} \right) dx = \int_\Omega \frac{|v_\varepsilon|^{2^*}}{|x|^{2^*b}} dx.$$

Also we have that

$$G_\mu = \inf_{u \in \mathcal{H}_\mu \setminus \{0\}} E(u) = E(v_\varepsilon)$$

is the best constant.

In this work we prove the existence of at least two critical points of J_λ . The first is found by the Ekeland Variational Principle [8] with negative energy and

the second by Mountain Pass Theorem without Palais Smale conditions [2] with positive energy. Now, we define

$$w_\varepsilon := \begin{cases} y_\varepsilon & \text{if (A2) holds} \\ v_\varepsilon & \text{if (A3) holds,} \end{cases} \quad S_\mu := \begin{cases} D_\mu & \text{if (A2) holds} \\ G_\mu & \text{if (A3) holds.} \end{cases}$$

Now we can state our main result.

Theorem 1.1. *Assume that $-\infty < a < \frac{N-2}{2}$, $a \leq b < a + 1$, $0 \leq \beta < \frac{N}{2_*+1}(2_* + \gamma - 1)$, $-\infty < \mu < \bar{\mu}_a = [\frac{N-2(a+1)}{2}]^2$, $0 < \gamma < 1$, (A1) holds and (A4) or (A5) are satisfied. Then there exists $\Lambda_0 > 0$ such that for each $0 < \lambda < \Lambda_0$, problem (1.1) has at least two nontrivial solutions.*

This article is organized as follows. In Section 2, we give some preliminaries. Section 3 is devoted to the proof of Theorems 1.1.

2. PRELIMINARIES

Definition 2.1. Let c a real number, E a Banach space, and I a function in $C^1(E, \mathbb{R})$.

(i) $(u_n)_n$ is a Palais-Smale sequence at level c (in short $(PS)_c$) in E for I if

$$I(u_n) = c + o_n(1) \text{ and } I'(u_n) = o_n(1),$$

where $o_n(1)$ tends to 0 as n goes at infinity.

(ii) We say that I satisfies the $(PS)_c$ condition if any $(PS)_c$ sequence in E for I has a convergent subsequence.

Lemma 2.2. *Assume that $-\infty < a < \frac{N-2}{2}$, $a \leq b < a + 1$, $0 \leq \beta < \frac{N}{2_*+1}(2_* + \gamma)$, $-\infty < \mu < \bar{\mu}_a = [\frac{N-2(a+1)}{2}]^2$ and let $(u_n) \subset \mathcal{H}_\mu$ be a Palais-Smale sequence $((PS)_c$ in short) of J_λ , i.e.,*

$$J_\lambda(u_n) \rightarrow c \text{ and } J'_\lambda(u_n) \rightarrow 0 \text{ in } \mathcal{H}'_\mu \text{ (the dual of } \mathcal{H}_\mu) \text{ as } n \rightarrow \infty \quad (2.1)$$

for some $c \in \mathbb{R}$. Then $u_n \rightharpoonup u$ in \mathcal{H}_μ and $J'_\lambda(u) = 0$.

Proof. Let $R_0 > 0$ such that $\Omega \subset B(0, R_0) = \{x \in \mathbb{R}^N : |x| < R_0\}$. Let $r = |x|$. By Hölder's inequality which states that

$$\int_\Omega |fg| dx \leq \left(\int_\Omega |f|^p dx \right)^{1/p} \left(\int_\Omega |g|^q dx \right)^{1/q},$$

for $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$, and from (1.3), we obtain

