BOUNDEDNESS ON GENERALIZED MORREY SPACES FOR THE SCHRÖDINGER OPERATOR WITH POTENTIAL IN A REVERSE HÖLDER CLASS

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ABSTRACT. In this article, we prove boundedness for the Hessian of a Schrödinger operator with weak regularity on the coefficients, and potentials satisfying the reverse Hölder condition. This is done in in generalized Morrey spaces, and in vanishing generalized Morrey spaces. On the Schrödinger operator $L = -a_{ij}(x)D_{ij} + V(x)$ it is assumed that $a_{ij} \in \mathrm{BMO}_\theta(\rho)$ (a generalized Morrey space) and that $V(x) \in B^*_{n/2}$ (a reverse Hölder class).

1. INTRODUCTION

This article presents local estimates in the framework of generalized Morrey spaces and vanishing generalized Morrey spaces for the Schrödinger operator with a weak assumption on the main coefficient and a singular potential satisfying the reverse Hölder class. More precisely, we consider the non-divergence Schrödinger operator with discontinuous coefficient

$$Lu = -a_{ij}(x)D_{ij}u + V(x)u \quad \text{for } x \in \mathbb{R}^n \text{ with } n \geq 3, \tag{1.1}$$

where $\nabla^2 u = (D_{ij}u)_{n \times n}$ is the Hessian matrix of u, and $V(x) \in B^*_{n/2}$ is a nonnegative singular potential belonging to the so-called reverse Hölder class defined below. Here, we assume that $A = (a_{ij}(x))_{n \times n}$ is a measurable symmetric matrix with $a_{ij} = a_{ji}$ defined on $\mathbb{R}^n$ and it satisfies the following uniform ellipticity and boundedness such that there exists a positive constant $\lambda \in (0, 1]$ satisfying

$$\lambda|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq \lambda a^{-1}|\xi|^2, \quad |a_{ij}| \leq \lambda^{-1} \quad \text{for all } \xi \in \mathbb{R}^n. \tag{1.2}$$

The Calderón-Zygmund theory of second-order elliptic equations with discontinuous coefficients has been studied extensively in the last three decades. Interior and boundary $W^{2,p}$-estimates were first established by Chiarenza, Frasca and Longo [8, 9] for nondivergence elliptic equations with VMO discontinuous coefficients, and they were extended to nondivergence parabolic equations by Bramanti and Cerutti [5]. Recently, Krylov [19] gave a unified approach to consider the L^p-solvability of elliptic and parabolic equations of divergence or nondivergence form with weak assumptions of the coefficients belonging to the VMO class in the spatial variables.

This is achieved by pointwise estimates for the sharp maximal functions for the spatial derivatives of solutions by way of the famous Fefferman-Stein theorem. Recently, Bramanti, Brandolini, Harboure and Viviani [4] gave global $W^{2,p}$-estimates for the Schrödinger operator with VMO discontinuous coefficient and a potential $V(x)$ satisfying a reverse Hölder class. We would like to point out that Byun and Wang [6] showed a global L^p regularity for elliptic equations with small BMO coefficients in the Reifenberg flat domains. For divergence elliptic cases, Liang and Zheng [20] proved a global Orlicz estimate of gradients to a class of nonlinear obstacle problems with partially regular nonlinearities in nonsmooth domains, while Liang, Zheng and Feng [21] further gave a global Calderón-Zygmund type estimate in the framework of Lorentz spaces for the variable power of gradient of solution pair to the generalized steady Stokes system over a bounded non-smooth domain. For nondivergence elliptic cases, Zhang and Zheng [31] proved weighted Lorentz estimates of the Hessian of solution to nondivergence linear elliptic equations with partially BMO coefficients, while Tian and Zheng [28] got global Lorentz estimates for a variable power of gradient to linear elliptic obstacle problems with small partially BMO coefficients over a nonsmooth domain. It is also worth noting that Bongioanni, Harboure and Salinas in [3] proved the L^p-boundedness for commutators of Riesz transforms associated with Schrödinger operators with small $BMO_\theta(\rho)$ coefficients which include the classical BMO functions. Guliyev and Softova [15] got global regularity in generalized Morrey spaces for the gradient of solutions of nondivergence elliptic equations with VMO coefficients. Guliyev, Omarova, and Ragusa [18] further derived the boundedness on local generalized Morrey spaces for Schrödinger type operators involved in certain nonnegative potentials.

On the other hand, the Morrey spaces were first introduced by Morrey [23] to study a local behavior of solutions for elliptic differential equations of second-order. Later, many researchers studied Morrey spaces from various points of view. For examples, Fazio, Palagachev and Ragusa [12, 13] got an interior and global Morrey regularity of strong solutions to the Dirichlet problem for elliptic equations with discontinuous coefficients, respectively. Fan, Lu and Yang [11] also gave the regularity in Morrey spaces of strong solutions to nondivergence elliptic equations with VMO coefficients. Chen and Song [7] established the boundedness of the commutator for Riesz potential associated with Schrödinger operator on Morrey spaces. Recently, Tian and Zheng [29] provide another approach to Morrey regularity for linear elliptic equations with partially BMO coefficients, and they in [30] further proved global Morrey regularity for nonlinear elliptic equations with controlled growth under weak assumption of partial BMO nonlinearities on Reifenberg domains. Zhang and Zheng [32] presented a local Morrey regularity for linear parabolic equations of divergence form under the assumption that the leading coefficient being independent of t and not necessarily symmetry. After studying Morrey spaces in detail, some researchers passed to generalized Morrey spaces, weighted Morrey spaces and generalized weighted Morrey spaces. Mizuhara in [22] introduced the generalized Morrey spaces and established the boundedness of some classical operators on generalized Morrey spaces, which was later extended and studied by many authors. Note that Guliyev [14] introduced the generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$ and

the weak generalized Morrey space $WM^{p,\varphi}(\mathbb{R}^n)$ with normalized norms, respectively, as

$$\|f\|_{M^{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}|B(x,r)|^{-1/p}\|f\|_{WL^p(B(x,r))} < \infty;$$

and he obtained the boundedness of the maximal, potential and singular operators in the generalized Morrey spaces by imposing certain condition on $\varphi(x,r)$. For more details regarding the boundedness of various classical linear operators on the generalized Morrey spaces we refer to Guliyev et al in [1, 15, 17, 18, 24]. It is well known that the Calderón-Zygmund theory plays an important role in applications of harmonic analysis and partial differential equations. In [2], Akbulut and Kuzu successfully used the idea and argument of Guliyev's works for the boundedness on generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$ for the Marcinkiewicz integrals associated to the Schrödinger operators $L_0 = -\Delta + V$ with Δ as a Laplacian. The present paper is actually inspired by the L^p-estimate for $L_0 = -\Delta + V$ from Pan and Tang's paper in [25], and the boundedness for the commutators associated with the Schrödinger operators L_0 on local generalized Morrey spaces from Guliyev, Guliyev, Omarova and Ragusa's research in [18]. In fact, our study for general Schrödinger operator $L = -a_{ij}(x)D_{ij}+V(x)$ will attract much attention due to its discontinuous coefficient $a_{ij}(x) \in VMO(\Omega)$. This leads to that a key point of our argument is in an effort how to handle the variable coefficient $a_{ij}(x)$ as a perturbation of tha usual Schrödinger operators L'_0 with constant coefficient in the sense of integral.

