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DIRICHLET PROBLEMS WITH ANISOTROPIC PRINCIPAL PART INVOLVING UNBOUNDED COEFFICIENTS

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Abstract. This article establishes the existence of solutions in a weak sense for a quasilinear Dirichlet problem exhibiting anisotropic differential operator with unbounded coefficients in the principal part and full dependence on the gradient in the lower order terms. A major part of this work focuses on the existence of a uniform bound for the solution set in the anisotropic setting. The unbounded coefficients are handled through an appropriate truncation and a priori estimates.

1. Introduction

The aim of this article is to study quasilinear elliptic equations driven by an anisotropic differential operator with unbounded coefficients and that have the reaction term in the form of convection (i.e., it jointly depends on the solution and its gradient). Specifically, we state the following Dirichlet problem

$$
\begin{gathered}
-\sum_{i=1}^{N} \partial_i(G_i(u)|\partial_i u|^{p_i-2}\partial_i u) = F(x,u,\nabla u) \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial\Omega
\end{gathered}
\tag{1.1}
$$

on a bounded domain Ω in $\mathbb{R}^N$ ($N \geq 2$) with a Lipschitz boundary $\partial\Omega$. The notation ∂_i stands for the distributional partial derivative with respect to the variable x_i, i.e., $\partial_i = \partial/\partial_{x_i}$, and $\nabla u = (\partial_1, \dots, \partial_N)$ is the gradient of u. In (1.1) there are given real numbers $p_i \in (1, +\infty)$ with $i = 1, \dots, N$, continuous functions $G_i : \mathbb{R} \to [a_i, +\infty)$, with $a_i > 0$ for $i = 1, \dots, N$, and a Carathéodory function $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ (i.e., $F(\cdot, t, \xi)$ is measurable on Ω for each $(t, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and $F(x, \cdot, \cdot)$ is continuous on $\mathbb{R} \times \mathbb{R}^N$ for almost all $x \in \Omega$). The notation $G_i(u)$ means the composition of the function $G_i : \mathbb{R} \to \mathbb{R}$ with the solution $u : \Omega \to \mathbb{R}$.

Set $\overrightarrow{p} := (p_1, \dots, p_N)$ and denote by $W_0^{1,\overrightarrow{p}}(\Omega)$ the completion of the set of smooth functions with compact support $C_c^\infty(\Omega)$ with respect to the norm

$$
\|u\| := \sum_{i=1}^{N} \|\partial_i u\|_{L^{p_i}}.
$$

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Therefore $W_0^{1,\overrightarrow{p}}(\Omega)$ is a reflexive Banach space. This is the natural underlying space associated to problem (1.1). We mention that the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the above norm is the important space $\mathcal{D}^{1,\overrightarrow{p}}$.

A significant feature of problem (1.1) is that the differential operator driving the equation is anisotropic, thus admitting to have unequal p_i. This causes lack of homogeneity and lack of radial scaling. In this respect, the simplest case in (1.1) is when $G_i \equiv 1$ for all $i = 1, \dots, N$, which reads

$$\begin{gathered} -\Delta_{\overrightarrow{p}} u = F(x, u, \nabla u) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.2}$$

The operator in the left-hand side of equation (1.2) is the (negative) anisotropic $\overrightarrow{p}$-Laplacian $-\Delta_{\overrightarrow{p}} : W_0^{1,\overrightarrow{p}}(\Omega) \to W_0^{1,\overrightarrow{p}}(\Omega)^*$ defined by

$$\langle -\Delta_{\overrightarrow{p}} u, v\rangle = \sum_{i=1}^N \int_\Omega |\partial_i u(x)|^{p_i-2}\partial_i u(x)\partial_i v(x)dx \tag{1.3}$$

for all $u, v \in W_0^{1,\overrightarrow{p}}(\Omega)$. We emphasize that the operator $-\Delta_{\overrightarrow{p}}$ in (1.3) has properties essentially different with respect to its isotropic counterpart which is the (negative) p-Laplacian $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$. For example, we point out the nonexistence of the first eigenvalue in the case of $-\Delta_{\overrightarrow{p}}$. The known results on anisotropic elliptic problems concern the case where the driven operator is $-\Delta_{\overrightarrow{p}}$ and the reaction term does not depend on the gradient of the solution, i.e., $F(x, u, \nabla u) = f(x, u)$. Due to these restrictions, a variational approach can be implemented. For relevant results in this direction we refer to [4, 5, 6, 9, 10]. The recent paper [3] deals with anisotropic elliptic problems with a leading operator more general than $-\Delta_{\overrightarrow{p}}$. In addition to the major mathematical interest, there is a strong physical motivation for such type of problems. We highlight for example the applications in fluid mechanics involving anisotropic media where the conductivity depends on the direction (we refer to [1] for a comprehensive description).

The degree of difficulty regarding problem (1.1) is even higher due to the fact that the variable coefficients $G_i(u)$ may be unbounded. Isotropic problems with unbounded coefficients complying with (1.1) have been considered in [7] and [8]. Given a real number $p \in (1, +\infty)$, in [7] it is investigated the problem

$$\begin{gathered} -\operatorname{div}(G(u)|\nabla u|^{p-2}\nabla u) = F(x, u, \nabla u) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered}$$

for a continuous function $G : \mathbb{R} \to [a_0, +\infty)$ with $a_0 > 0$, whereas [8] is concerned with the weighted problem

$$\begin{gathered} -\operatorname{div}(a(x)g(|u|)|\nabla u|^{p-2}\nabla u) = f(x, u, \nabla u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \end{gathered}$$

for a positive weight $a \in L^1_{loc}(\Omega)$ and a continuous function $g : [0, +\infty) \to [a_0, +\infty)$ with $a_0 > 0$.

