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SOLUTIONS FOR THE NAVIER-STOKES EQUATIONS WITH CRITICAL AND SUBCRITICAL FRACTIONAL DISSIPATION IN LEI-LIN AND LEI-LIN-GEVREY SPACES

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Abstract. In this article, we prove the existence of a unique global solution for the critical case of the generalized Navier-Stokes equations in Lei-Lin and Lei-Lin-Gevrey spaces, by assuming that the initial data is small enough. Moreover, we obtain a unique local solution for the subcritical case of this system, for any initial data, in these same spaces. It is important to point out that our main result is obtained by discussing some properties of the solutions for the heat equation with fractional dissipation.

1. Introduction

This work studies the existence of global and local in time solutions for the incompressible Navier-Stokes equations in Lei-Lin-Gevrey and Lei-Lin spaces $\mathcal{X}^s_{a,\sigma}(\mathbb{R}^3)$,

$$
\begin{gathered}
u_t + (-\Delta)^\alpha u + u\cdot\nabla u + \nabla p = 0, \quad x\in\mathbb{R}^3, \quad t>0,\\
\operatorname{div} u = 0, \quad x\in\mathbb{R}^3, \quad t>0,\\
u(x,0) = u_0(x), \quad x\in\mathbb{R}^3,
\end{gathered}
\tag{1.1}
$$

where $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t)) \in \mathbb{R}^3$ denotes the incompressible velocity field, and $p(x,t)\in\mathbb{R}$ the hydrostatic pressure; see [1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 18, 20, 26, 28, 31] and their references. Here $(-\Delta)^\alpha$, with $\alpha\geq 1/2$, is the fractional Laplacian, see (2.1). The initial data for the velocity field, given by u_0 in (1.1), is assumed to be divergence free, i.e., $\operatorname{div} u_0 = 0$.

The fractional Laplacian $(-\Delta)^\alpha$ has been studied in many works in the literature (see, for instance, [32, 34] and references therein). To cite some models involving this kind of operator, we refer: Diffusion-reaction, Quasi-geostrophic, Cahn-Hilliard, Porous medium, Schrödinger, Ultrasound, Magnetohydrodynamics (MHD), Magnetohydrodynamics-α (MHD-α) and Navier-Stokes itself (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 17, 18, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 33, 35] and references therein). It is important to recall that, by applying the Spectral Theorem, $(-\Delta)^\alpha$ assumes the diagonal form in the Fourier variable,

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i.e., this is a Fourier multiplier operator with symbol $|\xi|^{2\alpha}$ (which extends Fourier multiplier property of $-\Delta$).

It is also important to notice that system (1.1) becomes the usual Navier-Stokes equations by replacing the fractional Laplacian operator by the usual one. More precisely, these last equations are

$$\begin{gathered} u_t + u\cdot\nabla u + \nabla p = \Delta u, \quad x\in\mathbb{R}^3, \quad t>0, \\ \operatorname{div} u = 0, \quad x\in\mathbb{R}^3, \quad t>0, \\ u(x,0) = u_0(x), \quad x\in\mathbb{R}^3, \end{gathered}$$

and play an important role in continuum mechanics. It is necessary to show that singularities for the solutions of theses equations are not present in finite time (from smooth initial data with finite energy) in order to make this system well-posed. This is one of the most important open problems in Nonlinear Analysis. Thus, the fractional Laplacian must be an interesting mathematical tool to understand better this problem. In fact, Wu [35] showed that the generalized Navier-Stokes equations (1.1) admit global classical solution provided that the initial data u_0 is smooth and $\alpha \geq \frac{5}{4}$. More precisely, [35] assumes that $\alpha \geq 5/4$ and $u_0 \in H^s(\mathbb{R}^3)$, with $s > 2\alpha$, to obtain a unique global classical solution for (1.1) (see also [1, 4, 5, 7, 8, 9, 15, 16, 17, 18, 27, 28, 29, 31, 35] and references therein).

Physically, (1.1) are the equations that describe the motion of a fluid with internal friction interaction and such motion is a chain of particles that are connected by elastic springs (see, for example, [32] for more details).

Recently, some authors have published works that study the usual Navier-Stokes equations and their extensions in Lei-Lin and Lei-Lin-Gevrey spaces (see [1, 4, 7, 9, 18, 24, 28] and references therein). For example Melo, Souza and Santos [28], by studying the MHD-α equations, proved the existence of a unique global solution in $C_b([0,\infty);\mathcal{X}^s_{a,\sigma}(\mathbb{R}^3))$. In addition, [28] presents analyticity and decay rates for global solutions in this same context (for more information, see [28] and references therein).

Another motivating work was written by Melo and Rocha [24, 30]. This article proves the local existence, as well as blow-up criteria, for solutions of the generalized Magnetohydrodynamics equations in $[C_T(\mathcal{X}^s(\mathbb{R}^3))\cap L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))]\times[C_T(\mathcal{X}^s(\mathbb{R}^3))\cap L^1_T(\mathcal{X}^{s+2\beta}(\mathbb{R}^3))]$, where the fractional dissipations α and β belong to the interval $(1/2,1]$, and $s\in(\max\{1-2\alpha, 1-2\beta, \frac{\alpha(1-2\beta)}{\beta}, \frac{\beta(1-2\alpha)}{\alpha}\},0]$ (see [24, 30] for more information on the blow-up criteria proved in these works).

Motivated by these works, we present global and local solutions for the Navier-Stokes equations (1.1), with fractional dissipation of order $\alpha \geq 1/2$, in Lei-Lin and Lei-Lin-Gevrey spaces (we refer to [8, 28, 30, 33] and papers therein). Moreover, it is worth to point out that we have adapted some of the ideas applied in the paper [17].

Our main result proves the existence and uniqueness of solutions for the Navier-Stokes equations (1.1) in Lei-Lin-Gevrey and Lei-Lin spaces and can be written as follows.

Theorem 1.1. *The following statements hold:*

(i) Critical Case: global solution. Assume that $\alpha = 1/2$, $(a,s,\sigma)\in\big((0,+\infty)\times[-1,0)\times(1,+\infty)\big)\cup\big([0,+\infty)\times\{0\}\times[1,+\infty)\big)$ *and* $u_0\in\mathcal{X}^s_{a,\sigma}(\mathbb{R}^3)$. *Then there is a constant* $\mathbb{C}_{a,\sigma,s}>0$ *such that if* $\|u_0\|_{\mathcal{X}^s_{a,\sigma}} < \mathbb{C}_{a,\sigma,s}$, *then for all instant* $T>0$ *there*

is a unique global solution

$$u \in C_T(\mathcal{X}^s_{a,\sigma}(\mathbb{R}^3)) \cap L^1_T(\mathcal{X}^{s+1}_{a,\sigma}(\mathbb{R}^3))$$

to the Navier-Stokes equations (1.1)*, it satisfies*

$$\|u\|_{L^\infty_T(\mathcal{X}^s_{a,\sigma})} + \|u\|_{L^1_T(\mathcal{X}^{s+1}_{a,\sigma})} \leq 4\|u_0\|_{\mathcal{X}^s_{a,\sigma}}.$$

Moreover, $u \in L^p_T(\mathcal{X}^{s+\frac{1}{p}}_{a,\sigma}(\mathbb{R}^3))$ *for all* $p \geq 1$.