The rest of this article is organized as follows. We devote Section 2 to the related notations and statement of main results. In Section 3, we will give some auxiliary lemmas. In Section 4, we prove the the boundedness on generalized Morrey space and vanishing generalized Morrey space for Hessian of operators $\nabla^2 L^{-1}$, respectively.

2. Notation and main results

To state our problem we first recall the definition of the reserve Hölder class B_q. Let $V(x)$ be a locally L^q-integrable nonnegative function in $\mathbb{R}^n$. We say that the potential $V(x)$ belongs to *the reverse Hölder class*, denoted by $V(x) \in B_q$ for $1 < q \le \infty$; if there exists a positive constant C such that the reverse Hölder inequality holds:

$$\Big(\frac{1}{|B(x,r)|}\int_{B(x,r)} V^q(y)dy\Big)^{1/q} \le C\Big(\frac{1}{|B(x,r)|}\int_{B(x,r)} V(y)dy\Big) \tag{2.1}$$

for all $B(x,r)$ with centered at $x \in \mathbb{R}^n$ and the radius $0 < r < \infty$. In particular, if $V(x)$ is a nonnegative polynomial, then $V \in B_\infty$. As well known, while $V \in B_q$ with $q > 1$, we obtain a higher integrability of $V(x)$, which implies that there exists an $\varepsilon > 0$ such that $V \in B_{q+\varepsilon}$, where the constant ε depends only on n and C of (2.1). Furthermore, there exists the following double condition: for $V(x) \in B_q$ it holds

$$\int_{B(x,2r)} V(y)dy \le C \int_{B(x,r)} V(y)dy. \tag{2.2}$$

With the class of reverse Hölder for $V(x) \in B_q$ in hand, we introduce the auxiliary function associated with the potentials $V(x)$, by

$$\rho(x) = \frac{1}{m(x,V)} := \sup_{r>0}\Big\{r : \tfrac{1}{r^{n-2}}\int_{B(x,r)} V(y)dy \le 1\Big\}. \tag{2.3}$$

In the following context, we consider that potential function $V(x)$ satisfies the $B^*_{n/2}$-condition, if there is a positive constant C independent of $V(x)$ such that

- $V(x) \in B_{n/2}$;
- $V(x) \leq Cm^2(x,V)$, $|\nabla V| \leq Cm^3(x,V)$, and $|\nabla^2 V| \leq Cm^4(x,V)$.

Before stating our main results, let us first recall some related notation and basic facts. It is necessary to impose some weaker regularity assumptions on the leading coefficients of Schröndinger operators. To this end, let us recall the concepts of BMO-space and $\text{BMO}_\theta(\rho)$-space. In this context, we denote the integral average over a ball $B(x,r)$ by $g_B = \frac{1}{|B|}\int_B g(y)dy$ for a local integrable functions $g(x)$.

Definition 2.1 (classical BMO-space)**.** We say that $g \in \text{BMO}$ for a locally integrable function $g(x) \in L^1(\mathbb{R}^n)$, if

$$\|g\|_{\text{BMO}} := \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |g(y) - g_B|dy < \infty.$$

Definition 2.2 ($\text{BMO}_\theta(\rho)$-space)**.** We say that $g \in \text{BMO}_\theta(\rho)$ with $\theta \geq 0$ associated with the potentials $V(x) \in B_q$, if for a locally integrable function $g(x) \in L^1_{\text{loc}}(\mathbb{R}^n)$ there holds

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} |g(y) - g_B|dy \leq C\Big(1 + \frac{r}{\rho(x)}\Big)^\theta \tag{2.4}$$

for $x \in \mathbb{R}^n$ and $r > 0$, and the semi-norm of $g \in \text{BMO}_\theta(\rho)$ is defined by

$$[g]_\theta := \sup_{x\in\mathbb{R}^n, r>0} \Big(1 + \frac{r}{\rho(x)}\Big)^{-\theta} \frac{1}{|B(x,r)|} \int_{B(x,r)} |g(y) - g_B|dy < \infty.$$

As a direct consequence of $\text{BMO}_\theta(\rho)$-space, we obviously check that $\text{BMO} \subset \text{BMO}_\theta(\rho) \subset \text{BMO}_{\theta'}(\rho)$ for $0 < \theta \leq \theta'$, see [25, 18]. We are now in a position to introduce the notations of generalized Campanato space and generalized Morrey space.

Definition 2.3 (generalized Campanato space $\Lambda^\theta_\nu(\rho)$)**.** We say that $g \in \Lambda^\theta_\nu(\rho)$ with $\theta > 0$ and $0 < \nu < 1$ associated with the potentials $V(x)$, if a locally integrable function $g(x) \in L^1(\mathbb{R}^n)$ satisfies

$$\frac{1}{|B(x,r)|^{1+\nu/n}} \int_{B(x,r)} |g(y) - g_B|dy \leq C\Big(1 + \frac{r}{\rho(x)}\Big)^\theta$$

for all $x \in \mathbb{R}^n$ and $r > 0$. We denote the semi-norm of $g \in \Lambda^\theta_\nu(\rho)$ by

$$[g]^\theta_\nu := \sup_{x\in\mathbb{R}^n, r>0} \frac{\int_{B(x,r)} |g(y) - g_B|dy}{|B(x,r)|^{1+\nu/n}\big(1 + \frac{r}{\rho(x)}\big)^\theta} < \infty.$$

Remark 2.4. We would like to remark that if $\theta = 0$, then $\Lambda^\theta_\nu(\rho)$ is exactly the classical Campanato space; if $\nu = 0$, then $\Lambda^\theta_\nu(\rho)$ is the generalized BMO space denoted by $\text{BMO}_\theta(\rho)$ space; if $\theta = 0$ and $\nu = 0$, $\Lambda^\theta_\nu(\rho)$ is nothing but the so-called John-Nirenberg space, that is, the usual BMO space.

