SINGULAR p-BIHARMONIC PROBLEMS INVOLVING THE HARDY-SOBOLEV EXPONENT

AMOR DRISSI, ABDELJABBAR GHANMI, DUŠAN D. REPOVŠ

ABSTRACT. This article concerns the existence and multiplicity of solutions for the singular p-biharmonic problem involving the Hardy potential and the critical Hardy-Sobolev exponent. To this end we use variational methods combined with the Mountain pass theorem and the Ekeland variational principle. We illustrate the usefulness of our results with and example.

1. INTRODUCTION

Recently, a lot of attention has been paid to the study of problems involving the p-Laplacian operator and the p-biharmonic operator. We refer the reader to Alsaedi et al. [1], Cung et al. [10], Huang and Liu [15], and Sun and Wu [24, 25]. The reason for studying these problems is their applications in fields such as quantum mechanics, flame propagation, and traveling waves in suspension bridges; for more applications see Bucur and Valdinoci [6], and Lazer and McKenna [16]. Problems involving Hardy terms have been extensively investigated by several authors, see, e.g., Bhakta et al. [3, 4], Ghoussoub and Yuan [13], and Guan et al. [14]. Various problems involving the critical Hardy-Sobolev exponent have been widely studied, see, e.g., Chaharlang and Razani [7], Chen et al. [9], Perera and Zou [19], Pérez-Llanos and Primo [20], Wang [26], and Wang and Zhao [27].

In particular, Ghoussoub and Yuan [13] used variational methods to study the existence of solutions of the problem

$$-\Delta_p \varphi = \lambda |\varphi|^{r-2}\varphi + \mu \frac{|\varphi|^{q-2}\varphi}{|x|^\alpha} \quad \text{in } \Omega,$$
$$\varphi = 0 \quad \text{on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^n$ is a regular bounded domain, μ and λ are positive parameters, $\min(q, r) \geq p$, $q \leq p^*(\alpha)$, and $r \leq p^*$.

Perrera and Zou [19] investigated the critical Hardy p-Laplacian problem

$$-\Delta_p\varphi = \lambda|\varphi|^{p-2}\varphi + \frac{|\varphi|^{p^*(\alpha)-2}\varphi}{|x|^\alpha} \quad \text{in } \Omega,$$
$$\varphi = 0 \quad \text{on } \partial\Omega. \tag{1.1}$$

More precisely, they used variational methods to establish the multiplicity of solutions of problem (1.1). Recently, Wang [26] considered the problem

$$\Delta_p^2\varphi = h(x,\varphi) + \mu\frac{|\varphi|^{r-2}\varphi}{|x|^s} \quad \text{in } \Omega,$$
$$\varphi = \Delta\varphi = 0 \quad \text{on } \partial\Omega. \tag{1.2}$$

He used the Mountain pass theorem to establish the existence of solutions of problem (1.2). Moreover, the existence of multiple solutions was established by applying the Fountain Theorem.

Motivated by the above mentioned results, we study in the existence and multiplicity of solutions of the following singular p-biharmonic problem involving the Hardy potential and the critical Hardy-Sobolev exponent,

$$\Delta_p^2\varphi - \lambda\frac{|\varphi|^{p-2}\varphi}{|x|^{2p}} + \Delta_p\varphi = \mu f(x)h(\varphi) + \frac{|\varphi|^{p^*(\alpha)-2}\varphi}{|x|^\alpha} \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

where $0 \le \alpha < 2p$, $1 < p < \frac{N}{2}$, $\lambda > 0$, $\mu > 0$, and $p^*(\alpha) := \frac{p(N-\alpha)}{N-2p}$. Here, Δ_p and Δ_p^2 are the p-Laplacian and the p-biharmonic operator, respectively, defined by

$$\Delta_p\varphi := \text{div}(|\nabla\varphi|^{p-2}\nabla\varphi), \quad \Delta_p^2\varphi := \Delta(|\Delta\varphi|^{p-2}\Delta\varphi),$$

respectively, f is a positive function, h is a continuous function. We use the following hypotheses:

(H1) There exists $r \in (p, p^*)$ such that $f \in L^\infty(\mathbb{R}^N)$ and $|h(\varphi)| \le c_1|\varphi|^{r-1}$ for every $\varphi \in E$ and some positive constant c_1, where $p^* := \frac{pN}{N-2p}$ and the space $E := W^{2,p}(\mathbb{R}^N)$ is defined in Section 2.

(H2) There exists $\sigma > 0$ such that for every $y \in \mathbb{R}^N$, we have $0 < rH(\varphi) \le h(\varphi)\varphi$, $|\varphi| \ge \sigma > 0$, where $H(t) := \int_0^t h(s)ds$.

(H3) There exist $c_1 > 0$, $1 < r < p$, and $s \in (\frac{p^*}{p^*-r}, \frac{p}{p-r})$ such that

$$0 < f \in L^{\frac{p^*}{p^*-r}}(\mathbb{R}^N) \cap L^s_{\text{loc}}(\mathbb{R}^N),$$

and $|h(\varphi)| \le c_1|\varphi|^{r-1}$ for for every $\varphi \in E$.

The following are the main results of this article.