(ii) Subcritical Case: local solution. Assume that $\alpha > 1/2$, $(a, s, \sigma) \in \big((0, +\infty) \times [-1, 0) \times (1, +\infty)\big) \cup \big([0, +\infty) \times \{0\} \times [1, +\infty)\big)$*, and* $u_0 \in \mathcal{X}^s_{a,\sigma}(\mathbb{R}^3)$*. Then there exist an instant* $\overline{T} > 0$ *and a unique local solution*

$$u \in C_{\overline{T}}(\mathcal{X}^s_{a,\sigma}(\mathbb{R}^3)) \cap L^1_{\overline{T}}(\mathcal{X}^{s+2\alpha}_{a,\sigma}(\mathbb{R}^3))$$

to the Navier-Stokes equations (1.1)*, such that*

$$\|u\|_{L^\infty_{\overline{T}}(\mathcal{X}^s_{a,\sigma})} + \|u\|_{L^1_{\overline{T}}(\mathcal{X}^{s+2\alpha}_{a,\sigma})} \leq 4\|u_0\|_{\mathcal{X}^s_{a,\sigma}}.$$

Furthermore, $u \in L^p_{\overline{T}}(\mathcal{X}^{s+\frac{2\alpha}{p}}_{a,\sigma}(\mathbb{R}^3))$ *for all* $p \geq 1$.

Let us recall that the Navier-Stokes equations (1.1) are invariant under the change of time and space scaling. More precisely, if u and p solve (1.1); then, for any $\lambda > 0$, the functions

$$u_\lambda(x,t) = \lambda^{2\alpha-1}u(\lambda x, \lambda^{2\alpha}t), \quad p_\lambda(x,t) = \lambda^{4\alpha-2}p(\lambda x, \lambda^{2\alpha}t), \quad u_0^\lambda(x) = \lambda^{2\alpha-1}u_0(\lambda x)$$

also solve (1.1). By observing this same scaling, we say that $(X, \|\cdot\|)$ is a critical space for the Navier-Stokes equations (1.1) (see [19] and references therein for more details) if $\|f_\lambda\| = \|f\|$, for all $\lambda > 0$, where $f_\lambda(x) = \lambda^{2\alpha-1}f(\lambda x)$. Then, it is easy to check that $\mathcal{X}^{1-2\alpha}(\mathbb{R}^3)$ is a critical space for (1.1) [9]. In particular, for $\alpha = 1/2$, one has that $\mathcal{X}^0(\mathbb{R}^3)$ is also a critical space for (1.1).

It is important to point out that Theorem 1.1 presents some information for solutions of system (1.1) in the critical Lei-Lin space $\mathcal{X}^0(\mathbb{R}^3)$ and the specific Lei-Lin-Gevrey space $\mathcal{X}^{1-2\alpha}_{a,\sigma}(\mathbb{R}^3)$. More precisely, we have the statements below:

- Theorem 1.1 (i) (in Lei-Lin spaces) can be rewritten as follows: Assume that $\alpha = 1/2$, $a = 0$ and $u_0 \in \mathcal{X}^0(\mathbb{R}^3)$. Thus, there is a constant $\mathbb{C} > 0$ such that if $\|u_0\|_{\mathcal{X}^0} < \mathbb{C}$; then for all instant $T > 0$ there is a unique global solution

$$u \in C_T(\mathcal{X}^0(\mathbb{R}^3)) \cap L^1_T(\mathcal{X}^1(\mathbb{R}^3))$$

to the Navier-Stokes equations (1.1), and it satisfies

$$\|u\|_{L^\infty_T(\mathcal{X}^0)} + \|u\|_{L^1_T(\mathcal{X}^1)} \leq 4\|u_0\|_{\mathcal{X}^0}.$$

(See [9] for the Quasi-geostrophic case). Under the hypotheses above, it follows that $u \in L^p_T(\mathcal{X}^{1/p}(\mathbb{R}^3))$, for all $p \geq 1$.

- Theorem 1.1 (ii), with $s = 1 - 2\alpha$, can be rewritten as follows: Assume that $\alpha \in (\frac{1}{2}, 1]$, $a > 0, \sigma > 1$, and $u_0 \in \mathcal{X}^{1-2\alpha}_{a,\sigma}(\mathbb{R}^3)$. Then there exist an instant $\overline{T} > 0$ and a unique local solution

$$u \in C_{\overline{T}}(\mathcal{X}^{1-2\alpha}_{a,\sigma}(\mathbb{R}^3)) \cap L^1_{\overline{T}}(\mathcal{X}^1_{a,\sigma}(\mathbb{R}^3))$$

to the Navier-Stokes equations (1.1) such that

$$\|u\|_{L^\infty_{\overline{T}}(\mathcal{X}^{1-2\alpha}_{a,\sigma})} + \|u\|_{L^1_{\overline{T}}(\mathcal{X}^1_{a,\sigma})} \leq 4\|u_0\|_{\mathcal{X}^{1-2\alpha}_{a,\sigma}}.$$

See [30] for a study of the generalized Magnetohydrodynamics equations. In this case, one infers that $u \in L^p_T(\mathcal{X}^{1+\frac{2(1-p)\alpha}{p}}_{a,\sigma}(\mathbb{R}^3))$, for all $p \geq 1$.

Remark 1.2. To obtain mild solutions for the Navier-Stokes equations (1.1), we apply a standard fixed point theorem (see Lemma 3.2). To this end, we need to prove the continuity of a bilinear operator (see (4.5)) related to the nonlinear term of this same system (1.1) (see proof of Theorem 1.1). Since this statement is the key point in the proof of our main result, we present some preliminary lemmas that are useful to achieve this goal. More specifically, these results show us how to estimate the solutions of the heat equation (see systems (3.2) and (4.6)) in Lei-Lin-Gevrey and Lei-Lin spaces through its nonhomogeneous term and initial data (see Lemma 3.1), and help us to choose the values for a, σ and s such that $\mathcal{X}^s_{a,\sigma}(\mathbb{R}^3) \hookrightarrow \mathcal{X}^0_{\frac{a}{\sigma},\sigma}(\mathbb{R}^3)$ (see Lemma 3.3 and (4.7)). At last, it is also important to emphasize that this path is taken because of the advantages of the use of Fourier analysis and some usual techniques.

The outline of this article is as follows: Section 2 presents the most important definitions and notations that are applied in this paper, Section 3 presents some lemmas that play an important role in this work, and Section 4 presents the proof of our main result (see Theorem 1.1).

2. Notation

In this section, we list the most important definitions and notation that are used throughout this paper.