$$\begin{aligned} \int_\Omega \frac{(u^+)^{1-\gamma}}{r^\beta} dx &\leq \int_\Omega \frac{|u|^{1-\gamma}}{r^\beta} dx \\ &= \int_\Omega \left(\frac{|u|^{1-\gamma}}{r^{b(1-\gamma)}} \right) r^{b(1-\gamma)-\beta} dx \\ &\leq \text{Big} \left(\int_\Omega \frac{|u|^{2_*}}{r^{2_*b}} \right)^{\frac{1-\gamma}{2_*}} \left(\int_\Omega r^{(b(1-\gamma)-\beta)\frac{2_*}{2_*+\gamma-1}} \right)^{\frac{2_*+\gamma-1}{2_*}} \\ &\leq \left(C_{a,b}^{\frac{2_*}{2}} \|u\|_\mu^{2_*} \right)^{\frac{1-\gamma}{2_*}} \left\langle \sigma_N \int_0^{R_0} r^{N-1+(b(1-\gamma)-\beta)\frac{2_*}{2_*+\gamma-1}} dr \right\rangle^{\frac{2_*+\gamma-1}{2_*}} \\ &\leq \left(C_{a,b}^{\frac{2_*}{2}} \|u\|_\mu^{2_*} \right)^{\frac{1-\gamma}{2_*}} \left(\sigma_N \frac{R_0^{N+[b(1-\gamma)-\beta]\frac{2_*}{2_*+\gamma-1}}}{N+[b(1-\gamma)-\beta]\frac{2_*}{2_*+\gamma-1}} \right)^{\frac{2_*+\gamma-1}{2_*}} \end{aligned}$$

$$\leq AC_{a,b}^{\frac{1-\gamma}{2}} \|u\|_{\mu}^{1-\gamma},$$

where $\sigma_N = \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}$ is the area of the $(N - 1)$ -dimensional unit sphere, $C_{a,b}$ defined in (1.4), and

$$\begin{aligned} |f| &= \frac{|u|^{2_*}}{r^{2_*b}}, \quad |g| = r^{[b(1-\gamma)-\beta]\frac{2_*}{2_*+\gamma-1}}, \quad \frac{1}{p} = \frac{1-\gamma}{2_*}, \quad \frac{1}{q} = \frac{2_*+\gamma-1}{2_*}, \\ A &= \left[\frac{2\pi^{\frac{N}{2}}(2_*+\gamma-1)}{N\Gamma(\frac{N}{2})[(2_*+\gamma-1)-\beta(\gamma+1)]} \right]^{\frac{2_*+\gamma-1}{2_*+1}} R_0^{\frac{N}{2_*+1}(2_*+\gamma-1)-\beta} > 0. \end{aligned}$$

From (2.1), we have

$$\begin{aligned} J_{\lambda}(u_n) &:= (1/2)\|u_n\|_{\mu}^2 - (1/2_*) \int_{\Omega} h(x)|x|^{-2_*b}|u_n|^{2_*} dx \\ &\quad - \frac{\lambda}{1-\gamma} \int_{\Omega} \frac{(u_n^+ + \theta)^{1-\gamma} - \theta^{1-\gamma}}{|x|^{\beta}} dx \\ &= c + o_n(1) \end{aligned}$$

and

$$\begin{aligned} \langle J'_{\lambda}(u_n), u_n \rangle &= \|u_n\|_{\mu}^2 - \int_{\Omega} h(x)|x|^{-2_*b}|u_n|^{2_*} dx - \lambda \int_{\Omega} \frac{(u_n^+ + \theta)^{1-\gamma} - \theta^{1-\gamma}}{|x|^{\beta}} dx \\ &= o_n(1), \quad \text{for } n \text{ large,} \end{aligned}$$

where $o_n(1)$ denotes $o_n(1) \rightarrow 0$ as $n \rightarrow \infty$. Then

$$\begin{aligned} c + o_n(1) &= J_{\lambda}(u_n) - \frac{1}{2_*} \langle J'_{\lambda}(u_n), u_n \rangle \\ &\geq \left(\frac{1}{2} - \frac{1}{2_*}\right) \|u_n\|_{\mu}^2 - \lambda \left(\frac{1}{1-\gamma} - \frac{1}{2_*}\right) \int_{\Omega} \frac{(u_n^+ + \theta)^{1-\gamma} - \theta^{1-\gamma}}{|x|^{\beta}} dx \\ &\geq \left(\frac{1}{2} - \frac{1}{2_*}\right) \|u_n\|_{\mu}^2 - \lambda \left(\frac{1}{1-\gamma} - \frac{1}{2_*}\right) A \|u_n\|_{\mu}^{1-\gamma} C_{a,b}^{\frac{-(1-\gamma)}{p}} \end{aligned}$$

