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G-CONVERGENCE OF ELLIPTIC OPERATORS IN NON DIVERGENCE FORM IN $\mathbb{R}^n$

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Abstract. The aim of this note is to prove a characterization of the G-limit of a sequence of elliptic operators in non-divergence form. As we consider any dimension, for this class of operators, it is not enough to deal with measurable and bounded coefficients so we need extra regularity assumptions on them.

1. Introduction

The aim of this note is to prove characterization of the G-convergencce of elliptic operators in non divergence form. Let be $\Omega \subset \mathbb{R}^n$ be a bounded open domain and let us consider the Dirichlet problem

$$\begin{gathered} Lu = h \in L^2(\Omega) \\ u \in W^{2,2} \cap W_0^{1,2}\,, \end{gathered} \tag{1.1}$$

where

$$Lu = \sum_{i=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} = \operatorname{tr}(AD^2 u).$$

Where with "tr"we denote the trace of a square matrix; $A = (a_{ij})$ is a measurable $n \times n$ matrix-valued defined on Ω satisfying $a_{ij} = a_{ji}$ and the ellipticity bounds

$$\beta|\xi|^2 \le \langle A\xi, \xi\rangle \le \frac{|\xi|^2}{\beta} \tag{1.2}$$

for a.e. $x \in \Omega$, for all $\xi \in \mathbb{R}^n$ and $0 < \beta \le 1$. We denote by $\mathcal{M}(\beta)$ the set of such functions $A = A(x)$. The Dirichlet problem (1.1) has unique solution in the plane, but differently from elliptic operator in divergence form, we need extra assumptions to guarantee the solvability of (1.1). More precisely

- Talenti in 1966 proved the existence and uniqueness of (1.1) under the assumption that the coefficient matrix A satisfies the Cordes condition, that means that the scattering of the eigenvalue of the matrix is small. We observe that in the plane every 2×2 matrix satisfying (1.2) fulfills automatically the Cordes condition (see [8]).

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- Miranda in 1963 proved the existence and uniqueness of (1.1) under the assumption that the coefficient matrix $A \in W^{1,n}(\Omega)$ (see [5]).
- Chiarenza-Frasca-Longo in 1993 proved the existence and uniqueness of (1.1) under the assumption that the coefficient matrix $A \in VMO$, where VMO stands for the space of vanishing mean oscillation functions (see [4]).

In this note we focus our attention to the case in which $A \in W^{1,n}(\Omega)$. Now, let A_k, $k = 1, \dots$, and A be matrices in $\mathcal{M}(\beta)$ and consider the associated differential operators $L_k u = \operatorname{tr} A_k D^2 u$ and $Lu = \operatorname{tr} AD^2 u$. To define G-convergence, let u_k and u the solutions to the Dirichlet problems

$$\begin{gathered} L_k u_k = f \\ u_k \in W^{2,2} \cap W_0^{1,2} \end{gathered}$$

and

$$\begin{gathered} Lu = f \\ u \in W^{2,2} \cap W_0^{1,2}, \end{gathered}$$

respectively. We say that $\{L_k\}$ G-converges to L, and write $L_k \overset{G}{\to} L$, if for all $f \in L^2(\Omega)$ we have $u_k \rightharpoonup u$ weakly in $W^{2,2}$. The Definition of G-convergence is well posed (see [5]). We recall that the class of non divergence operators whose coefficient matrix are in $\mathcal{M}(\beta)$ is compact respect to the G-convergence. For properties of G-convergence of elliptic operator in non divergence form we refer to [9]