- $S'(\mathbb{R}^3)$ is the space of tempered distributions.
- The Fourier transform and its inverse are defined by

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) := \int_{\mathbb{R}^3} e^{-i\xi\cdot x} f(x)\,dx,$$
$$\mathcal{F}^{-1}(g)(x) := (2\pi)^{-3} \int_{\mathbb{R}^3} e^{i\xi\cdot x} g(\xi)\,d\xi,$$

- The fractional Laplacian $(-\Delta)^\alpha$ (see [34]), for $\alpha \geq 1/2$, is defined by

$$\mathcal{F}[(-\Delta)^\alpha f](\xi) = |\xi|^{2\alpha} \hat{f}(\xi), \quad \forall \xi \in \mathbb{R}^3, \tag{2.1}$$

where $f \in S'(\mathbb{R}^3)$ and $\hat{f} \in L^1_{\rm loc}(\mathbb{R}^3)$.

- The tensor product is

$$f \otimes g := (g_1 f, g_2 f, g_3 f),$$

where $f = (f_1, f_2, f_3)$ and $g = (g_1, g_2, g_3) \in S'(\mathbb{R}^3)$.

- Let $s \in \mathbb{R}$. The Lei-Lin spaces are

$$\mathcal{X}^s(\mathbb{R}^3) := \{f \in S'(\mathbb{R}^3) : \hat{f} \in L^1_{\rm loc}(\mathbb{R}^3) \text{ and } \int_{\mathbb{R}^3} |\xi|^s |\widehat{f}(\xi)|\,d\xi < \infty\}$$

and the $\mathcal{X}^s(\mathbb{R}^3)$-norm is

$$\|f\|_{\mathcal{X}^s} = \int_{\mathbb{R}^3} |\xi|^s |\widehat{f}(\xi)|\,d\xi.$$

- Let $a > 0, \sigma \geq 1$, and $s \in \mathbb{R}$. The Lei-Lin-Gevrey spaces are

$$\mathcal{X}^s_{a,\sigma}(\mathbb{R}^3) := \{f \in S'(\mathbb{R}^3) : \hat{f} \in L^1_{\rm loc}(\mathbb{R}^3) \text{ and } \int_{\mathbb{R}^3} |\xi|^s e^{a|\xi|^{1/\sigma}} |\widehat{f}(\xi)|\,d\xi < \infty\}$$

and the $\mathcal{X}^s_{a,\sigma}(\mathbb{R}^3)$-norm is

$$\|f\|_{\mathcal{X}^s_{a,\sigma}} = \int_{\mathbb{R}^3} |\xi|^s e^{a|\xi|^{1/\sigma}} |\widehat{f}(\xi)|\, d\xi.$$

• Let $s \in \mathbb{R}$. The homogeneous Sobolev space is

$$\dot{H}^s(\mathbb{R}^3) = \{f \in S'(\mathbb{R}^3) : \widehat{f} \in L^1_{\rm loc}(\mathbb{R}^3) \text{ and } \int_{\mathbb{R}^3} |\xi|^{2s} |\widehat{f}(\xi)|^2\, d\xi < \infty\}$$

and the $\dot{H}^s(\mathbb{R}^3)$-norm is

$$\|f\|_{\dot{H}^s} := \Big(\int_{\mathbb{R}^3} |\xi|^{2s} |\widehat{f}(\xi)|^2\, d\xi \Big)^{1/2}.$$

• Let $a > 0, \sigma \geq 1$ and $s \in \mathbb{R}$. The Sobolev-Gevrey space is

$$\dot{H}^s_{a,\sigma}(\mathbb{R}^3) = \{f \in S'(\mathbb{R}^3) : \widehat{f} \in L^1_{\rm loc}(\mathbb{R}^3) \text{ and } \int_{\mathbb{R}^3} |\xi|^{2s} e^{2a|\xi|^{1/\sigma}} |\widehat{f}(\xi)|^2\, d\xi < \infty\}$$

and the $\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$-norm is

$$\|f\|_{\dot{H}^s_{a,\sigma}} := \Big(\int_{\mathbb{R}^3} |\xi|^{2s} e^{2a|\xi|^{1/\sigma}} |\widehat{f}(\xi)|^2\, d\xi \Big)^{1/2}.$$

• Let $T > 0$, $(X, \|\cdot\|_X)$ a normed space and $I \subseteq \mathbb{R}$ an interval. We define

$$C(I; X) = \{f : I \to X \text{ continuous function}\},$$

and the $C(I; X)$-norm

$$\|f\|_{L^\infty(I;X)} := \sup_{t \in I} \{\|f(t)\|_X\}.$$

We denote $C_T(X) = C([0,T]; X)$ and $\|\cdot\|_{L^\infty_T(X)} = \|\cdot\|_{L^\infty([0,T];X)}$.
• Let $1 \leq p < \infty$, $T > 0$, $(X, \|\cdot\|_X)$ a normed space and $I \subseteq \mathbb{R}$ an interval. We define

$$L^p(I; X) = \{f : I \to X \text{ mensurable function} : \int_I \|f(t)\|^p_X\, dt < \infty\},$$

and the $L^p(I; X)$-norm is given by

$$\|f\|_{L^p(I;X)} := \Big(\int_I \|f(t)\|^p_X\, dt \Big)^{1/p}.$$

We denote $L^p_T(X) = L^p([0,T]; X)$.
• The constants in this paper may change their values from line to line without change of notation. For example, C_q denotes any constant that depends on q and C is always a positive constant.

3. Preliminary lemmas

The most important result presented in this article, Theorem 1.1, is a consequence of a study based on the solutions for the following heat equation with fractional dissipation and initial data $v_0 \in \mathcal{X}^s_{a,\sigma}(\mathbb{R}^3)$ (with $a \geq 0, \sigma \geq 1$ and $s \in \mathbb{R}$):

$$\begin{gathered} v_t + (-\Delta)^\alpha v = f, \quad t \in (0,T]; \\ v(\cdot, 0) = v_0, \end{gathered} \tag{3.1}$$

where $f \in L^1_T(\mathcal{X}^s_{a,\sigma}(\mathbb{R}^3))$ (provided that $T > 0$ is arbitrary).

It is worth to point out that the main ideas that will be presented below were firstly established by Orf [31] and generalized a few years later by Guterres, Melo, Rocha, and Santos [17], in the case of Sobolev-Gevrey spaces. See [36, Lemma 2.6] for examples of particular cases.