and so $\{u_n\}$ is bounded in $H_{\mu}(\Omega)$. Going if necessary to a subsequence there exists $u \in H_{\mu}(\Omega)$ such that

$$\begin{aligned} u_n &\rightharpoonup u, \quad \text{in } H_{\mu}(\Omega), \\ u_n &\rightharpoonup u, \quad \text{in } L_q(\Omega), \quad (1 \leq q < 2_*) \\ u_n &\rightharpoonup u, \quad \text{in } L_{2_*}(\Omega), \\ u_n &\rightarrow u, \quad \text{a.e. on } \Omega, \end{aligned}$$

there exists $\varphi \in L_q(\Omega)$ ($1 \leq q < 2_*$) such that $|u_n|$ and $|u| \leq |\varphi|$, a.e. in Ω , where the last conclusion is from [14, Lemma A.1]. From (1.4), we obtain

$$\left| \frac{u_n}{|x|^{\beta}|u_n^+ + \theta|^{\gamma}} \right| \leq \frac{|u_n|}{|x|^{\beta}\theta^{\gamma}} \leq \frac{|\varphi|}{|x|^{\beta}\theta^{\gamma}}.$$

Since $1 < \frac{2_*(N-\beta)+N}{2(N-\beta)} < 2_*$ we have $\varphi \in L^{\frac{2_*(N-\beta)+N}{2(N-\beta)}}(\Omega)$, then

$$\begin{aligned} &\frac{1}{\theta^{\gamma}} \int_{\Omega} \frac{\varphi}{|x|^{\beta}} dx \\ &\leq \frac{1}{\theta^{\gamma}} \left| \int_{\Omega} |\varphi| \frac{1}{|x|^{\beta}} dx \right| \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\theta^\gamma} \left(\int_{\Omega} |\varphi|^{\frac{2^*(N-\beta)+N}{2(N-\beta)}} dx \right)^{\frac{2(N-\beta)}{2^*(N-\beta)+N}} \left(\int_{B(0,R)} \left(\frac{1}{|x|^\beta} \right)^{\frac{2^*(N-\beta)+N}{2^*(N-\beta)-N+2\beta}} dx \right)^{\frac{2^*(N-\beta)-N+2\beta}{2^*(N-\beta)+N}} \\
&\leq \frac{1}{\theta^\gamma} \left(\int_{\Omega} |\varphi|^{\frac{2^*(N-\beta)+N}{2(N-\beta)}} dx \right)^{\frac{2(N-\beta)}{2^*(N-\beta)+N}} \\
&\quad \times \left(\frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \int_0^R r^{N-1-\frac{\beta(2^*(N-\beta)+N)}{2^*(N-\beta)-N+2\beta}} dr \right)^{\frac{2^*(N-\beta)-N+2\beta}{2^*(N-\beta)+N}} \\
&\leq \frac{1}{\theta^\gamma} \left(\frac{2\pi^{\frac{N}{2}}}{(N-\frac{\beta(2^*(N-\beta)+N)}{2^*(N-\beta)-N+2\beta})\Gamma(\frac{N}{2})} \right)^{\frac{2^*(N-\beta)-N+2\beta}{2^*(N-\beta)+N}} \left(\int_{\Omega} |\varphi|^{\frac{2^*(N-\beta)+N}{2(N-\beta)}} dx \right)^{\frac{2(N-\beta)}{2^*(N-\beta)+N}} \\
&\quad \times \left(R^{N-\frac{\beta(2^*(N-\beta)+N)}{2^*(N-\beta)-N+2\beta}} \right)^{\frac{2^*(N-\beta)-N+2\beta}{2^*(N-\beta)+N}} \\
&\leq \frac{C(\pi, R)}{\theta^\gamma} \left(\int_{\Omega} |\varphi|^{\frac{2^*(N-\beta)+N}{2(N-\beta)}} dx \right)^{\frac{2(N-\beta)}{2^*(N-\beta)+N}},
\end{aligned}$$