Theorem 1.1. *Suppose that* (H1), (H2) *hold. Then for every $\mu > 0$, the singular p-biharmonic problem* (1.3) *has at least one nontrivial weak solution, provided that $\lambda > 0$ is small enough.*

Theorem 1.2. *Suppose that* (H2), (H3) *hold. Then there exists $\mu_0 > 0$ such that for every $\mu \in (0, \mu_0)$, the singular p-biharmonic problem* (1.3) *has at least two nontrivial weak solutions, provided that $\lambda > 0$ is small enough.*

Note that the singular p-biharmonic problem (1.3) is very important since it contains the p-biharmonic operator, the p-Laplacian operator, the singular nonlinearity, and the Hardy potential. Moreover, it appears in many applications, such as non-Newtonian fluids, viscous fluids, traveling waves in suspension bridges, and

various other physical phenomena, see, e.g., Chen et al. [8], Lazer and McKenna [16], and Ružička [22].

The article is organized as follows: In Section 2, we present some variational framework related to problem (1.3). In Section 3, we prove Theorem 1.1. In Section 4 we combine the Mountain pass theorem with the Ekeland variational principle to prove the multiplicity of solutions of problem (1.3) (Theorem 1.2). In Section 5 we present an example that illustrate our main results. Finally, in Section 6 we summarize the main contributions of this article.

2. Preliminaries

We begin by recalling some necessary facts related to the Hardy-Sobolev exponent nonlinearity. We finish this section by presenting the variational framework related to problem (1.3). For other necessary background material we refer to the comprehensive monograph by Papageorgiou et al. [18].

It is well-known that the Hardy-Sobolev exponent is closely related to the Rellich inequality (see Davies and Hinz [11, p. 520])

$$\int_{\mathbb{R}^N} \frac{|\varphi(x)|^p}{|x|^{2p}} \, dx \le \Big(\frac{p^2}{N(p-1)(N-2p)} \Big)^p \int_{\mathbb{R}^N} |\Delta\varphi(x)|^p \, dx, \qquad (2.1)$$

for every $\varphi \in W^{2,p}(\mathbb{R}^N)$, where $W^{2,p}(\mathbb{R}^N)$ denotes the Sobolev space which is defined by

$$W^{2,p}(\mathbb{R}^N) := \big\{ \varphi \in L^p(\mathbb{R}^N) : \Delta\varphi, |\nabla\varphi| \in L^p(\mathbb{R}^N) \big\}.$$

For more details about this space, see Davies and Hinz [11], Mitidieri [17], and Rellich [21].

According to the Rellich inequality (2.1), $W^{2,p}(\mathbb{R}^N)$ can be endowed with the following norm

$$\|\varphi\| := \Big(\int_{\mathbb{R}^N} |\Delta\varphi(x)|^p - \lambda \frac{|\varphi(x)|^p}{|x|^{2p}} + |\nabla\varphi(x)|^p \, dx \Big)^{1/p},$$

provided that

$$0 < \lambda < \Big(\frac{N(p-1)(N-2p)}{p^2} \Big)^p. \qquad (2.2)$$

The space $W^{2,p}(\mathbb{R}^N)$ is continuously embeddable into $L^\sigma(\mathbb{R}^N)$ for every $p \le \sigma \le p^*$, and compactly embeddable into $L^\sigma_{\mathrm{loc}}(\mathbb{R}^N)$, for every $p \le \sigma < p^*$. Moreover, for every $\varphi \in W^{2,p}(\mathbb{R}^N)$, one has

$$|\varphi|_\sigma \le S_\sigma^{-1/p} \|\varphi\|, \qquad (2.3)$$

where $|\varphi|_p$ denotes the usual $L^p(\mathbb{R}^N)$-norm and S_σ is defined by

$$S_\sigma := \inf_{\varphi \in W^{2,p}(\mathbb{R}^N), \varphi \ne 0} \frac{\int_{\mathbb{R}^N} |\Delta\varphi(x)|^p - \lambda \frac{|\varphi(x)|^p}{|x|^{2p}} + |\nabla\varphi(x)|^p \, dx}{\big(\int_{\mathbb{R}^N} |x^{-\alpha}||\varphi(x)|^\sigma \, dx \big)^{p/\sigma}}. \qquad (2.4)$$

Hereafter, for simplicity, we shall denote $E := W^{2,p}(\mathbb{R}^N)$.

We define the weighted Lebesgue space $L^r(\mathbb{R}^{\mathbb{N}}, f)$ by

$$L^r(\mathbb{R}^{\mathbb{N}}, f) := \Big\{ \varphi : \mathbb{R}^N \to \mathbb{R} : \varphi \text{ is measurable and } \int_{\mathbb{R}^N} f(x)|\varphi(x)|^r \, dx < \infty \Big\},$$

and endow it with the norm

$$\|\varphi\|_{r,f} := \Big(\int_{\mathbb{R}^N} f(x)|\varphi(x)|^r \, dx \Big)^{1/r}.$$

Then $L^r(\mathbb{R}^{\mathbb{N}}, f)$ is a uniformly convex Banach space. Dhifli and Alsaedi [12] proved that under hypothesis (H3), the embedding $E \hookrightarrow L^r(\mathbb{R}^N, f)$ is continuous and compact. Moreover, one has the estimate

$$\|\varphi\|_{r,f}^r \leq S_{p^*}^{-r/p} |f|_{\frac{p^*}{p^*-r}} \|\varphi\|^r, \quad \text{for every } \varphi \in E. \tag{2.5}$$

Now, let us introduce the notion of weak solutions.