Lemma 3.1. *Assume that* $a \geq 0, \sigma \geq 1$, $T > 0$, $s \in \mathbb{R}$, $\alpha \in \mathbb{R}$, $f \in L^1_T(\mathcal{X}^s_{a,\sigma}(\mathbb{R}^3))$ *and* $v_0 \in \mathcal{X}^s_{a,\sigma}(\mathbb{R}^3)$. *Consider that* $v \in C_T(S'(\mathbb{R}^3))$ *solves the system*

$$\begin{gathered} v_t + (-\Delta)^\alpha v = f, \quad x \in \mathbb{R}^3, t \in (0,T]; \\ v(\cdot,0) = v_0, \quad x \in \mathbb{R}^3. \end{gathered} \tag{3.2}$$

Then, $v \in C_T(\mathcal{X}^s_{a,\sigma}(\mathbb{R}^3)) \cap L^p_T(\mathcal{X}^{s+\frac{2\alpha}{p}}_{a,\sigma}(\mathbb{R}^3))$ *for all* $p \geq 1$. *Furthermore,*

- (i) $\|v\|_{L^\infty_T(\mathcal{X}^s_{a,\sigma})} \leq \|v_0\|_{\mathcal{X}^s_{a,\sigma}} + \|f\|_{L^1_T(\mathcal{X}^s_{a,\sigma})}$,
- (ii) $\|v\|_{L^p_T(\mathcal{X}^{s+\frac{2\alpha}{p}}_{a,\sigma})} \leq \|v_0\|_{\mathcal{X}^s_{a,\sigma}} + \|f\|_{L^1_T(\mathcal{X}^s_{a,\sigma})}$.

Proof. At first, by applying the heat semigroup $e^{-(t-\tau)(-\Delta)^\alpha}$ (where $0 \leq \tau \leq t \leq T$) to the first equation of system (3.2), using the Fourier transform and integrating over $[0,t]$ the result obtained, one concludes that

$$|\widehat{v}(t)| \leq e^{-t|\xi|^{2\alpha}}|\widehat{v}_0| + \int_0^t e^{-(t-\tau)|\xi|^{2\alpha}}|\widehat{f}(\tau)|\,d\tau. \tag{3.3}$$

Thereby, we can write

$$|\widehat{v}(t)| \leq |\widehat{v}_0| + \int_0^T |\widehat{f}(\tau)|\,d\tau.$$

Now, by multiplying the inequality above by $|\xi|^s e^{a|\xi|^{1/\sigma}}$, we obtain

$$|\xi|^s e^{a|\xi|^{1/\sigma}}|\widehat{v}(t)| \leq |\xi|^s e^{a|\xi|^{1/\sigma}}|\widehat{v}_0| + |\xi|^s e^{a|\xi|^{1/\sigma}}\int_0^T |\widehat{f}(\tau)|\,d\tau.$$

Applying the $L^1(\mathbb{R}^3)$-norm, we have

$$\|v(t)\|_{\mathcal{X}^s_{a,\sigma}} \leq \|v_0\|_{\mathcal{X}^s_{a,\sigma}} + \|f\|_{L^1_T(\mathcal{X}^s_{a,\sigma})}, \quad \forall t \in [0,T].$$

As a result, one concludes that

$$\|v\|_{L^\infty_T(\mathcal{X}^s_{a,\sigma})} \leq \|v_0\|_{\mathcal{X}^s_{a,\sigma}} + \|f\|_{L^1_T(\mathcal{X}^s_{a,\sigma})}. \tag{3.4}$$

This proves (i) and, furthermore, this shows that $v \in C_T(\mathcal{X}^s_{a,\sigma}(\mathbb{R}^3))$ (it is enough to recall that the Fourier transform $\mathcal{F}$ is continuous, $v \in C_T(S'(\mathbb{R}^3))$ and apply Dominated Convergence Theorem) since $v_0 \in \mathcal{X}^s_{a,\sigma}(\mathbb{R}^3)$ and $f \in L^1_T(\mathcal{X}^s_{a,\sigma}(\mathbb{R}^3))$.

To show (ii), we multiply (3.3) by $|\xi|^{s+2\alpha}e^{a|\xi|^{1/\sigma}}$ and obtain that

$$\begin{aligned} &|\xi|^{s+2\alpha}e^{a|\xi|^{1/\sigma}}|\widehat{v}(t)| \\ &\leq |\xi|^{s+2\alpha}e^{a|\xi|^{1/\sigma}}e^{-t|\xi|^{2\alpha}}|\widehat{v}_0| + |\xi|^{s+2\alpha}e^{a|\xi|^{1/\sigma}}\int_0^t e^{-(t-\tau)|\xi|^{2\alpha}}|\widehat{f}(\tau)|\,d\tau. \end{aligned}$$

By using the $L^1([0,T])$-norm, one has

$$\int_0^T |\xi|^{s+2\alpha}e^{a|\xi|^{1/\sigma}}|\widehat{v}(t)|\,dt \leq |\xi|^s e^{a|\xi|^{1/\sigma}}|\widehat{v}_0| + |\xi|^{s+2\alpha}e^{a|\xi|^{1/\sigma}}\int_0^T [e^{-t|\xi|^{2\alpha}}] * [|\widehat{f}(t)|]\,dt.$$

Apply Young's inequality we obtain

$$\int_0^T |\xi|^{s+2\alpha}e^{a|\xi|^{1/\sigma}}|\widehat{v}(t)|\,dt \leq |\xi|^s e^{a|\xi|^{1/\sigma}}|\widehat{v}_0| + |\xi|^s e^{a|\xi|^{1/\sigma}}\int_0^T |\widehat{f}(t)|\,dt.$$

By taking the $L^1(\mathbb{R}^3)$-norm, it follows that

$$\|v\|_{L^1_T(\mathcal{X}^{s+2\alpha}_{a,\sigma})} \le \|v_0\|_{\mathcal{X}^s_{a,\sigma}} + \|f\|_{L^1_T(\mathcal{X}^s_{a,\sigma})}. \tag{3.5}$$

On the other hand, Hölder's inequality implies that

$$\begin{aligned}\|v\|_{\mathcal{X}^{s+\frac{2\alpha}{p}}_{a,\sigma}} &= \int_{\mathbb{R}^3} |\xi|^{s+\frac{2\alpha}{p}} e^{a|\xi|^{1/\sigma}} |\widehat{v}(\xi)|\, d\xi \\ &\le \Big(\int_{\mathbb{R}^3} |\xi|^s e^{a|\xi|^{1/\sigma}} |\widehat{v}(\xi)|\, d\xi\Big)^{1-\frac{1}{p}} \Big(\int_{\mathbb{R}^3} |\xi|^{s+2\alpha} e^{a|\xi|^{1/\sigma}} |\widehat{v}(\xi)|\, d\xi\Big)^{1/p}.\end{aligned}$$

Hence, one has

$$\|v\|_{\mathcal{X}^{s+\frac{2\alpha}{p}}_{a,\sigma}} \le \|v\|^{1-\frac{1}{p}}_{\mathcal{X}^s_{a,\sigma}} \|v\|^{1/p}_{\mathcal{X}^{s+2\alpha}_{a,\sigma}}.$$