with

$$\begin{aligned}
&C(\pi, R) \\
&= \left(\frac{2\pi^{\frac{N}{2}}}{(N-\frac{\beta(2^*(N-\beta)+N)}{2^*(N-\beta)-N+2\beta})\Gamma(\frac{N}{2})} \right)^{\frac{2^*(N-\beta)-N+2\beta}{2^*(N-\beta)+N}} \left(R^{N-\frac{\beta(2^*(N-\beta)+N)}{2^*(N-\beta)-N+2\beta}} \right)^{\frac{2^*(N-\beta)-N+2\beta}{2^*(N-\beta)+N}}
\end{aligned}$$

and

$$N - \frac{\beta[2^*(N-\beta)+N]}{2^*(N-\beta)-N+2\beta} = \frac{(N-\beta)[2^*(N-\beta)-N]}{2^*(N-\beta)-N+2\beta} > 0.$$

From the calculations above, we know that $\frac{\varphi}{\theta^\gamma|x|^\beta} \in L^1(\Omega)$. Thus, applying the Dominated Convergence Theorem, one has

$$\lim_{n \rightarrow +\infty} \int_{\Omega} \frac{u_n}{|x|^\beta |u_n^+ + \theta|^\gamma} dx = \int_{\Omega} \frac{u}{|x|^\beta |u^+ + \theta|^\gamma} dx.$$

Consequently, $J'_\lambda(u) = 0$. \square

Lemma 2.3. Assume that $-\infty < a < \frac{N-2}{2}$, $a \leq b < a+1$, $0 \leq \beta < \frac{N}{2_*+1}(2_*+\gamma-1)$, $-\infty < \mu < \bar{\mu}_a = [\frac{N-2(a+1)}{2}]^2$ and let $(u_n) \subset \mathcal{H}_\mu$ be a Palais-Smale sequence (PS) $_c$ of J_λ for some $c \in \mathbb{R}$. Then, $u_n \rightarrow u$ in \mathcal{H}_μ and either

$$u_n \rightarrow u \quad \text{or} \quad c \geq J_\lambda(u) + \left(\frac{1}{2} - \frac{1}{2_*} \right) (h_0^{-2/2_*} S_\mu)^{2_*/(2_*-2)}.$$

Proof. We know that (u_n) is bounded in \mathcal{H}_μ . Up to a subsequence if necessary, we have that

$$\begin{aligned}
&u_n \rightharpoonup u \quad \text{in } \mathcal{H}_\mu \\
&u_n(x) \rightarrow u(x) \quad \text{a.e. in } \Omega.
\end{aligned}$$

We denote $v_n = u_n - u$. Then $v_n \rightarrow 0$. As in Brézis and Lieb [3], we have

$$\lim_{n \rightarrow \infty} \int_{\Omega} h(x) \left(|x|^{-2_*b} |u_n|^{2_*} - |x|^{-2_*b} |u_n - u|^{2_*} \right) dx = \int_{\Omega} h(x) |x|^{-2_*b} |u|^{2_*} dx \quad (2.2)$$

and

$$\lim_{n \rightarrow \infty} \int_{\Omega} \frac{|u_n + \theta|^{1-\gamma}}{|x|^\beta} dx = \int_{\Omega} \frac{|u_0^+ + \theta|^{1-\gamma}}{|x|^\beta} dx. \quad (2.3)$$

On the other hand, we can prove that

$$\lim_{n \rightarrow \infty} \int_{\Omega} (h(x) - h_0) \frac{|v_n|^{2^*}}{|x|^{2^*b}} dx = 0.$$

Fix $\varepsilon > 0$. By assumption (A1), there exists $r_\varepsilon > 0$ such that

$$|h(x) - h_0| = h(x) - h_0 < \varepsilon \quad \text{for a.e. } x \in \Omega \setminus B(0, r_\varepsilon).$$