Definition 2.5 (generalized Morrey space $M^{\alpha,V}_{p,\varphi}$)**.** Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$, and $V \in B_q$ with $q > 1$. We say that $g \in M^{\alpha,V}_{p,\varphi} = M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ for $1 \leq p < \infty$ and $\alpha \geq 0$ is the generalized Morrey space associated with the potentials $V(x)$, if $g \in L^p_{\text{loc}}(\mathbb{R}^n)$ satisfies

$$\|g\|_{M^{\alpha,V}_{p,\varphi}} := \sup_{x\in\mathbb{R}^n, r>0} \Big(1 + \frac{r}{\rho(x)}\Big)^\alpha r^{-n/p}\varphi(x,r)^{-1}\|g\|_{L^p(B(x,r))} < \infty. \tag{2.5}$$

Definition 2.6 (vanishing generalized Morrey space $\mathrm{VM}_{p,\varphi}^{\alpha,V}$). We say that $g \in \mathrm{VM}_{p,\varphi}^{\alpha,V} = \mathrm{VM}_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ be the vanishing generalized Morrey spaces associated with the potentials $V(x)$, if $g \in M_{p,\varphi}^{\alpha,V}$ satisfies

$$\lim_{r \to 0} \sup_{x \in \mathbb{R}^n} \left(1 + \frac{r}{\rho(x)}\right)^\alpha r^{-n/p} \varphi(x,r)^{-1} \|g\|_{L^p(B(x,r))} = 0. \qquad (2.6)$$

As an immediate consequence of the above definitions, the generalized Morrey spaces and the vanishing generalized Morrey spaces associated with the potentials are Banach spaces with respect to their norm, respectively, see [22, 24].

Remark 2.7. (i) If $\alpha = 0$ and $\varphi(x,r) = r^{(\lambda-n)/p}$, then $M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ is actually the classical Morrey spaces denoted by $L_{p,\lambda}(\mathbb{R}^n)$, which was originally introduced by Morrey to study the local behavior of solutions to second order elliptic partial differential equations, see [23].

(ii) If $\alpha = 0$, then $M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ is the generalized Morrey spaces denoted by $M_{p,\varphi}(\mathbb{R}^n)$, which was first introduced by Mizuhara and Nakai in [22, 24], and later Guliyev made further study for it, see [15, 17, 2].

(iii) If $\varphi(x,r) = r^{(\lambda-n)/p}$, then $M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ is the Morrey spaces associated with the potentials, denoted by $L_{p,\lambda}^{\alpha,V}(\mathbb{R}^n)$ that was introduced by Tang and Dong in [27].

(iv) Here, the generalized Morrey spaces $M_{p,\varphi}^{\alpha,V}$ and the vanishing generalized Morrey spaces $VM_{p,\varphi}^{\alpha,V}$ associated with the singular potentials, respectively, were introduced by Guliyev to study the boundedness of some operators and their commutators, see [16].

We are now ready to present the main results of this paper, which is involved in the boundedness in the generalized Morrey spaces for the Hessian $\nabla^2 L^{-1}$ of solutions as follows.

Theorem 2.8. *Let $a_{ij}(x) \in \mathrm{BMO}_\theta(\rho)$ and $V \in B_{n/2}^*$. For $\alpha \geq 0$ and $1 < p < \infty$, we assume that $\varphi_1, \varphi_2 \in \Omega_p^{\alpha,V}$ satisfies*

$$\int_r^\infty \frac{ess\inf_{t<s<\infty} \varphi_1(x,s) s^{n/p}}{t^{n/p}} \frac{dt}{t} \leq c_0 \varphi_2(x,r) \quad \text{for all } x \in \mathbb{R}^n, \qquad (2.7)$$

where the constant $c_0 > 0$ is independent of x and r. If there exists a constant $\varepsilon > 0$ such that $[a_{ij}]_\theta < \varepsilon$, for the solutions of $Lu = f(x)$, then the Hessian $\nabla^2 L^{-1}$ is a bounded operator from the the generalized Morrey space $M_{p,\varphi_1}^{\alpha,V}$ to the generalized Morrey space $M_{p,\varphi_2}^{\alpha,V}$. Moreover, for any $1 < p < \infty$ we have the estimate

$$\|\nabla^2 L^{-1} f\|_{M_{p,\varphi_2}^{\alpha,V}} \leq C \|f\|_{M_{p,\varphi_1}^{\alpha,V}}. \qquad (2.8)$$

A more delicate result is stated in the vanishing generalized Morrey spaces. Let us consider it under the following assumptions on $\varphi_1 \in \Omega_{p,1}^{\alpha,V}$, there still holds the boundedness of Schrödinger operators on the vanishing generalized Morrey spaces.

Theorem 2.9. *Let $a_{ij}(x) \in \mathrm{BMO}_\theta(\rho)$ and $V \in B_{n/2}^*$. Under the same assumptions of Theorem 2.8 on φ_1, φ_2. If $f \in VM_{p,\varphi_1}^{\alpha,V}$, then the operators $\nabla^2 L^{-1}$ is bounded from $VM_{p,\varphi_1}^{\alpha,V}$ to $VM_{q,\varphi_2}^{\alpha,V}$ for $1 < p < \infty$. Moreover, we have the estimate*

$$\|\nabla^2 L^{-1} f\|_{VM_{q,\varphi_2}^{\alpha,V}} \leq C \|f\|_{VM_{p,\varphi_1}^{\alpha,V}}. \qquad (2.9)$$

We pointed out that our method in this paper is novel in some sense. It seems that we cannot obtain directly the regularity results for solutions of the Schrödinger equations with $BMO_\theta(\rho)$ coefficients since the arguments depend heavily on the regularity of solutions of the elliptic equations with BMO coefficients. Our argument is motivated by considering the operator $\nabla^2(-\Delta + V)^{-1}$ for $V(x) \in B_{n/2}$ as a standard Calderón-Zygmund operator so that its kernel $K(x,y)$ of this operator possesses the estimate

$$|K(x,y)| \leq \frac{C_N}{\left(1 + \frac{|x-y|}{\rho(x)}\right)^N} \frac{1}{|x-y|^n} \tag{2.10}$$

for all $N \in \mathbb{N}$. We would like to mention that our regularity of solutions are worked in the generalized Morrey space which depends heavily on singular potential $V(x)$.