Definition 2.1. A function $\varphi \in E$ is said to be a *weak solution* of problem (1.3), provided that

$$\Lambda(\varphi, \psi) = \mu \int_{\mathbb{R}^N} f(x) h(\varphi) \psi \, dx + \int_{\mathbb{R}^N} |x|^{-\alpha} \varphi^{p^*(\alpha)-2} \varphi \psi \, dx, \quad \text{for every } \psi \in E,$$

where

$$\Lambda(\varphi, \psi) := \int_{\mathbb{R}^N} |\Delta \varphi|^{p-2} \Delta \varphi \Delta \psi - \lambda \frac{|\varphi|^{p-2} \varphi \psi}{|x|^{2p}} + |\nabla \varphi|^{p-2} \nabla \varphi \nabla \psi \, dx.$$

We define the energy functional $J_\mu : E \to \mathbb{R}$, by

$$J_\mu(\varphi) := \frac{1}{p} \|\varphi\|^p - \mu \int_{\mathbb{R}^N} f(x) h(\varphi) \varphi \, dx - \frac{1}{p^*(\alpha)} \int_{\mathbb{R}^N} |x|^{-\alpha} \varphi^{p^*(\alpha)} \, dx.$$

Note that a function $\varphi \in E$ is a weak solution of (1.3), if it satisfies $J_\mu'(\varphi) = 0$, i.e., φ is a critical value for J_μ.

Definition 2.2. We say that a function $\Phi \in C^1(F, \mathbb{R})$, where F is a Banach space, satisfies the *Palais-Smale condition*, if every sequence $\{\varphi_n\} \subset F$, such that $\Phi(\varphi_n)$ is bounded and $\Phi'(\varphi_n) \to 0$ in F^*, as $n \to \infty$, contains a convergent subsequence.

To prove Theorem 1.1, we need the following result which is proved in Ambrosetti and Rabinowitz [2, Theorem 2.4].

Theorem 2.3 (Mountain pass theorem). *Let $\Phi \in C^1(F, \mathbb{R})$, where F is a Banach space, and suppose that $\varphi \in F$ is such that $\|\varphi\| > r$, for some $r > 0$, and $\inf_{\|\psi\|=r} \Phi(\psi) > \Phi(0) > \Phi(\varphi)$. If in addition, Φ satisfies the Palais-Smale condition at level c, then c is a critical value of Φ, where $c := \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} \Phi(\gamma(s))$ and $\Gamma = \{\gamma \in C([0,1], F) : (\gamma(0), \gamma(1)) = (0, \varphi)\}$.*

3. Proof of Theorem 1.1

In this section, we shall prove the first main result of this paper. More precisely, under suitable conditions, we shall prove that the functional energy associated with problem (1.3) satisfies the Mountain pass geometry. First, we shall prove several lemmas.

Lemma 3.1. *Under hypotheses* (H1) *and* (H2), *there exist $\rho > 0$ and $\eta > 0$ such that $\|\varphi\| = \rho$ implies $J_\mu(\varphi) \geq \eta > 0$.*

Proof. Let $\varphi \in E$. From (H1), (H2) and (2.3), we obtain

$$J_\mu(\varphi) = \frac{1}{p} \|\varphi\|^p - \mu \int_{\mathbb{R}^N} f(x) H(\varphi) \, dx - \frac{1}{p^*(\alpha)} \int_{\mathbb{R}^N} |x|^{-\alpha} u^{p^*(\alpha)} \, dx$$

$$\geq \frac{1}{p} \|\varphi\|^p - \frac{\mu}{r} \int_{\mathbb{R}^N} f(x) h(\varphi) \, dx - \frac{1}{p^*(\alpha)} S_{p^*(\alpha)}^{-\frac{p^*(\alpha)}{p}} \|\varphi\|^{p^*(\alpha)}$$

$$\geq \frac{1}{p} \|\varphi\|^p - \frac{\mu}{r} c_1 \|f\|_\infty |u|_r^r - \frac{1}{p^*(\alpha)} S_{p^*(\alpha)}^{-\frac{p^*(\alpha)}{p}} \|\varphi\|^{p^*(\alpha)}$$

$$\geq \frac{1}{p}\|\varphi\|^p - \frac{\mu}{r}c_1\|f\|_\infty S_r^{-r/p}\|\varphi\|^r - \frac{1}{p^*(\alpha)}S_{p^*(\alpha)}^{-\frac{p^*(\alpha)}{p}}\|\varphi\|^{p^*(\alpha)}$$

$$\geq \|\varphi\|^p\Big\{\frac{1}{p} - \frac{\mu}{r}c_1\|f\|_\infty S_r^{-r/p}\|\varphi\|^{r-p} - \frac{1}{p^*(\alpha)}S_{p^*(\alpha)}^{-\frac{p^*(\alpha)}{p}}\|\varphi\|^{p^*(\alpha)-p}\Big\},$$

