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GLOBAL LOW-ENERGY WEAK SOLUTIONS FOR COMPRESSIBLE MAGNETO-MICROPOLAR FLUIDS WITH DISCONTINUOUS INITIAL DATA IN $\mathbb{R}^3$

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Abstract. This article concerns the weak solutions of a 3D Cauchy problem of compressible magneto-micropolar fluids with discontinuous initial data. Under the assumption that the initial data are of small energy and the initial density is positive and essentially bounded, we establish the existence of weak solutions that are global-in-time. Moreover, we obtain the large-time behavior of such solutions.

1. Introduction

We consider the 3D compressible magneto-micropolar fluid equation

$$\begin{gathered}
\rho_t + \operatorname{div}(\rho u) = 0, \\
\begin{aligned}
&(\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) \\
&= (\mu + \zeta)\Delta u + (\mu + \lambda - \zeta)\nabla \operatorname{div}(u) + 2\zeta \nabla \times w + (\nabla \times H) \times H,
\end{aligned} \\
(\rho w)_t + \operatorname{div}(\rho u \otimes w) + 4\zeta w = \mu' \Delta w + (\mu' + \lambda')\nabla \operatorname{div}(w) + 2\zeta \nabla \times u, \\
H_t - \nabla \times (u \times H) = -\nabla \times (\nu \nabla \times H), \\
\operatorname{div}(H) = 0,
\end{gathered} \tag{1.1}$$

where the functions $\rho = \rho(x,t) \geq 0$, $u = u(x,t)$, $P(\rho) = a\rho^\gamma$ $(a > 0, \gamma > 1)$, $w = w(x,t)$ and $H = H(x,t)$ are density, velocity, γ-law pressure, micro-rotational velocity and magnetic field for $(x,t) \in \mathbb{R}^3 \times \mathbb{R}_+$. Furthermore, the unknown constants μ, ζ, λ and ν are the shear viscosity coefficient, dynamics micro-rotation viscosity, bulk viscosity coefficient and resistivity coefficient, respectively. μ' and λ' denote the angular viscosities which satisfy the conditions:

$$\mu, \nu, \zeta, \mu' > 0, 2\mu' + 3\lambda' \geq 0, 2\mu + 3\lambda - 4\zeta \geq 0.$$

1.1. **History of the problem.** Before introducing the mathematical theory of the system, we explain the significance of studying magneto-micropolar fluids system. Equation (1.1) is commonly used to model the motion of a compressible conducting micropolar fluid in any magnetic field (see [2]). Because of the great research

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value, the research challenge and the phenomenological significance of magnetic-micropolar fluids in physics and mathematics, more and more researchers in physics and mathematics have devoted themselves to the research of magnetic-micropolar fluid equations.

The purpose of this article is to prove the existence weak solutions, that are global-in-time and low energy, for the Cauchy problem of system(1.1) with the initial conditions

$$(\rho(\cdot,0),u(\cdot,0),w(\cdot,0),H(\cdot,0)) = (\rho_0,u_0,w_0,H_0). \tag{1.2}$$

So that $\rho_0(x)$ has an upper and lower bounds far from zero, and $u_0(x), w_0(x), H_0(x) \in L^p(\mathbb{R}^3)$ satisfy (1.8) and (1.9) for some $p > 6$. Since $(\rho_0(x), u_0(x), w_0(x), H_0(x))$ is small in $L^2(\mathbb{R}^3)$, the total initial energy is small. No other smallness or regularity conditions are imposed.

As is well known, if there is no micro-rotational velocity, magnetic field and dynamics micro-rotation viscosity, i.e. $w = \zeta = H = 0$, magnetic-micropolar system (1.1) reduces to the compressible Navier-Stokes equations. For initial data approaching non-vacuum equilibrium in the $H^3(\mathbb{R}^3)$ space, Matsumura-Nishida first established relevant results on global classical solutions in [18, 19]. Furthermore, when the initial density of the system is small L^2 and bounded in L^∞, and the initial velocity is small in L^2 and bounded in L^{2^n} (the norm of L^2 must be slightly weighted in two dimensions), Hoff [8] proved the global existence of weak solutions in two and three dimensions. Moreover, Hoff [9] extended the above results to general initial data. Later, Lions [14](see also Feireisl et al. [6]) made a major breakthrough in the existence of solutions in two and three dimensional space with arbitrary initial data.

In the absence of a micro-rotation velocity and dynamics micro-rotation viscosity $w = \zeta = 0$, (1.1) reduces to the Magnetohydrodynamic equations (MHD). Assuming that the initial data of system is small in L^2 and the initial density is nonnegative and essentially bounded, Hu-Wang [11] established the global-in-time existence of the weak solutions. In the vacuum case, assuming that the initial energy is suitably small in L^2, Liu-Yu-Zhang [16] investigated the global existence of weak solutions in three-dimensional space. Other related results of the MHD system can be found in [25, 26, 31] and references therein.

When there is no dynamic micro-rotational viscosity $\zeta > 0$ and magnetic field, the compressible micropolar fluids become the magneto-micropolar system. The theory of micropolar fluids was first proposed by Eringen [4] and Lukaszewicz [17]. The problems related to one-dimensional micropolar flow can be referred to the literature such as [20, 21, 22]. In addition, in the case of multi-dimensional micropolar flows, we refer to [23, 27, 15]. In recent years, for weak solutions to equations with discontinuous initial data, Chen has studied the global existence of compressible micropolar fluids in the case that the vacuum state may be included and the oscillations of the solutions can be arbitrarily large.

For the compressible magneto-micropolar fluids (1.1), [30] analyzes the global existence and optimal convergence rates of the solutions. In the framework of Lions [14], Amirat-Hamdache [3] proved the global existence of finite-energy weak solutions. Recently, for the case of the half-space $\mathbb{R}^n_+ (n = 2, 3)$, Xu-Tan-Wang-Tong [32] established the global low-energy of the weak solutions magnetic-micropolar fluids (1.1) with no slip boundary and discontinuous initial data. Other relevant results can be found in references [28, 24, 30] and their references.

However, to the best of our knowledge, there is no result on the low-energy weak solutions for the 3D compressible magneto-micropolar fluid systems with discontinuous initial data in the whole space. Compared to the NS equation [8] and the MHD system [26], the coupling of the micro-rotation velocity w with the momentum equation $(1.1)_2$ brings research difficulties and research challenges in proving time-independent global energy estimates. In the proof, the local existence theorem of the smooth solution of the system needs to be established first. Compared with [32], our results and methods are very different. The main differences can be outlined as follows. First, [32] established the local existence Theorem by using the results proved by Kagei and Kawashima on the local solvability of an initial boundary value problem for a quasilinear hyperbolic-parabolic system in [12], while we establish local existence Theorem by using Kawashima's results in [13]. Secondly, during the proof process, we establish more effective viscous flux and energy functionals, and the uniform estimate of the energy functionals is much more complicated and difficult. Thirdly, to obtain our main results, we derive pointwise bound for the density ρ which is independent both of time and initial smoothness. The proof process consists of a maximum-principle arguments applied to integral curves of the velocity field and Hölder-continuity of $u(\cdot,t)$. In two cases, we derive the upper and lower pointwise bounds of density. However, [32] only proved the boundness of the density ρ. Finally, compared with [32], we also prove low-energy estimates of effective viscous flux and vorticity and the large-time behavior of the weak solutions.

1.2. **Main results.** Before stating the principal result of the paper, we introduce the notation that we need in later sections. For the definition of Hölder seminorms: if $u:\mathbb{R}^3\to\mathbb{R}^m$ and $\gamma\in(0,1]$, we have

$$\langle u\rangle^{\gamma}=\sup_{y_1,y_2\in\mathbb{R}^3,\ y_1\neq y_2}\frac{|u(y_2)-u(y_1)|}{|y_2-y_1|^{\gamma}}; \tag{1.3}$$

and if $u:\mathcal{S}\subseteq\mathbb{R}^3\times[0,\infty)\to\mathbb{R}^3$ and $\gamma_1,\gamma_2\in(0,1]$,

$$\langle u\rangle_Q^{\gamma_1,\gamma_2}=\sup_{\substack{(y_1,t_1),(y_2,t_2)\in\mathcal{S}\\(y_1,t_1)\neq(y_2,t_2)}}\frac{|u(y_2,t_2)-u(y_1,t_1)|}{|y_2-y_1|^{\gamma_1}+|t_2-t_1|^{\gamma_2}}. \tag{1.4}$$

For simplicity, we abbreviate X^3 to X, where X is a Banach space. When $I\subset[0,\infty)$ is an interval, then $C^1(I;X)$ will be the elements $u\in C(I;X)$ such that the distribution derivative $u_t\in\mathcal{D}'(\mathrm{int}I;\mathbb{R}^3)$ is regarded as an element of $C(I;X)$.

As described in [8, 10], the effective viscous flux plays a crucial part in the study of compressible fluid dynamics. In the following G_1 and G_2 represent the "effective viscosity" of flux, and W_1, W_2 denote the vorticity of magneto-micropolar fluids:

$$\begin{gathered}G_1\triangleq(2\mu+\lambda)\operatorname{div}(u)-(P(\rho)-P(\tilde\rho)),\quad W_1\triangleq\nabla\times u,\\ G_2\triangleq(2\mu'+\lambda')\operatorname{div}(w),\quad W_2\triangleq\nabla\times w.\end{gathered} \tag{1.5}$$

In addition, for the pressure $P(\rho)=a\rho^{\gamma}(a>0,\gamma>1)$, we select two positive bounding densities $\underline{\rho}$ and $\bar\rho$ to fix a positive reference density $\tilde\rho$,

$$\underline{\rho}<\tilde\rho<\bar\rho, \tag{1.6}$$

then we define the nonnegative number

$$\delta=\min\{\tilde\rho-\rho,\frac{1}{2}(\bar\rho-\underline{\rho}),\bar\rho-\tilde\rho\}. \tag{1.7}$$

It should be mentioned that δ need not be "small". For the parameters μ, ζ, λ, μ', λ', and ν, we assume that

$$\begin{gathered} \nu > 0, \quad 0 \le \lambda' < (-\frac{1}{2} + \frac{\sqrt{21}}{6})\mu', \\ \frac{1}{4}(\mu+\zeta)(p-2) - \frac{[\frac{1}{4}(\mu+\lambda-\zeta)(p-2)]^2}{\frac{1}{3}(\mu+\zeta)+(\mu+\lambda-\zeta)} > 0. \end{gathered} \tag{1.8}$$

Hence, we obtain

$$\frac{1}{4}\mu'(p-2) - \frac{[\frac{1}{4}(\mu'+\lambda')(p-2)]^2}{\frac{1}{3}\mu'+(\mu'+\lambda')} > 0, \tag{1.9}$$

for $p = 6$ and thus for some $p > 6$, which we now fix. For (ρ_0, u_0, w_0, H_0), suppose that we have the nonnegative numbers $s < \delta$ and N satisfying

$$\|u_0\|_{L^p} + \|w_0\|_{L^p} + \|H_0\|_{L^p} \le N, \tag{1.10}$$

$$\rho + s < \operatorname{ess\,inf} \rho_0 \le \operatorname{ess\,sup} \rho_0 < \bar\rho - s, \tag{1.11}$$

where $N > 0$ can be arbitrarily large. At the same time, we assume that

$$\operatorname{div}(H)_0 = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^3), \tag{1.12}$$

$$E_0 \triangleq \int_{\mathbb{R}^3} \Big(\frac{1}{2}\rho_0|u_0|^2 + D(\rho_0) + \frac{1}{2}\rho_0|w_0|^2 + \frac{1}{2}|H_0|^2\Big)\,\mathrm{d}x. \tag{1.13}$$

where the potential energy density D can be represented by

$$D(\rho) \triangleq \rho\int_{\tilde\rho}^{\rho} \frac{P(s)-P(\tilde\rho)}{s^2}\,\mathrm{d}s. \tag{1.14}$$

It is easy to show that

$$C_1(\underline{\rho},\tilde\rho,\bar\rho)(\rho-\tilde\rho)^2 \le D(\rho) \le C_2(\underline{\rho},\tilde\rho,\bar\rho)(\rho-\tilde\rho)^2, \tag{1.15}$$

where C_1 and C_2 are nonnegative constants that depend only on $\tilde\rho$, $\bar\rho$ and $\underline{\rho}$.

Definition 1.1. The weak solutions (ρ, u, w, H) of system (1.1) is defined as follows: we assume that $(\rho - \tilde\rho, \rho u, w, H) \in C([0,\infty); H^{-1}(\mathbb{R}^3))$ with $(\rho, u, w, H)|_{t=0} = (\rho_0, u_0, w_0, H_0)$, $(\nabla u, \nabla w, \nabla H) \in L^2(\mathbb{R}^3 \times (0,\infty))$ and $\operatorname{div}(H) = 0$ in $\mathcal{D}'(\mathbb{R}^3)$ for $t > 0$. Furthermore, the following equations hold for $t_2 > t_1 \ge 0$ and C^1 test functions ψ having uniformly bounded support in x for $t \in [t_1, t_2]$:

$$\int_{\mathbb{R}^3} \rho\psi(t,x)\,\mathrm{d}x\Big|_{t_1}^{t_2} = \int_{t_1}^{t_2}\int_{\mathbb{R}^3} (\rho\psi_t + \rho u\cdot\nabla\psi)\,\mathrm{d}x\,\mathrm{d}t, \tag{1.16}$$

$$\begin{aligned} &\int_{\mathbb{R}^3} \rho u\psi(t,x)\,\mathrm{d}x\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_{\mathbb{R}^3} ((\mu+\zeta)\nabla u\cdot\nabla\psi + (\mu+\lambda-\zeta)(\operatorname{div}(u))\nabla\psi)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{t_1}^{t_2}\int_{\mathbb{R}^3} \Big(\rho u\psi_t + \rho u u\cdot\nabla\psi + P(\rho)\operatorname{div}\psi + 2\zeta w \operatorname{rot}\psi \\ &\quad + \frac{1}{2}\nabla|H|^2\operatorname{div}\psi - H^T H\nabla\psi\Big)\,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{1.17}$$

$$\begin{aligned}&\int_{\mathbb{R}^3}\rho w\psi(t,x)\,\mathrm{d}x\Big|_{t_1}^{t_2}+\int_{t_1}^{t_2}\int_{\mathbb{R}^3}(\mu'\nabla w\cdot\nabla\psi+(\mu'+\lambda')(\operatorname{div}(w))\operatorname{div}\psi+4\zeta w\psi)\,\mathrm{d}x\,\mathrm{d}t\\&=\int_{t_1}^{t_2}\int_{\mathbb{R}^3}(\rho w\psi_t+\rho uw\cdot\nabla\psi+2\zeta u\operatorname{rot}\psi)\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}\tag{1.18}$$

$$\begin{aligned}&\int_{\mathbb{R}^3}H(x,t)\psi(x,t)\,\mathrm{d}x\Big|_{t_1}^{t_2}+\nu\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\nabla H\nabla\psi\,\mathrm{d}x\,\mathrm{d}t\\&=\int_{t_1}^{t_2}\int_{\mathbb{R}^3}(H^Tu-u^TH)\nabla\psi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}\tag{1.19}$$

The main results of this article reads as follows.

Theorem 1.2. *Let the parameters of system* (1.1)-(1.2) *satisfy* (1.6)–(1.9)*, let the nonnegative numbers N and $s<\delta$ be given. Then,depending on N, a nonnegative lower bound for s, the parameters and hypothesis of* (1.6)-(1.9)*, there exist positive numbers ε, C, and τ, such that, if (ρ_0,u_0,w_0,H_0) satisfies* (1.16)-(1.19) *and*

$$E_0<\varepsilon.\tag{1.20}$$

Then we have a weak solution (ρ,u,w,H) which satisfies Definition 1.1. At the same time, the solution satisfies the following:

$$\rho-\tilde{\rho},\rho u,w,H\in C([0,\infty);H^{-1}(\mathbb{R}^3)),\tag{1.21}$$

$$\nabla u,\nabla w,\nabla H\in L^2([0,\infty);\mathbb{R}^3),\tag{1.22}$$

$$u(\cdot,t),w(\cdot,t),H(\cdot,t)\in H^1(\mathbb{R}^3),\quad t>0,\tag{1.23}$$

$$G_1(\cdot,t),G_2(\cdot,t),W_1(\cdot,t),W_2(\cdot,t)\in H^1(\mathbb{R}^3),\quad t>0,\tag{1.24}$$

$$\langle u\rangle^{1/2,1/8}_{\mathbb{R}^3\times[\epsilon,\infty)},\langle w\rangle^{1/2,1/8}_{\mathbb{R}^3\times[\epsilon,\infty)},\langle H\rangle^{1/2,1/8}_{\mathbb{R}^3\times[\epsilon,\infty)}\le C(\epsilon)C_0^{\tau},\tag{1.25}$$

where $C(\epsilon)$ may depand on the nonnegative lower bound of ϵ,

$$\underline{\rho}\le\rho(x,t)\le\bar{\rho}\quad a.e.\ on\ \mathbb{R}^3\times[0,\infty),\tag{1.26}$$

and

$$\begin{aligned}&\sup_{t>0}\int_{\mathbb{R}^3}\Big[|\rho-\tilde{\rho}|^2+|u|^2+|w|^2+|H|^2+\vartheta(|\nabla u|^2+|\nabla w|^2+|w|^2+|\nabla H|^2)\\&\quad+\vartheta^5(G_1^2+G_2^2+|\nabla W_1|^2+|\nabla W_2|^2)\Big]\,\mathrm{d}x\\&+\int_0^\infty\int_{\mathbb{R}^3}[|\nabla u|^2+|\nabla w|^2+|\nabla H|^2+\vartheta\Big(|\dot{u}|^2+|\dot{w}|^2\\&+|H_t|^2+|\nabla W_1|^2+|\nabla W_2|^2\Big)+\vartheta^5(|\nabla\dot{u}|^2+|\nabla\dot{w}|^2+|\nabla H_t|^2)]\,\mathrm{d}x\,\mathrm{d}s\\&\le CC_0^{\tau},\end{aligned}\tag{1.27}$$

where $\vartheta(t)=\min\{1,t\}$. Moreover, we also have the following large-time behavior:

$$\lim_{t\to\infty}(\|\rho-\tilde{\rho}\|_{L^l(\mathbb{R}^3)}+\|u\|_{W^{1,r}(\mathbb{R}^3)}+\|w\|_{W^{1,r}(\mathbb{R}^3)}+\|H\|_{W^{1,r}(\mathbb{R}^3)})=0,\tag{1.28}$$

holds for $l\in(2,\infty),r\in(2,6)$.