In this article we focus on equation (1.1) where the anisotropic leading operator incorporates the unbounded coefficients $G_i(u)$. By a weak solution to problem (1.1) we mean any element $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ such that $G_i(u(x))|\partial_i u(x)|^{p_i-2}\partial_i u(x)\partial_i v(x)$, with $i = 1, \dots, N$, and $F(x, u(x), \nabla u(x))v(x)$ are integrable on Ω, and it holds

$$\sum_{i=1}^{N} \int_{\Omega} G_i(u(x))|\partial_i u(x)|^{p_i-2}\partial_i u(x)\partial_i v(x)dx = \int_{\Omega} F(x, u(x), \nabla u(x))v(x)dx \quad (1.4)$$

for all $v \in W_0^{1,\overrightarrow{p}}(\Omega)$.

The main contribution of this work is to build a coherent approach allowing for the first time to study equations that are driven by an anisotropic differential operator with unbounded coefficients and that exhibit a convection term (meaning to have full dependence on the solution and its gradient). Our main result is Theorem 2.3 below that provides under verifiable hypotheses the existence of a weak solution to problem (1.1) in the sense of (1.4). Furthermore, we prove in Theorem 3.2 the existence of a uniform bound for the solution set of problem (1.1). An essential fact is that the uniform bound does not depend on the coefficients G_i entering (1.1) except on the lower bound a_i of G_i for every $i = 1, \dots, N$. The proof of the main result relies on a truncation argument dropping the unboundedness of the coefficients $G_i(u)$ as well as on related a priori estimates. Another important tool is an auxiliary problem for which the theory of pseudomonotone operators can be applied.

The rest of this article is organized as follows. Section 2 presents the hypotheses and the main result. In Section 3 it is carried out the proof that the solution set to problem (1.1) is uniformly bounded. Section 4 deals with an auxiliary truncated problem and an associated operator. Section 5 is devoted to the proof of the main result that provides the existence of solutions to problem (1.1).

2. Statements of hypotheses and main result

For the rest of the paper we assume that

$$\sum_{i=1}^{N} \frac{1}{p_i} > 1. \quad (2.1)$$

Recall the critical exponent

$$p^* := \frac{N}{\sum_{i=1}^{N} \frac{1}{p_i} - 1}. \quad (2.2)$$

If $p_i = p$ for all $i = 1, \dots, N$, then p^* in (2.2) becomes the ordinary Sobolev critical exponent when $N > p$. Under assumption (2.1), there are the continuous embeddings

$$W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^q(\Omega) \quad (2.3)$$

provided $1 \le q \le p^*$, which are compact if $1 \le q < p^*$ (see [6, Theorem 1]). We set

$$\overline{p} := \max\{p_1, \dots, p_N\} \quad \text{and} \quad \underline{p} := \min\{p_1, \dots, p_N\}$$

and further assume that

$$\overline{p} < p^*. \quad (2.4)$$

In view of (2.3), there is a constant $\theta > 0$ such that

$$\|u\|_{L^{\underline{p}}}^{\underline{p}} \le \theta \|u\|^{\underline{p}}, \quad \forall u \in W_0^{1,\overrightarrow{p}}(\Omega). \quad (2.5)$$

To simplify the presentation, for any real number $r > 1$ we denote $r' := r/(r-1)$ (the Hölder conjugate of r). The strong and weak convergence are denoted by $\to$ and $\rightharpoonup$, respectively.

The Carathéodory function $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ describing the reaction term in (1.1) is subject to the following hypotheses:

(H1) There exist constants $c_1 \geq 0$, $c_2 \geq 0$, $c_3 \geq 0$, and $r \in (\overline{p}, p^*)$ such that

$$|F(x,t,\xi)| \leq c_1\Big(\sum_{i=1}^N |\xi_i|^{p_i}\Big)^{1/r'} + c_2|t|^{r-1} + c_3$$

for a.e. $x \in \Omega$, all $t \in \mathbb{R}$, and $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N$.

(H2) There exist constants $d_1 \geq 0$ and $d_2 \geq 0$ with $d_1 + N^{\underline{p}-1} d_2 \theta < a_i$ for all $i = 1, \dots, N$, and a function $\sigma \in L^1(\Omega)$ such that

$$F(x,t,\xi)t \leq d_1 \sum_{i=1}^N |\xi_i|^{p_i} + d_2|t|^{\underline{p}} + \sigma(x)$$

for a.e. $x \in \Omega$, all $t \in \mathbb{R}$, and $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N$.

Next we focus on the Nemytskii operator determined by the function F.

Proposition 2.1. *Assuming* (2.1), (2.4) *and* (H1), *the map* $\mathcal{N} : W_0^{1,\overrightarrow{p}}(\Omega) \to L^{r'}(\Omega)$ *given by*

$$\mathcal{N}u = F(x,u,\nabla u), \quad \forall u \in W_0^{1,\overrightarrow{p}}(\Omega),$$

is well defined, bounded (in the sense it maps bounded sets into bounded sets) and continuous.