2. Main result

Before stating and proving the main result of this note we need to introduce some auxiliary notion. We have already seen that for $u \in W^{2,2}(\Omega)$ the operator

$$L(u) = \sum_{i,j=1}^{n} a_{i,j}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} = \operatorname{tr}(AD^2 u)$$

is well defined. Now for $p \in L^2$ we denote by N the adjoint operator of L,

$$N(p) = \sum_{i,j=1}^{n} a_{i,j}(x) \frac{\partial^2 a_{ij} p}{\partial x_i \partial x_j}$$

defined by the rule

$$\int_\Omega \varphi N(p) = \int_\Omega p L(\varphi) dx$$

for all $\varphi \in W^{2,2}(\Omega)$. By definition $N(v) = 0$ if and only if

$$\int_\Omega p L(\varphi) dx = 0\,.$$

The reason to study the solutions of the adjoint operator is that they are important for the solvability of $Lu = f$ and for the properties of the Green's function for L, see for example [7].

Now we are in a position to show a reminiscent of the Div-Curl lemma of Murat and Tartar [6], but we emphasize that our class of operator is of non divergence form and we are in any dimension.

Lemma 2.1. *Let L_k and L be operators with coefficients matrices $A_k \in W^{1,n}$ for $k = 1, \dots$. Let $A \in W^{1,n}$ be in $\mathcal{M}(\beta)$, let $p_k \in L^2(\Omega)$ satisfy*

$$N_k(p_k) = 0\,, \tag{2.1}$$

and let $u_k \in W^{2,2}(\Omega)$ be given. If

$$u_k \rightharpoonup u \quad \text{in } W^{2,2}_{\rm loc}(\Omega), \quad \text{and} \tag{2.2}$$

$$p_k A_k \rightharpoonup pA \quad \text{in } L^2_{\rm loc}(\Omega, \mathcal{M}), \tag{2.3}$$

then

$$\operatorname{tr}(p_k A_k D^2 u_k) \to \operatorname{tr}(pAD^2 u) \tag{2.4}$$

in the sense of distributions

Proof. First we note that, for a fixed smooth function φ, the commutator

$$D^2(\varphi u_k) - \varphi D^2 u_k = Du_k \otimes D\varphi + D\varphi \otimes Du_k + u_k D^2\varphi$$

only contains u_k and its first order derivative, hence in view of compact embeddings of $W^{2,2}$ in $W^{1,2}$ and in L^2, we have

$$D^2(\varphi u_k) - \varphi D^2 u_k \to D^2(\varphi u) - \varphi D^2 u$$

strongly in L^2. So we can apply (2.3) to have

$$\int_\Omega p_k A_k[D^2(\varphi u_k) - \varphi D^2 u_k]dx \to \int_\Omega pA[D^2(\varphi u) - \varphi D^2 u]\,. \tag{2.5}$$

Since $N_k(p_k) = 0$, for all k, it follows that

$$\int_\Omega p_k \operatorname{tr}(A_k D^2 \varphi u_k) = \int_\Omega p \operatorname{tr}(AD^2\varphi u) = 0 \tag{2.6}$$

for all $\varphi \in C_0^\infty$. Combining (2.5) and (2.6) we conclude that

$$\int_\Omega \varphi \operatorname{tr}(p_k A_k D^2 u_k) \to \int_\Omega \varphi \operatorname{tr}(pAD^2 u)$$

for all $\varphi \in C_0^\infty$. □

Theorem 2.2. *Let L_k and L be operators with coefficients matrices $A_k \in W^{1,n}$ for $k = 1, \dots$, and let $A \in W^{1,n}$ be in $\mathcal{M}(\beta)$. Assume that $p_k \in L^2(\Omega)$ are solutions to the adjoint operators $N_k(p_k) = 0$. If*

$$p_k \rightharpoonup p \quad \text{in } L^2_{\rm loc}(\Omega), \quad \text{and} \tag{2.7}$$

$$p_k A_k \rightharpoonup pA \quad \text{in } L^2_{\rm loc}(\Omega, \mathcal{M}), \tag{2.8}$$

where $p(x) > 0$ a.e. in Ω. Then $L_k \xrightarrow{G} L$.