As a result, we can write

$$\begin{aligned}\|v\|^p_{L^p_T(\mathcal{X}^{s+\frac{2\alpha}{p}}_{a,\sigma})} &= \int_0^T \|v\|^p_{\mathcal{X}^{s+\frac{2\alpha}{p}}_{a,\sigma}}\, dt \le \int_0^T \|v\|^{p-1}_{\mathcal{X}^s_{a,\sigma}} \|v\|_{\mathcal{X}^{s+2\alpha}_{a,\sigma}}\, dt \\ &\le \|v\|^{p-1}_{L^\infty_T(\mathcal{X}^s_{a,\sigma})} \|v\|_{L^1_T(\mathcal{X}^{s+2\alpha}_{a,\sigma})}.\end{aligned}$$

By (3.4) and (3.5), one infers that

$$\|v\|_{L^p_T(\mathcal{X}^{s+\frac{2\alpha}{p}}_{a,\sigma})} \le \|v_0\|_{\mathcal{X}^s_{a,\sigma}} + \|f\|_{L^1_T(\mathcal{X}^s_{a,\sigma})},$$

for all $p \ge 1$. This proves (ii). As a result, we can conclude that $v \in L^p_T(\mathcal{X}^{s+\frac{2\alpha}{p}}_{a,\sigma}(\mathbb{R}^3))$ since $f \in L^1_T(\mathcal{X}^s_{a,\sigma}(\mathbb{R}^3))$ and $v_0 \in \mathcal{X}^s_{a,\sigma}(\mathbb{R}^3)$. $\square$

In the proof of Theorem 1.1, we shall apply the following Fixed Point Theorem.

Lemma 3.2 ([13])**.** *Let $(X, \|\cdot\|)$ be a Banach space and $B : X \times X \to X$ a continuous bilinear operator, i.e., there exists a positive constant C such that*

$$\|B(w,v)\| \le C\|w\|\|v\|, \quad \forall w, v \in X. \tag{3.6}$$

Then, for each $x_0 \in X$ that satisfies $4C\|x_0\| < 1$, the equation $a = x_0 + B(a,a)$, with $a \in X$, admits a solution $u \in X$. Moreover, u solves the inequality $\|u\| \le 2\|x_0\|$ and it is the only one such that $\|u\| \le \frac{1}{2C}$.

The next lemmas will be useful in the proof of our main result, Theorem 1.1.

Lemma 3.3 ([28])**.** *Let a, σ and s be real numbers such that $(a,s,\sigma) \in \big((0,+\infty) \times (-\infty,0) \times (1,+\infty)\big) \cup \big([0,+\infty) \times \{0\} \times [1,+\infty)\big)$. Assume that $f \in \mathcal{X}^s_{a,\sigma}(\mathbb{R}^3)$. Then, $f \in \mathcal{X}^0_{\frac{a}{\sigma},\sigma}(\mathbb{R}^3)$. Moreover, there exists a positive constant $C_{a,s,\sigma}$ such that*

$$\|f\|_{\mathcal{X}^0_{\frac{a}{\sigma},\sigma}} \le C_{a,s,\sigma} \|f\|_{\mathcal{X}^s_{a,\sigma}}.$$

Lemma 3.4 ([30])**.** *Let $a \ge 0$, $\sigma \ge 1$ and $s \ge -1$. Assume that $f, g \in \mathcal{X}^{s+1}_{a,\sigma}(\mathbb{R}^3) \cap \mathcal{X}^0_{\frac{a}{\sigma},\sigma}(\mathbb{R}^3)$. Then, $fg \in \mathcal{X}^{s+1}_{a,\sigma}(\mathbb{R}^3)$. Moreover, there is a positive constant C_s such that*

$$\|fg\|_{\mathcal{X}^{s+1}_{a,\sigma}} \le C_s[\|f\|_{\mathcal{X}^0_{\frac{a}{\sigma},\sigma}} \|g\|_{\mathcal{X}^{s+1}_{a,\sigma}} + \|f\|_{\mathcal{X}^{s+1}_{a,\sigma}} \|g\|_{\mathcal{X}^0_{\frac{a}{\sigma},\sigma}}].$$

4. Proof of main result

Proof of Theorem 1.1. First of all, it is necessary to apply the operator $e^{-(t-\tau)(-\Delta)^\alpha}$ (with $\tau \in [0,t]$) to the first equation in (1.1) to obtain

$$e^{-(t-\tau)(-\Delta)^\alpha} u_\tau + e^{-(t-\tau)(-\Delta)^\alpha} P(u \cdot \nabla u) + e^{-(t-\tau)(-\Delta)^\alpha} (-\Delta)^\alpha u = 0, \tag{4.1}$$

where P is the usual Helmholtz's projector. It is known that this operator satisfies

$$|\mathcal{F}[P(f)](\xi)| \le |\widehat{f}(\xi)|, \quad \forall \xi \in \mathbb{R}^3. \tag{4.2}$$

Integrate (4.1) over $[0,t]$ to obtain

$$u(t) = e^{-t(-\Delta)^\alpha} u_0 - \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} P(u \cdot \nabla u)(\tau)\, d\tau, \tag{4.3}$$

On the other hand, (4.3) implies

$$u(t) = e^{-t(-\Delta)^\alpha} u_0 + B(u,u)(t), \tag{4.4}$$

where

$$B(w,v)(t) = -\int_0^t e^{-(t-\tau)(-\Delta)^\alpha} P(v \cdot \nabla w)(\tau)\, d\tau, \quad \forall w, v \in \mathcal{X}_T. \tag{4.5}$$

Here $\mathcal{X}_T := C_T(\mathcal{X}^s_{a,\sigma}(\mathbb{R}^3)) \cap L^1_T(\mathcal{X}^{s+2\alpha}_{a,\sigma}(\mathbb{R}^3))$ (for any arbitrary $T > 0$) denotes Banach space endowed with the norm

$$\|g\|_{\mathcal{X}_T} := \|g\|_{L^\infty_T(\mathcal{X}^s_{a,\sigma})} + \|g\|_{L^1_T(\mathcal{X}^{s+2\alpha}_{a,\sigma})}, \quad \forall g \in \mathcal{X}_T.$$

Notice that, from (4.5) it is easy to check that $B : \mathcal{X}_T \times \mathcal{X}_T \to \mathcal{X}_T$ is a bilinear operator. Thus, our next goal is to show that the operator B is also continuous. To prove this fact, we firstly observe that

$$\begin{aligned} \partial_t B(w,v)(t) &= (-\Delta)^\alpha \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} P(v \cdot \nabla w)(\tau)\, d\tau - P(v \cdot \nabla w)(t) \\ &= -(-\Delta)^\alpha B(w,v)(t) - P(v \cdot \nabla w)(t). \end{aligned}$$

As a consequence, we can write the following system related to the operator B:

$$\begin{gathered} \partial_t B(w,v)(t) + (-\Delta)^\alpha B(w,v)(t) = -P(v \cdot \nabla w)(t); \\ B(w,v)(0) = 0. \end{gathered} \tag{4.6}$$