Proof. By (H1) and the convexity of the function $t \mapsto t^{r'}$ for $t > 0$, we obtain the estimate

$$\int_\Omega |F(x,u,\nabla u)|^{r'} dx \leq C\Big(\sum_{i=1}^N \int_\Omega |\partial_i u|^{p_i} dx + \int_\Omega |u|^r dx + 1\Big), \quad \forall u \in W_0^{1,\overrightarrow{p}}(\Omega),$$

with a constant $C > 0$. Since $\partial_i u \in L^{p_i}(\Omega)$ for all $i = 1, \dots, N$ and $u \in L^r(\Omega)$ (note $r < p^*$) whenever $u \in W_0^{1,\overrightarrow{p}}(\Omega)$, we obtain that $\mathcal{N}u \in L^{r'}(\Omega)$. The obtained estimate shows that the mapping $\mathcal{N} : W_0^{1,\overrightarrow{p}}(\Omega) \to L^{r'}(\Omega)$ is well defined and bounded.

To show the continuity of the mapping $\mathcal{N}$, let $u_n \to u$ in $W_0^{1,\overrightarrow{p}}(\Omega)$. The definition of the space $W_0^{1,\overrightarrow{p}}(\Omega)$ and the continuous embedding $W_0^{1,\overrightarrow{p}}(\Omega) \subset L^r(\Omega)$ imply that $\partial_i(u_n) \to \partial_i u$ in $L^{p_i}(\Omega)$ for $i = 1, \dots, N$, and $u_n \to u$ in $L^r(\Omega)$. Since $r' > 1$, the growth condition in assumption (H1) yields

$$|F(x,t,\xi)| \leq c_1 \sum_{i=1}^N |\xi_i|^{\frac{p_i}{r'}} + c_2|t|^{r-1} + c_3$$

for a.e. $x \in \Omega$, all $t \in \mathbb{R}$, and $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N$. Taking into account that $u_n \to u$ in $L^r(\Omega)$ and $\partial_i(u_n) \to \partial_i u$ in $L^{p_i}(\Omega)$ for $i = 1, \dots, N$, Krasnoselkii's classical theorem concerning the continuity of a Nemytskii operator guarantees that $F(x,u_n,\nabla u_n) \to F(x,u,\nabla u)$ in $L^{r'}(\Omega)$. The stated conclusion follows. $\square$

Corollary 2.2. *Assume that conditions* (2.1), (2.4) *and* (H1) *are fulfilled. If* $u_n \to u$ *in* $W_0^{1,\overrightarrow{p}}(\Omega)$, *then*
$$\lim_{n\to\infty} \langle \mathcal{N}(u_n), u_n - u\rangle = 0.$$

Proof. We note that by Hölder's inequality,
$$\begin{aligned} |\langle \mathcal{N}(u_n), u_n - u\rangle| &= \Big| \int_\Omega F(x, u_n, \nabla u_n)(u_n - u)dx\Big| \\ &\le \|F(x, u_n, \nabla u_n)\|_{L^{r'}} \|u_n - u\|_{L^r}. \end{aligned}$$
By the continuous embedding (2.3) we know that $u_n \to u$ in $L^r(\Omega)$ while Proposition 2.1 entails that the sequence $F(x, u_n, \nabla u_n)$ is bounded in $L^{r'}(\Omega)$. Consequently, the thesis is valid. □

Now we state our main result providing the existence of a bounded weak solution to problem (1.1).

Theorem 2.3. *Assume that conditions* (2.1) *and* (2.4) *hold,* $G_i : \mathbb{R} \to [a_i, +\infty)$, *with* $a_i > 0$ *for* $i = 1, \dots, N$, *are continuous functions, and* $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *is a Carathéodory function satisfying hypotheses* (H1) *and* (H2). *Then problem* (1.1) *has at least a weak solution* $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ *in the sense of* (1.4). *Moreover, the solution* u *is bounded.*

The proof of Theorem 2.3 is given in Section 5.

3. Uniformly bounded solution set

Our first objective is to estimate the solutions in $W_0^{1,\overrightarrow{p}}(\Omega)$.

Lemma 3.1. *Assume that conditions* (2.1), (2.4), (H1) *and* (H2) *hold. Then the set of solutions to problem* (1.1) *is bounded in* $W_0^{1,\overrightarrow{p}}(\Omega)$ *with a bound that depends on the function* G_i *only through the lower bound* a_i *of* G_i *for* $i = 1, \dots, N$.

Proof. Let $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ be a weak solution of (1.1). Equality (1.4) with $v = u$ gives
$$\sum_{i=1}^N \int_\Omega G_i(u(x))|\partial_i u(x)|^{p_i}\, dx = \int_\Omega F(x, u, \nabla u)u\, dx.$$
Then hypothesis (H2) yields
$$\sum_{i=1}^N a_i \|\partial_i u\|_{L^{p_i}}^{p_i} \le d_1 \sum_{i=1}^N \|\partial_i u\|_{L^{p_i}}^{p_i} + d_2\|u\|_{L^{\underline{p}}}^{\underline{p}} + \|\sigma\|_{L^1}.$$
Using (2.5), we obtain
$$\begin{aligned} \sum_{i=1}^N (a_i - d_1)\|\partial_i u\|_{L^{p_i}}^{p_i} &\le d_2\theta(\sum_{i=1}^N \|\partial_i u\|_{L^{p_i}})^{\underline{p}} + \|\sigma\|_{L^1} \\ &\le N^{\underline{p}-1} d_2\theta \sum_{i=1}^N \|\partial_i u\|_{L^{p_i}}^{\underline{p}} + \|\sigma\|_{L^1} \\ &\le N^{\underline{p}-1} d_2\theta(N + \sum_{i=1}^N \|\partial_i u\|_{L^{p_i}}^{p_i}) + \|\sigma\|_{L^1}. \end{aligned}$$

From the condition $d_1 + N^{\underline{p}-1} d_2 \theta < a_i$ for all $i = 1, \dots, N$, imposed in assumption (H2), the conclusion is achieved. $\square$

Now we show that the solution set of problem (1.1) is uniformly bounded.