and since $\min(r, p^*(\alpha)) > p$, we obtain

$$\lim_{\|\varphi\|\to 0}\Big\{\frac{1}{p} - \frac{\mu}{r}c_1\|f\|_\infty S_r^{-r/p}\|\varphi\|^{r-p} - \frac{1}{p^*(\alpha)}S_{p^*(\alpha)}^{-\frac{p^*(\alpha)}{p}}\|\varphi\|^{p^*(\alpha)-p}\Big\} = \frac{1}{p} > 0,$$

therefore, for $\rho > 0$ small enough, if $\|\varphi\| = \rho$, we obtain

$$\eta := \rho^p\Big(\frac{1}{p} - \frac{\mu}{r}c_1\|f\|_\infty S_r^{-r/p}\rho^{r-p} - \frac{1}{p^*(\alpha)}S_{p^*(\alpha)}^{-\frac{p^*(\alpha)}{p}}\rho^{p^*(\alpha)-p}\Big) > 0,$$

thus $\|\varphi\| = \rho \implies J_\mu(\varphi) \geq \eta > 0$. This completes the proof. $\qquad\square$

Lemma 3.2. *Under the hypotheses of Lemma 3.1, there exists $e \in E$ such that $\|e\| > \rho$ and $J_\mu(e) < 0$.*

Proof. Let φ be a positive function in $\mathcal{C}_c^\infty(E)$. Then for every $s > 0$ we have

$$J_\mu(s\varphi) = \frac{s^p}{p}\|\varphi\|^p - \mu\int_{\mathbb{R}^N}f(x)H(s\varphi)\,dx - \frac{s^{p^*(\alpha)}}{p^*(\alpha)}\int_{\mathbb{R}^N}|x|^{-\alpha}\varphi^{p^*(\alpha)}\,dx$$

$$\leq \frac{s^p}{p}\|\varphi\|^p - \frac{s^{p^*(\alpha)}}{p^*(\alpha)}\int_{\mathbb{R}^N}|x|^{-\alpha}\varphi^{p^*(\alpha)}\,dx.$$

Since $p < p^*(\alpha)$, it follows that $J_\mu(s\varphi) \to -\infty$, as $s \to \infty$. Therefore there exists $s_0 > \frac{\rho}{\|\varphi\|}$ large enough, such that $J_\mu(s_0\varphi) < 0$. If we now set $e = s_0\varphi$, then $\|e\| > \rho$ and $J_\mu(e) < 0$. This completes the proof. $\qquad\square$

Lemma 3.3. *Under the hypotheses of Lemma 3.1, J_μ satisfies the Palais-Smale condition.*

Proof. Let $\{\varphi_n\}$ be a Palais-Smale sequence, which means that $J_\mu(\varphi_n)$ is bounded and $J'_\mu(\varphi_n) \to 0$, as $n \to \infty$. Therefore there exist $m_1 > 0$ and $m_2 > 0$ such that $J_\mu(\varphi_n) \leq m_1$ and $|J'_\mu(\varphi_n)| \leq m_2$. Letting $\theta := \min(r, p^*(\alpha))$, we obtain by hypothesis (H1) that

$$\theta m_1 + m_2 \geq \theta J_\mu(\varphi_n) - \langle J'_\mu(\varphi_n), \varphi_n\rangle$$

$$\geq \frac{\theta}{p}\|\varphi_n\|^p - \mu\theta\int_{\mathbb{R}^N}f(x)H(\varphi_n)\,dx - \frac{\theta}{p^*(\alpha)}\int_{\mathbb{R}^N}|x|^{-\alpha}|\varphi_n|^{p^*(\alpha)}\,dx$$

$$\quad - \|\varphi_n\|^p + \mu\int_{\mathbb{R}^N}f(x)h(\varphi_n)\varphi_n\,dx + \int_{\mathbb{R}^N}|x|^{-\alpha}\varphi_n^{p^*(\alpha)}\,dx$$

$$\geq (\frac{\theta}{p} - 1)\|\varphi_n\|^p + \mu(r - \theta)\int_{\mathbb{R}^N}f(x)H(\varphi_n)\,dx$$

$$\quad + (1 - \frac{\theta}{p^*(\alpha)})\int_{\mathbb{R}^N}|x|^{-\alpha}|\varphi_n|^{p^*(\alpha)}\,dx$$

$$\geq (\frac{\theta}{p} - 1)\|\varphi_n\|^p,$$

and since $\theta = \min(r, p^*(\alpha)) > p$, it follows that the sequence $\{\varphi_n\}$ is bounded in E. Therefore (up to a subsequence) there exists $\varphi \in E$ such that

$$\varphi_n \rightharpoonup \varphi \quad \text{weakly in } E,$$

$$\varphi_n \to \varphi \quad \text{strongly in } L^r(\mathbb{R}^N),$$
$$\varphi_n \to \varphi \quad \text{a.e. in } \mathbb{R}^n,$$

so, by (H1), (H2) and the Dominated convergence theorem,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f(x)H(\varphi_n)\,dx = \int_{\mathbb{R}^N} f(x)H(\varphi)\,dx. \tag{3.1}$$

One can now show by a standard argument that the weak limit u of $\{\varphi_n\}$ is a critical point of J_μ and thus $J'_\mu(\varphi) = 0$.