3. Technical lemmas

This section is devoted to some well-known facts about some fundamental inequalities and technical lemmas that we will use later. Throughout this paper, $C(n, \lambda, \dots)$ stands for a universal positive constant depending only on prescribed quantities and possibly varying from line to line. However, the ones we need to emphasize will be denoted with special symbols, such as $C_1, C_2, \dots$. First of all, let us recall an inequality concerning the auxiliary function, and the relationships between the generalized BMO space and $BMO_\theta(\rho)$ space.

Lemma 3.1 ([26, Lemma 1.4]). *Let $V(x) \in B_q$ for $q \geq \frac{n}{2}$. Then, for the associated function $\rho(x)$ there exist two positive constants $k_0 \geq 1$ and $C_0 > 0$ such that*

$$\frac{1}{C_0} \rho(x) \left(1 + \frac{|x-y|}{\rho(x)}\right)^{-k_0} \leq \rho(y) \leq C_0 \, \rho(x) \left(1 + \frac{|x-y|}{\rho(x)}\right)^{\frac{k_0}{k_0+1}}. \tag{3.1}$$

In particular, if $|x-y| \leq C\rho(x)$, then $\rho(x) \sim \rho(y)$.

Lemma 3.2 ([1]). *Let $x \in B(x_0, r)$. Then for any $k \in \mathbb{N}$ there exists a positive constant C such that*

$$\frac{1}{\left(1 + \frac{2^k r}{\rho(x)}\right)^N} \leq \frac{C}{\left(1 + \frac{2^k r}{\rho(x_0)}\right)^{N/(k_0+1)}}, \tag{3.2}$$

where k_0 is the constant as in Lemma 3.1.

Lemma 3.3 ([3, Proposition 3]). *Let $g \in \mathrm{BMO}_\theta(\rho)$. Then, for $\theta > 0$ and $1 \leq s < \infty$ there exists a positive constant C such that*

$$\left(\frac{1}{|B_r|} \int_{B_r} |g(y) - g_{B_r}|^s dy\right)^{1/s} \leq C \, [g]_\theta \left(1 + \frac{r}{\rho(x)}\right)^{\theta_1} \tag{3.3}$$

for any $B_r = B(x, r)$ with $x \in \mathbb{R}^n$, where $\theta_1 = (k_0 + 1)\theta$ and k_0 as in Lemma 3.1.

Lemma 3.4 ([3, Lemma 1]). *Let $\theta > 0$ and $1 \leq s < \infty$. If $g \in \mathrm{BMO}_\theta(\rho)$, then there exists a positive constant C such that*

$$\left(\frac{1}{|2^k B_r|} \int_{2^k B_r} |g(y) - g_{B_r}|^s dy\right)^{1/s} \leq C \, [g]_\theta \, k \left(1 + \frac{2^k r}{\rho(x)}\right)^{\theta_1} \tag{3.4}$$

for any $B_r = B(x, r)$ with $x \in \mathbb{R}^n$ and $r > 0$, where $k \in \mathbb{N}$ and $\theta_1 = (k_0 + 1)\theta$ with k_0 as in Lemma 3.1.

According to the definition of $m(\cdot, V)$ in (2.3), we have the following decomposition lemma associated with $m(\cdot, V)$ on $\mathbb{R}^n$.

Lemma 3.5 ([10, Lemma 2.3]). *There exists a sequence of points $\{x_k\}_{k=1}^\infty$ in $\mathbb{R}^n$ such that the collection of balls $B_k = B(x_k, \rho(x_k))$, $k \geq 1$ satisfies the following:*

(i) *$\cup_k B_k = \mathbb{R}^n$;*
(ii) *for any $k \in \mathbb{N}$, we conclude that $\operatorname{card}\{j : 4B_j \bigcap 4B_k \neq \phi\} \leq N$ with some positive integer N.*

We are now in a position to recall the following Hardy-Littlewood maximal functions and sharp maximal functions for $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. For given $\alpha > 0$ we define

$$M_{\rho,\alpha}(g)(x) = \sup_{x \in B \in \mathcal{B}_{\rho,\alpha}} \frac{1}{|B|} \int_B |g(y)| dy,$$

$$M^\sharp_{\rho,\alpha}(g)(x) = \sup_{x \in B \in \mathcal{B}_{\rho,\alpha}} \frac{1}{|B|} \int_B |g(y) - g_B| dy,$$

where $\mathcal{B}_{\rho,\alpha} = \{B(y, r) : y \in \mathbb{R}^n, r \leq \alpha \rho(y)\}$.

In what follows, we recall the relationship between Hardy-Littlewood functions $M_{\rho,\alpha}$ and sharp coefficients $M^\sharp_{\rho,\alpha}$, which is just the so-called famous Fefferman-Stein inequality.

Lemma 3.6 ([3, Lemma 2]). *Let $\{B_k\}_{k=1}^\infty$ be the collection of balls as in Lemma 3.5, and $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then, for $1 < p < \infty$ there exist positive constants C, β and γ such that*

$$\int_{\mathbb{R}^n} |M_{\rho,\beta}(g)(z)|^p dz \leq C \int_{\mathbb{R}^n} |M^\sharp_{\rho,\gamma}(g)(z)|^p dz + C \sum_k |B_k| \left(\frac{1}{|B_k|} \int_{2B_k} |g(z)| dz \right)^p.$$

Let us now consider the Schrödinger operator L_0 with constant coefficients. To this end, we let

$$L_0 u(x) = -a^0_{ij} D_{ij} u(x) + V(x) u(x),$$

where a^0_{ij} is an $n \times n$ symmetric constant matrix with uniformly elliptic condition (1.2). By a scaling argument this $a^0_{ij} D_{ij}$ is actually Laplacian in the new coordinate system. Note that $V(x) \in B_{n/2}$ with $V(x) \leq cm^2(x, V)$. Then, for the operator L_0 with constant coefficients we have the following regularity conclusion.

Lemma 3.7 ([26, Remark 2.9]). *If $V(x) \in B^*_{n/2}$, then for $1 < p < \infty$ there exists a positive constant $C > 0$ such that*

$$\|\nabla^2 L_0^{-1} f\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}, \tag{3.5}$$

where C is independent of f.