Theorem 3.2. *If conditions* (2.1), (2.4), (H1) *and* (H2) *are satisfied, then the solution set of problem* (1.1) *is uniformly bounded, which means that there exists a constant $C_0 > 0$ such that $\|u\|_{L^\infty} \le C_0$ for all weak solutions $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ to problem* (1.1). *The uniform bound C_0 depends on the function G_i only through its lower bound a_i for $i = 1, \dots, N$.*

Proof. Let $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ be a weak solution to problem (1.1). Writing $u = u^+ - u^-$ with $u^+ = \max\{u, 0\}$ (the positive part of u) and $u^- = \max\{-u, 0\}$ (the negative part of u), we are going to prove the uniform boundedness for u^+ and u^-. Since the arguments are similar, we only give the proof for u^+.

Given an arbitrary number $h > 0$ we pose $u_h := \min\{u^+, h\}$. Corresponding to any number $k > 0$ and any integer $1 \le j \le N$, we note that $u^+(u_h)^{kp_j} \in W_0^{1,\overrightarrow{p}}(\Omega)$. This follows from

$$
\begin{aligned}
&|\partial_i(u^+(u_h)^k)| = |(u_h)^k \partial_i(u^+) + k(u_h)^{k-1} u^+ \partial_i(u_h)| \\
&\le (k+1)(u_h)^k |\partial_i(u^+)|, \quad \text{for } i = 1, \dots, N.
\end{aligned}
\tag{3.1}
$$

Using $u^+(u_h)^{kp_j}$ as test function in (1.4) implies

$$
\sum_{i=1}^N \int_\Omega G_i(u) |\partial_i u|^{p_i-2} \partial_i u \partial_i(u^+(u_h)^{kp_j}) dx = \int_\Omega F(x, u, \nabla u) u^+(u_h)^{kp_j} dx. \tag{3.2}
$$

The following estimate of the left-hand side of (3.2) holds

$$
\begin{aligned}
&\sum_{i=1}^N \int_\Omega G_i(u) |\partial_i u|^{p_i-2} \partial_i u \partial_i(u^+(u_h)^{kp_j}) dx \\
&= \sum_{i=1}^N \int_\Omega G_i(u) |\partial_i u|^{p_i-2} \partial_i u (\partial_i(u^+)(u_h)^{kp_j} + kp_j (u_h)^{kp_j-1} u^+ \partial_i(u_h)) dx \\
&\ge \sum_{i=1}^N a_i \int_\Omega (u_h)^{kp_j} |\partial_i(u^+)|^{p_i} dx, \quad \text{for } j = 1, \dots, N.
\end{aligned}
\tag{3.3}
$$

Using (H1) we estimate the right-hand side of (3.2) as follows

$$
\begin{aligned}
&\int_\Omega F(x, u, \nabla u) u^+(u_h)^{kp_j} dx \\
&\le c_1 \int_\Omega \Big(\Big(\sum_{i=1}^N |\partial_i u|^{p_i} \Big)^{1/r'} (u_h)^{\frac{kp_j}{r'}} \Big) \Big((u_h)^{\frac{kp_j}{r}} u^+ \Big) dx \\
&\quad + c_2 \int_\Omega |u|^{r-1} (u_h)^{kp_j} u^+ dx + c_3 \int_\Omega (u_h)^{kp_j} u^+ dx.
\end{aligned}
$$

Then Young's inequality under the first integral with any $\varepsilon > 0$ provides a constant $c(\varepsilon) > 0$ such that

$$
\int_\Omega F(x, u, \nabla u) u^+(u_h)^{kp_j} dx
$$

$$
\begin{aligned}
&\le \varepsilon \sum_{i=1}^N \int_\Omega |\partial_i(u^+)|^{p_i}(u_h)^{kp_j}\,dx + c(\varepsilon)\int_\Omega (u_h)^{kp_j}(u^+)^r\,dx \\
&\quad + c_2\int_\Omega (u_h)^{kp_j}(u^+)^r\,dx + c_3\Big(\int_\Omega (u_h)^{kp_j}(u^+)^r\,dx + |\Omega|\Big),
\end{aligned}
$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. Consequently, we obtain the estimate

$$
\begin{aligned}
&\int_\Omega F(x,u,\nabla u)u^+(u_h)^{kp_j}\,dx \\
&\le \varepsilon \sum_{i=1}^N \int_\Omega |\partial_i(u^+)|^{p_i}(u_h)^{kp_j}\,dx + b\Big(\int_\Omega (u_h)^{kp_j}(u^+)^r\,dx + 1\Big),
\end{aligned}
\tag{3.4}
$$

with a constant $b > 0$.