Proof. Let us fix $f \in L^2(\Omega)$. By Miranda's Theorem we know that the Dirichlet problem

$$\begin{gathered} L_k u = f \in L^2(\Omega) \\ u_k \in W^{2,2} \cap W_0^{1,2} \end{gathered} \tag{2.9}$$

has a solution u_k. It is well known that the sequence u_k is bounded in $W^{2,2}$, hence there exists a subsequence such that

$$u_{k_h} \rightharpoonup u \tag{2.10}$$

weakly in $W^{2,2}$. Letting $\varphi \in C_0^\infty(\Omega)$, we have

$$\int_\Omega \varphi p_{k_h} f dx = \int_\Omega \varphi \operatorname{tr}(p_{k_h} A_{k_h} D^2 u_{k_h}) dx, \quad \forall h\,.$$

Passing to the limit in the previous expression and using Lemma 2.1, we obtain

$$\int_\Omega \varphi p f dx = \int_\Omega \varphi \operatorname{tr}(pAD^2 u) dx, \quad \forall h,$$

and hence $pf = \operatorname{tr}(pAD^2u)$ a.e. in Ω. □

3. Final comments

Theorem 2.2 was proved in the plane with the only ellipticity condition by [3]. As already explained in the introduction to have the notion of G-convergence in more than two dimension we need more assumption on the coefficients. In [9] they prove Theorem 2.2 under the extra-assumption that the A_k, $k = 1, \dots$, and A, satisfy the Cordes conditions; that is the operator is almost the Laplace operator as these coefficients matrix are "near "the identity matrix. We found these assumption too restrictive so we prove Theorem 2.2 requiring regularity on the coefficients. Similar results were proved in the planar case (see [1] and in dimension 3 (see [2] (requiring that the coefficients matrix be in VMO). However they use a different notion of G-convergence based on the fact that if the datum in the Dirichlet problem has less regularity they are able to solve uniquely the Dirichlet problem (1.1) in a suitable Sobolev space.

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References

[1] T. Alberico C. Capozzoli, L. D'Onofrio; *G-convergence for non-divergence second order operators in the plane*, Diff. Int. Equ. A, v. 26 (2013), pages 1127-1138.
[2] T. Alberico C. Capozzoli, L. D'Onofrio, R. Schiattarella; *G-convergence for non-divergence second order operators with VMO coefficients in* $\mathbb{R}^3$, Discrete and Continuous Dynamical Systems Series S, v. 12 (2019), pages 129-137.
[3] L. D'Onofrio, L. Greco; *A counter-example in G-convergence of non-divergence elliptic operators*, Proc. Roy. Math. Society Edinburgh Sect. A, v. 133 (2003), pages 1299-1310.
[4] F. Chiarenza, M. Frasca, P. Longo; *$W^{2,p}$-solvability of the Dirichlet problem for non-divergence elliptic equations with VMO- coefficients*, Trans. Amer. Math. Society, v. 336 (1993), pages 841-853.
[5] C. Miranda; *Sulle equazioni ellettiche del secondo ordine di tipo non variazionale a coefficienti discontinui*, Annali di Matematica Pura ed Applicata, v. 63 (1963), pages 355-386.
[6] F. Murat, L. Tartar; *H-convergence*, Topics in the mathematical modelling of composite materials, Progress in Nonlinear Differential Equations and Applications, v. 31 (1997), pages 21-43.
[7] P. Syogren; *On the adjoint of an elliptic linear differential operator and its potential theory*, Ark. Mat., v. 11 (1973), pages 153-165.
[8] G. Talenti; *Sopra una classe di equazioni ellittiche a coefficienti misurabili*, Annali di Matematica Pura ed Applicata, v. 69 (1965), pages 285-304 .
[9] V. V. Zhikov, M. M. Sirazhudinov; *On G-compactness of nondivergence elliptic operators of second order*, Math USSR Izv., v. 19 (1982), pages 27-40.

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