Let $w_n := \varphi_n - \varphi$. Then w_n converges weakly to zero. Moreover, by Brezis and Lieb [5, Lemma 3], we obtain

$$|w_n|_{p^*(\alpha)}^{p^*(\alpha)} = |\varphi_n|_{p^*(\alpha)}^{p^*(\alpha)} - |\varphi|_{p^*(\alpha)}^{p^*(\alpha)} + o(1),$$

therefore,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |x|^{-\alpha}|\varphi_n|^{p^*(\alpha)} - |x|^{-\alpha}|w_n|^{p^*(\alpha)}\,dx = \int_{\mathbb{R}^N} |x|^{-\alpha}|\varphi|^{p^*(\alpha)}\,dx,$$

and from (3.1) we have

$$\langle J'_\mu(\varphi_n), \varphi_n\rangle - \langle J'_\mu(\varphi), \varphi\rangle = \|w_n\|^p - \int_{\mathbb{R}^N} |x|^{-\alpha}|w_n|^{p^*(\alpha)}\,dx + o(1),$$

hence for n large enough,

$$\|w_n\|^p = \int_{\mathbb{R}^N} |x|^{-\alpha}|w_n|^{p^*(\alpha)}\,dx + o(1),$$

thus

$$\lim_{n\to\infty} \|w_n\|^p = \lim_{n\to\infty} \int_{\mathbb{R}^N} |x|^{-\alpha}|w_n|^{p^*(\alpha)} = l \geq 0. \tag{3.2}$$

If $l > 0$, then by combining equation (2.4) with (3.2) we obtain

$$l \geq S_{p^*(\alpha)}^{\frac{p}{p^*(\alpha)-p}}. \tag{3.3}$$

On the other hand, one has

$$J_\mu(\varphi_n) - J_\mu(\varphi) = \frac{1}{p}\|w_n\|^p - \frac{1}{p^*(\alpha)}\int_{\mathbb{R}^N} |x|^{-\alpha}|w_n|^{p^*(\alpha)}\,dx + o(1),$$

so by letting n tend to infinity, we obtain

$$c - J_\mu(\varphi) = \left(\frac{1}{p} - \frac{1}{p^*(\alpha)}\right)l,$$

and using the last equation and (3.3) we obtain

$$J_\mu(\varphi) + \left(\frac{1}{p} - \frac{1}{p^*(\alpha)}\right)l = c < \left(\frac{1}{p} - \frac{1}{p^*(\alpha)}\right)S_\lambda^{\frac{p}{p^*(\alpha)-p}},$$

which implies that

$$J_\mu(\varphi) < 0. \tag{3.4}$$

However, we have $\langle J'_\mu(\varphi), \varphi\rangle = 0$, for every $\varphi \in E$. So, from (H2) we obtain

$$\|\varphi\|^p = \mu \int_{\mathbb{R}^N} f(x)h(\varphi)\varphi\,dx + \int_{\mathbb{R}^N} |x|^{-\alpha}|\varphi|^{p^*(\alpha)}\,dx$$
$$\geq r\mu \int_{\mathbb{R}^N} f(x)H(\varphi)\,dx + \int_{\mathbb{R}^N} |x|^{-\alpha}|\varphi|^{p^*(\alpha)}\,dx;$$

therefore,

$$
\begin{aligned}
J_\mu(\varphi) &= \frac{1}{p}\|\varphi\|^p - \mu \int_{\mathbb{R}^N} f(x)H(\varphi)\,dx - \frac{1}{p^*(\alpha)}\int_{\mathbb{R}^N} |x|^{-\alpha}|u|^{p^*(\alpha)}\,dx \\
&\geq \frac{1}{p}\Big(r\mu \int_{\mathbb{R}^N} f(x)H(\varphi)\,dx + \int_{\mathbb{R}^N} |x|^{-\alpha}|u|^{p^*(\alpha)}\,dx\Big) \\
&\quad - \mu \int_{\mathbb{R}^N} f(x)H(\varphi)\,dx - \frac{1}{p^*(\alpha)}\int_{\mathbb{R}^N} |x|^{-\alpha}|u|^{p^*(\alpha)}\,dx \\
&\geq \mu\Big(\frac{r}{p}-1\Big)\int_{\mathbb{R}^N} f(x)H(\varphi)\,dx + \Big(\frac{1}{p}-\frac{1}{p^*(\alpha)}\Big)\int_{\mathbb{R}^N} |x|^{-\alpha}|u|^{p^*(\alpha)}\,dx,
\end{aligned}
$$

and since $r \in (p, p^*)$ and $p < p^*(\alpha)$, it follows that $J_\mu(\varphi) \geq 0$. This is in contradiction with (3.4). Since $l = 0$, we see by (3.2) that $\{\varphi_n\}$ converges strongly to φ in E. This completes the proof. $\square$

