Electronic Journal of Differential Equations, Vol. 2023 (2023), No. 62, pp. 1–16.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2023.62

GLOBAL CLASSICAL SOLUTIONS TO EQUATORIAL SHALLOW-WATER EQUATIONS

YUE FANG, KAIQIANG LI, XIN XU

Abstract. In this article, we study the equatorial shallow-water equations with slip boundary condition in a bounded domain. By exploring the dissipative structures of the system, we obtaining a priori estimates of the solution for small initial data. Then the existence of classical global solutions and exponential stability results are given.

1. Introduction

In geophysical fluid dynamics, the rotating shallow water system is a widely used 2D approximation of the 3D geophysical hydrodynamic equations such as the Boussinesq equations and Euler equations [13]. In mid-latitudes, the rotational Coriolis terms are bounded away from zero and the rotation frequency is usually regarded as constant because the variations of the Coriolis force due to the curvature of the Earth can be neglected in many cases. However, the constant rotation frequency assumption is no longer reasonable in the equatorial region [11] since the tangential projection of the Coriolis force from rotation vanishes identically.

In this article, we consider the equatorial shallow-water equations [6]

$$\begin{aligned} &H_t + U\cdot\nabla H + H\nabla\cdot U = 0,\\ &U_t + U\cdot\nabla U + \nabla H + yU^{\perp} = -U, \end{aligned} \tag{1.1}$$

in which $H = H(t,x,y)$ is the height, $U = (u,v)(t,x,y)$ is the velocity, x the longitude, y the distance to the equator, and $U^{\perp} = (-v,u)^{\top}$. System (1.1) is supplemented with the initial and boundary conditions

$$\begin{aligned} &H\big|_{t=0} = H_0,\\ &U\big|_{t=0} = U_0,\\ &U\cdot\mathbf{n}\big|_{\partial\Omega} = 0,\\ &\int_\Omega H_0\,d\mathbf{x} = \overline{H} > 0, \end{aligned} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$, $\mathbf{n}$ is the unit outward normal vector on the boundary of Ω.

2020 *Mathematics Subject Classification.* 35Q35, 76N10, 76U05, 76M45.
Key words and phrases. Equatorial shallow-water equations; global classical solution; exponential stability.

Submitted April 20, 2023. Published September 21, 2023.

There have been a number of studies on the rotating shallow water system by physicists and mathematicians for its physical importance and mathematical challenges. The rotating shallow water equations with constant rotation frequency can be modeled by

$$\begin{aligned} &H_t + U\cdot\nabla H + H\nabla\cdot U = 0,\\ &U_t + U\cdot\nabla U + \nabla H + fU^{\perp} = S(U), \end{aligned} \tag{1.3}$$

where f is a constant and S represents the dissipation term like viscosity, damping, etc. Wang and Xu [17] proved the existence of a local large solution, as well as the existence of a global small solution, to the viscous shallow water system by using Littlewood-Paley theory. By the energy method of Matsumura and Nishida [12], and Sundbye obtained the global well-posedness of the system with small initial data in a bounded domain [15] and in the whole space [16], respectively. If the initial data close to a constant equilibrium state away from the vacuum, the existence of a global solution in Besov spaces was shown by Hao et al. [9]. For inviscid rotating shallow water system, Cheng and Xie [4] established a global classical solution under the zero relative vorticity condition by using the dispersive effect of the system. Subsequently, Qu [3] also proved the formation of singularities when the solution crosses certain thresholds. Moreover, the chemotaxis-shallow water system also attracted a lot of attention; see [1, 2, 18, 19] and the references therein.

Geophysical equatorial flows are a rich source of new problems in applied mathematics and partial differential equation theory. Mathematically, the equatorial shallow-water system was first studied by Majda and his collaborators. Dutrifoy and Majda [5] considered the singular limit problem

$$\begin{aligned} &\tilde{H}_t + U\cdot\nabla\tilde{H} + \tilde{H}\nabla\cdot U + \frac{1}{\delta}\nabla\cdot U = S^U,\\ &U_t + U\cdot\nabla U + \frac{1}{\delta}(\nabla\tilde{H} + yU^{\perp}) = S^{\tilde{H}}, \end{aligned} \tag{1.4}$$

where the small parameter δ represents the Froude number (typical fluid velocity ratio to the gravity wave speed) and the height fluctuations of the fluid. S^U and $S^{\tilde{H}}$ are forcing terms. By exploiting the special structure of the system in suitable new variables, they obtained the uniform existence and the convergence of the solutions with unbalanced initial data. In [6], authors proved that, in a suitable limit, solutions of the equatorial shallow water equations would converge to zonal jets. Moreover, with Schochet, they [7] also gave a simpler proof to the above singular limit problem.

In this article, we study the global well-posedness to the initial boundary value problem of the equatorial shallow-water equations (1.1)-(1.2). On the one hand, the damping term $-U$ in the momentum equation (1.1) will make the solution of the system dissipative. On the other hand, by the non-penetrating boundary condition (1.2), it is clear that $\int_\Omega H(\mathbf{x},t)\,d\mathbf{x} = \int_\Omega H_0(\mathbf{x})\,d\mathbf{x} = \overline{H}$. Thus, it is natural to expect that as time goes to infinity the solution of the system will converge to its equilibrium state $(\overline{H}/|\Omega|,\mathbf{0})$ provided that the initial perturbation around this equilibrium state is small, which is the most important part in this article.

Before stating our main results, we give some notation. Throughout this paper, C will denote a generic constant which is independent of time. The norms in $L^2(\Omega)$

and in $H^s(\Omega)$ are denoted by

$$\|u\| \equiv \|u\|_{L^2(\Omega)} = \Big(\int_\Omega |u|^2\,d\mathbf{x}\Big)^{1/2},$$
$$\|u\|_s \equiv \|u\|_{H^s(\Omega)} = \Big(\sum_{|\alpha|\le s}\int_\Omega |D^\alpha u|^2\,d\mathbf{x}\Big)^{1/2},$$

where $\alpha = (\alpha_1, \alpha_2)$ is any multi-index with order $|\alpha| = \alpha_1 + \alpha_2$ and $D^\alpha = \partial_x^{\alpha_1}\partial_y^{\alpha_2}$. For any vector valued function $F = (f_1, f_2) : \Omega \to \mathbb{R}^2$, $\|F\|_s^2 \equiv \|f_1\|_s^2 + \|f_2\|_s^2$. The energy space under consideration is:

$$X_3([0,T],\Omega) \equiv \big\{F : \Omega\times[0,T] \to \mathbb{R} : \partial_t^l F \in L^\infty([0,T]; H^{3-l}(\Omega)),\ l = 0,1,2,3\big\},$$

equipped with norm

$$|||F|||_{3,T} \equiv \sup_{0\le t\le T} |||F(\cdot,t)||| \equiv \sup_{0\le t\le T}\Big[\sum_{l=0}^{3}\|F(\cdot,t)\|_{3-l}^2\Big]^{1/2}.$$

Our main result reads as follows.