Combining (3.2), (3.3) and (3.4) yields

$$
\sum_{i=1}^N (a_i - \varepsilon)\int_\Omega (u_h)^{kp_j}|\partial_i(u^+)|^{p_i}\,dx \le b\Big(\int_\Omega (u_h)^{kp_j}(u^+)^r\,dx + 1\Big).
$$

If we choose $\varepsilon > 0$ small enough, the preceding inequality reads

$$
\sum_{i=1}^N \int_\Omega (u_h)^{kp_j}|\partial_i(u^+)|^{p_i}\,dx \le b_0\Big(\int_\Omega (u_h)^{kp_j}(u^+)^r\,dx + 1\Big),
\tag{3.5}
$$

with a constant $b_0 > 0$.

By (3.1) and (3.5), for each $j = 1,\dots,N$ we infer that

$$
\|\partial_j(u^+(u_h)^k)\|_{L^{p_j}} \le b_0^{1/p_j}(k+1)\Big(\int_\Omega (u_h)^{kp_j}(u^+)^r\,dx + 1\Big)^{1/p_j}.
\tag{3.6}
$$

Since $r \in (\overline{p}, p^*)$ (see hypothesis (H1)), we are able to choose $q \in (p_j, r)$ with

$$
\frac{(r-p_j)q}{q-p_j} < p^*, \quad \text{for } j = 1,\dots,N.
\tag{3.7}
$$

By means of Hölder's inequality, (2.3), (3.7) and Lemma 3.1 we then derive the existence of a constant $K > 0$ such that

$$
\begin{aligned}
\int_\Omega (u_h)^{kp_j}(u^+)^r\,dx &= \int_\Omega (u^+)^{r-p_j}((u_h)^k u^+)^{p_j}\,dx \\
&\le \Big(\int_\Omega (u^+)^{\frac{(r-p_j)q}{q-p_j}}\,dx\Big)^{\frac{q-p_j}{q}}\Big(\int_\Omega \big(u^+(u_h)^k\big)^q dx\Big)^{p_j/q} \\
&\le K\|u^+(u_h)^k\|_{L^q}^{p_j}.
\end{aligned}
$$

In view of (3.6) we find a constant $b_1 > 0$ for which

$$
\|\partial_j(u^+(u_h)^k)\|_{L^{p_j}} \le b_1(k+1)(\|u^+(u_h)^k\|_{L^q} + 1),
$$

thus

$$
\|u^+(u_h)^k\| \le b_1 N(k+1)(\|u^+(u_h)^k\|_{L^q} + 1).
$$

From the continuous embedding (2.3) we obtain

$$
\|u^+(u_h)^k\|_{L^{p^*}} \le b_2(k+1)(\|u^+\|_{L^{q(k+1)}}^{k+1} + 1),
$$

with a constant $b_2 > 0$. Through Fatou's lemma, letting $h \to 0$ results in

$$
\|u^+\|_{L^{p^*(k+1)}}^{k+1} = \|(u^+)^{k+1}\|_{L^{p^*}} \le b_2(k+1)(\|u^+\|_{L^{q(k+1)}}^{k+1} + 1).
\tag{3.8}
$$

We note that if there is a sequence $k_n \to +\infty$ with $\|u^+\|_{L^{p^*(k_n+1)}} \leq 1$ for all n, then it holds $\|u^+\|_{L^\infty} \leq 1$ and we are done. It remains to examine two situations: (a) we have $\|u^+\|_{L^{p^*(k+1)}} > 1$ for all $k > 0$; (b) there is $k_0 > 0$ such that $\|u^+\|_{L^{p^*(k_0+1)}} \leq 1$ and $\|u^+\|_{L^{p^*(k+1)}} > 1$ for all $k > k_0$.

If case (a) occurs, (3.8) reduces to

$$\|u^+\|_{L^{p^*(k+1)}} \leq (2b_2)^{\frac{1}{k+1}}(k+1)^{\frac{1}{k+1}}\|u^+\|_{L^{q(k+1)}}. \quad \forall k > 0.$$

Taking into account that the function $k \mapsto (k+1)^{1/\sqrt{k+1}}$ is bounded on $(0, +\infty)$, the preceding inequality entails

$$\|u^+\|_{L^{p^*(k+1)}} \leq C^{1/\sqrt{k+1}}\|u^+\|_{L^{q(k+1)}}, \quad \forall k > 0, \tag{3.9}$$

with a constant $C > 0$. We are in a position to implement the following Moser iteration:

$$(k_n + 1)q = (k_{n-1} + 1)p^*, \quad \forall n \geq 2, \tag{3.10}$$

starting with $(k_1 + 1)q = p^*$. The successive application of (3.9) produces

$$\|u^+\|_{L^{p^*(k_n+1)}} \leq C^{\sum_{i=1}^n \frac{1}{\sqrt{k_i+1}}}\|u^+\|_{L^{p^*}}.$$

The definition of (k_n) ensures that $k_n \to +\infty$ and the series $\sum_{n=1}^\infty 1/\sqrt{k_n+1}$ converges. The application of Lemma 3.1 and letting $n \to \infty$ show that $\|u\|_{L^\infty} \leq C_0$, with a constant $C_0 > 0$ independent on the solution u, thus reaching the desired conclusion.