We shall apply Lemma 3.1 to (4.6) to prove that B is continuous. By observing (4.2), we conclude that

$$\begin{aligned} \|P(v \cdot \nabla w)\|_{L^1_T(\mathcal{X}^s_{a,\sigma})} &= \int_0^T \int_{\mathbb{R}^3} |\xi|^s e^{a|\xi|^{1/\sigma}} |\mathcal{F}[P(v \cdot \nabla w)(t)]|\, d\xi dt \\ &\le \int_0^T \int_{\mathbb{R}^3} |\xi|^{s+1} e^{a|\xi|^{1/\sigma}} |\mathcal{F}[(w \otimes v)(t)]|\, d\xi dt. \end{aligned}$$

Hence,

$$\|P(v \cdot \nabla w)\|_{L^1_T(\mathcal{X}^s_{a,\sigma})} \le \int_0^T \|(w \otimes v)(t)\|_{\mathcal{X}^{s+1}_{a,\sigma}}\, dt.$$

By using Lemma 3.4, it follows that

$$\|P(v \cdot \nabla w)\|_{L^1_T(\mathcal{X}^s_{a,\sigma})} \le C_s \int_0^T [\|v\|_{\mathcal{X}^0_{\frac{a}{\sigma},\sigma}} \|w\|_{\mathcal{X}^{s+1}_{a,\sigma}} + \|v\|_{\mathcal{X}^{s+1}_{a,\sigma}} \|w\|_{\mathcal{X}^0_{\frac{a}{\sigma},\sigma}}]\, dt,$$

since $a \geq 0$, $\sigma \geq 1$ and $s \geq -1$. Apply Lemma 3.3 and Hölder's inequality to obtain

$$\begin{aligned} \|P(v\cdot\nabla w)\|_{L^1_T(\mathcal{X}^s_{a,\sigma})} &\leq C_{a,\sigma,s}\|v\|_{L^\infty_T(\mathcal{X}^s_{a,\sigma})}\|w\|^{1-\frac{1}{2\alpha}}_{L^\infty_T(\mathcal{X}^s_{a,\sigma})}\int_0^T \|w\|^{\frac{1}{2\alpha}}_{\mathcal{X}^{s+2\alpha}_{a,\sigma}}\,dt \\ &\quad + C_{a,\sigma,s}\|w\|_{L^\infty_T(\mathcal{X}^s_{a,\sigma})}\|v\|^{1-\frac{1}{2\alpha}}_{L^\infty_T(\mathcal{X}^s_{a,\sigma})}\int_0^T \|v\|^{\frac{1}{2\alpha}}_{\mathcal{X}^{s+2\alpha}_{a,\sigma}}\,dt, \end{aligned} \tag{4.7}$$

where $(a,s,\sigma) \in \big((0,+\infty)\times(-\infty,0)\times(1,+\infty)\big) \cup \big([0,+\infty)\times\{0\}\times[1,+\infty)\big)$ and $\alpha \geq \frac{1}{2}$. By using Hölder's inequality once again, one infers that

$$\begin{aligned} \|P(v\cdot\nabla w)\|_{L^1_T(\mathcal{X}^s_{a,\sigma})} &\leq C_{a,\sigma,s}T^{1-\frac{1}{2\alpha}}\|v\|_{L^\infty_T(\mathcal{X}^s_{a,\sigma})}\|w\|^{1-\frac{1}{2\alpha}}_{L^\infty_T(\mathcal{X}^s_{a,\sigma})}\|w\|^{\frac{1}{2\alpha}}_{L^1_T(\mathcal{X}^{s+2\alpha}_{a,\sigma})} \\ &\quad + C_{a,\sigma,s}T^{1-\frac{1}{2\alpha}}\|w\|_{L^\infty_T(\mathcal{X}^s_{a,\sigma})}\|v\|^{1-\frac{1}{2\alpha}}_{L^\infty_T(\mathcal{X}^s_{a,\sigma})}\|v\|^{\frac{1}{2\alpha}}_{L^1_T(\mathcal{X}^{s+2\alpha}_{a,\sigma})}. \end{aligned}$$

As a result, one has

$$\|P(v\cdot\nabla w)\|_{L^1_T(\mathcal{X}^s_{a,\sigma})} \leq C_{a,\sigma,s}T^{1-\frac{1}{2\alpha}}\|w\|_{\mathcal{X}_T}\|v\|_{\mathcal{X}_T}, \quad \forall w,v\in\mathcal{X}_T. \tag{4.8}$$

By applying Lemma 3.1 (ii) (with $p=1$) to system (4.6) and, by using (4.8), we obtain

$$\|B(w,v)\|_{L^1_T(\mathcal{X}^{s+2\alpha}_{a,\sigma})} \leq C_{a,\sigma,s}T^{1-\frac{1}{2\alpha}}\|w\|_{\mathcal{X}_T}\|v\|_{\mathcal{X}_T}, \quad \forall w,v\in\mathcal{X}_T. \tag{4.9}$$

By using Lemma 3.1 (i),(4.6) and (4.8), one infers that

$$\|B(w,v)\|_{L^\infty_T(\mathcal{X}^s_{a,\sigma})} \leq C_{a,\sigma,s}T^{1-\frac{1}{2\alpha}}\|w\|_{\mathcal{X}_T}\|v\|_{\mathcal{X}_T}, \quad \forall w,v\in\mathcal{X}_T. \tag{4.10}$$

From (4.9) and (4.10), one obtains

$$\|B(w,v)\|_{\mathcal{X}_T} \leq C_{a,\sigma,s}T^{1-\frac{1}{2\alpha}}\|w\|_{\mathcal{X}_T}\|v\|_{\mathcal{X}_T}, \quad \forall w,v\in\mathcal{X}_T. \tag{4.11}$$

This inequality shows that the operator B is continuous.

Therefore, we only need to estimate the term $e^{-t(-\Delta)^\alpha}u_0$ given in (4.4), by considering the space $\mathcal{X}_T$, to apply Lemma 3.2. Thereby,

$$\|e^{-t(-\Delta)^\alpha}u_0\|_{\mathcal{X}^s_{a,\sigma}} = \int_{\mathbb{R}^3}|\xi|^s e^{a|\xi|^{1/\sigma}}e^{-t|\xi|^{2\alpha}}|\widehat{u}_0(\xi)|\,d\xi \leq \int_{\mathbb{R}^3}|\xi|^s e^{a|\xi|^{1/\sigma}}|\widehat{u}_0(\xi)|\,d\xi,$$

for all $t\in[0,T]$. Then, we conclude that

$$\|e^{-t(-\Delta)^\alpha}u_0\|_{L^\infty_T(\mathcal{X}^s_{a,\sigma})} \leq \|u_0\|_{\mathcal{X}^s_{a,\sigma}}. \tag{4.12}$$