Next we have

$$\begin{aligned} & \int_{\Omega} (h(x) - h_0) \frac{|v_n|^{2^*}}{|x|^{2^*b}} dx \\ &= \int_{\Omega \setminus B(0, r_\varepsilon)} (h(x) - h_0) \frac{|v_n|^{2^*}}{|x|^{2^*b}} dx + \int_{B(0, r_\varepsilon)} (h(x) - h_0) \frac{|v_n|^{2^*}}{|x|^{2^*b}} dx \\ &\leq \varepsilon \int_{\Omega \setminus B(0, r_\varepsilon)} \frac{|v_n|^{2^*}}{|x|^{2^*b}} dx + (|h|_\infty - h_0) \int_{B(0, r_\varepsilon)} \frac{|v_n|^{2^*}}{|x|^{2^*b}} dx. \end{aligned}$$

Since $v_n \rightharpoonup 0$ in \mathcal{H}_μ , the Caffarelli-Kohn-Nirenberg inequality implies that $\{v_n\}$ is bounded in L^{2^*} . Moreover by $u_n \rightharpoonup 0$ in \mathcal{H}_μ it follow that $v_n \rightharpoonup 0$ in $(\mathcal{H}_\mu \setminus \{0\})$. The above relations yield

$$\limsup_{n \rightarrow \infty} \int_{\Omega} (h(x) - h_0) \frac{|v_n|^{2^*}}{|x|^{2^*b}} dx \leq C_\varepsilon,$$

for some constant $c > 0$ independent of n and ε . Since $\varepsilon > 0$ was arbitrarily chosen, we conclude that

$$\lim_{n \rightarrow \infty} \int_{\Omega} (h(x) - h_0) \frac{|v_n|^{2^*}}{|x|^{2^*b}} dx = 0. \tag{2.4}$$

From (2.2), (2.3) and (2.4) we deduce that

$$J_\lambda(u_n) = J_\lambda(u) + (1/2)\|v_n\|_\mu^2 - (h_0/2^*) \int_{\Omega} |x|^{-2^*b} |v_n|^{2^*} + o_n(1) \tag{2.5}$$

and

$$0 \leftarrow \langle J'_\lambda(u_n), u_n \rangle = \|v_n\|_\mu^2 - h_0 \int_{\Omega} |x|^{-2^*b} |v_n|^{2^*} + o_n(1).$$

Then we have

$$\lim_{n \rightarrow \infty} \|v_n\|_\mu^2 = h_0 \lim_{n \rightarrow \infty} \int_{\Omega} |x|^{-2^*b} |v_n|^{2^*} = l \geq 0.$$

If $l = 0$ then $\|u_n - u\|_\mu \rightarrow 0$ as $n \rightarrow \infty$.

Otherwise if $l > 0$, by the definition of S_μ , we have

$$l \geq S_\mu (lh_0^{-1})^{2/2^*},$$

so that

$$l \geq (h_0^{-2/2^*} S_\mu)^{2^*/(2^*-2)}.$$

Thus we obtain

$$\begin{aligned} c &= J_\lambda(u) + \left(\frac{1}{2} - \frac{1}{2^*}\right)l \text{ taking limits in (2.5)} \\ &\geq J_\lambda(u) + \left(\frac{1}{2} - \frac{1}{2^*}\right)(h_0^{-2/2^*} S_\mu)^{2^*/(2^*-2)}. \end{aligned}$$

□

3. PROOF OF THEOREM 1.1

The proof is given in two parts.

3.1. Existence of a local minimizer. We prove that there exists

$$\lambda_1 = \frac{2_* - 2}{2_* 2} (|h|_\infty S_\mu)^{\frac{-(1+\gamma)}{2_* - 2}} \frac{1 - \gamma}{A} C_{a,b}^{\frac{1-\gamma}{2}} > 0,$$

with

$$A = \left[\frac{2\pi^{\frac{N}{2}} (2_* + \gamma - 1)}{N\Gamma(\frac{N}{2}) [(2_* + \gamma - 1) - \beta(\gamma + 1)]} \right]^{\frac{2_* + \gamma - 1}{2_* + 1}} R_0^{\frac{N}{2_* + 1} (2_* + \gamma - 1) - \beta} > 0$$

such that for any $\lambda \in (0, \lambda_1)$, J_λ achieves a local minimizer. First, we establish the following result.