Let $K(x, y)$ be the kernel function of the operator $T_0 = D_{ij} L_0^{-1}$. Then, we have the following estimates for the kernel $K(x, y)$.

Lemma 3.8 ([25, Lemma 3.6]). *If $V(x) \in B^*_{n/2}$, then for every $N \geq 0$ we have*

(i) *there exists a constant C_N such that*

$$|K(x, y)| \leq C_N \frac{\left(1 + \frac{|x-y|}{\rho(x)}\right)^{-N}}{|x - y|^n}. \tag{3.6}$$

(ii) *there exists a constant C_N such that*

$$|K(x,y) - K(x_0,y)| \leq C_N \frac{|x - x_0|\big(1 + \frac{|x_0 - y|}{\rho(x_0)}\big)^{-N}}{|x - y|^{n+1}}, \tag{3.7}$$

provided $|x - x_0| < \frac{1}{2}|x - y|$.

With Lemma 3.8 in hand, Pan and Tang concluded the following L^q-regularity of $\nabla^2 L^{-1}$.

Lemma 3.9 ([25, Theorem 3.3]). *Assume that u is the solutions of $Lu = f(x)$ with $V \in B_{n/2}^*$ and $a_{ij}(x) \in \mathrm{BMO}_\theta(\rho)$. Then there exist positive constants $\varepsilon > 0$ and C such that for all $1 < q < \infty$ it holds*

$$\|\nabla^2 L^{-1} f\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^q(\mathbb{R}^n)} \tag{3.8}$$

provided that $[a_{ij}]_\theta < \varepsilon$.

4. Boundedness on (vanishing) generalized Morrey spaces

We devote this section to local bounded estimates for the Hessian of the operators L^{-1}. First of all, let us show the following local L^p-estimate for the Hessian $\nabla^2 L^{-1}$.

Theorem 4.1. *For $1 < p < \infty$, let u be the solutions of $Lu = f(x)$ with $a_{ij}(x) \in \mathrm{BMO}_\theta(\rho)$ and $V \in B_{n/2}^*$. If $f \in L_{\mathrm{loc}}^p(\mathbb{R}^n)$, then there exists small constant $\varepsilon > 0$ such that*

$$\|\nabla^2 L^{-1} f\|_{L^p(B(x_0,r))} \leq Cr^{n/p} \int_{2r}^\infty \frac{\|f\|_{L^p(B(x_0,t))}}{t^{n/p}} \frac{dt}{t} \tag{4.1}$$

provided that $[a_{ij}]_\theta < \varepsilon$, where C is independent of u, f.

Proof. For any fixed $x_0 \in \mathbb{R}^n$, let $B_r = B(x_0, r)$ and $\lambda B_r = B(x_0, \lambda r)$ for any $\lambda > 0$. We now divide $f(x)$ into two items as $f(x) = f_1(x) + f_2(x)$, where $f_1(y) = f(y)\chi_{B(x_0,2r)}(y)$ with $\chi_{B(x_0,2r)}$ being the characteristic function on $B(x_0, 2r)$. Then we obtain

$$\|\nabla^2 L^{-1} f\|_{L^p(B(x_0,r))} \leq \|\nabla^2 L^{-1} f_1\|_{L^p(B(x_0,r))} + \|\nabla^2 L^{-1} f_2\|_{L^p(B(x_0,r))}. \tag{4.2}$$

To estimate the first term we use the L^p-boundedness of $\nabla^2 L^{-1}$ in Lemma 3.9 and obtain

$$\begin{aligned}
\|\nabla^2 L^{-1} f_1\|_{L^p(B(x_0,r))} &\leq C\|f\|_{L^p(B(x_0,2r))} \\
&\leq Cr^{n/p}\|f\|_{L^p(B(x_0,2r))} \int_{2r}^\infty \frac{dt}{t^{\frac{n}{p}+1}} \\
&\leq Cr^{n/p} \int_{2r}^\infty \frac{\|f\|_{L^p(B(x_0,t))}}{t^{n/p}} \frac{dt}{t}.
\end{aligned} \tag{4.3}$$

Next we estimate the second term on (4.2). To this end, for any $x \in B_r$ and $y \in (2B_r)^c$ we see that $\frac{1}{2}|x_0 - y| \leq |x - y| \leq \frac{3}{2}|x_0 - y|$. Let us consider an operator $\mathcal{L}_0 = -\overline{a_{ij}} D_{i,j} + V$ with a constant coefficient matrix $\overline{a_{ij}} = \frac{1}{|B_r|} \int_{B_r} a_{ij}(x)dx$. Note that $D_{ij}L^{-1}f_2 = D_{ij}u - D_{ij}L^{-1}f_1$ leads to $|D_{ij}L^{-1}f_2| \leq |D_{ij}u| + |D_{ij}L^{-1}f_1|$. With the estimate (4.3) for $D_{ij}L^{-1}f_1$ in hand, it suffice to only estimate the term $D_{ij}u$. To this end, we have

$$\begin{aligned}
D_{ij}u &= D_{ij}\mathcal{L}_0^{-1}\mathcal{L}_0 u \\
&= D_{ij}\mathcal{L}_0^{-1}\mathcal{L}_0(u\chi_{2B_r} + u\chi_{(2B_r)^c})
\end{aligned}$$

$$= D_{ij}\mathcal{L}_0^{-1}(\mathcal{L}_0 u\chi_{2B_r} + \mathcal{L}_0 u\chi_{(2B_r)^c})$$
$$= D_{ij}\mathcal{L}_0^{-1}(\mathcal{L}_0 u\chi_{2B_r} + \mathcal{L}_0 u\chi_{(2B_r)^c} - Lu\chi_{(2B_r)^c} + Lu\chi_{(2B_r)^c})$$
$$= D_{ij}\mathcal{L}_0^{-1}(\mathcal{L}_0 u\chi_{2B_r}) + D_{ij}\mathcal{L}_0^{-1}[(\mathcal{L}_0 u - Lu)\chi_{(2B_r)^c}] + D_{ij}\mathcal{L}_0^{-1}(Lu\chi_{(2B_r)^c}).$$

This leads to

$$\|D_{ij}u(x)\|_{L^p(B(x_0,r))}$$
$$\leq \Big(\int_{B_r}|D_{ij}\mathcal{L}_0^{-1}(\mathcal{L}_0 u\chi_{2B_r})|^p dx\Big)^{1/p} + \Big(\int_{B_r}|D_{ij}\mathcal{L}_0^{-1}[(\mathcal{L}_0 u - Lu)\chi_{(2B_r)^c}]|^p dx\Big)^{1/p}$$
$$+ \Big(\int_{B_r}|D_{ij}\mathcal{L}_0^{-1}(Lu\chi_{(2B_r)^c})|^p dx\Big)^{1/p}$$
$$:= I_1 + I_2 + I_3.$$