On the other hand,

$$\begin{aligned} \|e^{-t(-\Delta)^\alpha}u_0\|_{L^1_T(\mathcal{X}^{s+2\alpha}_{a,\sigma})} &= \int_{\mathbb{R}^3}|\xi|^{s+2\alpha}e^{a|\xi|^{1/\sigma}}|\widehat{u}_0(\xi)|\Big(\int_0^T e^{-t|\xi|^{2\alpha}}\,dt\Big)\,d\xi \\ &\leq \int_{\mathbb{R}^3}|\xi|^s e^{a|\xi|^{1/\sigma}}|\widehat{u}_0(\xi)|\,d\xi = \|u_0\|_{\mathcal{X}^s_{a,\sigma}}. \end{aligned}$$

Hence, we are able to write the inequality

$$\|e^{-t(-\Delta)^\alpha}u_0\|_{L^1_T(\mathcal{X}^{s+2\alpha}_{a,\sigma})} \leq \|u_0\|_{\mathcal{X}^s_{a,\sigma}}. \tag{4.13}$$

Therefore, by (4.12) and (4.13), one concludes that

$$\|e^{-t(-\Delta)^\alpha}u_0\|_{\mathcal{X}_T} \leq 2\|u_0\|_{\mathcal{X}^s_{a,\sigma}}. \tag{4.14}$$

Now, let us prove Theorem 1.1 (i) ($\alpha = 1/2$ in this case). Thus, assume that $\|u_0\|_{\mathcal{X}^s_{a,\sigma}} < [8C_{a,\sigma,s}]^{-1}$ (where $C_{a,\sigma,s}$ is given in (4.11)) to apply Lemma 3.2 and obtain a unique global solution $u \in \mathcal{X}_T$ for the equation (4.4) that satisfies

$$\|u\|_{\mathcal{X}_T} \leq 2\|e^{-t(-\Delta)^{1/2}}u_0\|_{\mathcal{X}_T},$$

and, consequently, by (4.14), we have

$$\|u\|_{L^\infty_T(\mathcal{X}^s_{a,\sigma})} + \|u\|_{L^1_T(\mathcal{X}^{s+1}_{a,\sigma})} \leq 4\|u_0\|_{\mathcal{X}^s_{a,\sigma}}.$$

Analogously to the proof above, we can prove that the solution u also belongs to $L^p_T(\mathcal{X}^{s+\frac{1}{p}}_{a,\sigma}(\mathbb{R}^3))$, for all $p \geq 1$. In fact, observe that the Navier-Stokes equations (1.1) (with $\alpha = 1/2$) can be rewritten as

$$\begin{gathered} u_t + (-\Delta)^{1/2}u = -P(u \cdot \nabla u); \\ u(\cdot, 0) = u_0. \end{gathered}$$

Hence, similarly to (4.8), we have

$$\|P(u \cdot \nabla u)\|_{L^1_T(\mathcal{X}^s_{a,\sigma})} \leq C_{a,\sigma,s}[\|u\|_{L^\infty_T(\mathcal{X}^s_{a,\sigma}(\mathbb{R}^3))} + \|u\|_{L^1_T(\mathcal{X}^{s+1}_{a,\sigma}(\mathbb{R}^3))}]^2.$$

Therefore, $P(u \cdot \nabla u) \in L^1_T(\mathcal{X}^s_{a,\sigma}(\mathbb{R}^3))$ since $u \in C_T(\mathcal{X}^s_{a,\sigma}(\mathbb{R}^3)) \cap L^1_T(\mathcal{X}^{s+1}_{a,\sigma}(\mathbb{R}^3))$. By using that $u_0 \in \mathcal{X}^s_{a,\sigma}(\mathbb{R}^3)$, and applying Lemma 3.1 (ii), the proof of Theorem 1.1 i) is complete.

We are ready to show Theorem 1.1 (ii) ($\alpha > 1/2$ in this case). Thereby, by taking $0 < \overline{T} < [8C_{a,\sigma,s}\|u_0\|_{\mathcal{X}^s_{a,\sigma}}]^{\frac{2\alpha}{1-2\alpha}}$ (where $C_{a,\sigma,s}$ is given in (4.11)), Lemma 3.2 provides a unique local solution $u \in \mathcal{X}_{\overline{T}}$ for equation (4.4) such that

$$\|u\|_{\mathcal{X}_{\overline{T}}} \leq 2\|e^{-t(-\Delta)^\alpha}u_0\|_{\mathcal{X}_{\overline{T}}}.$$

Therefore, by (4.14), one obtains

$$\|u\|_{L^\infty_{\overline{T}}(\mathcal{X}^s_{a,\sigma})} + \|u\|_{L^1_{\overline{T}}(\mathcal{X}^{s+2\alpha}_{a,\sigma})} \leq 4\|u_0\|_{\mathcal{X}^s_{a,\sigma}}.$$

This solution u belongs to $L^p_{\overline{T}}(\mathcal{X}^{s+\frac{2\alpha}{p}}_{a,\sigma}(\mathbb{R}^3))$, for all $p \geq 1$. In fact, at first, rewrite the Navier-Stokes equations (1.1) as

$$\begin{gathered} u_t + (-\Delta)^\alpha u = -P(u \cdot \nabla u); \\ u(\cdot, 0) = u_0. \end{gathered}$$

Secondly, applying similar arguments as in (4.8) we obtain

$$\|P(u \cdot \nabla u)\|_{L^1_{\overline{T}}(\mathcal{X}^s_{a,\sigma})} \leq C_{a,\sigma,s}\overline{T}^{1-\frac{1}{2\alpha}}[\|u\|_{L^\infty_{\overline{T}}(\mathcal{X}^s_{a,\sigma}(\mathbb{R}^3))} + \|u\|_{L^1_{\overline{T}}(\mathcal{X}^{s+2\alpha}_{a,\sigma}(\mathbb{R}^3))}]^2.$$

Thereby, $P(u \cdot \nabla u) \in L^1_{\overline{T}}(\mathcal{X}^s_{a,\sigma}(\mathbb{R}^3))$ since $u \in C_{\overline{T}}(\mathcal{X}^s_{a,\sigma}(\mathbb{R}^3)) \cap L^1_{\overline{T}}(\mathcal{X}^{s+2\alpha}_{a,\sigma}(\mathbb{R}^3))$. By using that $u_0 \in \mathcal{X}^s_{a,\sigma}(\mathbb{R}^3)$ and Lemma 3.1 ii), the proof of Theorem 1.1 ii) is complete. □

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References

[1] H. Bae; Existence and analyticity of Lei-Lin solution to the Navier-Stokes equation, *Proc. Amer. Math. Soc.*, **143** (2015), 2887–2892.

[2] J. Benameur; On the blow-up criterion of 3D Navier-Stokes equations, *J. Math. Anal. Appl.*, **371** (2010), 719-727.

[3] J. Benameur; On the exponential type explosion of Navier-Stokes equations, *Nonlinear Anal.*, **103** (2014), 87-97.

[4] J. Benameur; Long time decay to the Lei-Lin solution of 3D Navier-Stokes equations, *J. Math. Anal. Appl.*, **422** (2015), 424-434.