Theorem 1.1. *Suppose that the initial data satisfies the compatibility condition of order 2, i.e.,*

$$\partial_t^l U(0)\cdot \mathbf{n}\big|_{\partial\Omega} = 0, \quad 0 \le l \le 2,$$

where $\partial_t^l U(0)$ is the l-th time derivative at $t = 0$ for any solution of (1.1)-(1.2)*, as calculated from* (1.1) *to yield an expression in terms of H_0 and U_0. Then there exists a constant $\varepsilon > 0$ such that if*

$$\|(H_0 - \overline{H}/|\Omega|, U_0)\|_3 \le \varepsilon,$$

the initial-boundary value problem (1.1)-(1.2) *admits a unique global solution*

$$(H, U) \in X_3([0,\infty),\Omega).$$

Moreover, there exist positive constants C and η, which are independent of t, such that

$$\|(H - \overline{H}/|\Omega|)(\cdot,t)\|_3 + \|U(\cdot,t)\|_3 \le C\|(H_0 - \overline{H}/|\Omega|, U_0)\|_3 \exp\{-\eta t\}. \tag{1.5}$$

We note that the main result in this paper still holds when the physical domain Ω is replaced by $\mathbb{T}\times[0,L]$ for any $L > 0$.

The proof of Theorem 1.1 is mainly based on the existence theorem and the a priori estimates of a local solution. Since the existence of the local solution can be obtained by the classical local well-posedness theory, we will focus on the a priori estimates of the solution in the majority of this article. The energy method is used to derive these key estimates. Several difficulties need to be overcome during obtaining the energy estimate to the system. The first one is that we can not obtain the normal derivative estimate of the solution because of the presence of the boundary. Here it seems that the missed estimates can only be compensated by estimating the time derivative of the solution and the vorticity $\omega = \nabla\times U$. However, as we can see later, this will lead to a new difficulty when we try to estimate the vorticity for the non-constant rotation frequency. When taking y-derivatives to the vorticity equation, we will encounter with some trouble terms which need to be controlled properly. We overcome this difficulty by designing some delicate semi-norms to capture the full dissipation mechanism of the system. Finally, since the system is partial dissipative, we need to use the nonlinear interaction of the system to derive the dissipative estimate of h, which can be achieved through a Kawashima-type energy estimate [10].

Using the special structure of (1.1) together with induction on the number of spatial derivatives, the estimate of total energy is reduced to those for the vorticity and temporal derivatives. Actually, the method used here is simpler than the classical energy estimate.

The plan of the rest of this article is as follows. In Section 2, we reformulate the original system to get a quasi-linear symmetric hyperbolic system, the local existence result and some basic facts which will be used in this paper are given. In Section 3, we prove Theorem 1.1 by energy estimates.

2. Reformulation of the problem

To symmetrize the equations, we give a reformulation to the initial-boundary value problem (1.1)-(1.2) in this section. Without loss of generality, we assume $\overline{H}/|\Omega| = 1$.

First, multiplying $(1.1)_1$ by $1/H$, we have

$$\begin{gathered} \frac{1}{H}H_t + \frac{1}{H}U\cdot\nabla H + \nabla\cdot U = 0, \\ U_t + U\cdot\nabla U + \nabla H + yU^{\perp} = -U. \end{gathered}$$

Since equilibrium height is conjectured to be $\overline{H}/|\Omega| = 1$, we set $H = 1 + h$ and get the desired symmetric system

$$\begin{gathered} \frac{1}{1+h}(h_t + U\cdot\nabla h) + \nabla\cdot U = 0, \\ U_t + U\cdot\nabla U + \nabla h + yU^{\perp} = -U, \end{gathered} \tag{2.1}$$

with the initial and boundary conditions

$$\begin{gathered} h\big|_{t=0} = h_0, \\ U\big|_{t=0} = U_0, \\ U\cdot\mathbf{n}\big|_{\partial\Omega} = 0, \end{gathered} \tag{2.2}$$

where $h_0 = H_0 - 1$.

Now, for the initial boundary value problem (2.1)-(2.2), we can use the same idea as in [14] to establish the existence of classical local solutions.

Lemma 2.1. *If $(h_0, U_0) \in H^3(\Omega)$ and satisfies the compatibility condition $\partial_t^l U(0)\cdot \mathbf{n}\big|_{\partial\Omega} = 0, 0 \le l \le 2$, then there exists a unique local solution (h, U) of problem (2.1)-(2.2) in $C^1(\overline{\Omega}\times[0,T])\cap X_3([0,T],\Omega)$ for some finite $T > 0$. Moreover, there exist positive constants ε_0 and $C_0(T)$ such that if*

$$\|h(\cdot,0)\|_3 + \|U(\cdot,0)\|_3 \le \varepsilon_0,$$

then

$$\|h\|_{3,T} + \|U\|_{3,T} \le C_0(\|h(\cdot,0)\|_3 + \|U(\cdot,0)\|_3).$$

Now, we give some lemmas to be used later. The first one is an inequality of Sobolev type whose proof can be found in many textbooks [8].

Lemma 2.2. *Let Ω be any bounded domain in $\mathbb{R}^2$ with smooth boundary. Then*

(i) $\|f\|_{L^\infty(\Omega)} \le C\|f\|_{H^2(\Omega)}$,
(ii) $\|f\|_{L^p(\Omega)} \le C\|f\|_{H^1(\Omega)}$ *for* $2 \le p < \infty$,

for some constant $C > 0$ depending only on Ω.

To give the control of the velocity in terms of $\nabla \cdot U$ and the vorticity ω, we need the following lemma, see [20].

Lemma 2.3. *Let $U \in H^s(\Omega)$ be a vector-valued function satisfying $U \cdot \mathbf{n}\big|_{\partial\Omega} = 0$, where $\mathbf{n}$ is the unit outer normal of $\partial\Omega$. Then*

$$\|U\|_s \leq C(\|\nabla \times U\|_{s-1} + \|\nabla \cdot U\|_{s-1} + \|U\|_{s-1}) \tag{2.3}$$

for $s \geq 1$, and the positive constant C depends only on s and Ω.

3. Existence of global solutions and long time behavior

In this section, we shall prove the existence and the large time behavior of global solutions to (2.1)-(2.2). To this end, we need to derive a key a priori estimates, which are the main part of this section. For convenience, we set the total energy

$$W(t) \equiv |||h(t)|||^2 + |||U(t)|||^2 = \sum_{l=0}^{3} \Big(\|\partial_t^l h(t)\|_{3-l}^2 + \|\partial_t^l U(t)\|_{3-l}^2 \Big). \tag{3.1}$$

The main result in this section is as follows.

Theorem 3.1. *Suppose that the initial data $(h_0, U_0) \in H^3(\Omega)$ and satisfies the compatibility condition of order 2. If there exists a small enough positive constant ε such that $\|(h_0, U_0)\|_3 \leq \varepsilon$, then there is a unique global classical solution of (2.1)-(2.2) such that*

$$W(t) \leq CW(0)e^{-\eta t}, \tag{3.2}$$

where C and η are positive constants independent of t.

The proof of Theorem 3.1 is based on a detailed energy estimates. To simplify the presentation, we define

$$E(t) \equiv \sum_{l=0}^{3} \Big(\|\partial_t^l h\|^2 + \|\partial_t^l U\|^2 \Big), \quad V(t) \equiv \sum_{l=0}^{2} \|\partial_t^l \omega\|_{2-l}^2. \tag{3.3}$$

By the definition of $E(t)$, $V(t)$ and $W(t)$, we find that the total energy $W(t)$ can be controlled by $E(t)$ and $V(t)$ as long as (h, U) is a sufficiently small solution of (2.1)-(2.2).