If case (b) holds, (3.8) results in

$$\begin{gathered}
\|u^+\|_{L^{p^*(k+1)}} \leq (2b_2)^{\frac{1}{k+1}}(k+1)^{\frac{1}{k+1}}\|u^+\|_{L^{q(k+1)}}, \quad \forall k > k_0, \\
\|u^+\|_{L^{p^*(k_0+1)}} \leq (2b_2)^{\frac{1}{k_0+1}}(k_0+1)^{\frac{1}{k_0+1}}.
\end{gathered}$$

Arguing as above, we obtain

$$\|u^+\|_{L^{p^*(k+1)}} \leq C^{1/\sqrt{k+1}}\|u^+\|_{L^{q(k+1)}}, \quad \forall k > k_0, \tag{3.11}$$

with a constant $C > 0$. Now we carry out the Moser iteration (3.10) starting with $(k_1 + 1)q = p^*(k_0 + 1)$. Then the repeated application of (3.11) leads to

$$\|u^+\|_{L^{p^*(k_n+1)}} \leq C^{\sum_{i=1}^n \frac{1}{\sqrt{k_i+1}}}\|u^+\|_{L^{p^*(k_0+1)}}.$$

The same reasoning as in case (a) enables us to conclude that $\|u\|_{L^\infty(\Omega)} \leq C_0$ with a constant $C_0 > 0$ independent of the solution u.

Summarizing, we have shown that one can find a constant $C_0 > 0$ as stated in the theorem. A careful reading of the preceding proof reveals that the constant C_0 does not depend on G_i except on its lower bound a_i for $i = 1, \dots, N$, which completes the proof. $\square$

4. Truncated problem and associated operator

A major difficulty in handling problem (1.1) consists in the fact that the coefficients G_i are unbounded. This issue is resolved by truncation. In the case of isotropic problems (possibly with weights) the idea appears in [7] and [8].

Fix a real number $R > 0$. For each $i = 1, \dots, N$, we truncate the function G_i entering problem (1.1) as follows

$$G_{iR}(t) = \begin{cases} G_i(t) & \text{if } |t| \le R \\ G_i(R) & \text{if } t > R \\ G_i(-R) & \text{if } t < -R. \end{cases} \tag{4.1}$$

Notice that $G_{iR} : \mathbb{R} \to \mathbb{R}$ is a continuous and bounded function and has the range in $[a_i, +\infty)$ as the function G_i does.

With the truncated coefficients G_{iR} in (4.1) we state the auxiliary problem

$$\begin{gathered} -\sum_{i=1}^{N} \partial_i (G_{iR}(u) |\partial_i u|^{p_i - 2} \partial_i u) = F(x, u, \nabla u) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{4.2}$$

We associate to problem (4.2) an operator $\mathcal{A}_R : W_0^{1,\overrightarrow{p}}(\Omega) \to W_0^{1,\overrightarrow{p}}(\Omega)^*$ defined by

$$\langle \mathcal{A}_R(u), v \rangle = \sum_{i=1}^{N} \int_\Omega G_{iR}(u) |\partial_i u|^{p_i - 2} \partial_i u \partial_i v dx, \quad \forall u, v \in W_0^{1,\overrightarrow{p}}(\Omega). \tag{4.3}$$

The next proposition lists the main properties of the operator $\mathcal{A}_R$ in (4.3).

Proposition 4.1. *Assume conditions* (2.1) *and* (2.4). *For each $R > 0$, the mapping $\mathcal{A}_R : W_0^{1,\overrightarrow{p}}(\Omega) \to W_0^{1,\overrightarrow{p}}(\Omega)^*$ is well defined, bounded, continuous, and fulfills the S_+-property, meaning that if $u_n \rightharpoonup u$ in $W_0^{1,\overrightarrow{p}}(\Omega)$ and*

$$\limsup_{n \to \infty} \langle \mathcal{A}_R(u_n), u_n - u \rangle \le 0, \tag{4.4}$$

then $u_n \to u$ in $W_0^{1,\overrightarrow{p}}(\Omega)$.

Proof. By Hölder's inequality, for all $u, v \in W_0^{1,\overrightarrow{p}}(\Omega)$ and $i = 1, \dots, N$, we note that

$$\Big| \int_\Omega G_{iR}(u) |\partial_i u|^{p_i - 2} \partial_i u \partial_i v dx \Big| \le \max_{|t| \le R} G_i(t) \|\partial_i u\|_{L^{p_i}}^{p_i - 1} \|\partial_i v\|_{L^{p_i}}. \tag{4.5}$$

We infer for all $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ that $\mathcal{A}_R(u) \in W_0^{1,\overrightarrow{p}}(\Omega)^*$, so $\mathcal{A}_R$ is well defined. Moreover, (4.5) shows that the mapping $\mathcal{A}_R$ is bounded.

Now we verify that $\mathcal{A}_R$ is continuous. To this end, let $u_n \to u$ in $W_0^{1,\overrightarrow{p}}(\Omega)$. We have

$$\begin{aligned} &\|\mathcal{A}_R(u_n) - \mathcal{A}_R(u)\|_{W_0^{1,\overrightarrow{p}}(\Omega)^*} \\ &\le \sum_{i=1}^{N} \|(G_{iR}(u_n) - G_{iR}(u)) |\partial_i(u_n)|^{p_i - 2} \partial_i(u_n)\|_{L^{\frac{p_i}{p_i - 1}}} \\ &\quad + \sum_{i=1}^{N} \|G_{iR}(u) (|\partial_i(u_n)|^{p_i - 2} \partial_i(u_n) - |\partial_i(u)|^{p_i - 2} \partial_i(u))\|_{L^{\frac{p_i}{p_i - 1}}}. \end{aligned} \tag{4.6}$$

