

VARIETY OF SOLUTIONS AND DYNAMICAL BEHAVIOR FOR YTSF EQUATIONS

WEI CHEN

ABSTRACT. We construct non-homogeneous polynomial lump wave solutions of the Yu-Toda-Sasa-Fukuyama (YTSF) equation, based on a bilinear approach, enriching the formal diversity of lump waves. By studying the interaction between the lump solutions of the YTSF equation and the solitary wave solutions, we find a new aggregation effect and elastic collision effect. We obtain exact solutions, such as the solution of separated variables and periodic nonlinear wave solutions, by applying the Lie symmetry group method and the bilinear method.

1. INTRODUCTION

The integrability and exact solutions of nonlinear evolution equations play an important role in the field of nonlinear science, such as condensed matter physics, solid state physics, aerodynamics, plasma physics, fluid dynamics and many other fields [7, 13, 17]. This article will focus on exact solutions to the (3+1)-dimensional Yu-Toda-Sasa-Fukuyama (YTSF) equation [1], as well as the interactions between these exact solutions, using the bilinear method and Lie symmetry method [11, 12].

The standard form of the YTSF equation is

$$\begin{aligned}(-4u_t + \phi(u)u_z)_x + 3u_{yy} &= 0, \\ \phi(u) &= \partial_x^2 + 4u + 2u_x \partial_x^{-1},\end{aligned}\tag{1.1}$$

in which $u = u(t, x, y, z)$ is an analytic function of the scaled spatial coordinates x, y, z and the temporal coordinate t . The YTSF equation is derived by applying a strong symmetry to the 2-dimensional Bogoyavlenskii-Schiff equation, which describes an elastic quasi-plane wave in a lattice or an interfacial wave in a two-layer liquid [20]. The typical way to explore exact solutions of the YTSF equation is to use classical methods such as the Bäcklund transformation method [21], the bilinear methods [10], elliptic function expansions [15] and Lie symmetry group method [16], which can bypass integration to derive explicit solutions. Yan [22] used the auto-Bäcklund transformation to study 1.1 and found a nonlinear wave solution to 1.1. Chen [2] applied Hirota bilinear formulation to study the lump solution of 1.1

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with the finding that the lump solution consists of a 2-fold homogeneous polynomial, but the lump solution in the form of non-homogeneous polynomial has not been found yet [1]-[14].

The rest of this article is organized as follows. In Section 2, we use the bilinear method to construct the lump solution of the YTSF equation with a non-homogeneous multinomial function. After constructing the solution, we further study the interaction between lump solution and soliton solution. In Section 3, we apply the Lie symmetry method to obtain a new non-traveling wave solution with an arbitrary function.

2. LUMP SOLUTION AND VARIABLE SOLITON FOR THE YTSF EQUATION

In this section, we start with the lump solution of the special form of the YTSF equation, then we analyze the interaction between the lump solution and the soliton wave.

2.1. Two-lump solution to the YTSF equation. By the traveling wave transformation $\eta = x + cz - at$, the equation (1.1) reduces to

$$3u_{yy} + (4\alpha u_{\eta} + 3(u^2)_{\eta} + cu_{\eta\eta\eta})_{\eta} = 0. \quad (2.1)$$

It is easy to see that the real constant u_0 is the trivial solution of the YTSF equation, where u_0 can be arbitrary. By the second order logarithmic derivative transformation

$$u = u_0 + 2(\ln F)_{\eta\eta}, \quad (2.2)$$

the equation (2.1) can be changed into

$$(3D_y^2 + \kappa D_{\eta}^2 + cD_{\eta}^4)F \cdot F = 0, \quad (2.3)$$

where $\kappa = 2(2\alpha + 3cu_0)$. In order to search the two-lump solution, we choose

$$F = (\lambda_2\eta^2 + \lambda_3y^2)^3 + \lambda_4\eta^4 + \lambda_5y^4 + \lambda_6\eta^2y^2 + \lambda_7\eta^2 + \lambda_8y^2 + \lambda_1, \quad (2.4)$$

where λ_i ($i = 1, 2, \dots, 8$) are constants to be determined. Substituting (2.4) into (2.3), we obtain the nonlinear algebraic equations of undetermined parameters, from which we obtain

$$\begin{aligned} \lambda_1 &= -15\left(\frac{5c}{\kappa}\lambda_2\right)^3, & \lambda_3 &= \frac{\kappa}{3}\lambda_2, & \lambda_4 &= -\frac{25c}{\kappa}\lambda_2^3, \\ \lambda_5 &= -\frac{17\kappa}{9}\lambda_2^3, & \lambda_6 &= -30c\lambda_2^3, & \lambda_7 &= -125\frac{c^2}{\kappa^2}\lambda_2^3, & \lambda_8 &= \frac{19}{\kappa}(5c)^2\lambda_2^3, \end{aligned} \quad (2.5)$$

where $c\lambda_2\kappa \neq 0$. Substituting (2.5) into (2.4), we have

$$\begin{aligned} F &= \lambda_2^3 \left[(\eta^2 + \frac{\kappa}{3}y^2)^3 - \frac{25c}{\kappa}\eta^4 - \frac{17\kappa}{9}y^4 - 30c\eta^2y^2 \right. \\ &\quad \left. - \frac{125c^2}{\kappa^2}\eta^2 + \frac{19}{\kappa}(5c)^2y^2 - 15\left(\frac{5c}{\kappa}\right)^3 \right]. \end{aligned} \quad (2.6)$$

Thus, from (2.2)-(2.6), we know that equation 1.1 has the rational solution of the form

$$\begin{aligned}
u &= u_0 + 2\left(24\left(\eta^2 + \frac{\kappa}{3}y^2\right)\eta^2 + 6\left(\eta^2 + \frac{\kappa}{3}y^2\right)^2 - \frac{300c}{\kappa}\eta^2 - 60cy^2 - \frac{250c^2}{\kappa^2}\right) \\
&\div \left(\left(\eta^2 + \frac{\kappa}{3}y^2\right)^3 - \frac{25c}{\kappa}\eta^4 - \frac{17c\kappa}{9}y^4 - 30c\eta^2y^2\right. \\
&\quad \left. - \frac{125c^2}{\kappa^2}\eta^2 + \frac{19}{\kappa}(5c)^2y^2 - 15\left(\frac{5c}{\kappa}\right)^3\right) \\
&+ 2\left(6\left(\eta^2 + \frac{\kappa}{3}y^2\right)^2\eta - \frac{100c\eta^3}{\kappa} - 60c\eta y^2 - \frac{250c^2\eta}{\kappa^2}\right)^2 \\
&\div \left(\left(\eta^2 + \frac{\kappa}{3}y^2\right)^3 - \frac{25c}{\kappa}\eta^4 - \frac{17c\kappa}{9}y^4 - 30c\eta^2y^2\right. \\
&\quad \left. - \frac{125c^2}{\kappa^2}\eta^2 + \frac{19}{\kappa}(5c)^2y^2 - 15\left(\frac{5c}{\kappa}\right)^3\right)^2.
\end{aligned} \tag{2.7}$$

2.2. Aggregation effect and elastic collision of two-lump wave. To better understand these solutions, we now consider their dynamical behaviors. From graphs of solutions as shown in Figure 1-3, it is possible to see clearly the height and positional behavior of the waves. The corresponding solutions $u(x, y, z, t)$ in different planes are shown in Figure 1-3. We observe that the solution u contains three troughs and two peaks in non-standard formation. Therefore, this solution is also called two-lump solution. The physical behavior influenced by the parameter u_0 will be explored.