Lemma 3.2. *Let (h, U) be solution of (2.1)-(2.2). Suppose that there is a small constant $\bar{\delta} > 0$ such that $W(t) \leq \bar{\delta}$, then*

$$W(t) \leq C(E(t) + V(t)). \tag{3.4}$$

Proof. Rewriting $(2.1)_2$ as

$$\nabla h = -(U + U_t + U \cdot \nabla U + yU^{\perp}), \tag{3.5}$$

and taking the L^2 inner product with ∇h yields

$$\|\nabla h\|^2 = \int_{\Omega} -(U + U_t + U \cdot \nabla U + yU^{\perp}) \cdot \nabla h \, d\mathbf{x}.$$

Combining Lemma 2.2 with the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \|\nabla h\|^2 &\leq C(\|U\|^2 + \|U_t\|^2) + C\|U\|_{L^\infty}^2 \|\nabla U\|^2 + \|yU\|^2 \\ &\leq C(\|U\|^2 + \|U_t\|^2) + CW(t)^{3/2} \leq C(E(t) + V(t)) + CW(t)^{3/2}. \end{aligned} \tag{3.6}$$

Similarly, by rewriting $(2.1)_1$ as

$$\nabla\cdot U=-\frac{1}{1+h}(h_t+U\cdot\nabla h), \tag{3.7}$$

we have

$$\|\nabla\cdot U\|^2\le C(\|h_t\|^2+W(t)^{3/2}). \tag{3.8}$$

Using Lemma 2.3 with $s=1$ and (3.8), one has

$$\begin{aligned}\|U\|_1^2&\le C(\|\omega\|^2+\|\nabla\cdot U\|^2+\|U\|^2)\\&\le C(\|\omega\|^2+\|h_t\|^2+\|U\|^2+W(t)^{3/2})\\&\le C(E(t)+V(t))+CW(t)^{3/2}.\end{aligned} \tag{3.9}$$

Applying ∂_t^l with $1\le l\le 2$ to (3.5) and (3.7), taking inner product with $\partial_t^l\nabla h$ and $\partial_t^l\nabla\cdot U$ respectively, then using Lemma 2.3 to control $\|\partial_t^l U\|_1$, we find that

$$\sum_{l=0}^{2}\Big(\|\partial_t^l h(t)\|_1^2+\|\partial_t^l U(t)\|_1^2\Big)\le C(E(t)+V(t))+CW(t)^{3/2}, \tag{3.10}$$

where we used the smallness of $\bar\delta$ and Sobolev inequality. Now, following similar procedure as above and by an induction on the number of spatial derivatives, one has

$$\sum_{l=0}^{1}\Big(\|\partial_t^l h(t)\|_2^2+\|\partial_t^l U(t)\|_2^2\Big)\le C(E(t)+V(t))+CW(t)^{3/2}, \tag{3.11}$$

$$\|h(t)\|_3^2+\|U(t)\|_3^2\le C(E(t)+V(t))+CW(t)^{3/2}. \tag{3.12}$$

Combining estimates (3.10)–(3.12), we have

$$W(t)\le C(E(t)+V(t))+CW(t)^{3/2}. \tag{3.13}$$

Noting that $W(t)\le\bar\delta$, the proof is complete. □

Lemma 3.3. *Let (h,U) be a solution of (2.1)-(2.2). Suppose that there is a small constant $\bar\delta>0$ such that $W(t)\le\bar\delta$. Then*

$$\begin{aligned}&\frac{d}{dt}\Big(\|h\|^2+\|h_t\|^2+\|\sqrt{\frac{1}{1+h}}h_{tt}\|^2+\|\sqrt{\frac{1}{1+h}}h_{ttt}\|^2+\sum_{l=0}^{3}\|\partial_t^l U\|^2\Big)\\&+2\sum_{l=0}^{3}\|\partial_t^l U\|^2\le CW(t)^{3/2}.\end{aligned} \tag{3.14}$$

Proof. By calculating $h(1+h)\cdot(2.1)_1+U\cdot(2.1)_2$, we have

$$\begin{aligned}&\frac12\frac{d}{dt}(h^2+|U|^2)+|U|^2\\&=-\big[-hU\cdot\nabla h+U\cdot(U\cdot\nabla U)+\nabla\cdot(hU)+\nabla\cdot(h^2U)\big].\end{aligned} \tag{3.15}$$

Integrating (3.15) over Ω and using integration by parts and the non-penetration boundary condition (2.2), we obtain

$$\frac12\frac{d}{dt}(\|h\|^2+\|U\|^2)+\|U\|^2\le\|h\|_{L^2}\|U\|_{L^\infty}\|\nabla h\|_{L^2}+\|U\|_{L^2}\|U\|_{L^\infty}\|\nabla U\|_{L^2}. \tag{3.16}$$

Then we deduce that

$$\frac{d}{dt}(\|h\|^2+\|U\|^2)+2\|U\|^2\le CW(t)^{3/2}. \tag{3.17}$$

Differentiating $(1+h)\cdot(2.1)_1$ and $(2.1)_2$ with respect to t, multiplying by h_t, U_t respectively, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(h_t^2+|U_t|^2)+|U_t|^2 &= -h_tU_t\cdot\nabla h-h_tU\cdot\nabla h_t-h_t^2\nabla\cdot U-hh_t\nabla\cdot U_t\\ &\quad -U_t\cdot(U_t\cdot\nabla U)-U_t\cdot(U\cdot\nabla U_t)-\nabla\cdot(h_tU_t).\end{aligned}$$

Similarly,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|h_t\|^2+\|U_t\|^2)+\|U_t\|^2\\ &\le \|h_t\|_{L^4}\|U_t\|_{L^4}\|\nabla h\|_{L^2}+\|h_t\|_{L^4}\|U\|_{L^4}\|\nabla h_t\|_{L^2}\\ &\quad+\|h_t\|_{L^4}\|h_t\|_{L^4}\|\nabla U\|_{L^2}+\|h\|_{L^4}\|h_t\|_{L^4}\|\nabla U_t\|_{L^2}\\ &\quad+\|U_t\|_{L^4}\|U_t\|_{L^4}\|\nabla U\|_{L^2}+\|U_t\|_{L^4}\|U\|_{L^4}\|\nabla U_t\|_{L^2}.\end{aligned}\tag{3.18}$$

Using the Sobolev embedding theorem, one has

$$\frac{d}{dt}(\|h_t\|^2+\|U_t\|^2)+2\|U_t\|^2\le CW(t)^{3/2}.\tag{3.19}$$

In a similar way, we can obtain the following second order and third order estimates

$$\frac{d}{dt}(\|\sqrt{\frac{1}{1+h}}h_{tt}\|^2+\|U_{tt}\|^2)+2\|U_{tt}\|^2\le CW(t)^{3/2},\tag{3.20}$$

$$\frac{d}{dt}(\|\sqrt{\frac{1}{1+h}}h_{ttt}\|^2+\|U_{ttt}\|^2)+2\|U_{ttt}\|^2\le CW(t)^{3/2}.\tag{3.21}$$

Thus, combining (3.17), (3.19), (3.20), and (3.21), we obtain (3.14). This completes the proof. □

The estimate (3.14) contains the dissipation in velocity due to the friction term $-U$ in the equation. To close our a priori assumption $W(t)\le\bar\delta$, it is important to derive the dissipation of h, which will be done in the following lemma.