[5] J. Benameur, L. Jlali; Long time decay for 3D Navier-Stokes equations in Sobolev-Gevrey spaces, *Electron. J. Differential Equations*, (2016), 13 pp.

[6] J. Benameur, L. Jlali; On the blow-up criterion of 3D-NSE in Sobolev-Gevrey spaces, *J. Math. Fluid Mech.* **18** (2016), 805-822.

[7] J. Benameur, M. Bennaceur; Large time behaviour of solutions to the 3D-NSE in $\mathcal{X}^\sigma$ spaces, *J. Math. Anal. Appl.*, **482** (2020), 123566, 19 pp.

[8] J. Benameur, L. Jlali; Long time decay of 3D-NSE in Lei-Lin-Gevrey spaces. *Math. Slovaca*, **70** (2020), 877–892.

[9] J. Benameur, S. B. Abdallah; Asymptotic behavior of critical dissipative quasi-geostrophic equation in Fourier space, *J. Math. Anal. Appl.*, **497** (2021), Paper No. 124873, 30 pp.

[10] A. Biswas, J. Hudson, J. Tian; Persistence time of solutions of the three-dimensional Navier-Stokes equations in Sobolev-Gevrey classes, *J. Differential Equations*, **277** (2021), 191-233.

[11] P. Braz e Silva, W. G. Melo, P. R. Zingano; Some remarks on the paper "On the blow up criterion of 3D Navier-Stokes equation" by J. Benameur, *C. R. Acad. Sci. Paris*, **352** (2014), 913-915.

[12] P. Braz e Silva, W. G. Melo, N. F. Rocha; Existence, uniqueness and blow-up of solutions for the 3D Navier-Stokes equations in homogeneous Sobolev-Gevrey spaces, *Comput. Appl. Math.*, **39** (2020), Paper No. 66, 11 pp.

[13] M. Cannone; Ondelettes, paraproduits et Navier-Stokes, *Diderot Editeur, Paris*, (1995), x+191 pp.

[14] A. Chaabani; On well-posedness of the Cauchy problem for 3D MHD system in critical Sobolev–Gevrey space, *Partial Differ. Equ. Appl.*, **2** (2021), Paper No. 24, 20 pp.

[15] Y. Dai, Z. Tan, J. Wu; A class of global large solutions to the magnetohydrodynamic equations with fractional dissipation, *Z. Angew. Math. Phys.*, **70** (2019), Paper No. 153, 13 pp.

[16] R. Guterres, W. G. Melo, J. Nunes, C. Perusato; Large time decay for the magnetohydrodynamics equations in Sobolev-Gevrey spaces, *Monatsh. Math.*, **192** (2020), 591-613.

[17] R. H. Guterres, W. G. Melo, N. F. Rocha, T. S. R. Santos; Well-Posedness, blow-up criteria and stability for solutions of the generalized MHD equations in Sobolev-Gevrey spaces, *Acta Appl. Math.*, **176** (2021), Paper No. 4, 30 pp.

[18] Z. Lei, F. Lin; Global mild solutions of Navier-Stokes equations, *Comm. Pure Appl. Math.*, **64** (2011), 1297-1304.

[19] P. Li, Z. Zhai; Well-posedness and regularity of generalized Navier-Stokes equations in some critical Q-spaces, *J. Funct. Anal.*, **259** (2010), 2457–2519.

[20] J. Lorenz, P. R. Zingano; Properties at potential blow-up times for the incompressible Navier-Stokes equations, *Bol. Soc. Parana. Mat.*, **35** (2017), 127-158.

[21] J. Lorenz, W. G. Melo, N. F. Rocha; The Magneto-Hydrodynamic equations: Local theory and blow-up of solutions, *Discrete Contin. Dyn. Syst. Ser. B*, **24** (2019), 3819-3841.

[22] D. Marcon, L. Schutz, J. S. Ziebell; On the blow-up criterion of magnetohydrodynamics equations in homogeneous Sobolev spaces, *Appl. Anal.*, **97** (2018), 1677-1687.

[23] W. G. Melo; The magneto-micropolar equations with periodic boundary conditions: solution properties at potential blow-up times, *J. Math. Anal. Appl.*, **435** (2016), 1194-1209.

[24] W. G. Melo, C. Perusato, N. F. Rocha; On local existence, uniqueness and blow-up of solutions for the generalized MHD equations in Lei-Lin spaces, *Z. Angew. Math. Phys.*, **70** (2019), Paper No. 74, 24 pp.

[25] W. G. Melo, N. F. Rocha, P. R. Zingano; Local existence, uniqueness and lower bounds of solutions for the magnetohydrodynamics equations in Sobolev-Gevrey spaces, *J. Math. Anal. Appl.*, **482** (2020), 123524, 31 pp.

[26] W. G. Melo, N. F. Rocha, E. Barbosa; Navier-Stokes equations: local existence, uniqueness and blow-up of solutions in Sobolev-Gevrey spaces, *Appl. Anal.*, **100** (2021), 1905–1924.

[27] W. G. Melo, N. F. Rocha, P. R. Zingano; Asymptotic behavior of solutions for the 2D micropolar equations in Sobolev-Gevrey spaces, *Asymptot. Anal.*, **123** (2021), 157–179.

[28] W. G. Melo, M. de Souza, T. S. R. Santos; On the generalized Magnetohydrodynamics-α equations with fractional dissipation in Lei-Lin and Lei-Lin-Gevrey spaces, *Z. Angew. Math. Phys.*, **73** (2022), Paper No. 44, 37 pp.

[29] W. G. Melo, T. S. R. Santos; Time decay rates for the generalized MHD-α equations in Sobolev-Gevrey spaces, *Appl. Anal.*, **101** (2022), 6623–6644.

[30] W. G. Melo, N. F. Rocha; New blow-up criteria for local solutions of 3D generalized MHD equations in Lei-Lin-Gevrey spaces, *Math. Nachr.* **296** (2023), 757–778.

[31] H. Orf; Long time decay for global solutions to the Navier-Stokes equations in Sobolev-Gevery spaces, (2019), 1-14 (arXiv:1903.03034v1).

[32] C. Pozrikidis; The fractional Laplacian, *CRC Press, Boca Raton, FL,* (2016), xv+278 pp.

[33] R. Selmi, A. Chaabani, M. Zaabi; Blow-up of the maximal solution to 3D Boussinesq system in Lei-Lin-Gevrey spaces, *Math. Methods Appl. Sci.,* **43** (2020), 2945–2952.

[34] E. M. Stein; Singular integrals and differentiability properties of functions, *Princeton University Press, Princeton, NJ,* (1970), xiv+290 pp.

[35] J. Wu; Generalized MHD equations, *J. Differential Equations*, **195** (2003), 284-312.

[36] B. Yuan, Y. Xiao; The global well-posedness of strong solutions to 2D MHD equations in Lei-Lin space, *Acta Math. Appl. Sin. Engl. Ser.*, **39** (2023), 647–655.

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