Figure 1 shows the two-lump solution of the form (2.7) in the (t, y) plane. It can be seen that the middle trough changes correspondingly along the u axis with the change of the value of u_0 and the distance between the left trough and the right trough changes symmetrically at the same time. In Figure 1(g) and (h), it can be seen that increasing the parameter u_0 will cause the background plane wave to rise, which can be verified from the solution formula (2.7). In Figure 1(a), (b) and (c), it can be observed that increasing the value of u_0 will increase the width of the wave when $u_0 < 0$. When $u_0 > 0$, it appears that increasing the value of u_0 leads to a decrease of the width of the wave. When $u_0 = 0$, the distance between the left trough and the right trough is the widest.

Figures 2 and 3 mainly reflect the influence of variable t upon the two-lump solution. Figure 2(a) shows the two-lump solution of the YTSF equation in the (x, y) plane when $t = 0$. One can also see that when $(x, y) \rightarrow (\infty, \infty)$, $u \rightarrow -1$, and the two-lump solution is travelling horizontally on the (x, y) plane. The two-lump solution will not disappear as the time evolves. Figure 2(c) shows clearly how the waves move: the position of the two-lump solution changes correspondingly with time evolution while maintaining its structure.

Figure 3 shows the two-lump solution of the YTSF equation in the (z, y) plane. The motion behavior in Figure 3 is the same as the one observed in Figure 2.

From Figures 2 and 3, we can see that the two-lump solution in the (x, y) and (z, y) planes always preserve their shape, amplitude and velocity while traveling.

2.3. Interaction between two-lump wave and soliton wave. In this section we will study the interaction between two-lump solution and soliton wave solution

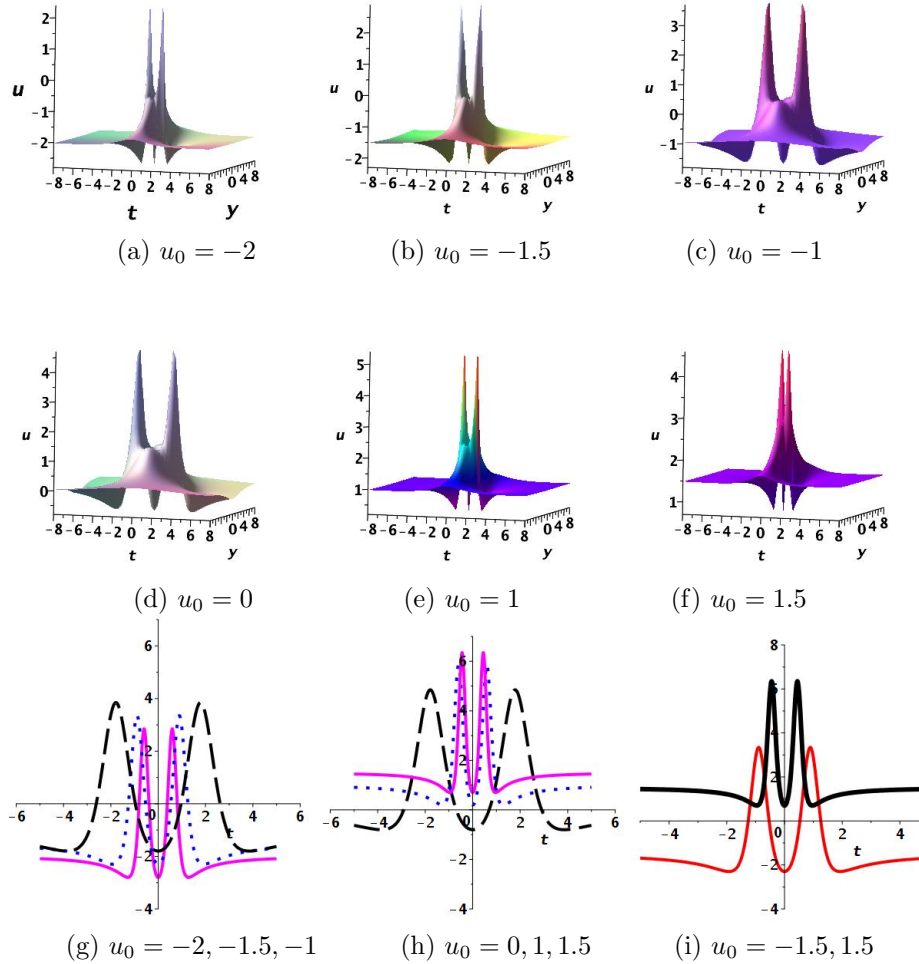


FIGURE 1. Two-lump solution of the form (2.7) in the (t, y) plane with $y = z = 1$ and $c = -1$, $\lambda_1 = \lambda_2 = 1$ with (a) $u_0 = -2$, (b) $u_0 = -1.5$, (c) $u_0 = -1$, (d) $u_0 = 0$, (e) $u_0 = 1$, (f) $u_0 = 1.5$. The black dashed line, the blue line and the magenta point of (g) curves correspond to (a), (b) and (c) respectively. The red line, the blue line and the black line of (h) curves correspond to (d), (e), and (f) respectively. The red and black lines of (i) curves correspond to (b) and (f).

by choosing

$$\begin{aligned}
 F = & \lambda_1 + (\lambda_2 \eta^2 + \lambda_3 y^2)^3 + \lambda_4 \eta^4 + \lambda_5 y^4 \\
 & + \lambda_6 \eta^2 y^2 + \lambda_7 \eta^2 + \lambda_8 y^2 + \lambda_9 e^{\lambda_{10} \eta + \lambda_{11} y + \lambda_{12}},
 \end{aligned} \tag{2.8}$$

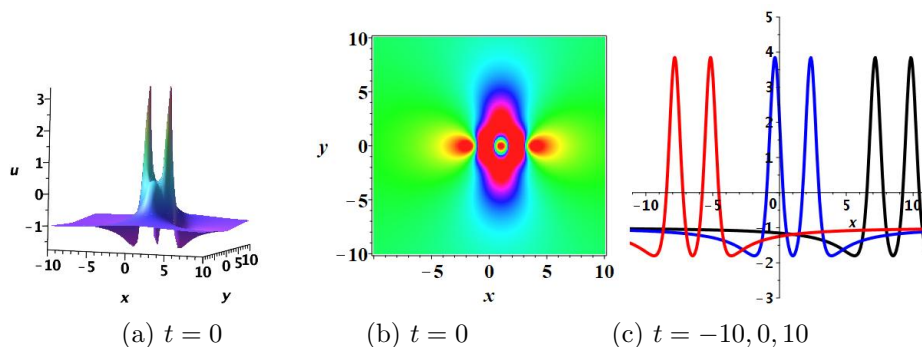


FIGURE 2. (a) Two-lump solution of the YTSEF equation in the (x, y) plane given by (2.7) with parameters $\lambda_2 = \lambda_3 = 1$, $c = u_0 = -1$, and $z = 1$. (b) is the density plot corresponding to (a). (c) is the evolution of the two-lump solution when $t = -10$ (red), $t = 0$ (blue), $t = 10$ (black).