Lemma 3.4. *Let (h,U) be the solution of (2.1)-(2.2). If there is a small constant $\bar\delta>0$ such that $W(t)\le\bar\delta$, then*

$$\frac{d}{dt}\Big(\sum_{l=1}^{3}\int_\Omega(-\partial_t^{l-1}h\partial_t^l h)\,d\mathbf{x}\Big)+\sum_{l=0}^{3}\|\partial_t^l h\|^2\le CW(t)^{3/2}+c_0\sum_{l=0}^{3}\|\partial_t^l U\|^2.\tag{3.22}$$

Proof. Since $\int_\Omega h\,d\mathbf{x}=\int_\Omega(H-1)\,d\mathbf{x}=0$, using Poincaré's inequality, we have $\|h\|^2\le C\|\nabla h\|^2$. This together with (3.6) yields

$$\|h\|^2\le C(\|U\|^2+\|U_t\|^2)+CW(t)^{3/2}.\tag{3.23}$$

A direct calculation of $\partial_t[(1+h)(2.1)_1]-(1+h)\nabla\cdot(2.1)_2$ gives

$$h_{tt}+(U\cdot\nabla h)_t+h_t(\nabla\cdot U)-(1+h)\nabla\cdot(U\cdot\nabla U+\nabla h+U+yU^\perp)=0.\tag{3.24}$$

Multiplying by h, we obtain

$$\begin{aligned} &h_{tt}h + (U\cdot\nabla h)_t h + h_t(\nabla\cdot U)h - (1+h)\nabla\cdot(U\cdot\nabla U + \nabla h + U + yU^{\perp})h \\ &= (hh_t)_t - h_t^2 + (U\cdot\nabla h)_t h + hh_t(\nabla\cdot U) + (1+h)h\nabla\cdot U_t \\ &= (hh_t)_t - h_t^2 + (U_t\cdot\nabla h)h + (U\cdot\nabla h_t)h + hh_t(\nabla\cdot U) + \nabla\cdot[(h^2+h)U_t] \\ &\quad - U_t\cdot\nabla(h^2+h) \\ &= (hh_t)_t - h_t^2 + (U_t\cdot\nabla h)h + \nabla\cdot(hh_tU) - hh_t(\nabla\cdot U) - h_t(U\cdot\nabla h) \\ &\quad + hh_t(\nabla\cdot U) + \nabla\cdot[(h^2+h)U_t] - U_t\cdot\nabla(h^2+h) \\ &= (hh_t)_t - h_t^2 + (U_t\cdot\nabla h)h - h_t(U\cdot\nabla h) - 2hU_t\cdot\nabla h - U_t\cdot\nabla h \\ &\quad + \nabla\cdot[(h^2+h)U_t + hh_tU] = 0. \end{aligned} \tag{3.25}$$

Integrating over Ω and using Cauchy-Schwarz inequality, we have

$$-\frac{d}{dt}\int_\Omega hh_t\,d\mathbf{x} + \|h_t\|^2 \le C\big(W(t)^{3/2} + \|\nabla h\|^2 + \|U_t\|^2\big). \tag{3.26}$$

Combining this with (3.6), we have

$$-\frac{d}{dt}\int_\Omega hh_t\,d\mathbf{x} + \|h_t\|^2 \le C\big(W(t)^{3/2} + \|U\|^2 + \|U_t\|^2\big), \tag{3.27}$$

Similarly, we have

$$-\frac{d}{dt}\int_\Omega h_th_{tt}\,d\mathbf{x} + \|h_{tt}\|^2 \le C\big(W(t)^{3/2} + \|U_t\|^2 + \|U_{tt}\|^2\big), \tag{3.28}$$

$$-\frac{d}{dt}\int_\Omega h_{tt}h_{ttt}\,d\mathbf{x} + \|h_{ttt}\|^2 \le C\big(W(t)^{3/2} + \|U_{tt}\|^2 + \|U_{ttt}\|^2\big). \tag{3.29}$$

Collecting (3.23), (3.27)-(3.29), the proof is complete. □

Let $c_1 \equiv \max\{2, c_0\}$, and

$$\begin{aligned} E_1(t) \equiv\ & c_1\big(\|h\|^2 + \|h_t\|^2 + \|\sqrt{\frac{1}{1+h}}h_{tt}\|^2 + \|\sqrt{\frac{1}{1+h}}h_{ttt}\|^2 + \sum_{l=0}^{3}\|\partial_t^lU\|^2\big) \\ & - \sum_{l=1}^{3}\int_\Omega(-\partial_t^{l-1}h\partial_t^lh)\,d\mathbf{x}. \end{aligned} \tag{3.30}$$

Then, from the estimates in Lemma 3.3 and Lemma 3.4, we have the following result.

Lemma 3.5. *There exist constants $C_2, C > 0$ such that*

$$\frac{d}{dt}E_1(t) + C_2E(t) \le CW(t)^{3/2}. \tag{3.31}$$

Proof. Taking $c_1 \times (3.14) + (3.22)$ yields

$$\frac{d}{dt}E_1(t) + c_0\sum_{l=0}^{3}\|\partial_t^lU\|^2 + \sum_{l=0}^{3}\|\partial_t^lh\|^2 \le CW(t)^{3/2}. \tag{3.32}$$

Letting $C_2 = \min\{c_0, 1\}$, inequality (3.31) follows directly from (3.32). □

Lemma 3.6. *Let* (h, U) *be the solution of* (2.1)-(2.2)*. Suppose that there is a small constant* $\bar{\delta} > 0$ *such that* $W(t) \le \bar{\delta}$*, then*

$$\frac{d}{dt}V_1(t) + C_5 V_1(t) \le CW(t)^{3/2} + C_6 E(t), \tag{3.33}$$

where $C_5, C_6, C > 0$ *are constants and* V_1 *is defined as* (3.66)*.*

Proof. Applying the curl to $(2.1)_2$ yields

$$\omega_t + \omega = -U \cdot \nabla\omega - \omega(\nabla \cdot U) - y\nabla \cdot U - v. \tag{3.34}$$

Multiplying the above equation by ω, we obtain

$$\omega\omega_t + \omega^2 = -\omega U \cdot \nabla\omega - \omega^2(\nabla \cdot U) - y\omega\nabla \cdot U - \omega v.$$

Integrating the resulting equation and using the boundary condition, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\omega\|^2 + \|\omega\|^2 \le C_3 W(t)^{3/2} - \int_\Omega y\omega\nabla \cdot U \, d\mathbf{x} - \int_\Omega \omega v \, d\mathbf{x}. \tag{3.35}$$

Dealing with the last two terms on the right-hand side of the above inequality,

$$\int_\Omega y\omega\nabla \cdot U \, d\mathbf{x} = \int_\Omega y\omega[-\frac{1}{1+h}(h_t + U \cdot \nabla h)] \, d\mathbf{x},$$

where we used the equality

$$\nabla \cdot U = -\frac{1}{1+h}(h_t + U \cdot \nabla h).$$

Combining Hölder's inequality, Lemma 2.2 and Cauchy-Schwarz inequality, we obtain

$$\int_\Omega y\omega\nabla \cdot U \, d\mathbf{x} \le C_3\big(W(t)^{3/2} + E(t)\big) + \frac{1}{8}\|\omega\|^2, \tag{3.36}$$

$$\int_\Omega \omega v \, d\mathbf{x} \le C_3 E(t) + \frac{1}{8}\|\omega\|^2. \tag{3.37}$$

Putting the above two estimates into (3.35), we obtain

$$\frac{d}{dt}\|\omega\|^2 + \frac{1}{2}\|\omega\|^2 \le C_4\big(W(t)^{3/2} + E(t)\big), \tag{3.38}$$

where $C_4 > 1$.