In section 2, we list a number of auxiliary inequalities and fundamental results that play an irreplaceable role in the proof. In section 3, we work on calculating energy estimates independent of time. In section 4, we establish the upper and lower limits of the key points of density. Finally, we complete the proof of the Theorem 1.2 in section 5.

2. Preliminaries

First, we introduce the celebrated Gagliardo-Nirenberg inequality [1, 33].

Lemma 2.1. *For each $\alpha \in [2,6]$ and $f \in H^1(\mathbb{R}^3)$, we can find a constant $C(\alpha)$ such that*

$$\|f\|_{L^\alpha(\mathbb{R}^3)} \le C(\alpha)\|f\|_{L^2(\mathbb{R}^3)}^{(6-\alpha)/2\alpha}\|\nabla f\|_{L^2(\mathbb{R}^3)}^{(3\alpha-6)/2\alpha}. \tag{2.1}$$

For any $\alpha \in (3,\infty)$, $q > 1$ and $f \in L^q(\mathbb{R}^3) \cap W^{1,\alpha}(\mathbb{R}^3)$, there exists a constant $C(\alpha, q)$ such that

$$\|f\|_{L^\infty(\mathbb{R}^3)} \le C(\alpha,q)\|f\|_{L^q(\mathbb{R}^3)}^{q(\alpha-3)/(3\alpha+q(\alpha-3))}\|\nabla f\|_{L^\alpha(\mathbb{R}^3)}^{3\alpha/(3\alpha+q(\alpha-3))}, \tag{2.2}$$

$$\langle f\rangle_{\mathbb{R}^3}^{\beta} \le C(\alpha)\|\nabla f\|_{L^\alpha(\mathbb{R}^3)}, \tag{2.3}$$

where $\beta = 1 - \frac{3}{\alpha}$.

The next Lemma can be found in [8].

Lemma 2.2. *Given $q_1 \in [1,3)$ and $q_2 \in (3,\infty]$, let Γ be the fundamental solution for the Laplace operator in $\mathbb{R}^3$, we can find a constant $C = C(n, q_1, q_2)$ such that*

$$\|\Gamma_{x_j} * g\|_{L^\infty(\mathbb{R}^3)} \le C(n,q_1,q_2)[\|g\|_{L^{q_1}(\mathbb{R}^3)} + \|g\|_{L^{q_2}(\mathbb{R}^3)}]. \tag{2.4}$$

In the process of proving Theorem 1.2, the existence of smooth solutions plays an indispensable role. Therefore, based on a direct generalization of the classical results of the Navier-Stokes equation [29] and Magnetohydrodynamics equation [13], we will give the following formal existence results:

Theorem 2.3. *Suppose that μ, μ', ζ and ν are positive constants and the pressure satisfy $P \in C^3((0,\infty))$. Then given $\tilde\rho > \underline{\rho} > 0$ and $C_3 > 0$, there exists a positive time T which depends on $\tilde\rho$, $\underline{\rho}$, and C_3 and on the system parameters μ, ζ, μ', ν, λ, λ' and P, such that, if the initial data $(\bar\rho_0 - \tilde\rho, u_0, w_0, H_0)$ satisfies*

$$\|(\rho_0 - \tilde\rho, u_0, w_0, H_0)\|_{H^3(\mathbb{R}^3)} < C_3, \tag{2.5}$$

$\inf \rho_0 \ge \underline{\rho}$, *and* $\operatorname{div}(H)_0 = 0$, *then in $\mathbb{R}^3 \times [0,T]$ the solution (ρ,u,w,H) of* (1.1)-(1.2) *satisfying*

$$\rho - \tilde\rho \in C^1([0,T]; H^2(\mathbb{R}^3)) \cap C([0,T]; H^3(\mathbb{R}^3)), \tag{2.6}$$

$$u, w, H \in C^1([0,T]; H^1(\mathbb{R}^3)) \cap C([0,T]; H^3(\mathbb{R}^3)) \cap L^2([0,T]; H^4(\mathbb{R}^3)). \tag{2.7}$$

Then the equations in (1.1) *are satisfied in the sense of equality of weak derivatives on $\mathbb{R}^3\times(0,T)$, and each weak derivative is treated as an element of $C([0,T];H^1(\mathbb{R}^3))$, and the weak form* (1.16)-(1.19) *hold.*

In addition, there exists a nonnegative number that depends on μ, ζ, μ', ν, λ, λ' and P. Then, if the above assumptions hold with $C_3 < \varepsilon$, the solution exists on $\mathbb{R}^3 \times [0,\infty)$.

Lemma 2.4. *Under the assumption that (ρ,u,w,H) is the smooth solution of* (1.1) *on $(0,T)\times\mathbb{R}^3$. Then, for any $2 \le p \le 6$, there exists a constant $C > 0$ depending on μ, λ, μ', λ' and ζ such that*

$$\|(\nabla G_1, \nabla G_2, \nabla W_1, \nabla W_2)\|_{L^p} \le C\|(\rho\dot u, \rho\dot w, \nabla u, \nabla w, w, H\cdot\nabla H, \nabla|H|^2)\|_{L^p}, \tag{2.8}$$

$$\|\nabla u(\cdot,t)\|_{L^p} \le C\|G_1(\cdot,t), W_1(\cdot,t), (P(\rho)-\tilde{P})(\cdot,t)\|_{L^p}, \tag{2.9}$$
$$\|\nabla w(\cdot,t)\|_{L^p} \le C\|G_2(\cdot,t), W_2(\cdot,t)\|_{L^p}, \tag{2.10}$$

where the definitions of G_1, G_2, W_1 and W_2 can be found (1.5) *and* $\|(f,g)\|_{L^p}$ *represents* $\|f\|_{L^p} + \|g\|_{L^p}$*.*

Proof. According to (1.5), one has

$$\begin{gathered}\rho\dot{u} = \nabla G_1 - (\mu+\zeta)\operatorname{rot} W_1 + 2\zeta \operatorname{rot} w - (\nabla\times H)\times H,\\ \rho\dot{w} + 4\zeta w = \nabla G_2 - \mu' \operatorname{rot} W_2 + 2\zeta \operatorname{rot} u,\end{gathered} \tag{2.11}$$

and thus we obtain the following equations

$$\begin{gathered}\Delta G_1 = \operatorname{div}(\rho\dot{u}) - \operatorname{div}[(\nabla\times H)\times H],\\ \Delta G_2 = \operatorname{div}(\rho\dot{w}) - 4\zeta\operatorname{div}(w),\\ (\mu+\zeta)\Delta W_1 = \nabla\times(\rho\dot{u}) - 2\zeta\nabla\times W_2 - \nabla\times[(\nabla\times H)\times H],\\ \mu'\Delta W_2 - 4\zeta W_2 = \nabla\times(\rho\dot{w}) - 2\zeta\nabla\times W_1,\end{gathered} \tag{2.12}$$

where $\dot{g} \triangleq g_t + u\cdot\nabla g$. Next, applying standard L^p-estimates for elliptic systems, we obtain (2.8). Meanwhile, by the definition $-\Delta u = \nabla\times W_1 - \nabla\operatorname{div} u$ we have

$$\nabla u = \nabla(-\Delta)^{-1}\nabla\times W_1 - \nabla(-\Delta)^{-1}\nabla\operatorname{div} u. \tag{2.13}$$

On the other hand, employing the Marcinkiewicz multiplier Theorem for the above relation, it holds that

$$\|\nabla u(\cdot,t)\|_{L^p} \le C[\operatorname{div} u + \|W_1(\cdot,t)\|_{L^p}] \le C\|G_1(\cdot,t), W_1(\cdot,t), (P-\tilde{P})(\cdot,t)\|_{L^p}. \tag{2.14}$$

Inequality (2.10) can be obtained by employing the similar method. The proof is complete. □

3. Energy estimates

Here we establish several prior bounds of the smooth solution described in Section 2, which can roughly correspond to (1.27). These estimates require quite complex and technical methods. Therefore we omit the identical and analogous parts of the proof as in [8] and [26]. Specifically, we define a new energy functional

$$\begin{aligned}A(t) = &\sup_{0<s\le t} \vartheta(\|\nabla u\|_{L^2}^2 + \|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + \|\nabla H\|_{L^2}^2)\\ &+ \sup_{0<s\le t} \vartheta^5(\|\dot{u}\|_{L^2}^2 + \|\dot{w}\|_{L^2}^2 + \|\nabla W_1\|_{L^2}^2 + \|\nabla W_2\|_{L^2}^2 + \|H_t\|_{L^2}^2)\\ &+ \int_0^t \vartheta(\|\dot{u}\|_{L^2}^2 + \|\dot{w}\|_{L^2}^2 + \|\nabla W_1\|_{L^2}^2 + \|\nabla W_2\|_{L^2}^2 + \|H_t\|_{L^2}^2)\,\mathrm{d}s\\ &+ \int_0^t \vartheta^5(\|\nabla\dot{u}\|_{L^2}^2 + \|\nabla\dot{w}\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2)\,\mathrm{d}s,\end{aligned} \tag{3.1}$$

where $\vartheta(t) = \min\{1,t\}$. Assuming that the initial energy E_0 in (1.20) is small and the upper and lower bound of the density is away from zero, then we obtain a prior bound for $A(t)$ in the following proposition.

Proposition 3.1. *Assume that the parameters in system* (1.1) *satisfy the conditions of* (1.6)-(1.9)*, and N and $s<\delta$ are nonnegative constants. If (ρ,u,w,H) is a solution of* (1.1) *on $\mathbb{R}^3\times[0,T]$ in the sense of Theorem 2.3 with initial data*

$(\rho_0-\tilde{\rho},u_0,H_0)\in H^3(\mathbb{R}^3))$ *satisfying* (1.10)-(1.13). *Then according to the hypotheses and notation in* (1.6)-(1.9), N *and the nonnegative lower bound of* s, *there exist nonnegative numbers* ε, M *and* τ *such that, if* $C_0<\varepsilon$ *and*

$$\underline{\rho}\le \rho(x,t)\le \bar{\rho}\quad \text{on } \mathbb{R}^3\times[0,T], \tag{3.2}$$

we have $A(T)\le MC_0^{\tau}$.

The proof will be given in the series of lemmas below, and more specifically, we estimate some auxiliary energy functionals. We start by reviewing the definition of p in (1.8) and (1.9), which is an open condition and thus allows us to select $q\in[6,\min\{p,12\})$ that satisfies both (1.8) and (1.9). So, we can define

$$\begin{aligned}\bar{A}(t)&=\sup_{1\le s\le t}\Big(\|\nabla u\|_{L^2}^2+\|w\|_{L^2}^2+\|\nabla w\|_{L^2}^2+\|\nabla H\|_{L^2}^2\Big)\\&\quad+\sup_{1\le s\le t}\Big(\|\dot u\|_{L^2}^2+\|\dot w\|_{L^2}^2+\|\nabla W_1\|_{L^2}^2+\|\nabla W_2\|_{L^2}^2+\|H_t\|_{L^2}^2\Big)\\&\quad+\int_1^t\Big(\|\dot u\|_{L^2}^2+\|\dot w\|_{L^2}^2+\|\nabla W_1\|_{L^2}^2+\|\nabla W_2\|_{L^2}^2+\|H_t\|_{L^2}^2\Big)\,\mathrm{d}s\\&\quad+\int_1^t(\|\nabla\dot u\|_{L^2}^2+\|\nabla\dot w\|_{L^2}^2+\|\nabla H_t\|_{L^2}^2)\,\mathrm{d}s,\end{aligned}\tag{3.3}$$

$$\begin{aligned}O_q(t)&=\sup_{0\le s\le t}(\|u\|_{L^q}^q+\|w\|_{L^q}^q+\|H\|_{L^q}^q)\\&\quad+\int_0^t\int_{\mathbb{R}^3}\Big(|u|^{q-2}|\nabla u|^2+|w|^{q-2}|\nabla w|^2+|H|^{q-2}|\nabla H|^2\Big)\,\mathrm{d}x\,\mathrm{d}s\\&\quad+\int_0^t\int_{\mathbb{R}^3}\Big(|u|^{q-4}|\nabla(|u|^2)|^2+|w|^{q-4}|\nabla(|w|^2)|^2\\&\quad+|H|^{q-4}|\nabla(|H|^2)|^2\Big)\,\mathrm{d}x\,\mathrm{d}s,\end{aligned}\tag{3.4}$$

$$P(t)=\sup_{0<s\le t}\int_{\mathbb{R}^3}\vartheta^5\Big(|u|^2|\nabla H|^2+|\nabla u|^2|H|^2+|H|^2|\nabla H|^2\Big)\,\mathrm{d}x,\tag{3.5}$$

$$\bar{P}(t)=\sup_{1\le s\le t}\int_{\mathbb{R}^3}\Big(|\nabla u|^2|H|^2+|u|^2|\nabla H|^2+|H|^2|\nabla H|^2\Big)\,\mathrm{d}x,\tag{3.6}$$

$$\begin{aligned}Q(t)&=\int_0^t\vartheta^{3/2}\Big(\|\nabla u\|_{L^3}^3+\|\nabla w\|_{L^3}^3+\|\nabla H\|_{L^3}^3\Big)\\&\quad+\vartheta^5\Big(\|\nabla u\|_{L^4}^4+\|\nabla w\|_{L^4}^4+\|\nabla H\|_{L^4}^4\Big)\,\mathrm{d}s\\&\quad+\Big|\sum_{1\le h_i,k_j\le 3}\int_0^t\int_{\mathbb{R}^3}\vartheta u^{k_1}_{x_{h_1}}u^{k_2}_{x_{h_2}}u^{k_3}_{x_{h_3}}\,\mathrm{d}x\,\mathrm{d}s\Big|,\end{aligned}\tag{3.7}$$

$$\begin{aligned}\bar{Q}(t)&=\int_1^t\Big(\|\nabla u\|_{L^3}^3+\|\nabla w\|_{L^3}^3+\|\nabla H\|_{L^3}^3+\|\nabla u\|_{L^4}^4+\|\nabla w\|_{L^4}^4\\&\quad+\|\nabla H\|_{L^4}^4\Big)\,\mathrm{d}s.\end{aligned}\tag{3.8}$$

The regularity (2.6)-(2.7) is sufficient to justify the following estimate. First, we prove the L^2 energy estimate of the solution.

Lemma 3.2. *Under the assumptions and notation of Proposition 3.1, we have*

$$\begin{aligned}&\sup_{0\le t\le T}(\|\rho-\tilde\rho\|_{L^2}^2+\|u\|_{L^2}^2+\|w\|_{L^2}^2+\|H\|_{L^2}^2)\\&+\int_0^T(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2+\|\nabla H\|_{L^2}^2)\,\mathrm{d}t\le MC_0.\end{aligned}\tag{3.9}$$

Proof. Adding $\langle(1.1)_2,u\rangle$ and $\langle(1.1)_3,w\rangle$ we obtain

$$\begin{aligned}&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\rho|u|^2+\rho|w|^2\,\mathrm{d}x+\int_{\mathbb{R}^3}\nabla P(\rho)u\,\mathrm{d}x+(\mu+\zeta)\|\nabla u\|_{L^2}^2\\&+(\mu+\lambda-\zeta)\|\mathrm{div}u\|_{L^2}^2+\mu'\|\nabla w\|_{L^2}^2+(\mu'+\lambda')\|\mathrm{div}w\|_{L^2}^2+4\zeta\|w\|_{L^2}^2\\&=2\zeta\int_{\mathbb{R}^3}(\nabla\times w)\cdot u\,\mathrm{d}x+\int_{\mathbb{R}^3}(\nabla\times H)\times H\cdot u\,\mathrm{d}x+2\zeta\int_{\mathbb{R}^3}(\nabla\times u)\cdot w\,\mathrm{d}x.\end{aligned}\tag{3.10}$$

First, based on the definition of $D(\rho)$ in (1.14) and the mass equation $(1.1)_1$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}D(\rho)\,\mathrm{d}x+\int_{\mathbb{R}^3}(P(\rho)-P(\tilde\rho))\,\mathrm{div}\,u\,\mathrm{d}x=0.\tag{3.11}$$

For H, we take the similar steps

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|H|^2\,\mathrm{d}x+\nu\|\nabla H\|_{L^2}^2=\int_{\mathbb{R}^3}\nabla\times(u\times H)\cdot H\,\mathrm{d}x.\tag{3.12}$$

Thus by summing up (3.10)-(3.12), we ahve

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\frac12\rho|u|^2+\frac12\rho|w|^2+D(\rho)+\frac12|H|^2\,\mathrm{d}x+(\mu+\zeta)\|\nabla u\|_{L^2}^2\\&+(\mu+\lambda-\zeta)\|\,\mathrm{div}\,u\|_{L^2}^2+\mu^{'}\|\nabla w\|_{L^2}^2+(\mu^{'}+\lambda^{'})\|\mathrm{div}w\|_{L^2}^2\\&+\nu\|\nabla H\|_{L^2}^2+4\zeta\|w\|_{L^2}^2=0.\end{aligned}\tag{3.13}$$

Using the fact $\int\nabla\times w\cdot u=\int\nabla\times u\cdot w$ and Young's inequality, we have

$$4\zeta\int_{\mathbb{R}^3}(\nabla\times u)\cdot w\,\mathrm{d}x\le(\zeta+\frac{\mu}{2})\|\nabla u\|_{L^2}^2+(4\zeta+\frac{2\zeta}{2\zeta+\mu})\|w\|_{L^2}^2.\tag{3.14}$$

Because of the divergence free condition, we have

$$\int_{\mathbb{R}^3}(\nabla\times H)\times H\cdot u\,\mathrm{d}x+\int_{\mathbb{R}^3}\nabla\times(u\times H)\cdot H\,\mathrm{d}x=0.\tag{3.15}$$

Substituting (3.14) and (3.15) into (3.13) and integrating the resultant inequality over $[0,t]$, we obtain (3.9). The proof is complete. □

Lemma 3.3. *Under the assumptions of Proposition 3.1, for $0<t\le1\wedge T$, one has*

$$\begin{aligned}&\sup_{0<s\le t}\vartheta\Big(\|\nabla u\|_{L^2}^2+\|w\|_{L^2}^2+\|\nabla w\|_{L^2}^2+\|\nabla H\|_{L^2}^2\Big)\\&+\int_0^t\vartheta\Big(\|\rho^{1/2}\dot u\|_{L^2}^2+\|\rho^{1/2}\dot w\|_{L^2}^2+\|H_t\|_{L^2}^2\Big)\,\mathrm{d}s\\&\le M\big[C_0+C_0^{\frac{q-6}{q-2}}O_q^{\frac{4}{q-2}}+Q\big],\end{aligned}\tag{3.16}$$

and for $T > 1$ and $1 \le t \le T$, one has

$$\begin{aligned} &\sup_{1\le s\le t} (\|\nabla u\|_{L^2}^2 + \|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + \|\nabla H\|_{L^2}^2) \\ &+ \int_1^t \Big(\|\rho^{1/2}\dot{u}\|_{L^2}^2 + \|\rho^{1/2}\dot{w}\|_{L^2}^2 + \|H_t\|_{L^2}^2\Big)\,\mathrm{d}s \\ &\le M\big[C_0 + C_0^{3/2}\bar{A}^{1/2} + \bar{Q}\big] + A(1). \end{aligned} \tag{3.17}$$

Here $1 \wedge T = \min\{1, T\}$.