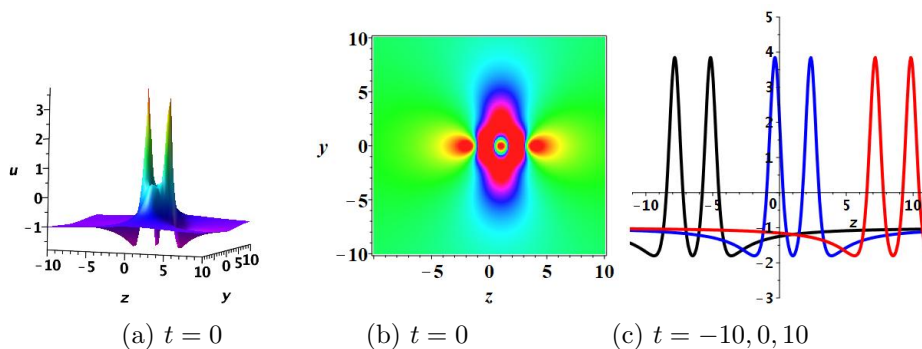


FIGURE 3. (a) Two-lump solution of the YTSEF equation in the (z, y) plane given by (2.7) with $\lambda_2 = \lambda_3 = 1$, $c = u_0 = -1$, and $z = 1$. (b) is the density plot corresponding to (a). (c) is the evolution of the two-lump solution when $t = -10$ (red), $t = 0$ (blue), $t = 10$ (black).

where λ_i ($i = 1, 2, \dots, 12$) are constants to be determined. Substituting (2.8) into (2.3), we obtain the nonlinear algebraic equations with undetermined parameters, from which we have

$$c = \lambda_3 = \lambda_5 = \lambda_6 = \lambda_8 = \lambda_{11} = 0, \quad u_0 = -\frac{2}{3}\alpha. \tag{2.9}$$

Substituting (2.9) in (2.8) yields

$$F = \lambda_2^3 \eta^6 + \lambda_4 \eta^4 + \lambda_7 \eta^2 + \lambda_1 + \lambda_9 e^{\lambda_{10} \eta + \lambda_{12}}. \tag{2.10}$$

Substituting (2.10) into (2.2) with $\eta = x + cz - \alpha t$, we obtain a hybrid solution to equation 1.1 as

$$\begin{aligned}
 u = & -\frac{2}{3}\alpha + 2\left(30\lambda_2^3(-\alpha t + x)^4 + 12\lambda_4(-\alpha t + x)^2 + 2\lambda_7 \right. \\
 & \left. + \lambda_9\lambda_{10}^2 e^{\lambda_{10}(-\alpha t + x) + \lambda_{12}}\right) \\
 & \div \left(\lambda_2^3(-\alpha t + x)^6 + \lambda_4(-\alpha t + x)^4 + \lambda_7(-\alpha t + x)^2 + \lambda_1 \right. \\
 & \left. + \lambda_9 e^{\lambda_{10}(-\alpha t + x) + \lambda_{12}}\right) \\
 & - 2\left(6\lambda_2^3(-\alpha t + x)^5 + 4\lambda_4(-\alpha t + x)^3 + 2\lambda_7(-\alpha t + x) \right. \\
 & \left. + \lambda_9\lambda_{10} e^{\lambda_{10}(-\alpha t + x) + \lambda_{12}}\right)^2 \\
 & \div \left(\lambda_2^3(-\alpha t + x)^6 + \lambda_4(-\alpha t + x)^4 + \lambda_7(-\alpha t + x)^2 + \lambda_1 + \lambda_9 e^{\lambda_{10}(-\alpha t + x) + \lambda_{12}}\right).
 \end{aligned}
 \tag{2.11}$$

The expression 2.11 is a linear superposition of two-lump wave and soliton wave, which is also an interaction solution to equation 1.1.

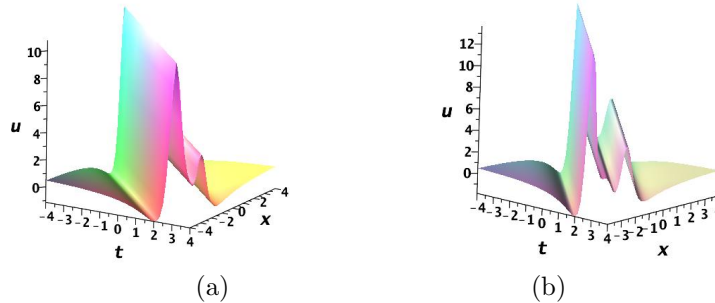


FIGURE 4. Interaction between two-lump wave and soliton wave of the YTSF equation given by (2.11). (a) $\alpha = -1$, $\lambda_1 = \lambda_2 = \lambda_4 = \lambda_7 = \lambda_9 = \lambda_{10} = \lambda_{12} = 1$. (b) $\alpha = \lambda_4 = -1$, $\lambda_1 = \lambda_2 = \lambda_7 = \lambda_9 = \lambda_{10} = \lambda_{12} = 1$.

Figure 4 illustrates the interaction between two-lump wave and soliton wave. As the exponential changes from negative to positive, the sum of the exponential function and the polynomial changes from a polynomial control to an exponential control. The results obtained here show that the two-lump soliton degenerates to a rouge wave and then to a soliton and finally disappears with a spatio-temporal variation, which has not been found in the previous literature.

3. VARIOUS SOLUTIONS AND DYNAMICAL BEHAVIOR OF THE YTSF EQUATION

Consider the potential form of the TYSF equation 1.1

$$v_{xxxz} + 4v_x v_{xz} + 2v_{xx} v_z + 3v_{yy} - 4v_{xt} = 0, \quad u = v_x. \tag{3.1}$$

This equation has a movable logarithmic branch point in the sense of WTC method [18]. Assume that equation (3.1) has the traveling wave solution in the form

$$v = w(\eta), \quad \eta = px + qy + rz - st, \quad (3.2)$$

where p, q, r and s are real constants to be determined. Substituting (3.2) into equation (3.1) and integrating once leads to

$$p^3rw_{\eta\eta\eta} + 3p^2rw_{\eta}^2 + 3q^2w_{\eta} + 4psw_{\eta} + C_1 = 0. \quad (3.3)$$