First order estimate. Differentiating (3.34) with respect to x and multiplying the resulting equation by ω_x, we obtain

$$\begin{aligned}\omega_x\omega_{tx} + \omega_x^2 &= -\omega_x U_x \cdot \nabla\omega - \omega_x U \cdot \nabla\omega_x - \omega_x^2(\nabla \cdot U)\\ &\quad - \omega_x\omega(\nabla \cdot U_x) - y\omega_x\nabla \cdot U_x - \omega_x v_x.\end{aligned}$$

Integrating the above equation over Ω and using Hölder's inequality and Lemma 2.2, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\omega_x\|^2 + \|\omega_x\|^2 \le C_3 W(t)^{3/2} - \int_\Omega y\omega_x\nabla \cdot U_x \, d\mathbf{x} - \int_\Omega \omega_x v_x \, d\mathbf{x}. \tag{3.39}$$

Similarly, we can estimate the last two terms on the right-hand side of the above inequality as follows:

$$\int_\Omega y\omega_x\nabla \cdot U_x \, d\mathbf{x} = \int_\Omega y\omega_x\big[\frac{1}{(1+h)^2}h_x(h_t + U \cdot \nabla h)\big] \, d\mathbf{x}$$

$$- \int_\Omega y\omega_x \frac{1}{1+h}(h_{tx} + U_x\cdot\nabla h + U\cdot\nabla h_x)\,d\mathbf{x}.$$

Using Hölder's inequality and Lemma 2.2, we have

$$\int_\Omega y\omega_x \nabla\cdot U_x\,d\mathbf{x} \le C_3 W(t)^{3/2} + c_3\int_\Omega \omega_x h_{tx}\,d\mathbf{x}. \tag{3.40}$$

Based on $(2.1)_2$, the Hölder's inequality, Lemma 2.2 and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
c_3\int_\Omega \omega_x h_{tx}\,d\mathbf{x} &= c_3\int_\Omega \omega_x(-u+yv-u_t-U\cdot\nabla u)_t\,d\mathbf{x}\\
&= c_3\int_\Omega \omega_x(-u_t+yv_t-u_{tt}-U_t\cdot\nabla u - U\cdot\nabla u_t)\,d\mathbf{x}\\
&\le c_3\big(\|\omega_x\|\|u_t\| + \|\omega_x\|\|v_t\| + \|\omega_x\|\|u_{tt}\|\big) + C_3W(t)^{3/2}\\
&\le \frac18\|\omega_x\|^2 + C_3\|u_t\|^2 + \frac18\|\omega_x\|^2 + C_3\|v_t\|^2 + \frac18\|\omega_x\|^2\\
&\quad + C_3\|u_{tt}\|^2 + C_3W(t)^{3/2}\\
&\le C_3\big(W(t)^{3/2}+E(t)\big) + \frac38\|\omega_x\|^2.
\end{aligned} \tag{3.41}$$

From (3.9), it is clear that

$$\begin{aligned}
\int_\Omega \omega_x v_x\,d\mathbf{x} &\le \|\omega_x\|\|v_x\| \le \|\omega_x\|\|U\|_1\\
&\le C_3\big(W(t)^{3/2}+E(t)+\|\omega\|^2\big) + \frac18\|\omega_x\|^2,
\end{aligned} \tag{3.42}$$

which together with (3.39)-(3.41) gives

$$\frac{d}{dt}\|\omega_x\|^2 + \frac12\|\omega_x\|^2 \le C_4\big(W(t)^{3/2}+E(t)+\|\omega\|^2\big). \tag{3.43}$$

Differentiating (3.34) with respect to y and multiplying the resulting equation by ω_y, we obtain

$$\begin{aligned}
\omega_y\omega_{ty} + \omega_y^2 &= -\omega_y U_y\cdot\nabla\omega - \omega_y U\cdot\nabla\omega_y - \omega_y^2(\nabla\cdot U)\\
&\quad - \omega_y\omega(\nabla\cdot U_y) - \omega_y\nabla\cdot U - y\omega_y\nabla\cdot U_y - \omega_y v_y.
\end{aligned}$$

Integrating the above equation over Ω, we have

$$\begin{aligned}
&\frac12\frac{d}{dt}\|\omega_y\|^2 + \|\omega_y\|^2\\
&\le C_3W(t)^{3/2} - \int_\Omega \omega_y\nabla\cdot U\,d\mathbf{x} - \int_\Omega y\omega_y\nabla\cdot U_y\,d\mathbf{x} - \int_\Omega \omega_y v_y\,d\mathbf{x}.
\end{aligned} \tag{3.44}$$

Now we estimate the last three terms on the right-hand side of the above inequality as follows:

$$\begin{aligned}
\int_\Omega y\omega_y\nabla\cdot U_y\,d\mathbf{x} &= \int_\Omega y\omega_y\Big[\frac{1}{(1+h)^2}h_y(h_t+U\cdot\nabla h)\Big]\,d\mathbf{x}\\
&\quad - \int_\Omega y\omega_y\frac{1}{1+h}(h_{ty} + U_y\cdot\nabla h + U\cdot\nabla h_y)\,d\mathbf{x}.
\end{aligned}$$

Using Lemma 2.2, we obtain

$$\int_\Omega y\omega_y\nabla\cdot U_y\,d\mathbf{x} \le C_3W(t)^{3/2} + c_3\int_\Omega \omega_y h_{ty}\,d\mathbf{x}. \tag{3.45}$$

By $(2.1)_2$, one has

$$
\begin{aligned}
c_3 \int_\Omega \omega_y h_{ty}\, d\mathbf{x} &= c_3 \int_\Omega \omega_y(-v - yu - v_t - U\cdot\nabla v)_t\, d\mathbf{x} \\
&\le C_3\big(W(t)^{3/2} + E(t)\big) + \frac{3}{8}\|\omega_y\|^2
\end{aligned} \tag{3.46}
$$

and

$$
\begin{aligned}
\int_\Omega \omega_y v_y\, d\mathbf{x} &\le \|\omega_y\|\|v_y\| \le \|\omega_y\|\|U\|_1 \\
&\le C_3\big(W(t)^{3/2} + E(t) + \|\omega\|^2\big) + \frac{1}{8}\|\omega_y\|^2.
\end{aligned} \tag{3.47}
$$

Similar derivations show that

$$
\begin{aligned}
\int_\Omega \omega_y \nabla\cdot U\, d\mathbf{x} &\le \|\omega_y\|\|\nabla\cdot U\| \le \|\omega_y\|\|U\|_1 \\
&\le C_3\big(W(t)^{3/2} + E(t) + \|\omega\|^2\big) + \frac{1}{8}\|\omega_y\|^2.
\end{aligned} \tag{3.48}
$$

Combining (3.44)-(3.48), we obtain

$$
\frac{d}{dt}\|\omega_y\|^2 + \frac{1}{2}\|\omega_y\|^2 \le C_4\big(W(t)^{3/2} + E(t) + \|\omega\|^2\big). \tag{3.49}
$$

Therefore,

$$
\frac{d}{dt}\|\partial\omega\|^2 + \frac{1}{2}\|\partial\omega\|^2 \le C_4\big(W(t)^{3/2} + E(t) + \|\omega\|^2\big), \tag{3.50}
$$

where ∂ denotes spatial derivatives ∂_x and ∂_y.