Proof. For the case of $0 \le t \le 1 \wedge T$, multiplying $(1.1)_2$-$(1.1)_3$ by $\vartheta\dot{u}$ and $\vartheta\dot{w}$, respectively, and integrating the resultant equations over $\mathbb{R}^3 \times [0, t]$, we have

$$\begin{aligned} &\int_0^t \vartheta\Big(\|\rho^{1/2}\dot{u}\|_{L^2}^2 + \|\rho^{1/2}\dot{w}\|_{L^2}^2\Big)\,\mathrm{d}s \\ &= -\int_0^t\int_{\mathbb{R}^3} \vartheta\dot{u}\cdot\nabla P(\rho)\,\mathrm{d}x + (\mu+\zeta)\int_0^t\int_{\mathbb{R}^3}\vartheta\dot{u}\cdot\Delta u\,\mathrm{d}x\,\mathrm{d}s \\ &\quad + (\mu+\lambda-\zeta)\int_0^t\int_{\mathbb{R}^3}\vartheta\dot{u}\cdot\nabla\operatorname{div}(u)\,\mathrm{d}x\,\mathrm{d}s + \mu'\int_0^t\int_{\mathbb{R}^3}\vartheta\dot{w}\cdot\Delta w\,\mathrm{d}x\,\mathrm{d}s \\ &\quad + (\mu'+\lambda')\int_0^t\int_{\mathbb{R}^3}\vartheta\dot{w}\cdot\nabla\operatorname{div}(w)\,\mathrm{d}x\,\mathrm{d}s + 2\zeta\int_0^t\int_{\mathbb{R}^3}\vartheta\operatorname{rot} w\cdot\dot{u}\,\mathrm{d}x\,\mathrm{d}s \\ &\quad - 4\zeta\int_0^t\int_{\mathbb{R}^3}\vartheta w\cdot\dot{w}\,\mathrm{d}x\,\mathrm{d}s + 2\zeta\int_0^t\int_{\mathbb{R}^3}\vartheta\operatorname{rot} u\cdot\dot{w}\,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \int_0^t\int_{\mathbb{R}^3}\vartheta(H\cdot\nabla H - \frac{1}{2}\nabla|H|^2)\cdot\dot{u}\,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{3.18}$$

From (1.1), we can infer that

$$(P(\rho) - P(\tilde{\rho}))_t + \gamma P(\rho)\operatorname{div}(u) + u\cdot\nabla(P(\rho) - P(\tilde{\rho})) = 0, \tag{3.19}$$

Then integrating by parts and (3.19) for the first term, we have

$$\begin{aligned} &-\int_0^t\int_{\mathbb{R}^3}\vartheta\dot{u}\cdot\nabla P(\rho)\,\mathrm{d}x \\ &= -\int_0^t\int_{\mathbb{R}^3}\vartheta(u_t + u\cdot\nabla u)\nabla P(\rho)\,\mathrm{d}x\,\mathrm{d}s \\ &= \vartheta(t)\int_{\mathbb{R}^3}\operatorname{div}(u)(P(\rho)-P(\tilde{\rho}))\,\mathrm{d}x - \int_0^t\int_{\mathbb{R}^3}\vartheta_t\operatorname{div}(u)(P(\rho)-P(\tilde{\rho}))\,\mathrm{d}x\,\mathrm{d}s \\ &\quad - \int_0^t\int_{\mathbb{R}^3}\vartheta((P(\rho)-P(\tilde{\rho}))_t\operatorname{div}(u) + \nabla P(\rho)u\cdot\nabla u)\,\mathrm{d}x\,\mathrm{d}s \\ &= \vartheta(t)\int_{\mathbb{R}^3}\operatorname{div}(u)(P(\rho)-P(\tilde{\rho}))\,\mathrm{d}x - \int_0^t\int_{\mathbb{R}^3}\vartheta_t\operatorname{div}(u)(P(\rho)-P(\tilde{\rho}))\,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \int_0^t\int_{\mathbb{R}^3}\vartheta((\gamma-1)(P(\rho)-P(\tilde{\rho}))(\operatorname{div}(u))^2 + \nabla P(\rho)u\cdot\nabla u)\,\mathrm{d}x\,\mathrm{d}s \\ &\le M\Big[\vartheta(t)\int_{\mathbb{R}^3}|\nabla u||(\rho-\tilde{\rho})|\,\mathrm{d}x + \int_0^{1\wedge t}\int_{\mathbb{R}^3}|\nabla u||\rho-\tilde{\rho}|\,\mathrm{d}x \\ &\quad + \int_0^t\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}s\Big]. \end{aligned} \tag{3.20}$$

For the second term in (3.18), we obtain

$$
\begin{aligned}
&(\mu+\zeta)\int_0^t\int_{\mathbb{R}^3}\vartheta(u_t+u\cdot\nabla u)\Delta u\,\mathrm{d}x\,\mathrm{d}s\\
&=-\frac{(\mu+\zeta)\vartheta(t)}{2}\|\nabla u\|_{L^2}^2\,\mathrm{d}x+\frac{(\mu+\zeta)}{2}\int_0^{1\wedge t}\|\nabla u\|_{L^2}^2\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+\sum_{1\le h_i,k_j\le 3}\int_0^t\int_{\mathbb{R}^3}\vartheta u_{x_{h_1}}^{k_1}u_{x_{h_2}}^{k_2}u_{x_{h_3}}^{k_3}\,\mathrm{d}s,
\end{aligned}\tag{3.21}
$$

Similarly, we have

$$
\begin{aligned}
&(\mu+\lambda-\zeta)\int_0^t\int_{\mathbb{R}^3}\vartheta(u_t+u\cdot\nabla u)\cdot\nabla\operatorname{div}(u)\,\mathrm{d}x\,\mathrm{d}s\\
&=-\frac{(\mu+\lambda-\zeta)\vartheta(t)}{2}\|\operatorname{div}(u)\|_{L^2}^2+\frac{(\mu+\lambda-\zeta)}{2}\int_0^{1\wedge t}\|\operatorname{div}(u)\|_{L^2}^2\,\mathrm{d}s\\
&\quad+\sum_{1\le h_i,k_j\le 3}\int_0^t\int_{\mathbb{R}^3}\vartheta u_{x_{h_1}}^{k_1}u_{x_{h_2}}^{k_2}u_{x_{h_3}}^{k_3}\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}\tag{3.22}
$$

Later, to deal with the terms $2\zeta\int_0^t\int_{\mathbb{R}^3}\vartheta\operatorname{rot}w\cdot\dot u\,\mathrm{d}x\,\mathrm{d}s$ and $2\zeta\int_0^t\int_{\mathbb{R}^3}\vartheta\operatorname{rot}u\cdot\dot w\,\mathrm{d}x\,\mathrm{d}s$, we use integration by parts to obtain

$$
\begin{aligned}
&2\zeta\int_0^t\int_{\mathbb{R}^3}\vartheta\operatorname{rot}w\cdot\dot u\,\mathrm{d}x\,\mathrm{d}s+2\zeta\int_0^t\int_{\mathbb{R}^3}\vartheta\operatorname{rot}u\cdot\dot w\,\mathrm{d}x\,\mathrm{d}s\\
&=2\zeta\int_0^t\int_{\mathbb{R}^3}\vartheta(w\cdot\operatorname{rot}u_t+w_t\cdot\operatorname{rot}u)\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+2\zeta\int_0^t\int_{\mathbb{R}^3}\vartheta(\operatorname{rot}w\cdot(u\cdot\nabla u)+\operatorname{rot}u\cdot(u\cdot\nabla w))\,\mathrm{d}x\,\mathrm{d}s\\
&\le 2\zeta\vartheta(t)\int_{\mathbb{R}^3}w\cdot\operatorname{rot}u\,\mathrm{d}x-2\zeta\int_0^{1\wedge t}\int_{\mathbb{R}^3}w\cdot\operatorname{rot}u\,\mathrm{d}x\,\mathrm{d}s+2\zeta\int_0^t\int_{\mathbb{R}^3}\vartheta|\nabla w|^2\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+2\zeta\int_0^t\int_{\mathbb{R}^3}\vartheta|u|^2|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}s+2\zeta\int_0^t\int_{\mathbb{R}^3}\vartheta|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+2\zeta\int_0^t\int_{\mathbb{R}^3}\vartheta|u|^2|\nabla w|^2\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}
$$

For the terms $\mu'\int_0^t\int_{\mathbb{R}^3}\vartheta\dot w\cdot\Delta w\,\mathrm{d}x\,\mathrm{d}s$ and $(\mu'+\lambda')\int_0^t\int_{\mathbb{R}^3}\vartheta\dot w\cdot\nabla\operatorname{div}(w)\,\mathrm{d}x\,\mathrm{d}s$, we take the similar proof of (3.21) and (3.22). At the same time, it is obvious that

$$
\begin{aligned}
&-4\zeta\int_0^t\int_{\mathbb{R}^3}\vartheta w\cdot\dot w\,\mathrm{d}x\,\mathrm{d}s\\
&=-2\zeta\vartheta(t)\int_{\mathbb{R}^3}w^2\,\mathrm{d}x+2\zeta\int_0^{1\wedge t}\int_{\mathbb{R}^3}w^2\,\mathrm{d}x\,\mathrm{d}s-4\zeta\int_0^t\int_{\mathbb{R}^3}\vartheta w\cdot(u\cdot\nabla w)\,\mathrm{d}x\\
&\le -2\zeta\vartheta(t)\int_{\mathbb{R}^3}w^2\,\mathrm{d}x+2\zeta\int_0^{1\wedge t}\int_{\mathbb{R}^3}w^2\,\mathrm{d}x\,\mathrm{d}s+2\zeta\int_0^t\int_{\mathbb{R}^3}\vartheta|w|^2\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+2\zeta\int_0^t\int_{\mathbb{R}^3}\vartheta|u|^2|\nabla w|^2\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}\tag{3.23}
$$

Then, for magnetic field H, multiplying the magnetic field equation by ϑH_t and integrating, we have

$$\begin{aligned}
&\int_0^t \vartheta\|H_t\|_{L^2}^2\,\mathrm{d}s + \frac{1}{2}\nu\int_0^t \vartheta\|\nabla H\|_{L^2}^2\,\mathrm{d}s \\
&= \frac{1}{2}\nu\int_0^{1\wedge t}\|\nabla H\|_{L^2}^2\,\mathrm{d}s + \int_0^t\int_{\mathbb{R}^3}\vartheta H_t[\nabla\times(u\times H)]\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned} \tag{3.24}$$

Plugging (3.20)-(3.24) into (3.18) and employing Cauchy inequality, we obtain

$$\begin{aligned}
&\sup_{0<s\le t}\vartheta\big(\|\nabla u\|_{L^2}^2 + \|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + \|\nabla H\|_{L^2}^2\big) \\
&+ \int_0^t \vartheta\Big(\|\rho^{1/2}\dot{u}\|_{L^2}^2 + \|\rho^{1/2}\dot{w}\|_{L^2}^2 + \|H_t\|_{L^2}^2\Big)\,\mathrm{d}s \\
&\le M\Big\{C_0 + Q + \int_0^t\int_{\mathbb{R}^3}\vartheta\big[|\nabla H|^2|H|^2 + |\nabla u|^2|H|^2 \\
&\quad + |\nabla H|^2|u|^2 + |u|^2|\nabla u|^2 + |u|^2|\nabla w|^2\big]\,\mathrm{d}x\,\mathrm{d}s\Big\}.
\end{aligned} \tag{3.25}$$

Utilizing Young's inequality, one has

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}^3}\vartheta|\nabla u|^2|H|^2\,\mathrm{d}x\,\mathrm{d}s \\
&\le M\Big(\int_0^t\int_{\mathbb{R}^3}\vartheta^{3/2}|\nabla u|^3\,\mathrm{d}x\,\mathrm{d}s\Big)^{2/3}\Big(\int_0^t\int_{\mathbb{R}^3}|H|^6\,\mathrm{d}x\,\mathrm{d}s\Big)^{1/3} \\
&\le M\Big[Q + \int_0^t\Big(\int_{\mathbb{R}^3}|H|^2\,\mathrm{d}x\Big)^{\frac{q-6}{q-2}}\Big(\int_{\mathbb{R}^3}|H|^q\,\mathrm{d}x\Big)^{\frac{4}{q-2}}\,\mathrm{d}s\Big] \\
&\le M\big(Q + C_0^{\frac{q-6}{q-2}}O_q^{\frac{4}{q-2}}\big).
\end{aligned} \tag{3.26}$$

The other terms of (3.25) can be estimated in a similar manner used in (3.17). For the case of $1\le t\le T$, employing similar argument used in (3.16), one has

$$\begin{aligned}
&\sup_{1\le s\le t}\big(\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + \|w\|_{L^2}^2 + \|\nabla H\|_{L^2}^2\big) \\
&+ \int_1^t(\|\rho^{1/2}\dot{u}\|_{L^2}^2 + \|\rho^{1/2}\dot{w}\|_{L^2}^2 + \|H_t\|_{L^2}^2)\,\mathrm{d}s \\
&\le M\Big\{C_0 + \bar{Q} + \int_1^t\int_{\mathbb{R}^3}\big[|\nabla H|^2|H|^2 + |\nabla u|^2|H|^2 + |\nabla H|^2|u|^2 + |u|^2|\nabla u|^2 \\
&\quad + |u|^2|\nabla w|^2\big]\,\mathrm{d}x\,\mathrm{d}s\Big\} + A(1).
\end{aligned} \tag{3.27}$$

The terms on the right-hand side of the above inequality can be estimated in a similar way

$$\begin{aligned}
\int_1^t\int_{\mathbb{R}^3}|\nabla H|^2|u|^2\,\mathrm{d}x\,\mathrm{d}s &\le \int_1^t\|\nabla H\|_{L^4}^4 + \|u\|_{L^4}^4\,\mathrm{d}s \\
&\le \bar{Q} + \int_1^t\|u\|_{L^2}\|\nabla u\|_{L^2}^3\,\mathrm{d}s \\
&\le M(\bar{Q} + C_0^{3/2}\bar{A}^{1/2}).
\end{aligned}$$

Substituting the above estimates into (3.27) yields (3.17) directly. The proof is complete. □

In the following lemma, we estimate $\dot{u}$, $\dot{w}$ and H_t.

Lemma 3.4. *Under the assumptions of Proposition 3.1, if* $0 < t \le 1 \wedge T$, *then*

$$
\begin{aligned}
&\sup_{0<s\le t} \vartheta^5 \Big(\|\dot{u}\|_{L^2}^2 + \|\dot{w}\|_{L^2}^2 + \|H_t\|_{L^2}^2 \Big) \\
&+ \int_0^t \vartheta^5 \Big(\|\nabla \dot{u}\|_{L^2}^2 + \|\nabla \dot{w}\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2 \Big)\, \mathrm{d}s \\
&\le M\Big[C_0 + Q + C_0^{\frac{q-6}{q-2}} O_q^{\frac{4}{q-2}} + \Big(C_0^{\frac{q-4}{q-2}} O_q^{\frac{2}{q-2}} \Big) \Big(Q + C_0^{\frac{q-4}{q-2}} O_q^{\frac{2}{q-2}} \Big) \\
&\quad + (C_0^{\frac{q-6}{q-2}} O_q^{\frac{4}{q-2}})^{1/3} A \Big],
\end{aligned} \tag{3.28}
$$

and for the case of $T > 1$ *and* $1 \le t \le T$, *it holds that*

$$
\begin{aligned}
&\sup_{1\le s\le t} \Big(\|\dot{u}\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \|\dot{w}\|_{L^2}^2 \Big) + \int_1^t \Big(\|\nabla \dot{u}\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2 + \|\nabla \dot{w}\|_{L^2}^2 \Big)\, \mathrm{d}s \\
&\le M\big\{ C_0 + C_0 \bar{A}\bar{Q} + C_0^{\frac{2(q-3)}{3(q-2)}} \bar{G}_q^{\frac{2}{3(q-2)}} \bar{A} + \bar{Q} \big\} + A(1).
\end{aligned} \tag{3.29}
$$

Proof. For the case of $0 \le t \le 1 \wedge T$, $(1.1)_2$ and $(1.1)_3$ can be rewritten as

$$
\begin{gathered}
\rho\dot{u} + \nabla P = (\mu+\zeta)\Delta u + 2\zeta\nabla\times w + (\mu+\lambda-\zeta)\nabla \operatorname{div}(u) + (\nabla\times H)\times H, \\
\rho\dot{w} + 4\zeta w = \mu'\Delta w + 2\zeta\nabla\times u + (\mu'+\lambda')\nabla \operatorname{div}(w).
\end{gathered} \tag{3.30}
$$

Next, for the magnetic field H, taking the derivative of $(1.1)_4$ with respect to t, multiplying the result by $\vartheta^5 H_t$ and integrating over $\mathbb{R}^3 \times [0,t]$, we have

$$
\begin{aligned}
&\frac{1}{2}\vartheta^5\|H_t\|_{L^2}^2 + \nu\int_0^t \vartheta^5\|\nabla H_t\|_{L^2}^2\, \mathrm{d}s \\
&= \frac{5}{2}\int_0^t \vartheta^4\vartheta'\|H_t\|_{L^2}^2\, \mathrm{d}s + \int_0^t\int_{\mathbb{R}^3} \vartheta^5 [\nabla\times(u\times H)]_t H_t\, \mathrm{d}x\, \mathrm{d}s.
\end{aligned} \tag{3.31}
$$