Setting $w_{\eta} = f(\eta)$, then (3.2) is reduced to a second-order nonlinear ordinary differential equation,

$$p^3rf_{\eta\eta} + 3p^2rf^2 + 3q^2f + 4psf + C_1 = 0. \quad (3.4)$$

Multiplying (3.4) by f_{η} , integrating once with respect to η and taking the integration constant to be C_2 , we have

$$p^3rf_{\eta}^2 + 2p^2rf^3 + 4pf^2s + 3f^2q^2 + 2C_1f + 2C_2 = 0. \quad (3.5)$$

Using the Jacobi elliptic function expansion method [15] to find the periodic wave solution of (3.5), we can expand the solution to equation (3.5) in the form

$$f(\eta) = a_1sn^2(b\eta, m) + a_0, \quad (3.6)$$

where $sn(b\eta, m)$ is the Jacobi elliptic sine function with the modulus $m \in (0, 1)$, and a_0, a_1 and b are non-zero constants to be determined. Substituting (3.6) into (3.5) leads to an algebraic system. By assuming the coefficients of sn^k ($k = 0, 2, 4, 6$) to be zero, we obtain

$$\begin{aligned} p^2ra_1^2(2b^2m^2p + a_1) &= 0, \\ a_1^2(4b^2m^2p^3r + 4b^2p^3r - 4p^3ra_0 - 4ps - 3q^2) &= 0, \\ a_1(2b^2p^3ra_1 + 3p^2ra_0^2 + 4psa_0 + 3q^2a_0 + C_1) &= 0, \\ 2p^2ra_0^3 + 4psa_0^2 + 3q^2a_0^2 + 2C_1a_0 + 2C_2 &= 0. \end{aligned} \quad (3.7)$$

If $16b^4p^6r^2(m^4 - m^2 + 1) - 16p^2s^2 - 24pq^2s - 9q^4 + 12C_1p^2r = 0$ and $(4b^2p^3r(m^2 + 1) - 4ps - 3q^2)(4b^2p^3r(2m^2 - 1) + 4ps + 3q^2)(4b^2p^3r(m^2 - 2) - 4ps - 3q^2) - 216C_2p^4r^2 = 0$, then the coefficients are

$$a_0 = \frac{4b^2p^3r(m^2 + 1) - 4ps - 3q^2}{6p^2r}, \quad a_1 = -2b^2m^2p. \quad (3.8)$$

Combining (3.6) and (3.8), we have

$$f_1(\eta) = -2b^2m^2psn^2(b\eta, m) + \frac{4b^2p^3r(m^2 + 1) - 4ps - 3q^2}{6p^2r}. \quad (3.9)$$

When $m \rightarrow 1$, equation (3.9) reduces to the shock wave solution

$$f_2(\eta) = -2b^2p \tanh^2(b\eta) + \frac{8b^2p^3r - 4ps - 3q^2}{6p^2r}. \quad (3.10)$$

In view of $w_{\eta} = f(\eta)$, we deduce

$$w_1 = -2bpE(sn(b\eta, m), m) + \rho_1\eta, \quad (3.11)$$

where E is an elliptic integral of the second kind and

$$w_2 = bp \ln(\tanh^2(b\eta) - 1) + \rho_2\eta, \quad (3.12)$$

where $\rho_1 = \frac{4b^2p^3r(m^2 - 2) - 4ps - 3q^2}{6p^2r}$ and $\rho_2 = \frac{8b^2p^3r - 4ps - 3q^2}{6p^2r}$.

In (3.7), replacing $sn(b\eta, m)$ by $cn(b\eta, m)$ and repeating the same process as above, we find a new triangular periodic solution

$$w_3 = 2bpE(sn(b\eta, m), m) + \rho_3\eta \quad (3.13)$$

and a new Jacobi doubly periodic solution

$$w_4 = 2bp \arctan(\sinh(b\eta)^2) + \rho_4\eta \quad (3.14)$$

where $\rho_3 = \frac{2pm^2(3pr-2s)-3q^2m^2-6p^2r+4b^2p^3rm^2(1-2m^2)}{6p^2rm^2}$ and $\rho_4 = -\frac{4b^2p^3r+4ps+3q^2}{6p^2r}$.

As $m \rightarrow 1$, the Jacobi biperiodic solution (3.13) degenerates to the triangular periodic solution (3.14). From (3.11)-(3.14), we obtain the following solutions to equation 3.1:

$$\begin{aligned} v_1 &= -2bpE(sn(b(px+qy+rz-st), m), m) + \rho_1(px+qy+rz-st), \\ v_2 &= bp \ln(\tanh^2 b(px+qy+rz-st) - 1) + \rho_2(px+qy+rz-st), \\ v_3 &= 2bpE(sn(b(px+qy+rz-st), m), m) + \rho_3(px+qy+rz-st), \\ v_4 &= 2bp \arctan(\sinh(b(px+qy+rz-st))^2) + \rho_4(px+qy+rz-st), \end{aligned}$$

where E is an elliptic integral of the second kind.

3.1. Lie symmetry reduction of the YTSF equation 3.1. In this subsection, we concentrate on finding the exact solution to the YTSF equation using the Lie symmetry method. According to the Lie group theory [14], we know that σ must satisfy the equation

$$\sigma_{x^3z} + 4\sigma_x v_{xz} + 4\sigma_{xz} v_x + 2\sigma_{xx} v_z + \sigma_z v_{xx} + \sigma_{yy} - 4\sigma_{xt} = 0. \quad (3.15)$$

To seek symmetry of (3.1), we take the function σ in the form

$$\sigma = f_1 v_x + f_2 v_y + f_3 v_z + f_4 v_t + f_5 v + f_6, \quad (3.16)$$

where f_i ($i = 1, 2, \dots, 6$) are functions to be determined and $v(t, x, y, z)$ is the solution to 3.1. Substituting (3.16) and (3.15) into 3.1, we have

$$\begin{aligned} f_{1z} v_{x^4} + f_{2z} v_{x^3y} + f_{4z} v_{x^3t} - (3f_{1xz} + f_{5z}) v_{x^3} \\ + 3f_{4x} v_{x^2zt} + 3f_{2x} v_{x^2yz} + 3f_{3x} v_{x^2z^2} + \dots = 0. \end{aligned} \quad (3.17)$$

Because of the linear independence of the derivatives of v in (3.17), we obtain

$$\begin{aligned} f_1 &= \frac{2}{3} p_2'(t)y + p_1'(t), \quad f_2 = p_2(t), \quad f_3 = p_3(t), \quad f_4 = \lambda, \quad f_5 = 0, \\ f_6 &= \left(\frac{4}{3} p_2''(t)y + 2p_1''(t)\right)z + p_3'(t)x + \frac{2}{3} p_3''(t)y^2 + p_4'(t)y + p_5'(t), \end{aligned} \quad (3.18)$$

where $p_i(t)$ ($i = 1, 2, 3, 4, 5$) are arbitrary functions of t . Substituting (3.18) into (3.16) yields

$$\begin{aligned} \sigma &= \left(\frac{2}{3} p_2'(t)y + p_1'(t)\right)v_x + p_2(t)v_y + p_3(t)v_z + \lambda v_t \\ &+ \left(\frac{4}{3} p_2''(t)y + 2p_1''(t)\right)z v + \frac{2}{3} p_3''(t)y^2 v + p_4'(t)y v + p_5'(t)v. \end{aligned} \quad (3.19)$$