Finally, differentiating (3.34) with respect to t and multiplying the resulting equation by ω_t, we obtain

$$
\begin{aligned}
\omega_t\omega_{tt} + \omega_t^2 &= -\omega_t U_t\cdot\nabla\omega - \omega_t U\cdot\nabla\omega_t - \omega_t^2(\nabla\cdot U) \\
&\quad - \omega_t\omega(\nabla\cdot U_t) - y\omega_t\nabla\cdot U_t - \omega_t v_t.
\end{aligned}
$$

Integrating the above equation over Ω gives

$$
\frac{1}{2}\frac{d}{dt}\|\omega_t\|^2 + \|\omega_t\|^2 \le C_3 W(t)^{3/2} - \int_\Omega y\omega_t\nabla\cdot U_t\, d\mathbf{x} - \int_\Omega \omega_t v_t\, d\mathbf{x}. \tag{3.51}
$$

Since

$$
\begin{aligned}
\int_\Omega y\omega_t\nabla\cdot U_t\, d\mathbf{x} &= \int_\Omega y\omega_t\Big[\frac{1}{(1+h)^2}h_t(h_t + U\cdot\nabla h)\Big]\, d\mathbf{x} \\
&\quad - \int_\Omega y\omega_t\frac{1}{1+h}(h_{tt} + U_t\cdot\nabla h + U\cdot\nabla h_t)\, d\mathbf{x},
\end{aligned}
$$

we have

$$
\begin{aligned}
\int_\Omega y\omega_t\nabla\cdot U_t\, d\mathbf{x} &\le C_3 W(t)^{3/2} + c_3\int_\Omega \omega_t h_{tt}\, d\mathbf{x} \\
&\le C_3 W(t)^{3/2} + c_3\|\omega_t\|\|h_{tt}\| \\
&\le C_3\big(W(t)^{3/2} + E(t)\big) + \frac{1}{8}\|\omega_t\|^2.
\end{aligned} \tag{3.52}
$$

By Cauchy-Schwarz inequality, we have

$$
\int_\Omega \omega_t v_t\, d\mathbf{x} \le \|\omega_t\|\|v_t\| \le \|\omega_t\|\|U_t\| \le C_3 E(t) + \frac{1}{8}\|\omega_t\|^2, \tag{3.53}
$$

which together with (3.51)-(3.52) implies

$$\frac{d}{dt}\|\omega_t\|^2 + \frac{1}{2}\|\omega_t\|^2 \le C_4\big(W(t)^{3/2} + E(t)\big). \tag{3.54}$$

Second order estimate. Firstly, differentiating (3.34) with respect to t for second order and multiplying the resulting equation by ω_{tt}, we obtain

$$\begin{aligned}\omega_{tt}\omega_{ttt} + \omega_{tt}^2 &= -\omega_{tt}U_{tt}\cdot\nabla\omega - 2\omega_{tt}U_t\cdot\nabla\omega_t - \omega_{tt}U\cdot\nabla\omega_{tt} - \omega_{tt}^2(\nabla\cdot U)\\ &\quad - 2\omega_{tt}\omega_t\nabla\cdot U_t - \omega_{tt}\omega(\nabla\cdot U_{tt}) - y\omega_{tt}\nabla\cdot U_{tt} - \omega_{tt}v_{tt}.\end{aligned}$$

Integrating over Ω gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\omega_{tt}\|^2 + \|\omega_{tt}\|^2\\ &\le C_3W(t)^{3/2} - \int_\Omega \omega_{tt}U\cdot\nabla\omega_{tt}\,d\mathbf{x} - \int_\Omega y\omega_{tt}\nabla\cdot U_{tt}\,d\mathbf{x} - \int_\Omega \omega_{tt}v_{tt}\,d\mathbf{x}.\end{aligned} \tag{3.55}$$

Since

$$\begin{aligned}\int_\Omega \omega_{tt}U\cdot\nabla\omega_{tt}\,d\mathbf{x} &= \int_\Omega \nabla\cdot\big(\frac{\omega_{tt}^2}{2}U\big) - \frac{\omega_{tt}^2}{2}\nabla\cdot U\,d\mathbf{x}\\ &= -\int_\Omega \frac{\omega_{tt}^2}{2}\nabla\cdot U\,d\mathbf{x} \le C_3W(t)^{3/2},\end{aligned} \tag{3.56}$$

and

$$\begin{aligned}\int_\Omega y\omega_{tt}\nabla\cdot U_{tt}\,d\mathbf{x} &= \int_\Omega y\omega_{tt}\Big[\frac{h_{tt}(1+h)^2 - 2h_t^2(1+h)}{(1+h)^4}(h_t + U\cdot\nabla h)\Big]\,d\mathbf{x}\\ &\quad + 2\int_\Omega y\omega_{tt}\frac{h_t}{(1+h)^2}(h_{tt} + U_t\cdot\nabla h + U\cdot\nabla h_t)\,d\mathbf{x}\\ &\quad - \int_\Omega y\omega_{tt}\frac{1}{1+h}(h_{ttt} + U_{tt}\cdot\nabla h + 2U_t\cdot\nabla h_t + U\cdot\nabla h_{tt})\,d\mathbf{x},\end{aligned}$$

we obtain

$$\begin{aligned}\int_\Omega y\omega_{tt}\nabla\cdot U_{tt}\,d\mathbf{x} &\le C_3W(t)^{3/2} - \int_\Omega \frac{y}{1+h}\omega_{tt}h_{ttt}\,d\mathbf{x}\\ &\le C_3W(t)^{3/2} + \frac{1}{8}\|\omega_{tt}\|^2 + C_3\|h_{ttt}\|^2\\ &\le C_3\big(W(t)^{3/2} + E(t)\big) + \frac{1}{8}\|\omega_{tt}\|^2.\end{aligned} \tag{3.57}$$

Since

$$\int_\Omega \omega_{tt}v_{tt}\,d\mathbf{x} \le \|\omega_{tt}\|\|U_{tt}\| \le C_3E(t) + \frac{1}{8}\|\omega_{tt}\|^2, \tag{3.58}$$

from (3.55)-(3.57) we obtain

$$\frac{d}{dt}\|\omega_{tt}\|^2 + \frac{1}{2}\|\omega_{tt}\|^2 \le C_4\big(W(t)^{3/2} + E(t)\big). \tag{3.59}$$

Secondly, applying ∂_x^2 to (3.34) and multiplying by ω_{xx}, we obtain

$$\begin{aligned}\omega_{xx}\omega_{txx} + \omega_{xx}^2 &= -\omega_{xx}U_{xx}\cdot\nabla\omega - 2\omega_{xx}U_x\cdot\nabla\omega_x - \omega_{xx}U\cdot\nabla\omega_{xx} - \omega_{xx}^2(\nabla\cdot U)\\ &\quad - 2\omega_{xx}\omega_x\nabla\cdot U_x - \omega_{xx}\omega(\nabla\cdot U_{xx}) - y\omega_{xx}\nabla\cdot U_{xx} - \omega_{xx}v_{xx}.\end{aligned}$$