By applying the derivative (3.30) with respect to t and $(1.1)_1$, we can infer that

$$
\begin{aligned}
&\rho\dot{u}_t + \rho u\cdot\nabla\dot{u} + \nabla(P(\rho)_t) - (2\zeta\nabla\times w + (\nabla\times H)\times H)_t \\
&= (\mu+\zeta)\Delta\dot{u} + (\mu+\lambda-\zeta)\nabla \operatorname{div}\dot{u} - [(\mu+\zeta)\Delta(u\cdot\nabla u) \\
&\quad + (\mu+\lambda-\zeta)\nabla \operatorname{div}(u\cdot\nabla u)] + \operatorname{div}[((\mu+\zeta)\Delta u + (\mu+\lambda-\zeta)\nabla \operatorname{div}(u))\otimes u \\
&\quad - \nabla P(\rho)\otimes u + (2\zeta\nabla\times w + (\nabla\times H)\times H)\otimes u],
\end{aligned} \tag{3.32}
$$

and

$$
\begin{aligned}
&\rho\dot{w}_t + \rho w\cdot\nabla\dot{w} + (4\zeta w)_t - (2\zeta\nabla\times u)_t \\
&= \mu'\Delta\dot{u} + (\mu'+\lambda')\nabla \operatorname{div}\dot{w} - [\mu'\Delta(u\cdot\nabla w) + (\mu'+\lambda')\nabla \operatorname{div}(u\cdot\nabla w)] \\
&\quad + \operatorname{div}[(\mu'\Delta w + (\mu'+\lambda')\nabla \operatorname{div}(w))\otimes u - (4\zeta w - 2\zeta\nabla\times u)\otimes u].
\end{aligned} \tag{3.33}
$$

Multiplying (3.32) and (3.33) by $\vartheta^5\dot{u}$ and $\vartheta^5\dot{w}$, respectively and integrating over $\mathbb{R}^3\times[0,t]$, we obtain

$$
\begin{aligned}
&\sup_{0<s\le t}\vartheta^5\Big(\|\dot{u}\|_{L^2}^2+\|H_t\|_{L^2}^2+\|\dot{w}\|_{L^2}^2\Big)+\int_0^t\vartheta^5\|\nabla H_t\|_{L^2}^2\,\mathrm{d}s\\
&\le \vartheta^4\vartheta'\int_0^t\Big(\|\rho^{1/2}\dot{u}\|_{L^2}^2+\|\rho^{1/2}\dot{w}\|_{L^2}^2\Big)\,\mathrm{d}s\\
&\quad+(\mu+\zeta)\int_0^t\int_{\mathbb{R}^3}\vartheta^5\dot{u}(\Delta u_t+\operatorname{div}(\Delta u\otimes u))\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+(\mu-\zeta+\lambda)\int_0^t\int_{\mathbb{R}^3}\vartheta^5\dot{u}(\nabla\operatorname{div}u_t+\operatorname{div}(u\partial_j\operatorname{div}u))\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+\mu'\int_0^t\int_{\mathbb{R}^3}\vartheta^5\dot{w}(\Delta w_t+\operatorname{div}(\Delta w\otimes u))\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+(\mu'+\lambda')\int_0^t\int_{\mathbb{R}^3}\vartheta^5\dot{w}[\nabla\operatorname{div}w_t+\operatorname{div}(u\partial_j\operatorname{div}(w))]\,\mathrm{d}x\,\mathrm{d}s\\
&\quad-\int_0^t\int_{\mathbb{R}^3}\vartheta^5\dot{u}(\nabla(P(\rho)_t)+\operatorname{div}(\nabla P(\rho)\otimes u))\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+2\zeta\int_0^t\int_{\mathbb{R}^3}\vartheta^5\dot{u}(\operatorname{rot}w_t+\operatorname{div}(\operatorname{rot}w\otimes u))+\vartheta^5\dot{w}(\operatorname{rot}u_t+\operatorname{div}(\operatorname{rot}u\otimes u))\,\mathrm{d}x\,\mathrm{d}s\\
&\quad-4\zeta\int_0^t\int_{\mathbb{R}^3}\vartheta^5\dot{w}(w_t+\operatorname{div}(w\otimes u))\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\int_{\mathbb{R}^3}\vartheta^5[\nabla\times(u\times H)]_tH_t\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+\int_0^t\int_{\mathbb{R}^3}\vartheta^5\dot{u}\cdot[((\nabla\times H)\times H)_t+\operatorname{div}((\nabla\times H)\times H)\otimes u)]\\
&\quad+\frac{5}{2}\int_0^t\vartheta^4\vartheta'\|H_t\|_{L^2}^2\,\mathrm{d}s.
\end{aligned}
\tag{3.34}
$$

Then, we estimate the second term of (3.34) as follows

$$
\begin{aligned}
&(\mu+\zeta)\int_0^t\int_{\mathbb{R}^3}\vartheta^5\dot{u}(\Delta u_t+\operatorname{div}(\Delta u\otimes u))\,\mathrm{d}x\,\mathrm{d}s\\
&=-(\mu+\zeta)\sum_{i,j=1}^n\int_0^t\int_{\mathbb{R}^3}\vartheta^5(\partial_j\dot{u}\partial_ju_t-\partial_{ij}^2\dot{u}u_i\partial_ju-\partial_i\dot{u}\partial_ju_i\partial_ju)\,\mathrm{d}x\,\mathrm{d}s\\
&=-(\mu+\zeta)\sum_{i,j=1}^n\int_0^t\int_{\mathbb{R}^3}\vartheta^5(|\nabla\dot{u}|^2+\partial_j\dot{u}\partial_iu_i\partial_ju-\partial_j\dot{u}\partial_ju_i\partial_iu-\partial_i\dot{u}\partial_ju_i\partial_ju)\,\mathrm{d}x\,\mathrm{d}s\\
&\le M\Big(-\int_0^t\vartheta^5\|\nabla\dot{u}\|_{L^2}^2\,\mathrm{d}s+\int_0^t\vartheta^5\|\nabla u\|_{L^4}^4\,\mathrm{d}s\Big).
\end{aligned}
\tag{3.35}
$$

Similarly, we can obtain the following estimates

$$
\begin{aligned}
&(\mu+\lambda-\zeta)\int_0^t\int_{\mathbb{R}^3}\vartheta^5\dot{u}(\nabla\operatorname{div}u_t+\operatorname{div}(u\partial_j\operatorname{div}u))\,\mathrm{d}x\,\mathrm{d}s\\
&\le M\Big(-\int_0^t\vartheta^5\|\operatorname{div}\dot{u}\|_{L^2}^2\,\mathrm{d}s+\int_0^t\vartheta^5\|\nabla u\|_{L^4}^4\,\mathrm{d}s\Big),
\end{aligned}
\tag{3.36}
$$

$$
\begin{aligned}
&\mu' \int_0^t \int_{\mathbb{R}^3} \vartheta^5 \dot{w}(\Delta w_t + \operatorname{div}(\Delta w \otimes u))\,\mathrm{d}x\,\mathrm{d}s \\
&\le M\Big(-\int_0^t \vartheta^5 \|\nabla \dot{w}\|_{L^2}^2\,\mathrm{d}s + \int_0^t \vartheta^5 \|\nabla u\|_{L^4}^4\,\mathrm{d}s + \int_0^t \vartheta^5 \|\nabla w\|_{L^4}^4\,\mathrm{d}s\Big),
\end{aligned} \tag{3.37}
$$

$$
\begin{aligned}
&(\mu' + \lambda') \int_0^t \int_{\mathbb{R}^3} \vartheta^5 \dot{w}[\nabla \operatorname{div} w_t + \operatorname{div}(u\partial_j \operatorname{div}(w))]\,\mathrm{d}x\,\mathrm{d}s \\
&\le M\Big(-\int_0^t \vartheta^5 \|\operatorname{div} \dot{w}\|_{L^2}^2\,\mathrm{d}s + \int_0^t \vartheta^5 \|\nabla u\|_{L^4}^4\,\mathrm{d}s + \int_0^t \vartheta^5 \|\nabla w\|_{L^4}^4\,\mathrm{d}s\Big).
\end{aligned} \tag{3.38}
$$

Integrating by parts and by (3.19), we obtain

$$
\begin{aligned}
&-\int_0^t \int_{\mathbb{R}^3} \vartheta^5 \dot{u}(\nabla(P(\rho)_t) + \operatorname{div}(\nabla P(\rho) \otimes u))\,\mathrm{d}x\,\mathrm{d}s \\
&= \sum_{i,j=1}^n \int_0^t \int_{\mathbb{R}^3} \vartheta^5 (-\gamma P(\rho) \operatorname{div} u \partial_j \dot{u} + \partial_j \dot{u} \partial_i u_i P(\rho) - P(\rho) \partial_j(\partial_i \dot{u} u_i))\,\mathrm{d}x\,\mathrm{d}s \\
&\le M\Big(-\int_0^t \vartheta^5 \|\nabla \dot{u}\|_{L^2}^2\,\mathrm{d}s + \int_0^t \vartheta^5 \|\nabla u\|_{L^2}^2\,\mathrm{d}s\Big),
\end{aligned} \tag{3.39}
$$

and

$$
\begin{aligned}
&2\zeta \int_0^t \int_{\mathbb{R}^3} \vartheta^5 [\dot{u}(\operatorname{rot} w_t + \operatorname{div}(\operatorname{rot} w \otimes u)) + \dot{w}(\operatorname{rot} u_t + \operatorname{div}(\operatorname{rot} u \otimes u))]\,\mathrm{d}x\,\mathrm{d}s \\
&= 4\zeta \int_0^t \int_{\mathbb{R}^3} \vartheta^5 \operatorname{rot}(\dot{u}\dot{w})\,\mathrm{d}x \\
&\quad - 2\zeta \int_0^t \int_{\mathbb{R}^3} [\vartheta^5 \operatorname{rot} \dot{u} \cdot (u \cdot \nabla w)\,\mathrm{d}x - \vartheta^5 \operatorname{rot} \dot{w} \cdot (u \cdot \nabla u)]\,\mathrm{d}x\,\mathrm{d}s \\
&\quad - 2\zeta \sum_{i=1}^n \int_0^t \int_{\mathbb{R}^3} \vartheta^5 u_i \partial_i \dot{u} \cdot \operatorname{rot} w\,\mathrm{d}x\,\mathrm{d}s - 2\zeta \sum_{i=1}^n \int_0^t \int_{\mathbb{R}^3} \vartheta^5 u_i \partial_i \dot{w} \cdot \operatorname{rot} u\,\mathrm{d}x\,\mathrm{d}s \\
&\le M \int_0^t \int_{\mathbb{R}^3} \vartheta^5 |\operatorname{rot} \dot{u}| |(\tilde{\rho} - \rho)\dot{w} + \rho \dot{u}|\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + M \int_0^t \vartheta^5 \|u\|_{L^6} \|\nabla w\|_{L^3} \|\nabla \dot{u}\|_{L^2}\,\mathrm{d}s + M \int_0^t \vartheta^5 \|u\|_{L^6} \|\nabla u\|_{L^3} \|\nabla \dot{w}\|_{L^2}\,\mathrm{d}s \\
&\le M\Big[\int_0^t \vartheta^5 \|\nabla \dot{u}\|_{L^2}^2\,\mathrm{d}s + \int_0^t \vartheta^5 \|\rho \dot{w}\|_{L^2}^2\,\mathrm{d}s + \int_0^t \vartheta^5 \|\nabla \dot{w}\|_{L^2}^2\,\mathrm{d}s \\
&\quad + \int_0^t \vartheta^5 \|u\|_{L^6}^6\,\mathrm{d}s + \int_0^t \vartheta^5 \|\nabla u\|_{L^3}^3\,\mathrm{d}s + \int_0^t \vartheta^5 \|\nabla w\|_{L^3}^3\,\mathrm{d}s\Big],
\end{aligned}
$$

and

$$
\begin{aligned}
&-4\zeta \int_0^t \int_{\mathbb{R}^3} \vartheta^5 \dot{w}(w_t + \operatorname{div}(w \otimes u))\,\mathrm{d}x\,\mathrm{d}s \\
&= -4\zeta \int \vartheta^5 |\dot{w}|^2\,\mathrm{d}x + 4\zeta \int \vartheta^5 ((u \cdot \nabla w) \cdot \dot{w} + (u \cdot \nabla \dot{w}) \cdot w)\,\mathrm{d}x \\
&\le M\Big[-\int_0^t \vartheta^5 \|\dot{w}\|_{L^2}^2\,\mathrm{d}s + \int_0^t \vartheta^5 \|\nabla u\|_{L^3}^3\,\mathrm{d}s + \int_0^t \vartheta^5 \|w\|_{L^6}^6\,\mathrm{d}s\Big].
\end{aligned} \tag{3.40}
$$

Plugging all the above estimates (3.35)-(3.40) into (3.34), using the definitions of P and O_q, as well as the Lemma 3.2, Lemma 3.3, Cauchy's and Gagliardo-Nirenberg inequalities, we finally derive

$$
\begin{aligned}
&\sup_{0<s\le t}\vartheta^5\int_{\mathbb{R}^3}\Big(|\dot u|^2+|H_t|^2+|\dot w|^2\Big)\,\mathrm{d}x\\
&+\int_0^t\int_{\mathbb{R}^3}\vartheta^5(|\nabla\dot u|^2+|\nabla H_t|^2+|\nabla\dot w|^2)\,\mathrm{d}x\,\mathrm{d}s\\
&\le M\Big[C_0+Q+C_0^{\frac{q-6}{q-2}}O_q^{\frac{4}{q-2}}+\int_0^t\int_{\mathbb{R}^3}\vartheta^5|H|^2|u|^2(|\nabla u|^2+|\nabla H|^2)\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+\int_0^t\int_{\mathbb{R}^3}\vartheta^5(|H|^2|\dot u|^2+|H|^2|H_t|^2+|H_t|^2|u|^2)\,\mathrm{d}x\,\mathrm{d}s\Big].
\end{aligned}\tag{3.41}
$$

Using Gagliardo-Nirenberg inequality and Cauchy inequality, we have

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}^3}\vartheta^5|H|^2|u|^2|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}s\\
&\le\int_0^t\int_{\mathbb{R}^3}\vartheta^5(|H|^8+|u|^8+|\nabla u|^4)\,\mathrm{d}x\,\mathrm{d}s\\
&\le Q+\sup_{0\le s\le t}\|(u,H)\|_{L^4(\mathbb{R}^3)}^4\int_0^t\vartheta^5\|(u,H)\|_{L^\infty(\mathbb{R}^3)}^4\,\mathrm{d}s\\
&\le Q+M[C_0^{\frac{q-4}{q-2}}O_q^{\frac{2}{q-2}}][Q+C_0^{\frac{q-4}{q-2}}O_q^{\frac{2}{q-2}}].
\end{aligned}\tag{3.42}
$$

Similarly, we obtain

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}^3}\vartheta^5|H|^2|\dot u|^2\,\mathrm{d}x\,\mathrm{d}s\\
&\le M\Big(\int_0^t\int_{\mathbb{R}^3}|H|^6\,\mathrm{d}x\,\mathrm{d}s\Big)^{1/3}\Big(\int_0^t\int_{\mathbb{R}^3}\vartheta^{\frac{15}{2}}|\dot u|^3\,\mathrm{d}x\,\mathrm{d}s\Big)^{2/3}\\
&\le M\big(C_0^{\frac{q-6}{q-2}}O_q^{\frac{4}{q-2}}\big)^{1/3}\Big(\int_0^t\vartheta^{15/2}\|\dot u\|_{L^2(\mathbb{R}^3)}^{3/2}\|\nabla\dot u\|_{L^2(\mathbb{R}^3)}^{3/2}\,\mathrm{d}s\Big)^{2/3}\\
&\le M\big(C_0^{\frac{q-6}{q-2}}O_q^{\frac{4}{q-2}}\big)^{1/3}\Big(\int_0^t\vartheta^{15}\|\dot u\|_{L^2(\mathbb{R}^3)}^6\,\mathrm{d}x\,\mathrm{d}s\Big)^{1/6}\\
&\quad\times\Big(\int_0^t\vartheta^5\|\nabla\dot u\|_{L^2(\mathbb{R}^3)}^2\,\mathrm{d}x\,\mathrm{d}s\Big)^{1/2}\\
&\le M\big(C_0^{\frac{q-6}{q-2}}O_q^{\frac{4}{q-2}}\big)^{1/3}A.
\end{aligned}\tag{3.43}
$$

The other integrals in (3.41) can be bounded similarly. Thus, we can obtain (3.28). For the case of $1\le t\le T$, similar to (3.28), one has

$$
\begin{aligned}
&\sup_{0<s\le t}\int_{\mathbb{R}^3}\Big(|\dot u|^2+|H_t|^2+|\dot w|^2\Big)\,\mathrm{d}x+\int_0^t\int_{\mathbb{R}^3}\Big(|\nabla\dot u|^2+|\nabla H_t|^2+|\nabla\dot w|^2\Big)\,\mathrm{d}x\,\mathrm{d}s\\
&\le M\Big[C_0+\bar P+\int_1^t\int_{\mathbb{R}^3}\big(|\nabla u|^2+|\nabla H|^2\big)|H|^2|u|^2\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+\int_1^t\int_{\mathbb{R}^3}\Big(|\dot u|^2|H|^2+|H_t|^2|H|^2+|u|^2|H_t|^2\Big)\,\mathrm{d}x\,\mathrm{d}s\Big]+A(1).
\end{aligned}\tag{3.44}
$$

Employing Cauchy's and Gagliardo-Nirenberg inequalities, we can arrive at

$$\begin{aligned}
&\int_1^t \int_{\mathbb{R}^3} |u|^2|\nabla u|^2|H|^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\le \int_1^t \int_{\mathbb{R}^3} (|u|^8 + |\nabla u|^4 + |H|^8) \,\mathrm{d}x\,\mathrm{d}s \\
&\le \bar{Q} + \sup_{1\le s\le t} \|(u,H)\|_{L^{10/3}(\mathbb{R}^3)}^{10/3} \int_1^t \|(u,H)\|_{L^\infty(\mathbb{R}^3)}^{14/3} \,\mathrm{d}s \\
&\le \bar{Q} + \sup_{1\le s\le t} \big[\|(u,H)\|_{L^2(\mathbb{R}^3)}^{4/3} \|\nabla(u,H)\|_{L^2(\mathbb{R}^3)}^2\big] \\
&\quad \times \int_1^t \|(u,H)\|_{L^2(\mathbb{R}^3)}^{2/3} \|\nabla(u,H)\|_{L^4(\mathbb{R}^3)}^4 \,\mathrm{d}s \\
&\le \bar{Q} + MC_0\bar{A}\bar{Q},
\end{aligned} \tag{3.45}$$

and

$$\begin{aligned}
\int_1^t \int_{\mathbb{R}^3} |H|^2|\dot{u}|^2 \,\mathrm{d}x\,\mathrm{d}s &\le \int_1^t \Big(\int_{\mathbb{R}^3} |H|^3 \,\mathrm{d}x\Big)^{2/3} \Big(\int_{\mathbb{R}^3} |\dot{u}|^6 \,\mathrm{d}x\Big)^{1/3} \mathrm{d}s \\
&\le \Big(\sup_{1\le s\le t} \int_{\mathbb{R}^3} |H|^3 \,\mathrm{d}x\Big)^{2/3} \int_1^t \int_{\mathbb{R}^3} |\nabla \dot{u}|^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\le C_0^{\frac{2(q-3)}{3(q-2)}} \bar{G}_q^{\frac{2}{3(q-2)}} \bar{A}.
\end{aligned} \tag{3.46}$$

Since the other integrals in (3.44) can be bounded similarly, we can prove (3.29). The proof is complete. □

Next, we shall estimate the effective viscous flux and vorticity.