Proposition 3.1. *Suppose that $-\infty < a < \frac{N-2}{2}$, $a \leq b < a + 1$, $0 \leq \beta < \frac{N}{2_* + 1} (2_* + \gamma - 1)$, $-\infty < \mu < \bar{\mu}_a = [\frac{N-2(a+1)}{2}]^2$, $0 < \gamma < 1$, (A1) hold and (A4) or (A5) hold. Then there exist positive reals λ_1, ϱ and δ such that for all $\lambda \in (0, \lambda_1)$, we have*

$$J_\lambda(u) \geq \delta > 0 \quad \text{for } \|u\|_\mu = \varrho \quad (3.1)$$

Proof. By the Holder inequality and the definition of S_μ , for all $u \in \mathcal{H}_\mu \setminus \{0\}$ we have

$$\begin{aligned} J_\lambda(u) &:= (1/2) \|u\|_\mu^2 - (1/2_*) \int_\Omega h(x) |x|^{-2_* b} |u|^{2_*} dx \\ &\quad - \frac{\lambda}{1 - \gamma} \int_\Omega \frac{(u^+ + \theta)^{1-\gamma} - \theta^{1-\gamma}}{|x|^\beta} dx \\ &\geq (1/2) \|u\|_\mu^2 - (|h|_\infty / 2_*) S_\mu \|u\|_\mu^{2_*} - \frac{\lambda A}{1 - \gamma} \|u\|_\mu^{1-\gamma} C_{a,b}^{\frac{-(1-\gamma)}{p}}. \end{aligned}$$

Taking $\varrho = \|u\|_\mu$, then there exist $\varrho > 0$ small enough and a positive constant λ_1 such that

$$J_\lambda(u) \geq \delta > 0 \quad \text{for } \|u\|_\mu = \varrho \text{ and } \lambda \in (0, \lambda_1). \quad (3.2)$$

This completes proof. \square

Since $\int_\Omega \frac{|u|^{1-\gamma}}{|x|^\beta} dx > 0$ and $0 < \gamma < 1$, it follows that for $t > 0$ small,

$$\begin{aligned} J_\lambda(t\phi) &:= (t^2/2) \|\phi\|^2 - (t^{2_*}/2_*) \int_\Omega h(x) |x|^{-2_* b} |\phi|^{2_*} dx \\ &\quad - \frac{\lambda t^{1-\gamma}}{1 - \gamma} \int_\Omega \frac{|u|^{1-\gamma}}{|x|^\beta} dx < 0. \end{aligned} \quad (3.3)$$

We also assume that t is so small enough such that $\|t\phi\|_\mu < \varrho$. Thus, we have

$$c_1 = \inf\{J_\lambda(u) : u \in B_\varrho\} < 0, \quad \text{where } B_\varrho = \{u \in \mathcal{H}_\mu, \|u\|_\mu \leq \varrho\}. \quad (3.4)$$

Using Ekeland's variational principle, for the complete metric space \bar{B}_ϱ with respect to the norm of \mathcal{H}_μ , there exists a $(PC)_{c_1}$ sequence $(u_n) \subset \bar{B}_\varrho$ such that $u_n \rightharpoonup u_1$ for some u_1 with $\|u_1\|_\mu \leq \varrho$.

Now, we claim that $u_n \rightarrow u_1$ in \mathcal{H}_μ , if not, by Lemma 2.3, we have

$$c_1 \geq J_\lambda(u_1) + \left(\frac{1}{2} - \frac{1}{2_*}\right) (h_0^{-2/2_*} S_\mu)^{2_*/(2_* - 2)}$$

$$\geq c_1 + \left(\frac{1}{2} - \frac{1}{2_*}\right) (h_0^{-2/2_*} S_\mu)^{2_*/(2_*-2)} > c_1,$$

which is a contradiction. Thus $u_n \rightarrow u_1$ in \mathcal{H}_μ .