Let us first give an estimate of I_1. By Lemma 3.7, Minkowski inequality, Hölder inequality, Lemma 3.4 we conclude that

$$I_1 = \Big(\int_{B_r}|D_{ij}\mathcal{L}_0^{-1}(\mathcal{L}_0 u\chi_{2B_r})|^p dx\Big)^{1/p}$$
$$\leq C\Big(\frac{1}{|B_r|}\int_{2B_r}|\mathcal{L}_0 u(x)|^p dx\Big)^{1/p}$$
$$\leq C\Big(\frac{1}{|2B_r|}\int_{2B_r}|\mathcal{L}_0 u(x) - Lu(x)|^p dx\Big)^{1/p} + \Big(\frac{1}{|2B_r|}\int_{2B_r}|Lu(x)|^p dx\Big)^{1/p}$$
$$\leq C\Big(\frac{1}{|2B_r|}\int_{2B_r}|a_{ij}(x) - \overline{a_{ij}}|^{pv'} dx\Big)^{\frac{1}{pv'}}\Big(\frac{1}{|2B_r|}\int_{2B_r}|D_{ij}u(x)|^{pv} dx\Big)^{\frac{1}{pv}}$$
$$+ \Big(\frac{1}{|2B_r|}\int_{2B_r}|Lu(x)|^p dx\Big)^{1/p}$$
$$\leq C[a_{ij}]_\theta\|D_{i,j}u(x)\|_{L^p(B(x_0,r))} + \|f\|_{L^p(B(x_0,2r))},$$

where $\frac{1}{v} + \frac{1}{v'} = 1$.

We now estimate I_2. We apply the boundedness of the kernel functions $K(x, y)$ in Lemma 3.8 to show that

$$I_2 \leq \frac{C}{|B_r|}\Big(\int_{B_r}|D_{ij}\mathcal{L}_0^{-1}[(\mathcal{L}_0 u - Lu)\chi_{(2B_r)^c}]|^p dx\Big)^{1/p}$$
$$\leq \frac{C}{|B_r|}\Big(\int_{B_r}\Big(\int_{(2B_r)^c}|K(x, y)(\mathcal{L}_0 u - Lu)|dy\Big)^p dx\Big)^{1/p}$$
$$\leq \frac{C}{|B_r|}\Big(\int_{B_r}\Big(\sum_{k=1}^{\infty}\int_{2^k r<|y-x_0|\leq 2^{k+1}r}\frac{\big(1+\frac{|x-y|}{\rho(x)}\big)^{-N}}{|x-y|^n}|\mathcal{L}_0 u - Lu|dy\Big)^p dx\Big)^{1/p}.$$

By considering $x \in B_r$ and $y \in (2B_r)^c$ we obtain that $|x - y| \sim |x_0 - y|$, which yields the following facts that

$$I_2 \leq C\frac{1}{|B_r|}\Big(\int_{B_r}\Big(\sum_{k=1}^{\infty}\int_{2^k r<|y-x_0|\leq 2^{k+1}r}\frac{\big(1+\frac{|x_0-y|}{\rho(x_0)}\big)^{-N}}{|x_0-y|^n}|\mathcal{L}_0 u - Lu|dy\Big)^p dx\Big)^{1/p}$$
$$\leq C\sum_{k=1}^{\infty}2^{-kn}\Big(1+\frac{2^k r}{\rho(x_0)}\Big)^{-N}\int_{2^k r<|y-x_0|\leq 2^{k+1}r}|\mathcal{L}_0 u - Lu|dy$$

$$\leq C \sum_{k=1}^{\infty} 2^{-kn} \Big(1 + \frac{2^k r}{\rho(x_0)} \Big)^{-N} \Big(\frac{1}{|2^{k+1} B_r|} \int_{2^{k+1} B_r} |\mathcal{L}_0 u - L u|^p dy \Big)^{1/p}.$$

It follows from Hölder's inequality and Lemma 3.4 that

$$I_2 \leq C \sum_{k=1}^{\infty} 2^{-kn} \Big(1 + \frac{2^k r}{\rho(x_0)} \Big)^{-N} \Big(\frac{1}{|2^{k+1} B_r|} \int_{2^{k+1} B_r} |\mathcal{L}_0 u - L u|^p dy \Big)^{1/p}$$

$$\leq C \sum_{k=1}^{\infty} 2^{-kn} \Big(1 + \frac{2^k r}{\rho(x_0)} \Big)^{-N} \Big(\frac{1}{|2^{k+1} B_r|} \int_{2^{k+1} B_r} |a_{ij}(x) - \overline{a_{ij}}|^{pv'} dx \Big)^{\frac{1}{pv'}}$$

$$\times \Big(\frac{1}{|2^{k+1} B_r|} \int_{2^{k+1} B_r} |D_{ij} u(y)|^{pv} dy \Big)^{\frac{1}{pv}}$$

$$\leq C \sum_{k=1}^{\infty} 2^{-kn} \Big(1 + \frac{2^k r}{\rho(x_0)} \Big)^{-N} (k+1) [a_{ij}]_\theta \Big(1 + \frac{2^{k+1} r}{\rho(x_0)} \Big)^{\theta_1}$$

$$\times \Big(\frac{1}{|2^{k+1} B_r|} \int_{2^{k+1} B_r} |D_{ij} u(y)|^{pv} dy \Big)^{\frac{1}{pv}}$$

$$\leq C [a_{ij}]_\theta \| D_{ij} u(x) \|_{L^p(B(x_0, r))},$$

where $\frac{1}{v} + \frac{1}{v'} = 1$ and $N > \theta_1$.