We can obtain many symmetries of (3.1). On the basis of the integrability of the reduced equation, we find two types of solutions

$$\lambda = 1, \quad p_2(t) = p_3(t) = 0; \quad (3.20)$$

$$\lambda = 1, \quad p_1(t) = p_2(t) = p_4(t) = p_5(t) = 0, \quad p_3(t) = p'(t), \quad (3.21)$$

where $p(t) = \int p_3(t)d(t)$ and $p_3(t)$ is an arbitrary functions of t . Substituting (3.20) into (3.19) yields

$$\sigma = p_1'v_x + v_t + 2p_1''z + p_4'y + p_5'. \quad (3.22)$$

Letting σ be 0 leads to

$$v = -2p_1'(t)z - p_4(t)y - p_5(t) + f(\xi, y, z), \quad \xi = x - p(t). \quad (3.23)$$

Substituting (3.23) into (3.1), we obtain a symmetry reduction as

$$4f_\xi f_{\xi z} + 2f_{\xi\xi} f_z + f_{z\xi\xi\xi} + 3f_{yy} = 0. \quad (3.24)$$

Repeating the above process, from (3.1), (3.19), and (3.21), we obtain another symmetry reduction:

$$\begin{aligned} v &= -p'(t)x - \frac{2}{3}p''(t)y^2 + f(x, y, \xi), \quad \xi = z - p(t), \\ 2f_{xx}f_\xi + 4f_{\xi x}f_x + 3f_{yy} + f_{\xi xxx} &= 0, \end{aligned} \quad (3.25)$$

where $p(t) = \int p_3(t)d(t)$ and $p_3(t)$ is an arbitrary functions of t .

3.2. Solution to the symmetry reduction equation. In this subsection, we use the consistent Riccati expansion (CRE) method [4] to solve the symmetry reduction equation 3.24. Balancing the highest derivative term with nonlinear terms in 3.24 gives the expansion

$$f = a_0 + a_1U(\eta), \quad (3.26)$$

where a_0 , a_1 and η are to be determined later and $U = U(\eta)$ is a solution to the Riccati equation

$$U_\eta' = \mu_0 + \mu_1U^2. \quad (3.27)$$

This equation has the following three sets of traveling wave solutions

$$\begin{aligned} U_1 &= \frac{1}{C - \mu_1\eta}, \mu_0 = 0, \\ U_2 &= \sqrt{\frac{\mu_0}{\mu_1}} \tan(\sqrt{\mu_0\mu_1}\eta + C), \mu_0\mu_1 > 0, \\ U_3 &= -\sqrt{-\frac{\mu_0}{\mu_1}} \tanh(\sqrt{-\mu_0\mu_1}\eta + C), \mu_0\mu_1 < 0, \end{aligned} \quad (3.28)$$

where μ_0 and μ_1 are constants. Substituting (3.28) with (3.26) into (3.24), we have

$$\begin{aligned} &12\eta_z\eta_\xi^2a_1\mu_1^3(2\eta_\xi\mu_1 + a_1)U^5 + 2\mu_1^3(3a_{1z}\eta_\xi^3\mu_1 + 9a_1\eta_{z\xi}\eta_\xi^2\mu_1 + 9a_1\eta_{\xi\xi}\eta_\xi\eta_z\mu_1 \\ &+ 9\eta_{\xi\xi}\eta_z a_1\xi\mu_1 + 4a_{1z}\eta_{\xi\xi}a_1 + 2\eta_\xi\eta_{z\xi}a_1^2 + \eta_z\eta_{\xi\xi}a_1^2 + 8\eta_z\eta_\xi a_1\xi a_1)U^4 + \dots \\ &= 0. \end{aligned} \quad (3.29)$$

By eliminating all the coefficients of U^i ($i = 0, 1, 2, \dots, 5$), we obtain

$$\begin{aligned} \eta &= (\xi C_1 + C_2)^k \\ a_0 &= h(\xi)y + q(\xi) \\ a_1 &= -\frac{3}{4}kC_1\mu_1(\xi C_1 + C_2)^{k-1}, \end{aligned} \quad (3.30)$$

where C_1 and C_2 are constants and function $h(\xi)$, $q(\xi)$ are arbitrary smooth function of ξ . Substituting (3.30) into (3.26) yields

$$f(\xi, y, z) = h(\xi)y + q(\xi) - \frac{3}{4}kC_1\mu_1(\xi C_1 + C_2)^{k-1}U((\xi C_1 + C_2)^k). \quad (3.31)$$

Lemma 3.1 ([4]). *If $U(\eta)$ is the solution to the Riccati equation (3.27), then the function defined in equation (3.26) is the solution to the symmetry reduction equation (3.19).*

Applying Lemma 3.1, the solution of the symmetry reduction equation (3.19) is

$$\begin{aligned} f_1 &= h(\xi)y + q(\xi) - \frac{3}{4}kC_1\mu_1 \frac{(\xi C_1 + C_2)^{k-1}}{C - \mu_1(\xi C_1 + C_2)^k}, \\ f_2 &= h(\xi)y + q(\xi) - \frac{3}{4}kC_1\sqrt{\mu_0\mu_1}(\xi C_1 + C_2)^{k-1} \tan(\sqrt{\mu_0\mu_1}(\xi C_1 + C_2)^k + C), \\ f_3 &= h(\xi)y + q(\xi) + \frac{3}{4}kC_1\sqrt{-\mu_0\mu_1}(\xi C_1 C_2)^{k-1} \tanh(\sqrt{-\mu_0\mu_1}(\xi C_1 + C_2)^k + C). \end{aligned}$$

By applying the symmetric transformation (3.23), the corresponding solution to the YTSF equation 3.1 is

$$\begin{aligned} v_1 &= -2p_1'(t)z - p_4(t)y - p_5(t) + h(\xi)y \\ &\quad + q(\xi) - \lambda\mu_1 \frac{(\xi C_1 + C_2)^{k-1}}{C - \mu_1(\xi C_1 + C_2)^k}, \\ v_2 &= -2p_1'(t)z - p_4(t)y - p_5(t) + h(\xi)y + q(\xi) \\ &\quad - \lambda\kappa_1(\xi C_1 + C_2)^{k-1} \tan(\kappa_1(\xi C_1 + C_2)^k + C), \\ v_3 &= -2p_1'(t)z - p_4(t)y - p_5(t) + h(\xi)y + q(\xi) \\ &\quad + \lambda\kappa_2(\xi C_1 + C_2)^{k-1} \tanh(\kappa_2(\xi C_1 + C_2)^k + C), \end{aligned}$$

where $\xi = x - p(t)$, $\lambda = \frac{3}{4}kC_1$, $\kappa_1 = \sqrt{\mu_0\mu_1}$, and $\kappa_2 = \sqrt{-\mu_0\mu_1}$.

We now solve the symmetry reduction equation (3.25).