Proof of Theorem 1.1. By Lemma 3.1, there exist $\rho \in (0, \infty)$ and $\eta \in (0, \infty)$ such that $\inf_{\|\varphi\|=\rho} J_\mu(\varphi) \geq \eta > 0$. On the other hand, by Lemma 3.2, there exists $e \in E$ such that

$$
\rho \leq \|e\| \quad \text{and} \quad J_\mu(e) < 0 < \inf_{\|\varphi\|=\rho} J_\mu(\varphi),
$$

hence, combining Lemma 3.3 and Theorem 2.3, we can establish the existence of a critical point φ_μ. Moreover, φ_μ is characterized by

$$
J_\mu(\varphi_\mu) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_\mu(\gamma(t)),
$$

where

$$
\Gamma := \{\gamma \in C([0,1], X) : (\gamma(0), \gamma(1)) = (0, e)\},
$$

so if we take $\gamma(s) = se$, then there exists $s_0 \in [0, 1]$ such that $\|s_0 e\| = \rho$, hence invoking Lemma 3.2, we obtain

$$
J_\mu(\varphi_\mu) \geq \eta > 0. \tag{3.5}
$$

This completes the proof. $\square$

4. Proof of Theorem 1.2

The proof is divided into several lemmas.

Lemma 4.1. *Under hypotheses* (H2) *and* (H3), *there exist positive constants* μ_0, ρ, *and* η *such that for every* $\mu \in (0, \mu_0)$, $\|\varphi\| = \rho$ *implies* $J_\mu(\varphi) \geq \eta > 0$.

Proof. Let $\varphi \in E$. Invoking hypotheses (H2), (H3), equations (2.3), (2.5), and the Hölder inequality, we obtain

$$
\begin{aligned}
J_\mu(\varphi) &= \frac{1}{p}\|\varphi\|^p - \mu \int_{\mathbb{R}^N} f(x) H(\varphi)\, dx - \frac{1}{p^*(\alpha)} \int_{\mathbb{R}^N} |x|^{-\alpha} \varphi^{p^*(\alpha)}\, dx \\
&\geq \frac{1}{p}\|\varphi\|^p - \frac{\mu}{r} \int_{\mathbb{R}^N} f(x) h(\varphi)\, dx - \frac{1}{p^*(\alpha)} S_{p^*}^{-\frac{p^*(\alpha)}{p}} \|\varphi\|^{p^*(\alpha)} \\
&\geq \frac{1}{p}\|\varphi\|^p - \frac{\mu}{r} c_1 \|\varphi\|_{r,f}^r - \frac{1}{p^*(\alpha)} S_{p^*}^{-\frac{p^*(\alpha)}{p}} \|\varphi\|^{p^*(\alpha)} \\
&\geq \frac{1}{p}\|\varphi\|^p - \frac{\mu}{r} c_1 \|f\|_{\frac{p^*}{p^*-r}} S_{p^*}^{-r/p} \|\varphi\|^r - \frac{1}{p^*(\alpha)} S_{p^*}^{-\frac{p^*(\alpha)}{p}} \|\varphi\|^{p^*(\alpha)} \\
&\geq \|\varphi\|^r \Big(\frac{1}{p}\|\varphi\|^{p-r} - \frac{\mu}{r} c_1 \|f\|_{\frac{p^*}{p^*-r}} S_{p^*}^{-r/p} - \frac{1}{p^*(\alpha)} S_{p^*}^{-\frac{p^*(\alpha)}{p}} \|\varphi\|^{p^*(\alpha)-r} \Big) \\
&\geq \|\varphi\|^r \Big(h(\|\varphi\|) - \frac{\mu}{r} c_1 \|f\|_{\frac{p^*}{p^*-r}} S_{p^*}^{-r/p} \Big),
\end{aligned}
\tag{4.1}
$$

where

$$
h(s) := \frac{1}{p} s^{p-r} - \frac{1}{p^*(\alpha)} S_{p^*}^{-\frac{p^*(\alpha)}{p}} s^{p^*(\alpha)-r}.
$$

It is not difficult to prove that h attains its global maximum at

$$
s_0 := \Big(\frac{p^*(\alpha)(p-r) S_{p^*}^{\frac{p^*(\alpha)}{p}}}{p(p^*(\alpha)-r)} \Big)^{\frac{1}{p^*-p}}.
$$

Set

$$
\mu_0 := \frac{r f(s_0)}{c_1 \|f\|_{\frac{p^*}{p^*-r}} S_{p^*}^{-r/p}}.
\tag{4.2}
$$

Then for every $\mu \in (0, \mu_0)$, we have

$$
h(s_0) - \frac{\mu}{r} c_1 \|f\|_{\frac{p^*}{p^*-r}} S_{p^*}^{-r/p} > 0,
$$

and since h is continuous, we can find $\rho > 0$ such that

$$
h(\rho) - \frac{\mu}{r} c_1 \|f\|_{\frac{p^*}{p^*-r}} S_{p^*}^{-r/p} > 0,
$$

thus for every $\varphi \in E$ with $\|\varphi\| = \rho$, we have

$$
J_\mu(\varphi) \geq \eta := \rho^r \Big(h(\rho) - \frac{\mu}{r} c_1 \|f\|_{\frac{p^*}{p^*-r}} S_{p^*}^{-r/p} \Big) > 0.
$$

This completes the proof. $\square$

Lemma 4.2. *There exists $e \in E$ such that $\|e\| > \rho$ and $J_\mu(e) < 0$.*

Since the proof of the above lemma is very similar to that of Lemma 3.2, we omit it.