Finally, we show the estimate of I_3. We use the boundedness of the kernel functions $K(x, y)$ in Lemma 3.8 and $|x - y| \sim |x_0 - y|$ to obtain

$$I_3 \leq \frac{C}{|B_r|} \Big(\int_{B_r} |D_{ij} \mathcal{L}_0^{-1} (L u \chi_{(2B_r)^c})|^p dx \Big)^{1/p}$$

$$\leq \frac{C}{|B_r|} \Big(\int_{B_r} \Big(\int_{(2B_r)^c} |K(x, y)(L u)| dy \Big)^p dx \Big)^{1/p}$$

$$\leq \frac{C}{|B_r|} \Big(\int_{B_r} \Big(\sum_{k=1}^{\infty} \int_{2^k r < |y - x_0| \leq 2^{k+1} r} \frac{\big(1 + \frac{|x-y|}{\rho(x)} \big)^{-N}}{|x - y|^n} |L u(y)| dy \Big)^p dx \Big)^{1/p}$$

$$\leq \frac{C}{|B_r|} \Big(\int_{B_r} \Big(\sum_{k=1}^{\infty} \int_{2^k r < |y - x_0| \leq 2^{k+1} r} \frac{\big(1 + \frac{|x_0-y|}{\rho(x_0)} \big)^{-N}}{|x_0 - y|^n} |L u| dy \Big)^p dx \Big)^{1/p}$$

$$\leq C \sum_{k=1}^{\infty} 2^{-kn} \Big(1 + \frac{2^k r}{\rho(x_0)} \Big)^{-N} \int_{2^k r < |y - x_0| \leq 2^{k+1} r} |L u| dy$$

$$\leq C \sum_{k=1}^{\infty} 2^{-kn} \Big(1 + \frac{2^k r}{\rho(x_0)} \Big)^{-N} \Big(\int_{2^{k+1} B} |L u|^p dy \Big)^{1/p}$$

$$\leq C \| f \|_{L^p(B(x_0, 2r))}.$$

Let us put the above estimates of I_1, I_2, I_3 together and deduce that

$$\| D_{ij} u(x) \|_{L^p(B(x_0, r))} \leq C [a_{ij}]_\theta \| D_{ij} u(x) \|_{L^p(B(x_0, r))} + C \| f \|_{L^p(B(x_0, 2r))}.$$

By considering the small $BMO_\theta(\rho)$ condition of a_{ij} with $[a_{ij}]_\theta < \varepsilon$, we now take $\varepsilon > 0$ small enough that $\varepsilon < \frac{1}{2Cn^2}$. Then we have

$$\| D_{ij} u(x) \|_{L^p(B(x_0, r))} \leq C \| f \|_{L^p(B(x_0, 2r))},$$

which implies

$$\| D_{ij} L^{-1} f_2 \|_{L^p(B(x_0, r))} \leq C \| f \|_{L^p(B(x_0, 2r))}$$

$$\leq Cr^{n/p}\|f\|_{L^p(B(x_0,2r))}\int_{2r}^{\infty}\frac{dt}{t^{\frac{n}{p}+1}}$$
$$\leq Cr^{n/p}\int_{2r}^{\infty}\frac{\|f\|_{L^p(B(x_0,t))}}{t^{n/p}}\frac{dt}{t},$$

which yields

$$\|\nabla^2 L^{-1}f_2\|_{L^p(B(x_0,r))}\leq Cr^{n/p}\int_{2r}^{\infty}\frac{\|f\|_{L^p(B(x_0,t))}}{t^{n/p}}\frac{dt}{t}. \tag{4.4}$$

Let us combine (4.2),(4.3) and (4.4) to yield the desired inequality (4.1).   □

Proof of Theorem 2.8. By Lemma 3.6 we have

$$\frac{1}{\operatorname{ess\,inf}_{t<s<\infty}\varphi_1(x_0,s)s^{n/p}}=\operatorname{ess\,sup}_{t<s<\infty}\frac{1}{\varphi_1(x_0,s)s^{n/p}}.$$

Note that $\|f\|_{L^p(B(x_0,t))}$ is a nondecreasing function with respect to t. Since $f \in M_{p,\varphi_1}^{\alpha,V}$, we have

$$\frac{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha}\|f\|_{L^p(B(x_0,t))}}{\operatorname{ess\,inf}_{t<s<\infty}\varphi_1(x_0,s)s^{n/p}}=\operatorname{ess\,sup}_{t<s<\infty}\frac{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha}\|f\|_{L^p(B(x_0,t))}}{\varphi_1(x_0,s)s^{n/p}}$$
$$\leq C\sup_{0<s<\infty}\frac{\left(1+\frac{s}{\rho(x_0)}\right)^{\alpha}\|f\|_{L^p(B(x_0,s))}}{\varphi_1(x_0,s)s^{n/p}}$$
$$\leq C\|f\|_{M_{p,\varphi_1}^{\alpha,V}}.$$

Thanks to $\alpha \geq 0$ and (φ_1,φ_2) satisfying (2.7), we conclude that

$$\int_{2r}^{\infty}\frac{\|f\|_{L^p(B(x_0,t))}}{t^{n/p}}\frac{dt}{t}$$
$$=\int_{2r}^{\infty}\frac{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha}\|f\|_{L^p(B(x_0,t))}}{\operatorname{ess\,inf}_{t<s<\infty}\varphi_1(x_0,s)s^{n/p}}\frac{\operatorname{ess\,inf}_{t<s<\infty}\varphi_1(x_0,s)s^{n/p}}{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha}t^{n/p}}\frac{dt}{t}$$
$$\leq C\|f\|_{M_{p,\varphi_1}^{\alpha,V}}\int_{2r}^{\infty}\frac{\operatorname{ess\,inf}_{t<s<\infty}\varphi_1(x_0,s)s^{n/p}}{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha}t^{n/p}}\frac{dt}{t}$$
$$\leq C\|f\|_{M_{p,\varphi_1}^{\alpha,V}}\left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha}\int_{r}^{\infty}\frac{\operatorname{ess\,inf}_{t<s<\infty}\varphi_1(x_0,s)s^{n/p}}{t^{n/p}}\frac{dt}{t}$$
$$\leq C\|f\|_{M_{p,\varphi_1}^{\alpha,V}}\left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha}\varphi_2(x_0,r).$$

Finally, it follows from Theorem 4.1 that

$$\|\nabla^2 L^{-1}f\|_{M_{p,\varphi_2}^{\alpha,V}}$$
$$=\sup_{x_0\in\mathbb{R}^n,r>0}\left(1+\frac{r}{\rho(x_0)}\right)^{\alpha}r^{-n/p}\varphi_2(x_0,r)^{-1}\|[b,T]f\|_{L^p(B(x_0,r))}$$
$$\leq C\sup_{x_0\in\mathbb{R}^n,r>0}\left(1+\frac{r}{\rho(x_0)}\right)^{\alpha}r^{-n/p}\varphi_2(x_0,r)^{-1}r^{n/p}\int_{2r}^{\infty}\frac{\|f\|_{L^p(B(x_0,t))}}{t^{n/p}}\frac{dt}{t}$$
$$\leq C\|f\|_{M_{p,\varphi_1}^{\alpha,V}}.$$