3.2.1. *Periodic solution of (3.25).* Using the wave transform

$$f = p(\eta), \quad \eta = \alpha x + \beta y + \gamma \xi, \quad (3.32)$$

where α, β, γ are real constants and substituting (3.32) into (3.25), we find that the function $p(\eta)$ satisfies the fourth-order nonlinear ordinary differential equation

$$6\alpha^2\gamma p''p' + \alpha^3\gamma p^{(4)} + 3\beta^2 p'' = 0. \quad (3.33)$$

Integrating (3.33) once with respect to η and taking the integration constant to be B , we obtain

$$3\alpha^2\gamma p'(\eta)^2 + \alpha^3\gamma p^{(3)}(\eta) + 3\beta^2 p'(\eta) + B = 0. \quad (3.34)$$

By setting $\frac{dp(\eta)}{d\eta} = q(\eta)$, the second order nonlinear ordinary differential equation reduces to

$$3\alpha^2\gamma q^2 + \alpha^3\gamma q'' + 3\beta^2 q + B = 0. \quad (3.35)$$

Multiplying the equation (3.35) by q' and integrating with respect to η , we will have

$$\frac{1}{2}\alpha^3\gamma q'^2 + \alpha^2\gamma q^3 + \frac{3}{2}\beta^2 q^2 + Bq + C = 0. \quad (3.36)$$

Here we use the same idea as that in the method of expanding the elliptic function mentioned above to obtain the following two types of solutions to equation (3.36).

(1) If the arbitrary constants B and C satisfy

$$9\beta^4 - 16\alpha^6 b^4 \gamma^2 (m^4 - m^2 + 1) - B12\alpha^2 r = 0,$$

$$(4\alpha^3 b^2 \gamma (m^2 + 1) - 3\beta^2) (4\alpha^3 b^2 \gamma (m^2 - 2) - 3\beta^2) = \frac{216B\gamma^2 \alpha^4}{(4\alpha^3 b^2 \gamma (2m^2 - 1) + 3\beta^2)},$$

then the generalized periodic solution can be expressed by the elliptic Jacobi sine function in the form

$$q_1(\eta) = -2\alpha b^2 m^2 \operatorname{sn}^2(b\eta, m) + \frac{4\alpha^3 b^2 \gamma (m^2 + 1) - 3\beta^2}{6\alpha^2 r}. \quad (3.37)$$

When $m \rightarrow 1$, the solution (3.37) reduces to the following shock wave solution

$$q_2(\eta) = -2\alpha b^2 \tanh^2(b\eta) + \frac{8\alpha^3 b^2 \gamma - 3\beta^2}{6\alpha^2 r}. \quad (3.38)$$

In particular, letting $b = bi$ with $i^2 = -1$, the periodic solution is

$$q_3(\eta) = 2\alpha b^2 \tan^2(b\eta) + \frac{8\alpha^3 b^2 \gamma + 3\beta^2}{6\alpha^2 r}. \quad (3.39)$$

(2) If the arbitrarily integral constant satisfies

$$9\beta^4 - 16\alpha^6 b^4 \gamma^2 (m^4 - m^2 + 1) - B12\alpha^2 r = 0,$$

$$(4\alpha^3 b^2 \gamma (m^2 + 1) - 3\beta^2) (4\alpha^3 b^2 \gamma (m^2 - 2) - 3\beta^2) = \frac{216C\gamma^2 \alpha^4}{(4\alpha^3 b^2 \gamma (2m^2 - 1) + 3\beta^2)},$$

then the generalized periodic solution in terms of the Jacobi elliptic cosine function is

$$q_4(\eta) = 2\alpha b^2 m^2 \operatorname{cn}^2(b\eta, m) - \frac{4\alpha^3 b^2 \gamma (2m^2 - 1) + 3\beta^2}{6\alpha^2 r}. \quad (3.40)$$

When $m \rightarrow 1$, solution (3.40) reduces to the traveling solitary wave solution

$$q_5(\eta) = 2\alpha b^2 \operatorname{sech}^2(b\eta) - \frac{4\alpha^3 b^2 \gamma + 3\beta^2}{6\alpha^2 r}. \quad (3.41)$$

Let $b = bi$ with $i^2 = -1$, the periodic solution of traveling wave is given by

$$q_6(\eta) = -2\alpha b \operatorname{sec}^2(b\eta) + \frac{4\alpha^3 b^2 \gamma - 3\beta^2}{6\alpha^2 r}. \quad (3.42)$$

Substituting (3.37)-(3.42) with $p(\eta) = \int q(\eta) d\eta$ into (3.23), we deduce the corresponding solution to YTSF equation 3.1 as

$$v_5 = 2\alpha b E(\operatorname{sn}(b(\alpha x + \beta y + \gamma(z - p(t))), m), m) - p'(t)x - \frac{2}{3}p''(t)y^2 + \rho_5(\alpha x + \beta y + \gamma(z - p(t))), \quad (3.43)$$

$$v_6 = \alpha b (2 \tanh b(\alpha x + \beta y + \gamma(z - p(t))) + \ln \frac{\tanh b(\alpha x + \beta y + \gamma(z - p(t))) - 1}{\tanh b(\alpha x + \beta y + \gamma(z - p(t))) + 1}) - p'(t)x - \frac{2}{3}p''(t)y^2 + \rho_6(\alpha x + \beta y + \gamma(z - p(t))), \quad (3.44)$$

$$v_7 = 2\alpha b \tan b(\alpha x + \beta y + \gamma(z - p(t))) - p'(t)x - \frac{2}{3}p''(t)y^2 + \rho_7(\alpha x + \beta y + \gamma(z - p(t))), \quad (3.45)$$

$$v_8 = 2\alpha b E(\operatorname{sn}(b(\alpha x + \beta y + \gamma(z - p(t))), m), m) - p'(t)x - \frac{2}{3}p''(t)y^2 - \rho_8(\alpha x + \beta y + \gamma(z - p(t))), \quad (3.46)$$

$$v_9 = 2\alpha b \tanh b(\alpha x + \beta y + \gamma(z - p(t))) - p'(t)x - \frac{2}{3}p''(t)y^2 - \rho_9(\alpha x + \beta y + \gamma(z - p(t))), \quad (3.47)$$

$$v_{10} = -2\alpha \tan b(\alpha x + \beta y + \gamma(z - p(t))) - p'(t)x - \frac{2}{3}p''(t)y^2 + \rho_{10}(\alpha x + \beta y + \gamma(z - p(t))), \quad (3.48)$$

$$(3.49)$$

where E is an elliptic integral of the second kind, $p(t) = \int p_3(t)d(t)$, and

$$\begin{aligned} \rho_5 &= \frac{4\alpha^3 b^2(\gamma(m^2 + 1) - 3r) - 3\beta^2}{6\alpha^2 r}, & \rho_6 &= \frac{8\alpha^3 b^2 \gamma - 3\beta^2}{6\alpha^2 r}, \\ \rho_7 &= \frac{4\alpha^3 b^2(2\gamma - 3r) + 3\beta^2}{6\alpha^2 r}, & \rho_8 &= \frac{4b^2 \alpha^3(m^2(2\gamma - 3r) - \gamma + 3r) + 3\beta^2}{6\alpha^2 r}, \\ \rho_9 &= \frac{4\alpha^3 b^2 \gamma + 3\beta^2}{6\alpha^2 r}, & \rho_{10} &= \frac{4\alpha^3 b^2 \gamma - 3\beta^2}{6\alpha^2 r}. \end{aligned}$$