Lemma 3.5. *For $0 < t \le 1 \wedge T$, under the hypotheses of Proposition 3.1, one has*

$$\begin{aligned}
&\sup_{0<s\le t} \vartheta^5 (\|\nabla G_1\|_{L^2}^2 + \|\nabla G_2\|_{L^2}^2 + \|\nabla W_1\|_{L^2}^2 + \|\nabla W_2\|_{L^2}^2) \\
&+ \int_0^t \vartheta (\|\nabla G_1\|_{L^2}^2 + \|\nabla G_2\|_{L^2}^2 + \|\nabla W_1\|_{L^2}^2 + \|\nabla W_2\|_{L^2}^2) \,\mathrm{d}s \\
&\le M\Big(A + P + Q^{2/3} C_0^{\frac{q-6}{3q-6}} O_q^{\frac{4}{3q-6}} + \sup_{0<s\le t} \int_{\mathbb{R}^3} \vartheta^5 |\dot{u}|^2 \,\mathrm{d}x \\
&\quad + \int_0^t \int_{\mathbb{R}^3} \vartheta |\dot{u}|^2 \,\mathrm{d}x\,\mathrm{d}s + \sup_{0<s\le t} \int_{\mathbb{R}^3} \vartheta^5 |\dot{w}|^2 \,\mathrm{d}x + \int_0^t \int_{\mathbb{R}^3} \vartheta |\dot{w}|^2 \,\mathrm{d}x\,\mathrm{d}s\Big),
\end{aligned} \tag{3.47}$$

and for the case of $1 \le t \le T$ and $T > 1$, one has

$$\begin{aligned}
&\sup_{1\le s\le t} (\|\nabla G_1|^2 + \|\nabla G_2|^2 + \|\nabla W_1|^2 + \|\nabla W_2|^2) \\
&+ \int_1^t (\|\nabla G_1|^2 + \|\nabla G_2|^2 + \|\nabla W_1|^2 + \|\nabla W_2|^2) \,\mathrm{d}s \\
&\le M\Big(\bar{A} + \bar{Q} + \sup_{1\le s\le t} \int_{\mathbb{R}^3} |\dot{u}|^2 \,\mathrm{d}x + \int_1^t \int_{\mathbb{R}^3} |\dot{u}|^2 \\
&\quad + \sup_{1\le s\le t} \int_{\mathbb{R}^3} |\dot{w}|^2 \,\mathrm{d}x + \int_1^t \int_{\mathbb{R}^3} |\dot{w}|^2 \,\mathrm{d}x\,\mathrm{d}s\Big).
\end{aligned} \tag{3.48}$$

Proof. From the definition of G_1 and G_2, we obtain $\Delta G_1 = \operatorname{div}(\rho\dot{u}) - \operatorname{div}[(\nabla \times H)\times H]$ and $\Delta G_2 = \operatorname{div}(\rho\dot{w}) - 4\zeta\operatorname{div}(w)$. Then

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}^3}\vartheta(|\nabla G_1|^2+|\nabla G_2|^2+|\nabla W_1|^2+|\nabla W_2|^2)\,\mathrm{d}x\,\mathrm{d}s\\
&\le M\int_0^t\int_{\mathbb{R}^3}\vartheta(|\dot{u}|^2+|\dot{w}|^2+|\nabla u|^2+|\nabla w|^2+|w|^2+|\nabla H|^2|H|^2)\,\mathrm{d}x\,\mathrm{d}s\\
&\le M\Big[A+\Big(\int_0^t\int_{\mathbb{R}^3}|H|^6\,\mathrm{d}x\,\mathrm{d}s\Big)^{1/3}\Big(\int_0^t\int_{\mathbb{R}^3}\vartheta^3|\nabla H|^2\,\mathrm{d}x\,\mathrm{d}s\Big)^{2/3}\Big]\\
&\quad+\int_0^t\int_{\mathbb{R}^3}\vartheta|\dot{u}|^2\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\int_{\mathbb{R}^3}\vartheta|\dot{w}|^2\,\mathrm{d}x\,\mathrm{d}s\\
&\le M\big[A+Q^{2/3}C_0^{\frac{q-6}{3q-6}}O_q^{\frac{4}{3q-6}}\big]+\int_0^t\int_{\mathbb{R}^3}\vartheta|\dot{u}|^2\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\int_{\mathbb{R}^3}\vartheta|\dot{w}|^2\,\mathrm{d}x\,\mathrm{d}s,
\end{aligned}$$

and

$$\begin{aligned}
&\vartheta^5\int_{\mathbb{R}^3}(|\nabla G_1|^2+|\nabla G_2|^2+|\nabla W_1|^2+|\nabla W_2|^2)\,\mathrm{d}x\,\mathrm{d}s\\
&\le M\Big[\sup_{0\le s\le t}\int_{\mathbb{R}^3}\vartheta^5(|\dot{u}|^2+|\dot{w}|^2+|\nabla u|^2+|\nabla w|^2+|w|^2+|\nabla H|^2|H|^2)\,\mathrm{d}x\,\mathrm{d}s\Big]\\
&\le M(A+P)+\sup_{0<s\le t}\int_{\mathbb{R}^3}\vartheta^5|\dot{u}|^2\,\mathrm{d}x+\sup_{0<s\le t}\int_{\mathbb{R}^3}\vartheta^5|\dot{w}|^2\,\mathrm{d}x.
\end{aligned}$$

For the case of $1\le t\le T$, we take a similar approach to (3.47). □

Now we estimate P and $\bar{P}$.

Lemma 3.6. *Under the assumptions of Proposition 3.1, for $0<t\le 1\wedge T$ one has*

$$P\le MA\big[C_0^{\frac{q-4}{2q-4}}O_q^{\frac{2}{2q-4}}+A+A^4\big], \tag{3.49}$$

and for the case $T>1$ and $1\le t\le T$, one has

$$\bar{P}\le M\big[C_0^{1/4}\bar{A}^{7/4}+C_0\bar{A}^4+\bar{A}^2+\bar{A}^5\big]. \tag{3.50}$$

Proof. We only focus on the proof of (3.49), since the proof of (3.50) is similar. Using Gagliardo-Nirenberg inequality and Cauchy inequality, we can bound the first term of P as

$$\begin{aligned}
&\int_{\mathbb{R}^3}\vartheta^5|\nabla u|^2|H|^2\,\mathrm{d}x\\
&\le\vartheta^4\|H(\cdot,t)\|_{L^\infty}^2[\vartheta\|\nabla u(\cdot,t)\|_{L^2}^2]\\
&\le MA\big[\vartheta^4\|H(\cdot,t)\|_{L^4}^2+\vartheta^4\|\nabla H(\cdot,t)\|_{L^4}^2\big]\\
&\le MA\Big[C_0^{\frac{q-4}{2q-4}}O_q^{\frac{2}{2q-4}}+\|\vartheta^{1/2}\nabla H(\cdot,t)\|_{L^2}^{1/2}\|\vartheta^{5/2}\nabla^2H(\cdot,t)\|_{L^2}^{3/2}\Big].
\end{aligned}\tag{3.51}$$

From $(1.1)_4$, we have

$$\begin{aligned}
\|\vartheta^{5/2}\nabla^2H(\cdot,t)\|_{L^2}&\le M\Big[\int_{\mathbb{R}^3}\vartheta^5(|H_t|^2+|\nabla H\cdot u|^2+|\nabla u\cdot H|^2)\,\mathrm{d}x\Big]^{1/2}\\
&\le M(A+P)^{1/2},
\end{aligned}\tag{3.52}$$

which implies

$$\int_{\mathbb{R}^3} \vartheta^5 |\nabla u|^2 |H|^2 \,\mathrm{d}x \le MA\big[C_0^{\frac{q-4}{2q-4}} O_q^{\frac{2}{2q-4}} + A^{1/4}(A+P)^{3/4}\big]. \tag{3.53}$$

The terms $\int_{\mathbb{R}^3} \vartheta^5 |\nabla H|^2 |H|^2 \,\mathrm{d}x$ and $\int_{\mathbb{R}^3} \vartheta^5 |\nabla H|^2 |u|^2 \,\mathrm{d}x$ in P can be estimated in the same method, and we obtain (3.49). $\square$

Now we estimate the energy functional O_q.

Lemma 3.7. *Under the assumptions of Proposition 3.1, for $0 < t \le T$ one has*

$$O_q \le M\Big[C_0^{\frac{p-q}{p-2}} N^{\frac{q-2}{p-2}} + C_0^{\frac{q-6}{6q-12}} O_q^{\frac{2}{3q-6}+1} + C_0^{\frac{q+3}{3(q-2)}} O_q^{\frac{3q-11}{3(q-2)}} + C_0^{\frac{q}{6q-12}} O_q^{\frac{3q-7}{3q-6}}\Big]. \tag{3.54}$$

Proof. Multiplying $(1.1)_2$ and $(1.1)_3$ by $|u|^{q-2}u$ and $|w|^{q-2}w$, respectively and integrating by parts, we obtain

$$\begin{aligned}
& q^{-1}\int_{\mathbb{R}^3} \rho|u|^q \,\mathrm{d}x\Big|_0^t + q^{-1}\int_{\mathbb{R}^3} \rho|w|^q \,\mathrm{d}x\Big|_0^t \\
& + \int_0^t\int_{\mathbb{R}^3} \mu'|w|^{q-2}|\nabla w|^2 + (\mu+\zeta)|u|^{q-2}|\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}s \\
& + \int_0^t\int_{\mathbb{R}^3} \Big[\frac14(\mu+\zeta)(q-2)|u|^{q-4}|\nabla(|u|^2)|^2 \\
& + (\mu+\lambda-\zeta)|u|^{q-2}(\operatorname{div} u)^2\Big] \,\mathrm{d}x\,\mathrm{d}s \\
& + \int_0^t\int_{\mathbb{R}^3} \Big[\frac14\mu'(q-2)|\nabla(|w|^2)|^2|w|^{q-4} + (\mu'+\lambda')|w|^{q-2}(\operatorname{div} w)^2\Big] \,\mathrm{d}x\,\mathrm{d}s \\
& = \int_0^t\int_{\mathbb{R}^3} \Big[\operatorname{div}(|u|^{q-2}u)(P-\tilde P) + |u|^{q-2}u\cdot((\nabla\times H)\times H) \\
& \quad + 2\zeta|u|^{q-2}u\cdot(\nabla\times w)\Big] \,\mathrm{d}x\,\mathrm{d}s \\
& \quad - \int_0^t\int_{\mathbb{R}^3} \frac12(\mu+\lambda-\zeta)(q-2)|u|^{q-4}(\operatorname{div} u)u\cdot\nabla(|u|^2) \,\mathrm{d}x\,\mathrm{d}s \\
& \quad - \int_0^t\int_{\mathbb{R}^3} \frac12(\mu'+\lambda')(q-2)|w|^{q-4}(\operatorname{div} w)w\cdot\nabla(|w|^2) \,\mathrm{d}x\,\mathrm{d}s \\
& \quad + \int_0^t\int_{\mathbb{R}^3} [2\zeta|w|^{q-2}w\cdot(\nabla\times u)] \,\mathrm{d}x\,\mathrm{d}s \\
& := \sum_{i=1}^4 J_i.
\end{aligned} \tag{3.55}$$

Firstly, we estimate the second integral of (3.55). For $\varrho > 0$ we have

$$\begin{aligned}
J_2 &\le \frac12(\mu+\lambda-\zeta)(q-2)\int_0^t\int_{\mathbb{R}^3} |u|^{\frac{q-2}{2}}|\operatorname{div} u||u|^{\frac{q-4}{2}}|\nabla(|u|^2)| \,\mathrm{d}x\,\mathrm{d}s \\
&\le \frac14(\mu+\lambda-\zeta)(q-2)\Big[\varrho\int_0^t\int_{\mathbb{R}^3} |u|^{q-2}|\operatorname{div} u|^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \varrho^{-1}\int_0^t\int_{\mathbb{R}^3} |u|^{q-4}|\nabla(|u|^2)|^2 \,\mathrm{d}x\,\mathrm{d}s\Big].
\end{aligned}$$

If we select

$$\frac{1}{4}(\mu+\lambda-\zeta)(q-2)\varrho = \beta(\mu+\zeta)+(\mu+\lambda-\zeta), \tag{3.56}$$

for a fixed $\beta > 0$, then

$$\begin{aligned} J_2 &\le 3\beta(\mu+\zeta)\int_0^t\int_{\mathbb{R}^3}|u|^{q-2}|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}s + (\mu+\lambda-\zeta)\int_0^t\int_{\mathbb{R}^3}|u|^{q-2}(\operatorname{div}u)^2\,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \frac{[\frac{1}{4}(\mu+\lambda-\zeta)(q-2)]^2}{\beta(\mu+\zeta)+(\mu+\lambda-\zeta)}\int_0^t\int_{\mathbb{R}^3}|u|^{q-4}|\nabla(|u|^2)|^2\,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{3.57}$$

Similarly, for any $\varrho' > 0$, we have

$$\begin{aligned} J_3 &\le \frac{1}{2}(\mu'+\lambda')(q-2)\int_0^t\int_{\mathbb{R}^3}|w|^{\frac{q-2}{2}}|\operatorname{div}w||w|^{\frac{q-4}{2}}|\nabla(|w|^2)|\,\mathrm{d}x\,\mathrm{d}s \\ &\le \frac{1}{4}(\mu'+\lambda')(q-2)\Big[\varrho'\int_0^t\int_{\mathbb{R}^3}|w|^{q-2}|\operatorname{div}w|^2\,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \varrho'^{-1}\int_0^t\int_{\mathbb{R}^3}|w|^{q-4}|\nabla(|w|^2)|^2\,\mathrm{d}x\,\mathrm{d}s\Big]. \end{aligned}$$

If we select

$$\frac{1}{4}(\mu'+\lambda')(q-2)\varrho' = \beta'\mu'+(\mu'+\lambda'), \tag{3.58}$$

for a fixed $\beta' > 0$,

$$\begin{aligned} J_3 &\le 3\beta'\mu'\int_0^t\int_{\mathbb{R}^3}|w|^{q-2}|\nabla w|^2\,\mathrm{d}x\,\mathrm{d}s + (\mu'+\lambda')\int_0^t\int_{\mathbb{R}^3}|w|^{q-2}(\operatorname{div}w)^2\,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \frac{[\frac{1}{4}(\mu'+\lambda')(q-2)]^2}{\beta'\mu'+(\mu'+\lambda')}\int_0^t\int_{\mathbb{R}^3}|w|^{q-4}|\nabla(|w|^2)|^2\,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{3.59}$$

Substituting (3.57) and (3.59) into (3.55), we obtain

$$\begin{aligned} &q^{-1}\int_{\mathbb{R}^3}\rho|u|^q\,\mathrm{d}x\Big|_0^t + (\mu+\zeta)(1-3\beta)\int_0^t\int_{\mathbb{R}^3}|\nabla u|^2|u|^{q-2}\,\mathrm{d}x\,\mathrm{d}s \\ &+ q^{-1}\int_{\mathbb{R}^3}\rho|w|^q\,\mathrm{d}x\Big|_0^t + \mu'(1-3\beta')\int_0^t\int_{\mathbb{R}^3}|w|^{q-2}|\nabla w|^2\,\mathrm{d}x\,\mathrm{d}s \\ &+ \Big[\frac{1}{4}(\mu+\zeta)(q-2) - \frac{[\frac{1}{4}(\mu+\lambda-\zeta)(q-2)]^2}{\beta(\mu+\zeta)+(\mu+\lambda-\zeta)}\Big]\int_0^t\int_{\mathbb{R}^3}|\nabla(|u|^2)|^2|u|^{q-4}\,\mathrm{d}x\,\mathrm{d}s \\ &+ \Big[\frac{1}{4}\mu'(q-2) - \frac{[\frac{1}{4}(\mu'+\lambda')(q-2)]^2}{\beta'\mu'+(\mu'+\lambda')}\Big]\int_0^t\int_{\mathbb{R}^3}|\nabla(|w|^2)|^2|w|^{q-4}\,\mathrm{d}x\,\mathrm{d}s \\ &\le \Big|\int_0^t\int_{\mathbb{R}^3}(P-\tilde{P})\operatorname{div}(|u|^{q-2}u)\,\mathrm{d}x\,\mathrm{d}s\Big| + \Big|\int_0^t\int_{\mathbb{R}^3}2\zeta|u|^{q-2}u\cdot(\nabla\times w)\,\mathrm{d}x\,\mathrm{d}s\Big| \\ &\quad + \Big|\int_0^t\int_{\mathbb{R}^3}|u|^{q-2}u\cdot(H\cdot\nabla H)\,\mathrm{d}x\,\mathrm{d}s\Big| + \Big|\int_0^t\int_{\mathbb{R}^3}|u|^{q-2}u\cdot\nabla(\frac{1}{2}|H|^2)\,\mathrm{d}x\,\mathrm{d}s\Big| \\ &\quad + \Big|\int_0^t\int_{\mathbb{R}^3}2\zeta|w|^{q-2}w\cdot(\nabla\times u)\,\mathrm{d}x\,\mathrm{d}s\Big|. \end{aligned}$$

Because $q \in [6, p)$, inequalities (1.8) and (1.9) remain unchanged when p is replaced by q. When $\beta = 1/3$, the condition int the left bracket is nonnegative. Then for some $\beta \in (0, \frac{1}{3})$, this term which we now fix is positive. Next, we have

$$\begin{aligned} &q^{-1}\int_{\mathbb{R}^3}\rho|u|^q\,\mathrm{d}x\Big|_0^t + q^{-1}\int_{\mathbb{R}^3}\rho|w|^q\,\mathrm{d}x\Big|_0^t \\ &+\int_0^t\int_{\mathbb{R}^3}|\nabla(|u|^2)|^2|u|^{q-4}\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\mathbb{R}^3}|w|^{q-4}|\nabla(|w|^2)|^2\,\mathrm{d}x\,\mathrm{d}s \\ &\le M\Big[\Big|\int_0^t\int_{\mathbb{R}^3}\operatorname{div}(|u|^{q-2}u)(P-\tilde{P})\,\mathrm{d}x\,\mathrm{d}s\Big| + \Big|\int_0^t\int_{\mathbb{R}^3}2\zeta|u|^{q-2}u\cdot(\nabla\times w)\,\mathrm{d}x\,\mathrm{d}s\Big| \\ &\quad+\Big|\int_0^t\int_{\mathbb{R}^3}|u|^{q-2}u\cdot(H\cdot\nabla H)\,\mathrm{d}x\,\mathrm{d}s\Big| + \Big|\int_0^t\int_{\mathbb{R}^3}|u|^{q-2}u\cdot\nabla(\frac{1}{2}|H|^2)\,\mathrm{d}x\,\mathrm{d}s\Big| \\ &\quad+\Big|\int_0^t\int_{\mathbb{R}^3}2\zeta|w|^{q-2}w\cdot(\nabla\times u)\,\mathrm{d}x\,\mathrm{d}s\Big|\Big]. \end{aligned} \tag{3.60}$$