This completes the proof.   □

Proof of Theorem 2.9. By an argument similar to the one in Theorem 2.8, we conclude the boundedness for $\nabla^2 L^{-1} f$ in the vanishing generalized Morrey spaces. In fact, it suffices to only prove

$$f \in VM_{p,\varphi_1}^{\alpha,V}(\mathbb{R}^n) \Rightarrow \nabla^2 L^{-1} f \in VM_{q,\varphi_2}^{\alpha,V}(\mathbb{R}^n), \tag{4.5}$$

which yields

$$\lim_{r \to 0} \sup_{x \in \mathbb{R}^n} \left(1 + \frac{r}{\rho(x)}\right)^\alpha r^{-n/q} \varphi_2(x,r)^{-1} \|\nabla^2 L^{-1} f\|_{L^q(B(x,r))} = 0.$$

It suffices to only prove that for any $\varepsilon > 0$ we find sufficient small $r > 0$ such that

$$\sup_{x \in \mathbb{R}^n} \left(1 + \frac{r}{\rho(x)}\right)^\alpha r^{-n/q} \varphi_2(x,r)^{-1} \|\nabla^2 L^{-1} f\|_{L^q(B(x,r))} < \varepsilon.$$

To this end, by the local L^q-boundedness of the operator $\nabla^2 L^{-1}$ as in Lemma (3.9) and taking $\delta_0 > r$ with δ_0 being specially determined later, then we obtain

$$\begin{aligned}
&\left(1 + \frac{r}{\rho(x)}\right)^\alpha r^{-n/q} \varphi_2(x,r)^{-1} \|\nabla^2 L^{-1} f\|_{L^q(B(x,r))} \\
&\leq C_0 \left(1 + \frac{r}{\rho(x)}\right)^\alpha \varphi_2(x,r)^{-1} \int_{2r}^\infty \frac{\|f\|_{L^p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t} \\
&\leq C_0 \left(1 + \frac{r}{\rho(x)}\right)^\alpha \varphi_2(x,r)^{-1} \int_r^\infty \frac{\|f\|_{L^p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t} \\
&= C_0 \Big[\left(1 + \frac{r}{\rho(x)}\right)^\alpha \varphi_2(x,r)^{-1} \int_r^{\delta_0} \frac{\|f\|_{L^p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t} \\
&\quad + \left(1 + \frac{r}{\rho(x)}\right)^\alpha \varphi_2(x,r)^{-1} \int_{\delta_0}^\infty \frac{\|f\|_{L^p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t} \Big] \\
&:= C_0 \left(A + B\right).
\end{aligned} \tag{4.6}$$

To estimate A, by condition (2.7), we have

$$\begin{aligned}
A &:= \left(1 + \frac{r}{\rho(x)}\right)^\alpha \varphi_2(x,r)^{-1} \int_r^{\delta_0} \varphi_1(x,t) \frac{1}{\varphi_1(x,t)} \frac{\|f\|_{L^p(B(x_0,t))}}{t^{n/p}} \frac{dt}{t} \\
&\leq C_1 \varphi_2(x,r) \left(1 + \frac{r}{\rho(x)}\right)^\alpha \varphi_2(x,r)^{-1} r^{-\frac{n}{p}} \varphi_1(x,r)^{-1} \|f\|_{L^p(B(x_0,r))} \\
&= C_1 \left(1 + \frac{r}{\rho(x)}\right)^\alpha r^{-\frac{n}{p}} \varphi_1(x,r)^{-1} \|f\|_{L^p(B(x_0,r))}.
\end{aligned}$$

Note that $f \in VM_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ for $p > 1$. We find a fixed $\delta_0 > 0$ such that if $0 < r < \delta_0$ it holds

$$\sup_{x \in \mathbb{R}^n} \left(1 + \frac{r}{\rho(x)}\right)^\alpha r^{-n/p} \varphi_2(x,r)^{-1} \|f\|_{L^p(B(x,r))} < \frac{\varepsilon}{2C_1}.$$

It follows from (4.6) that

$$\sup_{x \in \mathbb{R}^n} C_0 A < \frac{\varepsilon}{2} \tag{4.7}$$

for any $0 < r < \delta_0$.

For the estimate of B, we also take r small enough. Then it follows from condition (2.7) that

$$B := \left(1 + \frac{r}{\rho(x)}\right)^\alpha \varphi_2(x,r)^{-1} \int_{\delta_0}^\infty \varphi_1(x,t) \frac{1}{\varphi_1(x,t)} \frac{\|f\|_{L^p(B(x_0,t))}}{t^{n/p}} \frac{dt}{t}$$

$$\leq \varphi_2(x,r)^{-1} \int_{\delta_0}^{\infty} \Big(1 + \frac{t}{\rho(x)}\Big)^{\alpha} \varphi_1(x,t) \frac{1}{\varphi_1(x,t)} \frac{\|f\|_{L^p(B(x_0,t))}}{t^{n/p}} \frac{dt}{t}$$

$$\leq \varphi_2(x,r)^{-1} \int_{\delta_0}^{\infty} \|f\|_{VM_{p,\varphi_1}^{\alpha,V}} \varphi_1(x,t) \frac{dt}{t^1}$$

$$\leq c_{\delta_0} \varphi_2(x,r)^{-1} \|f\|_{VM_{p,\varphi_1}^{\alpha,V}}$$

$$\leq c_{\delta_0} \varphi_2(x,r)^{-1} \Big(1 + \frac{r}{\rho(x)}\Big)^{\alpha} \|f\|_{VM_{p,\varphi_1}^{\alpha,V}}.$$

By the definition of $\Omega_{p,1}^{\alpha,V}$, it suffices to choose r small enough that

$$\sup_{x\in\mathbb{R}^n} \Big(1 + \frac{r}{\rho(x)}\Big)^{\alpha} \varphi_2(x,r)^{-1} \leq \frac{\varepsilon}{2c_{\delta_0}\|f\|_{VM_{p,\varphi_1}^{\alpha,V}}},$$

which implies

$$\sup_{x\in\mathbb{R}^n} C_0 B < \frac{\varepsilon}{2}. \tag{4.8}$$

Let us now put (4.6), (4.7) and (4.8) together to yield the inequality (4.5), which completes the proof. $\qquad\square$

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