3.2.2. *Variable separation solution to (3.25).* Suppose that (3.25) has a solution in the form

$$f(\xi, x, y) = g(\xi) + h(x, y), \quad (3.50)$$

where $g(\xi)$ and $h(x, y)$ are functions to be undetermined. Substituting (3.50) into (3.25) yields

$$2h_{xx}g_\xi + 3h_{yy} = 0, \quad (3.51)$$

from (3.51) we can obtain

$$2g_\xi = -3\frac{h_{yy}}{h_{xx}}. \quad (3.52)$$

Take $2g_\xi = -\alpha$. Then (3.52) will reduce into $3h_{yy} - \alpha h_{xx} = 0$, where α is an arbitrarily non-zero real number. After finding the results of g_ξ and $h(x, y)$ in (3.50), we obtain

$$g(\xi) = -\frac{1}{2}\alpha\xi + C_1, \quad h(x, y) = r\left(\frac{y\sqrt{\alpha} + \sqrt{3}x}{\sqrt{\alpha}}\right) + s\left(\frac{y\sqrt{\alpha} - \sqrt{3}x}{\sqrt{\alpha}}\right), \quad (3.53)$$

where $r\left(\frac{y\sqrt{\alpha} + \sqrt{3}x}{\sqrt{\alpha}}\right)$ is an arbitrary smooth function of $\frac{y\sqrt{\alpha} + \sqrt{3}x}{\sqrt{\alpha}}$ and $s\left(\frac{y\sqrt{\alpha} - \sqrt{3}x}{\sqrt{\alpha}}\right)$ is an arbitrary smooth function of $\frac{y\sqrt{\alpha} - \sqrt{3}x}{\sqrt{\alpha}}$. Substituting (3.53) into (3.50) reduces to

$$f(x, y, \xi) = -\frac{1}{2}\alpha\xi + C_1 + r\left(\frac{y\sqrt{\alpha} + \sqrt{3}x}{\sqrt{\alpha}}\right) + s\left(\frac{y\sqrt{\alpha} - \sqrt{3}x}{\sqrt{\alpha}}\right). \quad (3.54)$$

Substituting (3.54) into (3.25), we obtain

$$v_{11} = -p'(t)x - \frac{2}{3}p''(t)y^2 - \frac{1}{2}\alpha(z - p(t)) + C_1 + r\left(\frac{y\sqrt{\alpha} + \sqrt{3}x}{\sqrt{\alpha}}\right) + s\left(\frac{y\sqrt{\alpha} - \sqrt{3}x}{\sqrt{\alpha}}\right).$$

3.2.3. *Two-wave solution to (3.25)*. We suppose that

$$f(x, y, \xi) = g(\eta, y), \quad \eta = kx - c\xi. \quad (3.55)$$

Then the second equation in (3.25) is reduced to

$$3g_{yy} - 3ck^2(g_\eta^2)_\eta - ck^3g_{\eta\eta\eta} = 0. \quad (3.56)$$

We make the variable transformation

$$g = 2k(\ln \varphi)_\eta, \quad (3.57)$$

where $\varphi = \varphi(\eta, y)$ is a function to be undetermined. Substituting (3.57) into (3.56), we obtain the Hirota bilinear form

$$(3D_{yy}^2 - ck^3D_\eta^4 + 2B)(\varphi \cdot \varphi) = 0, \quad (3.58)$$

where B is an integral arbitrary constant.

Then we will seek the two-wave solution to (3.58) in the form

$$\varphi = e^{-p_1(\lambda_1\eta + \lambda_2y + \lambda_3)} + \delta_1 \cos(p_2(\lambda_4\eta + \lambda_5y + \lambda_6)) + \delta_2 e^{p_1(\lambda_1\eta + \lambda_2y + \lambda_3)}, \quad (3.59)$$

where δ_i, p_i ($i = 1, 2$) are real constants. If we set the integral constant $B = 0$, then we obtain

$$\begin{aligned} p_1p_2\delta_1\delta_2(2ck^3\lambda_1\lambda_4(\lambda_1^2p_1^2 - \lambda_4^4p_2^2) - 3\lambda_2\lambda_5) &= 0, \\ \delta_1\delta_2(ck^3(\lambda_1^2p_1^2(\lambda_1^2p_1^2 - 6\lambda_4^2p_2^2) + \lambda_4^4p_2^4) - 3\lambda_2^2p_1^2 + 3\lambda_5^2p_2^2) &= 0, \\ (\delta_1^2p_2^2(4ck^3\lambda_4^4p_2^2 + 3\lambda_5^2) + 4\delta_2p_1^2(4ck^3\lambda_1^4p_1^2 - 3\lambda_2^2)) &= 0, \\ \delta_1p_1p_2(2ck^3\lambda_1^3\lambda_4p_1^2 - 2ck^3\lambda_1\lambda_4^3p_2^2 - 3\lambda_2\lambda_5) &= 0, \\ \delta_1(ck^3(\lambda_1^4p_1^4 - 6\lambda_1^2\lambda_4^2p_1^2p_2^2 + \lambda_4^4p_2^4) - 3\lambda_2^2p_1^2 + 3\lambda_5^2p_2^2) &= 0, \\ \delta_1p_1p_2(2ck^3\lambda_1^3\lambda_4p_1^2 - 2ck^3\lambda_1\lambda_4^3p_2^2 - 3\lambda_2\lambda_5) &= 0, \\ \delta_1(ck^3(\lambda_1^4p_1^4 - 6\lambda_1^2\lambda_4^2p_1^2p_2^2 + \lambda_4^4p_2^4) - 3\lambda_2^2p_1^2 + 3\lambda_5^2p_2^2) &= 0. \end{aligned} \quad (3.60)$$

Solving (3.60), we find $c = 12^{-2}k^{-3}\lambda_1^{-4}p_1^{-6}$ and

$$\begin{aligned} \delta_2 &= \delta_1^2\tau, \quad \lambda_2 = \sqrt{3}\left(18 + \frac{\tau}{18}\right), \quad \lambda_4 = \sqrt{3}\lambda_1p_1, \\ \lambda_5 &= \frac{1}{6p_1}, \quad p_2 = \sqrt{1 - 36p_1^2}, \end{aligned} \quad (3.61)$$

with $\tau = 1 - (6p_1)^{-2} \neq 0$.