Thus

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\omega_{xx}\|^2 + \|\omega_{xx}\|^2 \le C_3 W(t)^{3/2} - \int_\Omega \omega_{xx} U\cdot\nabla\omega_{xx}\,d\mathbf{x} \\ - \int_\Omega y\omega_{xx}\nabla\cdot U_{xx}\,d\mathbf{x} - \int_\Omega \omega_{xx}v_{xx}\,d\mathbf{x}. \end{aligned} \tag{3.60}$$

In a similar fashion, we find that

$$\int_\Omega \omega_{xx} U\cdot\nabla\omega_{xx}\,d\mathbf{x} = \int_\Omega \nabla\cdot(\frac{\omega_{xx}^2}{2}U) - \frac{\omega_{xx}^2}{2}\nabla\cdot U\,d\mathbf{x} \le C_3 W(t)^{3/2} \tag{3.61}$$

and

$$\begin{aligned} &\int_\Omega y\omega_{xx}\nabla\cdot U_{xx}\,d\mathbf{x} \\ &= \int_\Omega y\omega_{xx}\big[\frac{h_{xx}(1+h)^2 - 2h_x^2(1+h)}{(1+h)^4}(h_t + U\cdot\nabla h)\big]\,d\mathbf{x} \\ &\quad + 2\int_\Omega y\omega_{xx}\frac{h_x}{(1+h)^2}(h_{tx} + U_x\cdot\nabla h + U\cdot\nabla h_x)\,d\mathbf{x} \\ &\quad - \int_\Omega y\omega_{xx}\frac{1}{1+h}(h_{txx} + U_{xx}\cdot\nabla h + 2U_x\cdot\nabla h_x + U\cdot\nabla h_{xx})\,d\mathbf{x}. \end{aligned}$$

Straightforward calculation gives

$$\begin{aligned} &\int_\Omega y\omega_{xx}\nabla\cdot U_{xx}\,d\mathbf{x} \\ &\le C_3 W(t)^{3/2} - \int_\Omega \frac{y}{1+h}\omega_{xx}h_{txx}\,d\mathbf{x} \\ &\le C_3 W(t)^{3/2} - \int_\Omega \frac{y}{1+h}\omega_{xx}(-u_{tx} + yv_{tx} - u_{ttx} \\ &\quad - U_{tx}\cdot\nabla u - U_t\cdot\nabla u_x - U_x\cdot\nabla u_t - U\cdot\nabla u_{tx})\,d\mathbf{x} \\ &\le C_3 W(t)^{3/2} + \frac{3}{8}\|\omega_{xx}\|^2 + C_3(\|u_{tx}\|^2 + \|v_{tx}\|^2 + \|u_{ttx}\|^2) \\ &\le C_3 W(t)^{3/2} + \frac{3}{8}\|\omega_{xx}\|^2 + C_3(\|U_t\|_1^2 + \|U_t\|_1^2 + \|U_{tt}\|_1^2) \\ &\le C_3\big(W(t)^{3/2} + E(t) + \|\omega_t\|^2 + \|\omega_{tt}\|^2\big) + \frac{3}{8}\|\omega_{xx}\|^2. \end{aligned} \tag{3.62}$$

Similarly,

$$\begin{aligned} \int_\Omega \omega_{xx}v_{xx}\,d\mathbf{x} &\le \|\omega_{xx}\|\|U\|_2 \\ &\le C_3\big(W(t)^{3/2} + E(t) + \|\omega\|^2 + \|\omega\|_1^2\big) + \frac{1}{8}\|\omega_{xx}\|^2. \end{aligned} \tag{3.63}$$

Combining this with (3.60)-(3.63), we obtain

$$\begin{aligned} \frac{d}{dt}\|\omega_{xx}\|^2 + \frac{1}{2}\|\omega_{xx}\|^2 \le C_4\big(W(t)^{3/2} + E(t) + \|\omega_t\|^2 + \|\omega_{tt}\|^2 + \|\omega\|^2 + \|\omega\|_1^2\big), \\ \frac{d}{dt}\|\omega_{yy}\|^2 + \frac{1}{2}\|\omega_{yy}\|^2 \le C_4\big(W(t)^{3/2} + E(t) + \|\omega_t\|^2 + \|\omega_{tt}\|^2 + \|\omega\|^2 + \|\omega\|_1^2\big), \\ \frac{d}{dt}\|\omega_{xy}\|^2 + \frac{1}{2}\|\omega_{xy}\|^2 \le C_4\big(W(t)^{3/2} + E(t) + \|\omega_t\|^2 + \|\omega_{tt}\|^2 + \|\omega\|^2 + \|\omega\|_1^2\big), \end{aligned}$$

$$\frac{d}{dt}\|\omega_{tx}\|^2+\frac{1}{2}\|\omega_{tx}\|^2\le C_4\big(W(t)^{3/2}+E(t)+\|\omega_t\|^2\big),$$
$$\frac{d}{dt}\|\omega_{ty}\|^2+\frac{1}{2}\|\omega_{ty}\|^2\le C_4\big(W(t)^{3/2}+E(t)+\|\omega_t\|^2\big).$$

Therefore,

$$\begin{aligned}\frac{d}{dt}\|\partial^2\omega\|^2+\frac{1}{2}\|\partial^2\omega\|^2\le C_4\big(&W(t)^{3/2}+E(t)+\|\omega_t\|^2+\|\omega_{tt}\|^2\\&+\|\omega\|^2+\|\omega\|_1^2\big),\end{aligned}\tag{3.64}$$

$$\frac{d}{dt}\|\partial\omega_t\|^2+\frac{1}{2}\|\partial\omega_t\|^2\le C_4\big(W(t)^{3/2}+E(t)+\|\omega_t\|^2\big).\tag{3.65}$$

Calculating $20C_4^2(3.38)+8C_4[(3.50)+(3.54)+(3.59)]+(3.64)+(3.65)$, letting $C_5\equiv\min\{\frac{C_4-1}{10C_4},\frac{1}{4}\}$, and defining

$$V_1(t)\equiv 20C_4^2\|\omega\|^2+8C_4(\|\partial\omega\|^2+\|\omega_t\|^2)+8C_4\|\omega_{tt}\|^2+\|\partial^2\omega\|^2+\|\partial\omega_t\|^2,\tag{3.66}$$

we obtain

$$\frac{d}{dt}V_1(t)+C_5V_1(t)\le CW(t)^{3/2}+C_6E(t).$$

This completes the proof. □

With the above lemmas, we can give the following proof.