Let us notice that

$$\begin{aligned} &\|(G_{iR}(u_n) - G_{iR}(u)) |\partial_i(u_n)|^{p_i - 2} \partial_i(u_n)\|_{L^{\frac{p_i}{p_i - 1}}}^{\frac{p_i}{p_i - 1}} \\ &\le \int_\Omega |G_{iR}(u_n) - G_{iR}(u)|^{\frac{p_i}{p_i - 1}} |\partial_i(u_n)|^{p_i} dx. \end{aligned}$$

By the continuity and boundedness of the function G_{iR}, in conjunction with $u_n \to u$ in $W_0^{1,\overrightarrow{p}}(\Omega)$, we are able to apply Lebesgue's dominated convergence theorem obtaining

$$\lim_{n\to\infty} \|(G_{iR}(u_n) - G_{iR}(u))|\partial_i(u_n)|^{p_i-2}\partial_i(u_n)\|_{L^{\frac{p_i}{p_i-1}}} = 0, \tag{4.7}$$

$$\lim_{n\to\infty} \|G_{iR}(u)(|\partial_i(u_n)|^{p_i-2}\partial_i(u_n) - |\partial_i(u)|^{p_i-2}\partial_i(u))\|_{L^{\frac{p_i}{p_i-1}}} = 0. \tag{4.8}$$

Combining (4.6), (4.7) and (4.8) shows that $\mathcal{A}_R(u_n) \to \mathcal{A}_R(u)$ in $W_0^{1,\overrightarrow{p}}(\Omega)^*$, which establishes the continuity of $\mathcal{A}_R$.

It remains to prove the S_+-property. Let $u_n \rightharpoonup u$ in $W_0^{1,\overrightarrow{p}}(\Omega)$ such that (4.4) is satisfied. Hence it is assured the validity for

$$\limsup_{n\to\infty} \langle \mathcal{A}_R(u_n) - \mathcal{A}_R(u), u_n - u\rangle \le 0. \tag{4.9}$$

We observe that

$$\begin{aligned} &\langle \mathcal{A}_R(u_n) - \mathcal{A}_R(u), u_n - u\rangle \\ &\ge \sum_{i=1}^N a_i \int_\Omega (|\partial_i(u_n)|^{p_i-2}\partial_i(u_n) - |\partial_i(u)|^{p_i-2}\partial_i(u))\partial_i(u_n-u)dx \\ &\quad + \sum_{i=1}^N \int_\Omega (G_{iR}(u_n) - G_{iR}(u))|\partial_i(u)|^{p_i-2}\partial_i(u)\partial_i(u_n-u)dx. \end{aligned} \tag{4.10}$$

As for (4.7) we can prove that

$$\lim_{n\to\infty} \int_\Omega (G_{iR}(u_n) - G_{iR}(u))|\partial_i(u)|^{p_i-2}\partial_i(u)\partial_i(u_n-u)dx = 0. \tag{4.11}$$

Then (4.9), (4.10), (4.11) and Hölder's inequality imply

$$\lim_{n\to\infty} (\|\partial_i(u_n)\|_{L^{p_i}} - \|\partial_i(u)\|_{L^{p_i}})\Big(\|\partial_i(u_n)\|_{L^{p_i}}^{p_i-1} - \|\partial_i(u)\|_{L^{p_i}}^{p_i-1}\Big) = 0$$

for all $i = 1, \dots, N$, from which it follows

$$\lim_{n\to\infty} \|\partial_i(u_n)\|_{L^{p_i}} = \|\partial_i u\|_{L^{p_i}}, \quad \forall i = 1, \dots, N.$$

Since the space $L^{p_i}(\Omega)$ is uniformly convex, we infer the strong convergence $u_n \to u$ in $W_0^{1,\overrightarrow{p}}(\Omega)$. The S_+-property of the operator $\mathcal{A}_R$ ensues, which completes the proof. □

The next result points out the properties of the operator $\mathcal{A}_R - \mathcal{N}$, with $\mathcal{A}_R$ and $\mathcal{N}$ introduced in (4.3) and Proposition 2.1, respectively.

Proposition 4.2. *Assume* (2.1), (2.4), (H1) *and* (H2). *Then, for each real number* $R > 0$, *the mapping* $\mathcal{A}_R - \mathcal{N} : W_0^{1,\overrightarrow{p}}(\Omega) \to W_0^{1,\overrightarrow{p}}(\Omega)^*$ *has the properties:*

(i) $\mathcal{A}_R - \mathcal{N}$ *is bounded (i.e., it maps bounded sets into bounded sets).*

(ii) $\mathcal{A}_R - \mathcal{N}$ *is pseudomonotone, that is, if* $u_n \rightharpoonup u$ *in* $W_0^{1,\overrightarrow{p}}(\Omega)$ *and*

$$\limsup_{n\to\infty} \langle (A_R - \mathcal{N})(u_n), u_n - u\rangle \le 0, \tag{4.12}$$

then

$$\liminf_{n\to\infty} \langle (A_R - \mathcal{N})(u_n), u_n - u\rangle \ge \liminf_{n\to\infty} \langle (\mathcal{A}_R - \mathcal{N})(u), u - v\rangle \tag{4.13}$$

for all $v \in W_0^{1,\overrightarrow{p}}(\Omega)$.