For the magnetic field H, in a similar way as for (3.55), one has

$$\begin{aligned} &q^{-1}\int_{\mathbb{R}^3}|H|^q\,\mathrm{d}x\Big|_0^t + \int_0^t\int_{\mathbb{R}^3}\nu|H|^{q-2}|\nabla H|^2 + \frac{1}{4}v(q-2)|H|^{q-4}|\nabla(|H|^2)|^2\,\mathrm{d}x\,\mathrm{d}s \\ &= -\int_0^t\int_{\mathbb{R}^3}|H|^{q-2}H\cdot(\nabla\times(u\times H))\,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{3.61}$$

By adding (3.60) and (3.61), and using the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^3}(|u|^q + |w|^q + |H|^q)\,\mathrm{d}x \\ &+\int_0^t\int_{\mathbb{R}^3}(|u|^{q-2}|\nabla u|^2 + |w|^{q-2}|\nabla w|^2 + |H|^{q-2}|\nabla H|^2)\,\mathrm{d}x\,\mathrm{d}s \\ &+\int_0^t\int_{\mathbb{R}^3}\Big[|\nabla(|u|^2)|^2|u|^{q-4} + |w|^{q-4}|\nabla(|w|^2)|^2 + |H|^{q-4}|\nabla(|H|^2)|^2)\Big]\,\mathrm{d}x\,\mathrm{d}s \\ &\le M\Big[\int_{\mathbb{R}^3}(|u_0|^q + |w_0|^q + |H_0|^q)\,\mathrm{d}x + \int_0^t\int_{\mathbb{R}^3}|\rho-\tilde{\rho}|\,|\operatorname{div}(|u|^{q-2}u)|\,\mathrm{d}x\,\mathrm{d}s \\ &\quad+\Big|\int_0^t\int_{\mathbb{R}^3}|u|^{q-2}u\cdot(H\cdot\nabla H)\,\mathrm{d}x\,\mathrm{d}s\Big| + \Big|\int_0^t\int_{\mathbb{R}^3}|u|^{q-2}u\cdot\nabla(\frac{1}{2}|H|^2)\,\mathrm{d}x\,\mathrm{d}s\Big| \\ &\quad+\Big|\int_0^t\int_{\mathbb{R}^3}|H|^{q-2}H\cdot(\nabla\times(u\times H))\,\mathrm{d}x\,\mathrm{d}s\Big| \\ &\quad+\Big|\int_0^t\int_{\mathbb{R}^3}2\zeta|w|^{q-2}w\cdot(\nabla\times u)\,\mathrm{d}x\,\mathrm{d}s\Big| \\ &\quad+\Big|\int_0^t\int_{\mathbb{R}^3}2\zeta|u|^{q-2}u\cdot(\nabla\times w)\,\mathrm{d}x\,\mathrm{d}s\Big|\Big] := \sum_{i=1}^7 J_i. \end{aligned} \tag{3.62}$$

By Hölder's and Sobolev's inequalities, one has

$$\begin{aligned} J_1 &\le \Big(\int_{\mathbb{R}^3}|u_0|^2 + |w_0|^2 + |H_0|^2\,\mathrm{d}x\Big)^{\frac{p-q}{p-2}}\Big(\int_{\mathbb{R}^3}|u_0|^p + |w_0|^p + |H_0|^p\,\mathrm{d}x\Big)^{\frac{q-2}{p-2}} \\ &\le MC_0^{\frac{p-q}{p-2}}N^{\frac{q-2}{p-2}}, \end{aligned} \tag{3.63}$$

Employing Gagliardo-Nirenberg inequality, one has

$$
\begin{aligned}
J_2 &\le \Big[\int_0^t\int_{\mathbb{R}^3}|u|^{2q-4}\,\mathrm{d}x\,\mathrm{d}s\Big]^{1/2}\Big[\int_0^t\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}s\Big]^{1/2}\\
&\le C_0^{1/2}\Big[\int_0^t\Big(\int_{\mathbb{R}^3}|u|^{3q}\,\mathrm{d}x\Big)^{1/3}\Big(\int_{\mathbb{R}^3}|u|^{\frac{3}{2}(q-4)}\,\mathrm{d}x\Big)^{2/3}\mathrm{d}s\Big]^{1/2}\\
&\le C_0^{1/2}\Big[\int_0^t\Big(\int_{\mathbb{R}^3}|u|^{q-2}|\nabla u|^2\,\mathrm{d}x\Big)\Big(\int_{\mathbb{R}^3}|u|^{\frac{3}{2}(q-4)}\,\mathrm{d}x\Big)^{2/3}\mathrm{d}s\Big]^{1/2}\\
&\le C_0^{1/2}O_q^{1/2}\sup_{0<s\le t}\Big[\int_{\mathbb{R}^3}|u(s)|^2\,\mathrm{d}x\Big]^{\frac{12-q}{6(q-2)}}\Big[\int_{\mathbb{R}^3}|u(s)|^q\,\mathrm{d}x\Big]^{\frac{3q-16}{6(q-2)}}\\
&\le MC_0^{\frac{q+3}{3(q-2)}}O_q^{\frac{3q-11}{3(q-2)}}.
\end{aligned}
$$

For the other terms, it can be deduced that

$$
\begin{aligned}
&J_3+J_4+J_5\\
&\le \Big[\int_0^t\int_{\mathbb{R}^3}\Big(|\nabla(|u|^2)|^2|u|^{q-4}+|\nabla u|^2|u|^{q-2}\Big)\,\mathrm{d}x\,\mathrm{d}s\Big]^{1/2}\Big[\int_0^t\int_{\mathbb{R}^3}|u|^{q-2}|H|^4\,\mathrm{d}x\,\mathrm{d}s\Big]^{1/2}\\
&\quad+\Big[\int_0^t\int_{\mathbb{R}^3}(|\nabla(|H|^2)|^2|H|^{q-4}+|\nabla H|^2|H|^{q-2})\,\mathrm{d}x\,\mathrm{d}s\Big]^{1/2}\Big[\int_0^t\int_{\mathbb{R}^3}|u|^2|H|^q\,\mathrm{d}x\,\mathrm{d}s\Big]^{1/2}.
\end{aligned}
$$

The terms $\int_0^t\int_{\mathbb{R}^3}|H|^4|u|^{q-2}\,\mathrm{d}x\,\mathrm{d}s$ and $\int_0^t\int_{\mathbb{R}^3}|H|^q|u|^2\,\mathrm{d}x\,\mathrm{d}s$ are also bounded. We will estimate the first term of the above inequality, the other terms are estimated similarly. By Hölder inequality,

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}^3}|H|^4|u|^{q-2}\,\mathrm{d}x\,\mathrm{d}s\\
&\le\Big(\int_0^t\int_{\mathbb{R}^3}|H|^6\,\mathrm{d}x\,\mathrm{d}s\Big)^{1/3}\Big(\int_0^t\int_{\mathbb{R}^3}|H|^{\frac{3q}{2}}\,\mathrm{d}x\,\mathrm{d}s\Big)^{\frac{4}{3q}}\Big(\int_0^t\int_{\mathbb{R}^3}|u|^{\frac{3q}{2}}\,\mathrm{d}x\,\mathrm{d}\,\mathrm{d}s\Big)^{\frac{2(q-2)}{3q}}\\
&\le\big(C_0^{\frac{q-6}{q-2}}O_q^{\frac{4}{q-2}}\big)^{1/3}\Big[\int_0^t\Big(\int_{\mathbb{R}^3}|H|^{\frac{3q}{2}}\,\mathrm{d}x\Big)\,\mathrm{d}s\Big]^{\frac{4}{3q}}\Big[\sup_{0\le s\le t}\int_{\mathbb{R}^3}|u|^q\,\mathrm{d}x\Big]^{1/q}\\
&\quad\times\Big[\int_0^t\int_{\mathbb{R}^3}|\nabla u|^2|u|^{q-2}\,\mathrm{d}x\,\mathrm{d}s\Big]^{\frac{q-2}{2q}}\\
&\le\big(C_0^{\frac{q-6}{q-2}}O_q^{\frac{4}{q-2}}\big)^{1/3}\Big[\sup_{0\le s\le t}\int_{\mathbb{R}^3}|H|^q\,\mathrm{d}x\Big]^{1/q}\Big[\int_0^t(\int_{\mathbb{R}^3}|\nabla H|^2|H|^{q-2}\,\mathrm{d}x)^{3/4}\,\mathrm{d}s\Big]^{\frac{4}{3q}}O_q^{2/q}\\
&\le\big(C_0^{\frac{q-6}{q-2}}O_q^{\frac{4}{q-2}}\big)^{1/3}\Big[\sup_{0\le s\le t}\int_{\mathbb{R}^3}|H|^q\,\mathrm{d}x\Big]^{1/q}\Big[\int_0^t\int_{\mathbb{R}^3}|H|^{q-2}|\nabla H|^2\,\mathrm{d}x\,\mathrm{d}s\Big]^{1/q}O_q^{2/q}\\
&\le\big(C_0^{\frac{q-6}{q-2}}O_q^{\frac{4}{q-2}}\big)^{1/3}O_q^{\frac{q-2}{q}}O_q^{2/q}\\
&\le C_0^{\frac{q-6}{6q-12}}O_q^{\frac{2}{3q-6}+1}.
\end{aligned}
\tag{3.64}
$$

Thus, we have

$$
J_3+J_4+J_5\le C_0^{\frac{q-6}{3q-6}}O_q^{\frac{4}{3q-6}+1}. \tag{3.65}
$$

For the last term, by employing the fact $\int \nabla\times w\cdot u=\int \nabla\times u\cdot w$, one has

$$\begin{aligned} J_6+J_7 \le& \Big[\int_0^t\int_{\mathbb{R}^3}|w|^{q-2}|\nabla w|^2\,\mathrm{d}x\,\mathrm{d}s\Big]^{1/2}\Big[\int_0^t\int_{\mathbb{R}^3}|u|^2|w|^{q-2}\,\mathrm{d}x\,\mathrm{d}s\Big]^{1/2}\\ &+\Big[\int_0^t\int_{\mathbb{R}^3}|u|^{q-2}|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}s\Big]^{1/2}\Big[\int_0^t\int_{\mathbb{R}^3}|w|^2|u|^{q-2}\,\mathrm{d}x\,\mathrm{d}s\Big]^{1/2}. \end{aligned}\tag{3.66}$$

Next, we have the estimate

$$\begin{aligned} &\int_0^t\int_{\mathbb{R}^3}|w|^2|u|^{q-2}\,\mathrm{d}x\,\mathrm{d}s\\ &\le \Big(\int_0^t\int_{\mathbb{R}^3}|w|^4\,\mathrm{d}x\,\mathrm{d}s\Big)^{1/2}\Big(\int_0^t\int_{\mathbb{R}^3}|u|^{2(q-2)}\,\mathrm{d}x\,\mathrm{d}s\Big)^{1/2}\\ &\le M\Big[C_0^{\frac{q-4}{q-2}}O_q^{\frac{2}{q-2}}\Big]^{1/2}\Big[\int_0^t(\int_{\mathbb{R}^3}|u|^{3q}\,\mathrm{d}x)^{1/3}\Big(\int_{\mathbb{R}^3}|u|^{\frac{3}{2}(q-4)}\,\mathrm{d}x\Big)^{2/3}\mathrm{d}s\Big]^{1/2}\\ &\le M\big[C_0^{\frac{q-4}{q-2}}O_q^{\frac{2}{q-2}}\big]^{1/2}\Big[\int_0^t\Big(\int_{\mathbb{R}^3}|u|^{q-2}|\nabla u|^2\,\mathrm{d}x\Big)\Big(\int_{\mathbb{R}^3}|u|^{\frac{3}{2}(q-4)}\,\mathrm{d}x\Big)^{2/3}\mathrm{d}s\Big]^{1/2}\\ &\le M\big[C_0^{\frac{q-4}{q-2}}O_q^{\frac{2}{q-2}}\big]^{1/2}O_q^{1/2}\sup_{0<s\le t}\Big[\int_{\mathbb{R}^3}|u(s)|^2\,\mathrm{d}x\Big]^{\frac{12-q}{6(q-2)}}\Big[\int_{\mathbb{R}^3}|u(s)|^q\,\mathrm{d}x\Big]^{\frac{3q-16}{6(q-2)}}\\ &\le MC_0^{\frac{q}{3(q-2)}}O_q^{\frac{3q-8}{3(q-2)}}. \end{aligned}$$

Thus, we obtain

$$J_6+J_7\le C_0^{\frac{q}{6q-12}}O_q^{\frac{3q-7}{3q-6}}.\tag{3.67}$$

Substituting (3.63)-(3.67) into (3.62), we obtain (3.54). The proof is complete. □

Next, we estimate u in $W^{1,r}$, which plays an important role in proving the estimation of the auxiliary energy functionals Q and $\bar{Q}$.

Lemma 3.8. *Under the assumptions of Proposition 3.1, there exist polynomials φ_1 and φ_2 whose degrees and coefficients depend on M defined by Proposition 3.1 such that: for the case $0<t\le 1\wedge T$, it holds that*

$$Q\le M[\varphi_1(C_0)+\varphi_2(A+O_q)],\tag{3.68}$$

and for the case $1\le t\le T$ and $T>1$, it holds that

$$\bar{Q}\le M[\varphi_1(C_0+A(1))+\varphi_2(\bar{A})].\tag{3.69}$$

The polynomial φ_1 contains no constant term, and the degrees of the monomials in φ_2 are strictly greater than 1.

Proof. We only give the proof of (3.68), and we only need to estimate the term

$$\int_0^t\int_{\mathbb{R}^3}[\vartheta^{3/2}(|\nabla u|^3+|\nabla w|^3+|\nabla H|^3)+\vartheta^5(|\nabla u|^4+|\nabla w|^4+|\nabla H|^4)]\,\mathrm{d}x\,\mathrm{d}s$$

occurring in the definition (3.3) of P. The term

$$\sum_{1\le k_i,j_m\le 3}\Big|\int_0^t\int_{\mathbb{R}^3}\vartheta u^{j_1}_{x_{k_1}}u^{j_2}_{x_{k_2}}u^{j_3}_{x_{k_3}}\,\mathrm{d}x\,\mathrm{d}s\Big|$$

has been bounded in [8]. First, from Lemma 2.4, we have

$$\int_0^t\int_{\mathbb{R}^3}\vartheta^5|\nabla u|^4\,\mathrm{d}x\,\mathrm{d}s\le M\Big[\int_0^t\int_{\mathbb{R}^3}\vartheta^5(|\rho-\tilde\rho|^4+|G_1|^4+|W_1|^4)\,\mathrm{d}x\,\mathrm{d}s\Big].\tag{3.70}$$

The first term in (3.70) can be bounded by MC_0. For the second term, applying Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}^3}\vartheta^5|G_1|^4\,\mathrm{d}x\,\mathrm{d}s\\
&\le\Big(\sup_{0\le s}\int_{\mathbb{R}^3}\vartheta|G_1|^2\,\mathrm{d}x\Big)^{1/2}\Big(\vartheta^5\int_{\mathbb{R}^3}|\nabla G_1|^2\,\mathrm{d}x\Big)^{1/2}\Big(\int_0^t\int_{\mathbb{R}^3}\vartheta|\nabla G_1|^2\,\mathrm{d}x\,\mathrm{d}s\Big).
\end{aligned}\tag{3.71}$$

From the definition of G_1 and Lemma 3.2, we have

$$\Big(\sup_{0\le s\le t}\int_{\mathbb{R}^3}\vartheta|G_1|^2\,\mathrm{d}x\Big)^{1/2}\le M(C_0+A)^{1/2}.\tag{3.72}$$

Meanwhile, from the defintion $\Delta G_1=\operatorname{div}(\rho\dot u)-\operatorname{div}[(\nabla\times H)\times H]$, we obtain

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}^3}\vartheta|\nabla G_1|^2\,\mathrm{d}x\,\mathrm{d}s\\
&\le M\int_0^t\int_{\mathbb{R}^3}\vartheta|g|^2\,\mathrm{d}x\,\mathrm{d}s\\
&\le M\Big[\int_0^t\int_{\mathbb{R}^3}\vartheta(|\dot u|^2+|\nabla H|^2|H|^2)\,\mathrm{d}x\,\mathrm{d}s\Big]\\
&\le M\Big[A+\Big(\int_0^t\int_{\mathbb{R}^3}\vartheta^{3/2}|\nabla H|^2\,\mathrm{d}x\,\mathrm{d}s\Big)^{2/3}\Big(\int_0^t\int_{\mathbb{R}^3}|H|^6\,\mathrm{d}x\,\mathrm{d}s\Big)^{1/3}\Big]\\
&\le M\big[A+P^{2/3}C_0^{\frac{q-6}{3q-6}}O_q^{\frac{4}{3q-6}}\big],
\end{aligned}\tag{3.73}$$

and

$$\vartheta^5\int_{\mathbb{R}^3}|\nabla G_1|^2\,\mathrm{d}x\le M\Big[\sup_{0\le s\le t}\int_{\mathbb{R}^3}\vartheta^5(|\dot u|^2+|\nabla H|^2|H|^2)\,\mathrm{d}x\Big]\le M(A+Q).\tag{3.74}$$

Thus

$$\int_0^t\int_{\mathbb{R}^3}\vartheta^5|G_1|^4\le M(C_0+A)^{1/2}(A+Q)^{1/2}\Big(A+P^{2/3}C_0^{\frac{q-6}{3q-6}}O_q^{\frac{4}{3q-6}}\Big).\tag{3.75}$$