If $|p_1| > \frac{1}{6}$, we derive an exact solution to (3.60) as

$$\begin{aligned} \varphi_1 &= \delta_1 \cosh\left(h_1\left(\sqrt{3}\lambda_1p_1\eta + \frac{1}{6p_1}y + \lambda_6\right)\right) \\ &+ 2\delta_1\sqrt{\tau} \cosh(p_1(\lambda_1\eta + \lambda_2y + \lambda_3) + \omega_1). \end{aligned} \quad (3.62)$$

If $|p_1| < 1/6$, we derive an exact solution to (3.60) as

$$\begin{aligned} \varphi_2 &= \delta_1 \cos\left(h_2\left(\sqrt{3}\lambda_1p_1\eta + \frac{1}{6p_1}y + \lambda_6\right)\right) \\ &- 2\delta_1\sqrt{-\tau} \sinh(p_1(\lambda_1\eta + \lambda_2y + \lambda_3) + \omega_2), \end{aligned} \quad (3.63)$$

where $\omega_1 = \frac{1}{2} \ln(\delta_1^2 \tau)$, $\omega_2 = \frac{1}{2} \ln(-\delta_1^2 \tau)$, $h_1 = \sqrt{36p_1^2 - 1}$, and $h_2 = \sqrt{1 - 36p_1^2}$, λ_2 are given in equation (3.61). Substituting (3.62) and (3.63) into (3.57) we deduce

$$\begin{aligned} g_1 = & A \left(\sqrt{3} h_1 \sinh \left(h_1 \left(\sqrt{3} \lambda_1 p_1 \eta + \frac{1}{6p_1} y + \lambda_6 \right) \right) \right. \\ & \left. + 2\sqrt{\tau} \sinh \left(p_1 (\lambda_1 \eta + \lambda_2 y + \lambda_3) + \omega_1 \right) \right) \\ & \div \left(\cosh \left(h_1 \left(\sqrt{3} \lambda_1 p_1 \eta + \frac{1}{6p_1} y + \lambda_6 \right) \right) \right. \\ & \left. + 2\sqrt{\tau} \cosh \left(p_1 (\lambda_1 \eta + \lambda_2 y + \lambda_3) + \omega_1 \right) \right), \end{aligned} \quad (3.64)$$

$$\begin{aligned} g_2 = & -A \left(\sqrt{3} h_2 \sin \left(h_2 \left(\sqrt{3} \lambda_1 p_1 \eta + \frac{1}{6p_1} y + \lambda_6 \right) \right) \right. \\ & \left. + 2\sqrt{-\tau} \cosh \left(p_1 (\lambda_1 \eta + \lambda_2 y + \lambda_3) + \omega_2 \right) \right) \\ & \div \left(\cos \left(h_2 \left(\sqrt{3} \lambda_1 p_1 \eta + \frac{1}{6p_1} y + \lambda_6 \right) \right) \right. \\ & \left. - 2\sqrt{-\tau} \sinh \left(p_1 (\lambda_1 \eta + \lambda_2 y + \lambda_3) + \omega_2 \right) \right), \end{aligned} \quad (3.65)$$

where $A = 2k\lambda_1 p_1$. Substituting (3.64) and (3.65) into (3.57), we obtain two-wave solution of YTSF equation 3.1 as

$$v_{12} = -p'(t)x - \frac{2}{3}p''(t)y^2 + A \frac{\sqrt{3}h_1 \sinh(M) + 2\sqrt{\tau} \sinh(N_1)}{\cosh(M) + 2\sqrt{\tau} \cosh(N_1)}, \quad (3.66)$$

$$v_{13} = -p'(t)x - \frac{2}{3}p''(t)y^2 - A \frac{\sqrt{3}h_2 \sin(M) + 2\sqrt{-\tau} \cosh(N_2)}{\cos(M) - 2\sqrt{-\tau} \sinh(N_2)}, \quad (3.67)$$

where $M = h_1(\sqrt{3}\lambda_1 p_1(kx - c(z - p(t))) + \frac{1}{6p_1}y + \lambda_6)$, $N_1 = p_1(\lambda_1(kx - c(z - p(t))) + \lambda_2 y + \lambda_3) + \omega_1$, $N_2 p_1(\lambda_1(kx - c(z - p(t))) + \lambda_2 y + \lambda_3) + \omega_2$, $p(t) = \int p_3(t)d(t)$, and $p_3(t)$ is an arbitrary function of t . Equations (3.66) and (3.67) are quasi traveling wave solutions with respect to the spatial variable x . Figure 5 demonstrates the evolutionary patterns of the interaction between the kink wave with periodic wave and the kink wave with Gaussian wave described by equation 3.66.

Selecting $p(t) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}$ in 3.66, we obtain a composite solution with the Gauss wave and the kink wave, as is shown in Figure 5(a). The Gauss wave is combined with the kink wave in the x -direction, and the Gauss wave travel along the t -direction.

If we choose $p(t) = \cos t$ in 3.66, we obtain a composite solution with the periodic wave and the kinked wave, as is shown in Figure 5(b), in which the periodic wave is combined with the kink wave in the x -direction, and the periodic wave moves along the t -direction.

4. CONCLUSION

In this study, we constructed a series of exact solutions such as the two-wave solution, the periodic solution, and the lump solution by using the Lie symmetry method, the Jacobi elliptic function expansion and the bilinear method. Among them, the lump solution has a non-singular structure and we have simulated its dynamical behaviors with the help of computer graphics (see Figures 1–3). From exact solutions we have found, there are two special interactions: one of which is

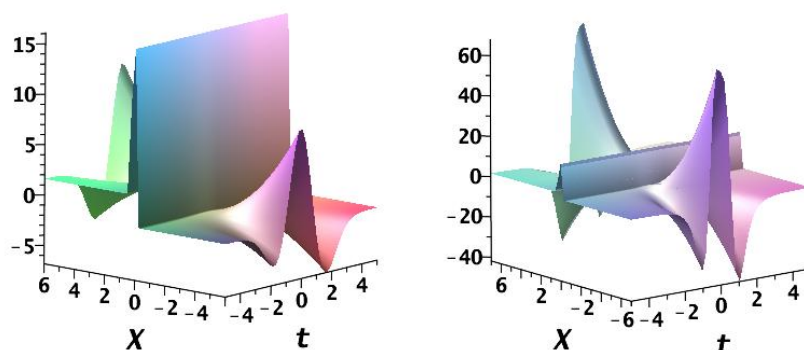


FIGURE 5. Kink wave solution (3.66) with $k = 0.8$, $\delta_1 = \frac{6\sqrt{35}}{35}$, $\lambda_1 = \lambda_6 = p_1 = 1$, $\lambda_3 = -1$.

the aggregation effect and elastic collision effect between the lump solution and the soliton wave (see Figure 4), and the other is the interaction between the kink wave with the periodic wave and the kink wave with the Gaussian wave (see Figure 5). Both of them are of great interest in the community of physics and engineering [16]–[23].

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WEI CHEN

SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA, HEFEI, ANHUI, 230026, CHINA.

SCHOOL OF MATHEMATICS AND PHYSICS, SOUTHWEST UNIVERSITY OF SCIENCE AND TECHNOLOGY, MIANYANG, SICHUAN, 621010, CHINA

Email address: chenwei7@mail.ustc.edu.cn