Proof of Theorem 3.1. Calculating $\frac{2C_6}{C_2}(3.31)+(3.33)$ we have

$$\frac{d}{dt}\big(\frac{2C_6}{C_2}E_1(t)+V_1(t)\big)+\big(C_6E(t)+C_5V_1(t)\big)\le CW(t)^{3/2}.\tag{3.67}$$

Set

$$E_2(t)\equiv\frac{2C_6}{C_2}E_1(t).\tag{3.68}$$

Then according to (3.3), (3.30), (3.68) and the definition of c_1, C_2 and C_6, we can easily see that $E(t)$ and $E_2(t)$ are equivalent, i.e., there exist constants $c_4,c_5>0$ such that

$$c_4E_2(t)\le E(t)\le c_5E_2(t).\tag{3.69}$$

Using (3.67) and (3.69), we have

$$\frac{d}{dt}\big(E_2(t)+V_1(t)\big)+\big(c_4C_6E_2(t)+C_5V_1(t)\big)\le CW(t)^{3/2}.\tag{3.70}$$

Let $C_7\equiv\min\{c_4C_6,C_5\}$. Then

$$\frac{d}{dt}\big(E_2(t)+V_1(t)\big)+C_7\big(E_2(t)+V_1(t)\big)\le CW(t)^{3/2}.\tag{3.71}$$

On the other hand, from (3.66) and the definition of C_4 we can see that $V(t)$ and $V_1(t)$ are also equivalent, i.e., there exist constants $c_6,c_7>0$ such that

$$c_6V_1(t)\le V(t)\le c_7V_1(t).\tag{3.72}$$

From (3.4), (3.69) and (3.72), we see that

$$W(t)\le C_1\big(c_5E_2(t)+c_7V_1(t)\big).\tag{3.73}$$

Letting $C_8\equiv\max\{c_5C_1,c_7C_1\}$, then we obtain

$$W(t)\le C_8\big(E_2(t)+V_1(t)\big).\tag{3.74}$$

Using the a priori assumption $W(t) \le \bar{\delta}$, from (3.71), (3.74) and the smallness of $\bar{\delta}$ we obtain

$$\frac{d}{dt}\big(E_2(t)+V_1(t)\big)+\frac{C_7}{2}\big(E_2(t)+V_1(t)\big)\le 0. \tag{3.75}$$

Integrating over $[0,t]$ for any $t>0$ gives

$$\big(E_2(t)+V_1(t)\big)+\frac{C_7}{2}\int_0^t\big(E_2(t')+V_1(t')\big)dt'\le\big(E_2(0)+V_1(0)\big). \tag{3.76}$$

Using $\|(h_0,U_0)\|_3\le\varepsilon$, we can choose sufficiently small ε such that

$$W(t)\le C\varepsilon<\bar{\delta}.$$

This justifies the a priori assumption.

Finally, the exponential decay of $E_2(t)+V_1(t)$ follows directly from (3.75). Thus $W(t)$ also decays exponentially because of (3.74). This completes the proof. □

Acknowledgments. The authors would like to thank the anonymous referees whose comments help us improve the contain of this article. K. Li was supported by Natural Science Foundation of China (Grant No. 12101534) and by the Natural Science Foundation of Shandong Province (Grant No. ZR2021QA052). X. Xu was supported by Project funded by China Postdoctoral Science Foundation (Grant Nos. 2021T140633 and 2021M693028), by the Natural Science Foundation of China (Grant No. 12001506) and by the Natural Science Foundation of Shandong Province (Grant No. ZR2020QA014).

References

[1] D. Bresch; *Shallow-water equations and related topics*, in: Handbook of Differential Equations: Evolutionary Equations, vol. V, in: Handb. Differ. Equ., Elsevier, North-Holland, Amsterdam, 2009, pp. 1-104.

[2] D. Bresch, B. Desjardins, G. Métivier; *Recent mathematical results and open problems about shallow water equations*, in: Analysis and Simulation of Fluid Dynamics, i n: Adv. Math. Fluid Mech., Birkhäuser, Basel, 2007, pp. 15–31.

[3] B. Cheng, P. Qu, C. Xie; Singularity formation and global existence of classical solutions for one-dimensional rotating shallow water system, *SIAM Journal on Mathematical Analysis*, **50(3)** (2018), 2486-2508.

[4] B. Cheng, C. Xie; On the classical solutions of two dimensional inviscid rotating shallow water system, *Journal of Differential Equations*, **250(2)** (2011), 690-709.

[5] A. Dutrifoy, A. Majda; The dynamics of equatorial long waves: a singular limit with fast variable coefficients, *Communications in Mathematical Sciences*, **4(2)** (2006), 375-397.

[6] A. Dutrifoy, A. Majda; Fast wave averaging for the equatorial shallow water equations, *Communications in Partial Differential Equations*, **32(10)** (2007), 1617-1642.

[7] A. Dutrifoy, S. Schochet, A. Majda; A simple justification of the singular limit for equatorial shallow-water dynamics, *Communications on Pure and Applied Mathematics*, **62(3)** (2009), 322-333.

[8] L. C. Evans; *Partial Differential Equations*, American Mathematical Soc., 2010.

[9] C. Hao, L. Hsiao, H. L. Li, Cauchy problem for viscous rotating shallow water equations, *Journal of Differential Equations*, **247(12)** (2009), 3234-3257.

[10] S. Kawashima; *System of a Hyperbolic-Parabolic Composite Type, with Applications to the Equations of Magnetohydrodynamics*. Ph. D. thesis, Kyoto University, Kyoto, 1983.

[11] A. Majda, Introduction to PDEs and Waves for the Atmosphere and Ocean, American Mathematical Soc., 2003.

[12] A. Matsumura, T. Nishida; The initial value problem for the equations of motion of viscous and heat-conductive gases, *Journal of Mathematics of Kyoto University*, **20(1)** (1980), 67-104.

[13] J. Pedlosky; *Geophysical Fluid Dynamics*, Springer-Verlag, Berlin, 1992.

[14] S. Schochet; The compressible Euler equations in a bounded domain: existence of solutions and the incompressible limit, *Communications in Mathematical Physics*, **104(1)** (1986), 49-75.

[15] L. Sundbye; Global existence for Dirichlet problem for the viscous shallow water equations, *J. Math. Anal. Appl.*, **202** (1996), 236-258.

[16] L. Sundbye; Global existence for the Cauchy problem for the viscous shallow water equations, *Rocky Mountain J. Math.*, **28** (1998), 1135-1152.

[17] W. K. Wang, C. J. Xu; The Cauchy problem for viscous shallow water equations, *Revista Matemática Iberoamericana*, **21(1)** (2005), 1-24.

[18] W. K. Wang, Y. C. Wang; Global existence of strong solution to the chemotaxis-shallow water system with vacuum in a bounded domain, *Journal of Differential Equations*, **307** (2022), 517-555.

[19] W. K. Wang, Y. C. Wang; Global existence and large time behavior for the chemotaxis-shallow water system in a bounded domain, *Discrete and Continuous Dynamical Systems*, **40(11)** (2020), 6379-6409.

[20] Y. Wang, Z. Xin, Y. Yong; Uniform regularity and vanishing viscosity limit for the compressible Navier-Stokes with heneral Navier-Slip boundary conditions in Three-dimensional domains, *SIAM Journal on Mathematical Analysis*, **47(6)** (2015), 4123-4191.

Yue Fang
School of Mathematical Sciences, Ocean University of China, Qingdao, China
Email address: yyfang0929@163.com

Kaiqiang Li (corresponding author)
School of Mathematics and Information Sciences, Yantai University, Yantai, China
Email address: kaiqiangli19@163.com

Xin Xu
School of Mathematical Sciences, Ocean University of China, Qingdao, China
Email address: xx@ouc.edu.cn