Similarly, using Lemma 3.5, we obtain

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}^3}\vartheta^5|W_1|^4\\
&\le\Big(\sup_{0<s\le t}\int_{\mathbb{R}^3}\vartheta|W_1|^2\,\mathrm{d}x\Big)^{1/2}\Big(\vartheta^5\int_{\mathbb{R}^3}|\nabla W_1|^2\,\mathrm{d}x\Big)^{1/2}\Big(\int_0^t\int_{\mathbb{R}^3}\vartheta|\nabla W_1|^2\,\mathrm{d}x\Big)\\
&\le MA^{1/2}(A+Q)^{1/2}\Big(A+P^{2/3}C_0^{\frac{q-6}{3q-6}}O_q^{\frac{4}{3q-6}}\Big),
\end{aligned}\tag{3.76}$$

which together with (3.70) gives

$$\int_0^t\int_{\mathbb{R}^3}\vartheta^5|\nabla u|^4\le M\Big[(C_0+A)^{1/2}(A+Q)^{1/2}\Big(A+P^{2/3}C_0^{\frac{q-6}{3q-6}}O_q^{\frac{4}{3q-6}}\Big)+C_0\Big],\tag{3.77}$$

as required. Using Lemmas 2.1, 2.4 and 3.5, we have

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}^3} \vartheta^5 |\nabla w|^4 \,\mathrm{d}x \,\mathrm{d}s \\
&\le M\Big[\int_0^t \int_{\mathbb{R}^3} \vartheta^5 (|G_2|^4 + |W_2|^4) \,\mathrm{d}x \,\mathrm{d}s\Big] \\
&\le \Big(\sup_{0\le s} \int_{\mathbb{R}^3} \vartheta |G_2|^2 + \vartheta |W_2|^2 \,\mathrm{d}x\Big)^{1/2} \Big(\vartheta^5 \int_{\mathbb{R}^3} |\nabla G_2|^2 + \vartheta |\nabla W_2|^2 \,\mathrm{d}x\Big)^{1/2} \\
&\quad \times \Big(\int_0^t \int_{\mathbb{R}^3} \vartheta |\nabla G_2|^2 + \vartheta |\nabla W_2|^2 \,\mathrm{d}x \,\mathrm{d}s\Big) \\
&\le M\big[A^{1/2}(A + C_0)^{1/2}\big(A + Q + P^{2/3} C_0^{\frac{q-6}{3q-6}} O_q^{\frac{4}{3q-6}}\big)\big].
\end{aligned} \tag{3.78}
$$

Next, we bound the term $\int_0^t \int_{\mathbb{R}^3} \vartheta^5 |\nabla H|^4 \,\mathrm{d}x \,\mathrm{d}s$. From the magnetic field equation, we have

$$
\begin{aligned}
\sup_{0<s\le t} \int_{\mathbb{R}^3} \vartheta^5 |\nabla^2 H|^2 \,\mathrm{d}x &\le \sup_{0\le s} M \int_{\mathbb{R}^3} \vartheta^5 (|H_t|^2 + |\nabla H|^2 |w|^2 + |\nabla w|^2 |H|^2) \,\mathrm{d}x \\
&\le M(A + P),
\end{aligned}
$$

and from Lemma 3.3 and (3.26), one has

$$
\begin{aligned}
\int_0^t \int_{\mathbb{R}^3} \vartheta |\nabla^2 H|^2 \,\mathrm{d}x \,\mathrm{d}s &\le M \int_0^t \int_{\mathbb{R}^3} \vartheta \big(|H_t|^2 + |\nabla H|^2 |w|^2 + |\nabla w|^2 |H|^2\big) \,\mathrm{d}x \,\mathrm{d}s \\
&\le M\big(A + Q + C_0^{\frac{q-6}{q-2}} O_q^{\frac{4}{q-2}}\big).
\end{aligned}
$$

Applying Lemma 2.1, we obtain

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}^3} \vartheta^5 |\nabla H|^4 \,\mathrm{d}x \,\mathrm{d}s \\
&\le \int_0^t \vartheta^5 \Big(\int_{\mathbb{R}^3} |\nabla H|^2 \,\mathrm{d}x\Big)^{1/2} \Big(\int_{\mathbb{R}^3} |\nabla^2 H|^2 \,\mathrm{d}x\Big)^{3/2} \,\mathrm{d}s \\
&\le \sup_{0<s\le t} \Big(\int_{\mathbb{R}^3} \vartheta |\nabla H|^2 \,\mathrm{d}x\Big)^{1/2} \Big(\int_{\mathbb{R}^3} \vartheta^5 |\nabla^2 H|^2 \,\mathrm{d}x\Big)^{1/2} \int_0^t \int_{\mathbb{R}^3} \vartheta |\nabla^2 H|^2 \,\mathrm{d}x \,\mathrm{d}s \\
&\le M A^{1/2} (A + P)^{1/2} \big(A + Q + C_0^{\frac{q-6}{q-2}} O_q^{\frac{4}{q-2}}\big),
\end{aligned} \tag{3.79}
$$

as required. The term $\int_1^t \int_{\mathbb{R}^3} (|\nabla u|^3 + |\nabla w|^3 + |\nabla H|^3) \,\mathrm{d}x \,\mathrm{d}s$ can be bounded in a similarway. Then employing Lemma 3.6, we can bound Q and thus proves (3.68). We completed the proof. □

Taking Lemmas 3.2-3.8 together, we obtain the following bound for $A + O_q$.

Lemma 3.9. *Under the assumptions and concepts of Proposition 3.1, for $0 < t \le 1 \wedge T$, there are polynomials φ_1 and φ_2 as described in Lemma 3.8 such that*

$$
A + O_q \le M[\varphi_1(C_0) + \varphi_2(A + O_q)], \tag{3.80}
$$

and for $1 \le t \le T$ and $T > 1$, it holds that

$$
\bar{A} \le M[\varphi_1(A(1) + C_0) + \varphi_2(\bar{A})]. \tag{3.81}
$$

3.1. **Proof of Proposition 3.1.** The statement can be derived directly from the bounds (3.80) and (3.81) and the fact that the functions $A, \bar{A}, O_q$ are continuous in time.

4. Point wise bounds for the density

We establish pointwise estimates for the density ρ, which are independent both of time and of initial smoothness. At the same time, this will close the estimates of Proposition 3.1, and provide an uncontingent estimate for the energy functional A defined in (3.1). We next list two auxiliary lemmas. The first lemma in Hoff [8] is a maximum-principle argument applied to integral curves of the velocity field.

Lemma 4.1. *Suppose that (ρ, u, w, H) is the solution satisfying Proposition 3.1 and $0 < C_1 \le \rho \le C_2$ on $\mathbb{R}^3 \times [0, T]$. Then, we fix $t_0 \ge 0$ and define the particle trajectories $x : [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ by*

$$\begin{aligned} \dot{x}(t, y) &= u(x(t, y), t), \\ x(t_0, y) &= y. \end{aligned} \tag{4.1}$$

Therefore, when $f \in L^1(\mathbb{R}^3)$ is non-negative and $t \in [0, T]$, there exists a constant C only depending on C_1 and C_2 such that each of the integrals $\int_{\mathbb{R}^3} f(x(t, y))\,\mathrm{d}y$ and $\int_{\mathbb{R}^3} f(x)\,\mathrm{d}x$ is bounded by C times the other.

In the second lemma, we derive a result concerning Hölder-continuity of $u(\cdot, t)$ to various norms appearing in the definition (3.1) of the functional A:

Lemma 4.2. *Let (ρ, u, w, H) be the solution satisfying Proposition 3.1. If $t \in (0, T]$ and $\alpha \in (0, 1/2]$, one has*

$$\begin{aligned} \langle u(\cdot, t)\rangle^\alpha \le M\Big[&(\|\nabla H \cdot H(\cdot, t)\|_{L^2}^2 + \|\dot{u}(\cdot, t)\|_{L^2}^2)^{\frac{1+2\alpha}{4}}(C_0 + \|\nabla u(\cdot, t)\|_{L^2}^2)^{\frac{1-2\alpha}{4}} \\ &+ \|\nabla u(\cdot, t)\|_{L^2}^{\frac{1-2\alpha}{2}}\|\nabla W_1(\cdot, t)\|_{L^2}^{\frac{1+2\alpha}{2}} + C_0^{\frac{1-\alpha}{3}}\Big], \end{aligned} \tag{4.2}$$

$$\begin{aligned} \langle w(\cdot, t)\rangle^\alpha \le M\Big[&(\|\nabla w(\cdot, t)\|_{L^2}^2)^{\frac{1-2\alpha}{4}}(\|\dot{w}(\cdot, t)\|_{L^2}^2 + \|w(\cdot, t)\|_{L^2}^2)^{\frac{1+2\alpha}{4}} \\ &+ \|\nabla w(\cdot, t)\|_{L^2}^{\frac{1-2\alpha}{2}}\|\nabla W_2(\cdot, t)\|_{L^2}^{\frac{1+2\alpha}{2}}\Big]. \end{aligned} \tag{4.3}$$

Proof. Let $\alpha \in (0, 1/2]$ and define $r \in (3, 6]$ by $r = 3/(1-\alpha)$. Using 2.3 and 2.10, we have

$$\begin{aligned} \langle u(\cdot, t)\rangle^\alpha &\le M[\|G_1(\cdot, t)\|_{L^r} + \|W_1(\cdot, t)\|_{L^r} + \|(\rho - \tilde{\rho})(\cdot, t)\|_{L^r}], \\ \langle w(\cdot, t)\rangle^\alpha &\le M[\|G_2(\cdot, t)\|_{L^r} + \|W_2(\cdot, t)\|_{L^r}]. \end{aligned} \tag{4.4}$$

Applying Gagliardo-Nirenberg inequality, we have

$$\begin{aligned} &\|G_1(\cdot, t)\|_{L^r} \\ &\le M\Big(\|\nabla G_1(\cdot, t)\|_{L^2}^{(3r-6)/2r}\|G_1(\cdot, t)\|_{L^2}^{(6-r)/2r}\Big) \\ &\le M(\|(\rho - \tilde{\rho})(\cdot, t)\|_{L^2}^2 + \|\nabla u(\cdot, t)\|_{L^2}^2)^{\frac{1-2a}{4}}(\|\dot{u}(\cdot, t)\|_{L^2}^2 + \|\nabla H \cdot H(\cdot, t)\|_{L^2}^2)^{\frac{1+2a}{4}} \\ &\le M(C_0 + \|\nabla u(\cdot, t)\|_{L^2(\mathbb{R}^3)}^2)^{\frac{1-2\alpha}{4}}(\|\dot{u}(\cdot, t)\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla H \cdot H(\cdot, t)\|_{L^2(\mathbb{R}^3)}^2)^{\frac{1+2\alpha}{4}}, \end{aligned} \tag{4.5}$$

$$\begin{aligned} \|G_2(\cdot, t)\|_{L^r} &\le M(\|\nabla G_2(\cdot, t)\|_{L^2}^{(3r-6)/2r}\|G_2(\cdot, t)\|_{L^2}^{(6-r)/2r}) \\ &\le M(\|\nabla w(\cdot, t)\|_{L^2}^2)^{\frac{1-2a}{4}}(\|\dot{w}(\cdot, t)\|_{L^2}^2 + \|w(\cdot, t)\|_{L^2}^2)^{\frac{1+2a}{4}}, \end{aligned} \tag{4.6}$$

$$\begin{aligned}\|W_1(\cdot,t)\|_{L^r} &\le M(\|W_1(\cdot,t)\|_{L^2(\mathbb{R}^3)}^{(6-r)/2r}\|\nabla W_1(\cdot,t)\|_{L^2(\mathbb{R}^3)}^{(3r-6)/2r})\\ &\le M(\|\nabla u(\cdot,t)\|_{L^2}^{\frac{1-2a}{2}}\|\nabla W_1(\cdot,t)\|_{L^2}^{\frac{1+2a}{2}}),\end{aligned}\tag{4.7}$$

$$\begin{aligned}\|W_2(\cdot,t)\|_{L^r} &\le M(\|W_2(\cdot,t)\|_{L^2(\mathbb{R}^3)}^{(6-r)/2r}\|\nabla W_2(\cdot,t)\|_{L^2(\mathbb{R}^3)}^{(3r-6)/2r})\\ &\le M(\|\nabla w(\cdot,t)\|_{L^2}^{\frac{1-2a}{2}}\|\nabla W_2(\cdot,t)\|_{L^2}^{\frac{1+2a}{2}}).\end{aligned}\tag{4.8}$$

Consequently, substituting (4.5)-(4.8) into (4.4), we can complete the proof. □

Proof of Proposition 3.1. First, we select two positive numbers Φ and Φ' satisfying

$$\underline{\rho} < \Phi < \underline{\rho} + s < \bar{\rho} - s < \Phi' < \bar{\rho}.$$

Then recall the definition of ρ_0 that takes values in $[\underline{\rho}+s, \bar{\rho}-s]$. Thus according to the time regularity (2.6), for some positive ϵ, we have $\rho \in [\underline{\rho}, \bar{\rho}]$ on $\mathbb{R}^3 \times [0,\epsilon]$. Then, from the proposition 3.1, we obtain $A(\epsilon) \le MC_0^{\tau}$, where M is now fixed. If C_0 is further restricted, we have $\Phi < \rho < \Phi'$ on $\mathbb{R}^3 \times [0,T]$, and therefore $A(T) \le MC_0^{\tau}$. We will establish the required upper bound, since the proof of the lower bound is parallel. For $y \in \mathbb{R}^3$, the corresponding particle path $x(t)$ can be defined by

$$\begin{aligned}\dot{x}(t,y) &= u(x(t,y),t),\\ x(t_0,y) &= y.\end{aligned}\tag{4.9}$$

Suppose that there exists a time $t_1 \le \epsilon$ such that $\rho(x(t_1),t_1) = \Phi'$. Then we take t_1 minimal and select $t_0 < t_1$ maximal such that $\rho(x(t_0),t_0) = \bar{\rho}-s$. Therefore, for $t \in [t_0,t_1]$, $\rho(x(t),t) \in [\bar{\rho}-s, \Phi']$. We consider the following two cases:

Case 1: For $t_0 < t_1 \le T \wedge 1$, according to the mass equation and the definition (1.5), we obtain

$$(2\mu+\lambda)\frac{d}{dt}[\log\rho(x(t),t) - \log(\tilde{\rho})] + P(\rho(x(t),t)) - \tilde{P} = -G_1(x(t),t),$$

$$(2\mu'+\lambda')\frac{d}{dt}[\log\rho(x(t),t) - \log(\tilde{\rho})] = -G_2(x(t),t).$$

Integrating over $[t_0,t_1]$ and simplifying $\rho(x(t),t)$ to $\rho(t)$, then we have

$$(2\mu+\lambda)[\log\rho(s) - \log(\tilde{\rho})]\Big|_{t_0}^{t_1} + \int_{t_0}^{t_1}[P(s)-\tilde{P}]\,\mathrm{d}s = -\int_{t_0}^{t_1} G_1(s)\,\mathrm{d}s, \tag{4.10}$$

$$(2\mu'+\lambda')[\log\rho(s) - \log(\tilde{\rho})]|_{t_0}^{t_1} = -\int_{t_0}^{t_1} G_2(s)\,\mathrm{d}s. \tag{4.11}$$

We will show that

$$\int_{t_0}^{t_1} G_1(s)\,\mathrm{d}s \le \tilde{M}C_0^{\tau}, \quad \int_{t_0}^{t_1} G_2(s)\,\mathrm{d}s \le \tilde{M}C_0^{\tau}, \tag{4.12}$$

for a constant $\tilde{M}$ which depends on the same quantities as M defined by Proposition 3.1. If so, using (4.10) and (4.11), we have

$$(2\mu+\lambda)[\log\Phi' - \log(\bar{\rho}-s)] \le \int_{t_0}^{t_1}[P(s)-\tilde{P}]\,\mathrm{d}s + \tilde{M}C_0^{\tau} \le \tilde{M}C_0^{\tau}, \tag{4.13}$$

$$(2\mu'+\lambda')[\log\Phi' - \log(\bar{\rho}-s)] \le \tilde{M}C_0^{\tau}. \tag{4.14}$$

Because $\rho(t)$ takes values in $[\bar{\rho}-s, \Phi'] \subset [\tilde{\rho}, \bar{\rho}]$, and P is an increasing function on $[\tilde{\rho}, \bar{\rho}]$, (4.13)-(4.14) hold. But, if C_0 is small depending on $\tilde{M}$, Φ', and $\bar{\rho}-s$, then (4.13) and (4.14) cannot hold. Specifying the smallness condition, therefore we can infer that there is no time t_1 such that $\rho(t_1) = \rho(x(t_1), t_1) = \Phi'$. Due to the arbitrariness of $y \in \mathbb{R}^3$, we have $\rho < \Phi'$ on $\mathbb{R}^3 \times [0, \epsilon]$. The method of proving $\rho > \Phi$ is analogous.

To prove (4.12), supposing that Γ is the fundamental solution of the Laplace operator in $\mathbb{R}^3$ and employing (2.12), we have

$$\begin{aligned} \int_{t_0}^{t_1} G_1(s)\,\mathrm{d}s &= \int_{t_0}^{t_1}\int_{\mathbb{R}^3} \rho\dot{u}(y,s)(\nabla_x\Gamma(x(s)-y))\,\mathrm{d}y\,\mathrm{d}s \\ &\quad + \int_{t_0}^{t_1}\int_{\mathbb{R}^3} (\nabla_x\Gamma(x(s)-y))((\nabla\times H)\times H)(y,s)\,\mathrm{d}y\,\mathrm{d}s, \end{aligned} \tag{4.15}$$

$$\begin{aligned} \int_{t_0}^{t_1} G_2(s)\,\mathrm{d}s &= \int_{t_0}^{t_1}\int_{\mathbb{R}^3} \rho\dot{w}(y,s)(\nabla_x\Gamma(x(s)-y))\,\mathrm{d}y\,\mathrm{d}s \\ &\quad + \int_{t_0}^{t_1}\int_{\mathbb{R}^3} (\nabla_x\Gamma(x(s)-y))w(y,s)\,\mathrm{d}y\,\mathrm{d}s. \end{aligned} \tag{4.16}$$

We notice that the first term in (4.15) is identical to Hoff [8, Lemma 4.2]. Thus, we can arrive at

$$\begin{aligned} &\Big|\int_{t_0}^{t_1}\int_{\mathbb{R}^3} \rho\dot{u}(y,s)(\nabla_x\Gamma(x(s)-y))\,\mathrm{d}y\,\mathrm{d}s\Big| \\ &\le \|\nabla\Gamma * (\rho u)(\cdot,t_1)\|_{L^\infty(\mathbb{R}^3)} + \|\nabla\Gamma * (\rho u)(\cdot,t_2)\|_{L^\infty(\mathbb{R}^3)} \\ &\quad + \int_0^t\int_{\mathbb{R}^3} [u^k(x(s),s) - u^k(y,s)]\Gamma_{x_jx_k}(x(s)-y)(\rho u^j)(y,s)\,\mathrm{d}y\,\mathrm{d}s \\ &\le \tilde{M}C_0^\tau + \tilde{M}C_0^\tau \int_0^1 \langle u(\cdot,s)\rangle^\alpha\,\mathrm{d}s \le \tilde{M}C_0^\tau. \end{aligned} \tag{4.17}$$

The last inequality is derived from Proposition 3.1 and Lemma 4.2. Note that (3.54) holds for $q = 6$, therefore if $2 < r < \frac{3q}{q+3}$, the second integral of (4.15) can be estimated by

$$\begin{aligned} &\Big|\int_0^{t_1}\int_{\mathbb{R}^3} (\nabla_x\Gamma(x(s)-y))((\nabla\times H)\times H)(y,s)\,\mathrm{d}y\,\mathrm{d}s\Big| \\ &\le \tilde{M}\int_0^1 \|((\nabla\times H)\times H)(s)\|_{L^2(\mathbb{R}^3)} + \|((\nabla\times H)\times H)(s)\|_{L^r(\mathbb{R}^3)}\,\mathrm{d}s \\ &\le \tilde{M}\int_0^1 \|(|H|^4|\nabla H|^2)(s)\|_{L^2(\mathbb{R}^3)}^{1/2} \||\nabla H|^2(s)\|_{L^2(\mathbb{R}^3)}^{1/2}\,\mathrm{d}s \\ &\quad + \int_0^1 \|\nabla H(s)\|_{L^3(\mathbb{R}^3)}\|H(s)\|_{L^{\frac{3r}{3-r}}(\mathbb{R}^3)}\,\mathrm{d}s \le \tilde{M}C_0^\tau. \end{aligned} \tag{4.18}$$

Similarly, the terms of (4.16) can be bounded. Thus we achieve the proof of (4.12).