Then we obtain a critical point u_1 of J_λ for all $\lambda \in (0, \lambda_1)$ satisfying $c_1 = J_\lambda(u_1) < 0$. Thus u_1 is a nontrivial solution of (1.5) with negative energy.

3.2. Existence of mountain pass type solution. We use the mountain pass theorem without Palais-Smale conditions to prove the existence of a nontrivial solution with positive energy. For this, we need the following Lemma. Set

$$c_\lambda^* = -\psi_2 \left(\frac{1+\gamma}{2}\right) \left[\frac{\psi_2(1-\gamma)}{2\psi_1}\right]^{\frac{1-\gamma}{1+\gamma}} + \left(\frac{1}{2} - \frac{1}{2_*}\right) (h_0^{-2/2_*} S_\mu)^{2_*/(2_*-2)},$$

with

$$\psi_1 = \left(\frac{1}{2} - \frac{1}{2_*}\right) \quad \text{and} \quad \psi_2 = \lambda \left(\frac{1}{1-\gamma} - \frac{1}{2_*}\right) AC_{a,b}^{\frac{-(1-\gamma)}{2}}.$$

Lemma 3.2. *Let $\lambda^* > 0$ such that $c_\lambda^* > 0$ for all $\lambda \in (0, \lambda^*)$. Then, there exist $\Lambda \in (0, \lambda^*)$ and $\varphi_\varepsilon \in \mathcal{H}_\mu$ for $\varepsilon > 0$ such that*

$$\sup_{t \geq 0} J_\lambda(t\varphi_\varepsilon) < c_\lambda^*, \quad \text{for all } \lambda \in (0, \Lambda).$$

Proof. Let

$$\varphi_\varepsilon(x) = \omega_\varepsilon(x) = \begin{cases} y_\varepsilon & \text{if (A2) holds} \\ v_\varepsilon & \text{if (A3) holds,} \end{cases} \tag{3.5}$$

where $y_\varepsilon, v_\varepsilon$ are defined in (1.6) and (1.7) respectively.

Now, we consider the functions

$$f(t) = J_\lambda(t\varphi_\varepsilon),$$

$$\tilde{f}(t) = (t^2/2) \|\varphi_\varepsilon(x)\|_\mu^2 - (t^{2_*}/2_*) h_0 \int_\Omega |x|^{-2_*b} |\varphi_\varepsilon(x)|^{2_*} dx.$$

Then, for all $\lambda \in (0, \lambda^*)$ we obtain that $0 = f(0) < c_\lambda^*$. By the continuity of $f(t)$, there exists t_1 a sufficiently small positive quantity such that $f(t) < c_\lambda^*$ for all $t \in (0, t_1)$. On the other hand, we have

$$\max_{t \geq 0} \tilde{f}(t) = \left(\frac{1}{2} - \frac{1}{2_*}\right) (h_0^{-2/2_*} S_\mu)^{2_*/(2_*-2)},$$

then, we obtain

$$\sup_{t \geq 0} J_\lambda(t\varphi_\varepsilon) < \left(\frac{1}{2} - \frac{1}{2_*}\right) (h_0^{-2/2_*} S_\mu)^{2_*/(2_*-2)} - \lambda \frac{t_1^{1-\gamma}}{1-\gamma} \int_\Omega \frac{|\varphi_\varepsilon|^{1-\gamma}}{|x|^\beta} dx.$$