Lemma 4.3. *Under hypotheses* (H2) *and* (H3)*, the functional J_μ satisfies the Palais-Smale condition.*

Proof. Let $\{\varphi_n\}$ be a Palais-Smale sequence. By the argument from the previous section, it follows that there exist $m_1 > 0$ and $m_2 > 0$, such that $J_\mu(\varphi_n) \leq m_1$ and $|J'_\mu(\varphi_n)| \leq m_2$.

Let us prove that $\{\varphi_n\}$ is bounded. If not, then up to a subsequence we can assume that $\|\varphi_n\| \to \infty$, as $n \to \infty$. By hypotheses (H2) and (H3), we obtain

$$
\begin{aligned}
p^*(\alpha)m_1 + m_2 &\geq p^*(\alpha)J_\mu(\varphi_n) - \langle J'_\mu(\varphi_n), \varphi_n \rangle \\
&\geq \frac{p^*(\alpha)}{p}\|\varphi_n\|^p - \mu p^*(\alpha) \int_{\mathbb{R}^N} f(x)H(\varphi_n)\,dx \\
&\quad - \frac{p^*(\alpha)}{p^*(\alpha)} \int_{\mathbb{R}^N} |x|^{-\alpha}\varphi_n^{p^*(\alpha)}\,dx - \|\varphi_n\|^p \\
&\quad + \mu \int_{\mathbb{R}^N} f(x)h(\varphi_n)\varphi_n\,dx + \int_{\mathbb{R}^N} |x|^{-\alpha}\varphi_n^{p^*(\alpha)}\,dx \\
&\geq \big(\frac{p^*(\alpha)}{p} - 1\big)\|\varphi_n\|^p + \mu(r - p^*(\alpha)) \int_{\mathbb{R}^N} f(x)H(\varphi_n)\,dx \\
&\geq \big(\frac{p^*(\alpha)}{p} - 1\big)\|\varphi_n\|^p - \mu(p^*(\alpha) - r)c_1\|f\|_{\frac{p^*}{p^*-r}} S_{p^*}^{-r/p}\|\varphi_n\|^r.
\end{aligned}
$$

Since $r < p$, a contradiction is obtained by letting n in the last inequality tend to infinity, therefore $\{\varphi_n\}$ is indeed bounded. The rest of the proof is analogous to the proof of Lemma 3.3. This completes the proof. $\qquad \square$

Proof of Theorem 1.2. Let $\mu \in (0, \mu_0)$, where μ_0 is defined in (4.2). Combining Lemmas 4.1, 4.2, and 4.3 with Theorem 2.3, we can deduce that problem (1.3) has a weak solution ψ_μ as a critical point for J_μ. Moreover, as in the proof of (3.5), one has

$$
J_\mu(\psi_\mu) \geq \eta > 0. \tag{4.3}
$$

Now, by Lemma 4.1, we can see that $\inf_{\psi \in \partial B(0,\rho)} J_\mu(\psi) > 0$. Moreover, by Lemma 4.2, and equation (4.1), we obtain

$$
-\infty < \underline{c} := \inf_{\psi \in \overline{B(0,\rho)}} (J_\mu(\psi)) < 0.
$$

Let $\varepsilon > 0$ be such that

$$
0 < \varepsilon < \inf_{\psi \in \partial B(0,\rho)} J_\mu(\psi) - \inf_{\psi \in B(0,\rho)} J_\mu(\psi). \tag{4.4}
$$

If we consider the functional $J_\mu : \overline{B(0,\rho)} \to \mathbb{R}$, then by the Ekeland variational principle there exists $\psi_\varepsilon \in \overline{B(0,\rho)}$, such that

$$
\begin{aligned}
\underline{c} &\leq J_\mu(\psi_\varepsilon) \leq \underline{c} + \varepsilon \\
J_\mu(\psi_\varepsilon) &< J_\mu(\psi) + \varepsilon\|\psi - \psi_\varepsilon\|, \ \psi \neq \psi_\varepsilon,
\end{aligned} \tag{4.5}
$$

so by (4.4), we have

$$
J_\mu(\psi_\varepsilon) \leq \inf_{\psi \in \overline{B(0,\rho)}} J_\mu(\psi) + \varepsilon \leq \inf_{\psi \in B(0,\rho)} J_\mu(\psi) + \varepsilon < \inf_{\psi \in \partial B(0,\rho)} J_\mu(\psi), \tag{4.6}
$$

which implies that $\psi_\varepsilon \in B(0,\rho)$.

On the other hand, if we define the functional $\Phi_\mu : \overline{B(0,\rho)} \to \mathbb{R}$ by $\Phi_\mu(\psi) := J_\mu(\psi) + \varepsilon\|\psi - \psi_\varepsilon\|$, then ψ_ε is a global minimum of Φ_μ. Therefore, for $s \in (0,1)$ small enough, we have

$$
\frac{\Phi_\mu(\psi_\varepsilon + s\psi) - \Phi_\mu(\psi_\varepsilon)}{s} \geq 0, \quad \text{for every } \psi \in B(0,1),
$$

i.e.,

$$\frac{J_\mu(\psi_\varepsilon + s\psi) - J_\mu(\psi_\varepsilon)}{s} + \varepsilon\|\psi\| \geq 0.$$

By letting s tend to zero, we obtain $\langle J'_\mu(\psi_\varepsilon), \psi\rangle + \varepsilon\|\psi\| \geq 0$. This implies that $\|J'_\mu(\psi_\varepsilon)\| \leq \varepsilon$.