Case 2: For $1 \le t_0 < t_1$, similar to case 1, we obtain

$$\frac{d}{dt}(\rho(t) - \tilde{\rho}) + (2\mu+\lambda)^{-1}\rho(t)(P(t) - \tilde{P}) = -(2\mu+\lambda)^{-1}\rho(t)G_1(t), \tag{4.19}$$

$$\frac{d}{dt}(\rho(t) - \tilde{\rho}) = -(2\mu'+\lambda')^{-1}\rho(t)G_2(t). \tag{4.20}$$

We multiply the above equations by $(\rho(t)-\tilde{\rho})$ to obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\rho(t)-\tilde{\rho})^2+(2\mu+\lambda)^{-1}f(t)\rho(t)(\rho(t)-\tilde{\rho})^2\\&=-(2\mu+\lambda)^{-1}\rho(t)(\rho(t)-\tilde{\rho})G_1(t),\end{aligned}\tag{4.21}$$

$$\frac{1}{2}\frac{d}{dt}(\rho(t)-\tilde{\rho})^2=-(2\mu'+\lambda')^{-1}\rho(t)(\rho(t)-\tilde{\rho})G_2(t).\tag{4.22}$$

Here $f(t)=(P(t)-\tilde{P})(\rho(t)-\tilde{\rho})^{-1}$. Since $f(t)\geq f(t_0)>0$ on $[t_0,t_1]$, it is easy to deduce that

$$(2\mu+\lambda)^{-1}f(t)\rho(t)(\rho(t)-\tilde{\rho})^2\geq(2\mu+\lambda)^{-1}f(t_0)\underline{\rho}(\rho(t)-\tilde{\rho})^2 \text{ for } t\in[t_0,t_1].$$

Thus integrating (4.21) and (4.22) over $[t_0,t_1]$, we arrive at

$$(\rho(t_1)-\tilde{\rho})^2-(\rho(t_0)-\tilde{\rho})^2\leq\tilde{M}\int_{t_0}^{t_1}\|G_1(\cdot,s)\|_{L^\infty(\mathbb{R}^3)}^2\,\mathrm{d}s,\tag{4.23}$$

$$(\rho(t_1)-\tilde{\rho})^2-(\rho(t_0)-\tilde{\rho})^2\leq\tilde{M}\int_{t_0}^{t_1}\|G_2(\cdot,s)\|_{L^\infty(\mathbb{R}^3)}^2\,\mathrm{d}s.\tag{4.24}$$

We shall show that

$$\int_{t_0}^{t_1}\|G_1(\cdot,s)\|_{L^\infty}^2\,\mathrm{d}s\leq\bar{M}C_0^\tau,\quad\int_{t_0}^{t_1}\|G_2(\cdot,s)\|_{L^\infty}^2\,\mathrm{d}s\leq\bar{M}C_0^\tau.\tag{4.25}$$

Therefore, from (4.23) and (4.24), we have

$$|\Phi'-\tilde{\rho}|^2-|\bar{\rho}-s-\tilde{\rho}|^2\leq\tilde{M}\int_{t_0}^{t_1}\|G_1(\cdot,s)\|_{L^\infty(\mathbb{R}^3)}^2\,\mathrm{d}s,\tag{4.26}$$

$$|\Phi'-\tilde{\rho}|^2-|\bar{\rho}-s-\tilde{\rho}|^2\leq\tilde{M}\int_{t_0}^{t_1}\|G_2(\cdot,s)\|_{L^\infty(\mathbb{R}^3)}^2\,\mathrm{d}s.\tag{4.27}$$

By employing similar argument used in Case 1, (4.26) and (4.27) cannot hold if C_0 is sufficiently small. Because $y\in\mathbb{R}^3$ is arbitrary, we obtain $\rho<\Phi'$ on $\mathbb{R}^3\times[0,\epsilon]$. To prove (4.25), applying (2.12) and Lemma 2.2, we have

$$\begin{aligned}&\int_{t_0}^{t_1}\|G_1(\cdot,s)\|_{L^\infty}^2\,\mathrm{d}s\\&\leq\int_{t_0}^{t_1}[\|\rho\dot{u}(\cdot,s)\|_{L^2}^2+\|\nabla H\cdot H(\cdot,s)\|_{L^2}^2]\,\mathrm{d}s\\&\quad+\int_{t_0}^{t_1}[\|\rho\dot{u}(\cdot,s)\|_{L^4}^2+\|\nabla H\cdot H(\cdot,s)\|_{L^4}^2]\,\mathrm{d}s\\&\leq\int_{t_0}^{t_1}[\|\rho\dot{u}(\cdot,s)\|_{L^2}^2+\|\nabla H\cdot H(\cdot,s)\|_{L^2}^2]\,\mathrm{d}s\\&\quad+\int_{t_0}^{t_1}[\|\dot{u}(\cdot,s)\|_{L^2}^{1/2}\|\nabla\dot{u}(\cdot,s)\|_{L^2}^{3/2}+\|\nabla H\cdot H(\cdot,s)\|_{L^2}^{1/2}\|\nabla(\nabla H\cdot H)(\cdot,s)\|_{L^2}^{3/2}]\,\mathrm{d}s\\&\leq\tilde{M}C_0^\tau+\Big(\int_1^{t_1}\int_{\mathbb{R}^3}|\dot{u}|^2\,\mathrm{d}x\,\mathrm{d}s\Big)^{1/4}\Big(\int_1^{t_1}\int_{\mathbb{R}^3}|\nabla\dot{u}|^2\,\mathrm{d}x\,\mathrm{d}s\Big)^{3/4}\\&\quad+\Big(\int_1^{t_1}\int_{\mathbb{R}^3}|\nabla H\cdot H|^2\,\mathrm{d}x\,\mathrm{d}s\Big)^{1/4}\Big(\int_1^{t_1}\int_{\mathbb{R}^3}|\nabla(\nabla H\cdot H)|^2\,\mathrm{d}x\,\mathrm{d}s\Big)^{3/4}\\&\leq\tilde{M}C_0^\tau,\end{aligned}$$

and

$$\begin{aligned}
&\int_{t_0}^{t_1} \|G_2(\cdot,s)\|_{L^\infty}^2 \,\mathrm{d}s \\
&\le \int_{t_0}^{t_1} [\|\rho\dot{w}(\cdot,s)\|_{L^2}^2 + \|w(\cdot,s)\|_{L^2}^2]\,\mathrm{d}s \\
&\quad + \int_{t_0}^{t_1} [\|\rho\dot{w}(\cdot,s)\|_{L^4}^2 + \|w(\cdot,s)\|_{L^4}^2]\,\mathrm{d}s \\
&\le \tilde{M}C_0^\tau + \int_{t_0}^{t_1} [\|\dot{w}(\cdot,s)\|_{L^2}^{1/2}\|\nabla\dot{w}(\cdot,s)\|_{L^2}^{3/2} + \|w(\cdot,s)\|_{L^2}^{1/2}\|\nabla w(\cdot,s)\|_{L^2}^{3/2}]\,\mathrm{d}s \\
&\le \tilde{M}C_0^\tau + \Big(\int_1^{t_1}\int_{\mathbb{R}^3} |\dot{w}|^2\,\mathrm{d}x\,\mathrm{d}s\Big)^{1/4}\Big(\int_1^{t_1}\int_{\mathbb{R}^3} |\nabla\dot{w}|^2\,\mathrm{d}x\,\mathrm{d}s\Big)^{3/4} \\
&\quad + \Big(\int_1^{t_1}\int_{\mathbb{R}^3} |w|^2\,\mathrm{d}x\,\mathrm{d}s\Big)^{1/4}\Big(\int_1^{t_1}\int_{\mathbb{R}^3} |\nabla w|^2\,\mathrm{d}x\,\mathrm{d}s\Big)^{3/4} \\
&\le \tilde{M}C_0^\tau.
\end{aligned}$$

The last inequality is derived from the Proposition 3.1. This completes the proof.

5. Proof of Theorem 1.2

The proof is done by constructing the weak solution as the limits of the smooth solutions. More specifically, provided that the assumptions of Proposition 3.1 be valid, and the initial data (ρ_0, u_0, w_0, H_0) of the solution satisfy the assumptions (1.10)-(1.13) and (1.20). By convolving (ρ_0, u_0, w_0, H_0) with a standard mollifying kernel of width $\xi > 0$, we can obtain the smooth approximate initial data $(\rho_0^\xi, u_0^\xi, w_0^\xi, H_0^\xi)$. Then using the local existence result of Theorem 2.3, (1.1)-(1.2) has a unique local solution $(\rho^\xi, u^\xi, w^\xi, H^\xi)$ on $\mathbb{R}^3 \times [0,T]$. Applying a conventional energy estimate similar to [26, Theorem 4.1], the local solution $(\rho^\xi, u^\xi, w^\xi, H^\xi)$ can be extended to any $T > 0$. We establish the global-in-time existence of smooth solutions with the initial data (1.2) satisfying low energy condition (1.20). Then

$$\sup_{0\le s\le T} \|(\rho-\tilde{\rho}, u, w, H)(\cdot,s)\|_{H^3} + \int_0^T \|(u,w,H)(\cdot,s)\|_{H^4}^2 \,\mathrm{d}s \le M'(T). \tag{5.1}$$

By Theorem 2.3, for every ξ we have a global solution that satisfies

$$A(t) \le MC_0^\tau, \quad \text{and} \quad \underline{\rho} \le \rho^\xi(x,t) \le \bar{\rho}. \tag{5.2}$$

Then (ρ, u, w, H) of $A(t)$ can be replaced by $(\rho^\xi, u^\xi, w^\xi, H^\xi)$ in (3.1). These estimates will offer the compactness needed to extract the required solution (ρ, u, w, H) in the limit as $\xi \to 0$. Next, we establish the uniform Hölder continuity away from $t = 0$.

Lemma 5.1. *Taking $\epsilon > 0$, there exists a constant $C = C(\epsilon)$ for all $\xi > 0$ such that*

$$\langle u(\cdot,t)\rangle^{1/2,1/8}_{\mathbb{R}^3\times[\epsilon,\infty)}, \langle w(\cdot,t)\rangle^{1/2,1/8}_{\mathbb{R}^3\times[\epsilon,\infty)}, \langle H(\cdot,t)\rangle^{1/2,1/8}_{\mathbb{R}^3\times[\epsilon,\infty)} \le C(\epsilon)C_0^\tau. \tag{5.3}$$

Proof. First, notice that we proved the Hölder-$\frac{1}{2}$ continuity of u^ξ in (3.2). To obtain the Hölder continuity in time, we need to fix x and $t_2 \ge t_1 \ge \epsilon$,

$$|u^\xi(x,t_2) - u^\xi(x,t_1)| \le \frac{1}{|B_{\breve{R}}(x)|}\int_{B_{\breve{R}}(x)} |u^\xi(z,t_2) - u^\xi(z,t_1)|\,\mathrm{d}z + C(\epsilon)C_0^\tau \breve{R}^{1/2}$$

$$\begin{aligned}&\le \breve{R}^{-\frac{3}{2}}|t_2-t_1|^{1/2}\sup_{t\ge\epsilon}\int |u_t(z,t)^\xi|^2\,\mathrm{d}x + C(\epsilon)C_0^\tau \breve{R}^{1/2}\\ &\le C(\epsilon)C_0^\tau[\breve{R}^{-\frac{3}{2}}|t_2-t_1|^{1/2}+\breve{R}^{1/2}],\end{aligned}$$

by the estimates in (5.2). Given $\breve{R}=|t_2-t_1|^{1/4}$, we obtain the bound of u^ξ in (5.3). The proofs for w^ξ and H^ξ are similar. □

Compactness of the approximate solutions $(\rho^\xi,u^\xi,w^\xi,H^\xi)$ now follows. From Lemma 5.1 and Ascoli-Arzela Theorem, we obtain

$$u^{\xi_\sigma}, w^{\xi_\sigma}, H^{\xi_\sigma}\to u,w,H \quad \text{uniformly on compact sets in } \mathbb{R}^3\times(0,\infty); \tag{5.4}$$

for a sequence $\xi_\sigma\to 0$. Then, according to the same sequence from (5.1) and based on the elementary consideration that the weak-L^2 derivative and the distribution derivatives are equal, it follows that

$$\begin{aligned}&\nabla u^{\xi_\sigma}(\cdot,t),\nabla w^{\xi_\sigma}(\cdot,t),\nabla H^{\xi_\sigma}(\cdot,t),\nabla W_1^{\xi_\sigma}(\cdot,t),\nabla W_2^{\xi_\sigma}(\cdot,t)\\ &\to \nabla u(\cdot,t),\nabla w(\cdot,t),\nabla H(\cdot,t),\nabla W_1(\cdot,t),\nabla W_2(\cdot,t)\end{aligned}\tag{5.5}$$

weakly in $L^2(\mathbb{R}^3)$ for every $t>0$; and

$$\begin{aligned}&\vartheta^{1/2}\dot{u}^{\xi_\sigma},\vartheta^{1/2}\dot{w}^{\xi_\sigma},\vartheta^{1/2}H_t^{\xi_\sigma},\vartheta^{5/2}\nabla\dot{u}^{\xi_\sigma},\vartheta^{5/2}\nabla\dot{w}^{\xi_\sigma},\vartheta^{5/2}\nabla H_t^{\xi_\sigma}\\ &\to \vartheta^{1/2}\dot{u},\vartheta^{1/2}\dot{w},\vartheta^{1/2}H_t,\vartheta^{5/2}\nabla\dot{u},\vartheta^{5/2}\nabla\dot{w},\vartheta^{5/2}\nabla H_t\end{aligned}\tag{5.6}$$

weakly in $L^2(\mathbb{R}^3\times[0,\infty))$.

Using [7] and [14], we can obtain the convergence of approximate densities (5.7) as

$$\rho^{\xi_\sigma}(\cdot,t)\to\rho(\cdot,t)\tag{5.7}$$

strongly in $L^2_{\rm loc}(\mathbb{R}^3)$ for all $t\ge 0$.

Proof of Theorem 1.2. Obviously, the definition of limiting functions (ρ,u,w,H) in (5.4)-(5.7) inherits the bounds from (5.1) and (5.2) (but please notice that there is no representations in (1.26) about $\dot{u}(\cdot,t)$, $\dot{w}(\cdot,t)$ or $H_t(\cdot,t)$). It is also explicit from the convergence pattern described in (1.22)-(1.27) that (ρ,u,w,H) meets the weak forms (1.16)-(1.19) of the differential equations in (1.1)-(1.2). Furthermore, the continuity statement (1.21) is easily derived from these weak forms and bounds (1.27). Therefore, (1.27) is achieved.

Next we study the large-time behavior of (ρ,u,w,H) in (1.28). Taking a similar methods and proofs as for [5] and [8], we have

$$\lim_{t\to\infty}\|\rho-\tilde{\rho}\|_{L^l(\mathbb{R}^3)}=0\tag{5.8}$$

for all $l\in(2,\infty)$. Then, following the same argument as in [5], we take a sequence

$$u^l(t,x):=u(t+l,x),$$

for all integer l, and $(x,t)\in\mathbb{R}^3\times[1,2]$. Then by (1.27), we can arrive at

$$\lim_{l\to\infty}\int_0^1\|\nabla u^l\|_{L^2(\mathbb{R}^3)}=0.$$

Similarly, we have

$$\|u^l\|_{H^1(\mathbb{R}^3)}\le C \text{ uniformly for } t,l.$$

Thus we obtain $\lim_{l\to\infty}\|u^l\|_{L^2(\mathbb{R}^3)}=0$ uniformly for t, which indicates

$$\lim_{t\to\infty}\|u(t)\|_{L^2(\mathbb{R}^3)}=0.\tag{5.9}$$

For $t \geq 1$, from Lemma 2.4, it is easy to deduce that

$$\begin{aligned}
&\|\nabla u(t)\|_{L^6(\mathbb{R}^3)} \\
&\leq C(\|G_1(t)\|_{L^6(\mathbb{R}^3)}) + \|W_1(t)\|_{L^6(\mathbb{R}^3)} + \|(P(\rho) - \tilde{P})(t)\|_{L^6(\mathbb{R}^3)} \\
&\leq C(1 + \|\nabla G_1(t)\|_{L^2(\mathbb{R}^3)} + \|\nabla W_1(t)\|_{L^2(\mathbb{R}^3)}) \\
&\leq C(1 + \|\dot{u}(t)\|_{L^2(\mathbb{R}^3)} + \|\nabla w(t)\|_{L^2(\mathbb{R}^3)} + \|H \cdot \nabla H\|_{L^2(\mathbb{R}^3)} \\
&\quad + \|\nabla |H|^2\|_{L^2(\mathbb{R}^3)}) \leq C.
\end{aligned} \tag{5.10}$$

Taking the summation of (1.27), (5.9) and (5.10), we can infer that

$$\lim_{t\to\infty} \|u\|_{W^{1,r}(\mathbb{R}^3)} = 0, \tag{5.11}$$

for $r \in (2, 6)$. Similarly, we have

$$\lim_{t\to\infty} \|(w, H)\to\|_{W^{1,r}(\mathbb{R}^3)} = 0, \tag{5.12}$$

for $r \in (2, 6)$. Combining (5.8),(5.11) and (5.12), we obtain (1.28). The proof of Theorem 1.2 is complete.

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