Taking $\lambda > 0$ such that

$$\lambda \frac{t_1^{1-\gamma}}{1-\gamma} \int_\Omega \frac{|\varphi_\varepsilon|^{1-\gamma}}{|x|^\beta} dx > (1/2_*2)(2_* - 1)(2_* - 2)^{-1/2} \lambda^2 AC_{a,b}^{\frac{-(1-\gamma)}{2}} \|\varphi_\varepsilon\|_\mu^{1-\gamma},$$

we obtain

$$0 < \lambda < \Gamma^*,$$

where

$$\Gamma^* := 2_*2(2_* - 2)^{1/2}(2_* - 1)^{-1} \frac{t_1^{1-\gamma}}{1-\gamma} \int_\Omega \frac{|\varphi_\varepsilon|^{1-\gamma}}{|x|^\beta} dx.$$

Setting $\Lambda = \min\{\lambda^*, \Gamma^*\}$ we deduce that

$$\sup_{t \geq 0} J_\lambda(t\varphi_\varepsilon) < c_\lambda^* \quad \text{for all } \lambda \in (0, \Lambda).$$

This completes the proof. \square

Proof of Theorem 1.1. Since $\lim_{t \rightarrow \infty} J_\lambda(t\varphi_\varepsilon) = -\infty$, we can choose $T > 0$ large enough such that $J_\lambda(T\varphi_\varepsilon) < 0$. From Proposition 3.1, we have $J_\lambda|_{\partial B_\rho} \geq \delta > 0$ for all $\lambda \in (0, \lambda_1)$. By the mountain pass theorem without the Palais-Smale condition [2], there exists a $(PC)_{c_2}$ sequence (u_n) in \mathcal{H}_μ which is characterized by

$$c_2 = \inf_{\theta \in \Gamma} \max_{t \in [0,1]} J_\lambda(\theta(t)) > \delta > 0 \quad \text{with } \delta \text{ independent of } \theta$$

with

$$\Gamma = \{\theta \in C([0,1], \mathcal{H}_\mu), \theta(0) = 0, \theta(1) = T\varphi_\varepsilon\}.$$

Then, (u_n) has a subsequence, still denoted by (u_n) such that $u_n \rightharpoonup u_2$ in \mathcal{H}_μ . By Lemma (2.3), if u_n does not converge to u_2 , we obtain

$$c_2 \geq J_\lambda(u_2) + \left(\frac{1}{2} - \frac{1}{2_*}\right) (h_0^{-2/2_*} S_\mu)^{2_*/(2_*-2)} \geq c_\lambda^*,$$

what contradicts the fact that, by Lemma 3.2, we have

$$\sup_{t \geq 0} J_\lambda(t\varphi_\varepsilon) < c_\lambda^*,$$

for all $\lambda \in (0, \Lambda)$. Thus $u_n \rightarrow u_2$ in \mathcal{H}_μ . Thus, we obtain a critical point u_2 of J_λ for all $\lambda \in (0, \lambda_1)$ with

$$\Lambda_0 := \min\{\lambda_1, \Lambda\}$$

satisfying $J_\lambda(u_2) > 0$. Now we prove that $u_1 \neq u_2$.

We have u_1 is the first solution of (1.1) where

$$J_\lambda(u_1)|_{\theta=0} = \inf\{J_\lambda(u)|_{\theta=0} : u \in B_\rho\} = c_1 < 0.$$

On the other hand, for $\theta \in (0, 1)$, (1.5) has at least a mountain pass solution $\{u_\theta\}$ with $J_\lambda(u_\theta) > \delta > 0$. Thus, there exists $\{\theta_n\} \subset (0, 1)$ with $\theta_n \rightarrow 0$ as $n \rightarrow \infty$, such that (u_{θ_n}) is a sequence mountain pass solutions of (1.5) with

$$J_\lambda(u_\theta) > \delta > 0, \text{ by Proposition 1,}$$

then, $\lim_{n \rightarrow \infty} u_{\theta_n} = u_2$ is the second solution of (1.1) and

$$J_\lambda(u_2)|_{\theta=0} = \lim_{n \rightarrow \infty} J_\lambda(u_{\theta_n}) \geq \delta > 0.$$

So,

$$J_\lambda(u_1)|_{\theta=0} < 0 < J_\lambda(u_2)|_{\theta=0},$$

which implies that $u_1 \neq u_2$. \square

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