If we put $w_n := \psi_{\frac{1}{n}}$, we obtain $\{w_n\} \subset B(0, \rho)$. Moreover, $J_\mu(w_n) \to \underline{c} < 0$, and $J'_\mu(w_n) \to 0$, as $n \to \infty$. Since $\{w_n\} \subset B(0, \rho)$, it follows that $\{w_n\}$ is bounded in E. So, up to a subsequence still denoted by w_n, there exists $\psi_\mu \in E$, such that $\{w_n\}$ converges weakly to $\psi_\mu \in E$. Invoking Lemma 4.3, we see that $w_n \to \psi_\mu$ strongly in E.

Now, from the fact that $J_\mu \in C^1(E, \mathbb{R})$ implies t $J'_\mu(w_n) \to J_\mu(\psi_\mu)$, as $n \to \infty$, we have

$$J'_\mu(\psi_\mu) = 0 \quad \text{and} \quad J_\mu(\psi_\mu) < 0, \tag{4.7}$$

hence ψ_μ is a nontrivial weak solution of (1.3). Moreover, by combining (4.3) with (4.7), we obtain that $J_\mu(\varphi_\mu) < 0 < J_\mu(\psi_\mu)$, i.e., u_μ and ψ_μ are distinct. This completes the proof.   □

5. An Application

As an application of our results, we shall consider the problem

$$\Delta_p^2 \varphi - \lambda \frac{|\varphi|^{p-2}\varphi}{|x|^{2p}} + \Delta_p \varphi = \mu f(x)|\varphi|^{r-2}\varphi + \frac{|\varphi|^{p^*(p)-2}\varphi}{|x|^p} \quad \text{in } \mathbb{R}^N, \tag{5.1}$$

where $1 < p < \frac{N}{2}$ and $\lambda > 0$. We note that problems of type (5.1) describe the deformations of an elastic beam. Also, they give a model for considering traveling waves in suspension bridges.

It is not difficult to see that $1 < \alpha = p < 2p$ and $h(\varphi) = |\varphi|^{r-2}\varphi$ satisfies the second inequality of hypotheses (H1) and (H3), with $c_1 = 1 > 0$. Moreover, a simple calculation shows that $H(\varphi) = \frac{1}{r}|\varphi|^r$ which satisfies $rH(\varphi) = h(\varphi)\varphi$, so hypothesis (H2) is also satisfied for every $\sigma > 0$.

Hence if $r \in (p, p^*)$ and $f \in L^\infty(\mathbb{R}^N)$, then Theorem 1.1 implies that for every $\mu > 0$, there exists $\lambda_0 > 0$ such that for every $\lambda \in (0, \lambda_0)$, problem (5.1) has a nontrivial solution. Moreover, if $1 < r < p$ and

$$0 < f \in L^{\frac{p^*}{p^*-r}}(\mathbb{R}^N) \cap L^s_{\text{loc}}(\mathbb{R}^N), \quad \text{for some } s \in \left(\frac{p^*}{p^*-r}, \frac{p}{p-r}\right),$$

then Theorem 1.2 implies the existence of $\lambda_0 > 0$ and $\mu_0 > 0$ such that for every $\lambda \in (0, \lambda_0)$ and $\mu \in (0, \mu_0)$, problem (5.1) has at least two nontrivial solutions.

6. Conclusion

The variational method has a long and rich history, and it has given rise to the functional energy. The Mountain pass theorem is used in the first part of this paper to prove the existence of a nontrivial solution for a p-biharmonic problem involving the Hardy-Sobolev exponent. Our first main result generalizes the paper of Ghoussoub and Yuan [13].

In the second part of the paper, the Mountain pass theorem is combined with the Ekeland variational principle to prove the existence of two nontrivial solutions. Our second main result of this paper generalizes the work of Perrera and Zou [19].

We note that the manipulation of the critical Hardy nonlinearity is more compli-
cated and the improvement method used here is an application of the Brezis-Lieb
lemma. As the foundation for further improvements, we aim to obtain even stronger
results for problems with discontinuous nonlinearities.

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Amor Drissi

Department of Mathematics, Faculty of Sciences, University of Tunis El Manar, 2092 Tunis, Tunisia

Email address: `amor.drissi@ipeiem.utm.tn`

Abdeljabbar Ghanmi

Department of Mathematics, Faculty of Sciences, University of Tunis El Manar, 2092 Tunis, Tunisia

Email address: `abdeljabbar.ghanmi@lamsin.rnu.tn`

Dušan D. Repovš

Faculty of Education and Faculty of Mathematics and Physics, University of Ljubljana, and Institute of Mathematics, Physics and Mechanics, SI-1000, Slovenia

Email address: `dusan.repovs@guest.arnes.si`