(iii) $\mathcal{A}_R - \mathcal{N}$ *is coercive, that is,*

$$\lim_{\|u\|\to\infty} \frac{\langle (A_R - \mathcal{N})(u), u\rangle}{\|u\|} = +\infty. \tag{4.14}$$

Proof. (i) This is a direct consequence of Propositions 2.1 and 4.1.

(ii) Suppose that $u_n \rightharpoonup u$ in $W_0^{1,\overrightarrow{p}}(\Omega)$ and (4.12) is satisfied. Corollary 2.2 and (4.12) ensure that (4.4) holds true. We are allowed to apply the S_+-property in Proposition 4.1, which provides the strong convergence $u_n \to u$ in $W_0^{1,\overrightarrow{p}}(\Omega)$. By Proposition 2.1 and the continuous embedding (2.3) with $q = r$ we know that $\mathcal{N}(u_n) \to \mathcal{N}(u)$ in $W_0^{1,\overrightarrow{p}}(\Omega)^*$. In addition, Proposition 4.1 guarantees $\mathcal{A}_R(u_n) \to \mathcal{A}_R(u)$ in $W_0^{1,\overrightarrow{p}}(\Omega)^*$. This enables us to conclude that (4.13) holds true.

(iii) According to hypothesis (H2) and (2.5) we have

$$\begin{aligned}
&\langle (\mathcal{A}_R - \mathcal{N})(u), u\rangle \\
&\geq \sum_{i=1}^N \int_\Omega G_{iR}(u)|\partial_i(u)|^{p_i}dx - d_1 \sum_{i=1}^N \|\partial_i u\|_{L^{p_i}}^{p_i} - d_2\|u\|_{L^{\underline{p}}}^{\underline{p}} - \|\sigma\|_{L^1} \\
&\geq \sum_{i=1}^N (a_i - d_1 - N^{\underline{p}-1}d_2\theta)\|\partial_i u\|_{L^{p_i}}^{p_i} - N^{\underline{p}}d_2\theta - \|\sigma\|_{L^1}.
\end{aligned}$$

Since $a_i - d_1 - N^{\underline{p}-1}d_2\theta > 0$ and $p_i > 1$ for $i = 1, \dots, N$, the inequality

$$\frac{\langle (\mathcal{A}_R - \mathcal{N})(u), u\rangle}{\|u\|} \geq \sum_{i=1}^N (a_i - d_1 - N^{\underline{p}-1}d_2\theta)\|\partial_i u\|_{L^{p_i}}^{p_i-1} - \frac{N^{\underline{p}}d_2\theta + \|\sigma\|_{L^1}}{\|u\|}$$

allows us to establish (4.14). $\square$

5. Existence of solutions

First we deal with the solvability of auxiliary problem (4.2).

Theorem 5.1. *Assume that conditions* (2.1) *and* (2.4) *hold,* $G_i : \mathbb{R} \to [a_i, +\infty)$*, with* $a_i > 0$ *for* $i = 1, \dots, N$*, are continuous functions, and* $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *is a Carathéodory function satisfying hypotheses* (H1) *and* (H2)*. Then, for every* $R > 0$*, problem* (4.2) *has at least a weak solution* $u_R \in W_0^{1,\overrightarrow{p}}(\Omega)$ *which means*

$$\sum_{i=1}^N \int_\Omega G_{iR}(u_R(x))|\partial_i u(x)|^{p_i-2}\partial_i u(x)\partial_i v(x)dx = \int_\Omega F(x,u,\nabla u)vdx \tag{5.1}$$

for all $v \in W_0^{1,\overrightarrow{p}}(\Omega)$*. Moreover, the solution set of problem* (4.2) *is uniformly bounded with the bound* $C_0 > 0$ *in Theorem 3.2. In particular, one has* $\|u_R\|_{L^\infty} \leq C_0$ *for every* $R > 0$*, with* $C_0 > 0$ *in Theorem 3.2.*

Proof. Fix $R > 0$. We note that auxiliary problem (4.2) in $W_0^{1,\overrightarrow{p}}(\Omega)$ is equivalent to the operator equation

$$(\mathcal{A}_R - \mathcal{N})(u) = 0. \tag{5.2}$$

Proposition 4.2 entails that the operator $\mathcal{A}_R - \mathcal{N} : W_0^{1,\overrightarrow{p}}(\Omega) \to W_0^{1,\overrightarrow{p}}(\Omega)^*$ is pseudomonotone, bounded and coercive. Hence we are entitled to apply the main theorem for pseudomonotone operators (see, e.g., [2, Theorem 2.99]) ensuring that equation (5.2) admits at least a weak solution $u_R \in W_0^{1,\overrightarrow{p}}(\Omega)$. Consequently, u_R is a weak solution of auxiliary problem (4.2).

Theorem 3.2 can be applied with G_{iR} in place of G_i for each $i = 1, \dots, N$ because the same set of assumptions is required to be verified. Since the range of G_{iR} is contained in $[a_i, +\infty)$ as it is the case of G_i, we can infer that the solution u_R of (4.2) fulfills the a priori estimate $\|u_R\|_{L^\infty} \le C_0$, where $C_0 > 0$ is the uniform bound given in Theorem 3.2. This completes the proof. $\square$

Finally, we will prove that $u_R \in W_0^{1,\overrightarrow{p}}(\Omega)$ found in Theorem 5.1 is a weak solution of the original problem (1.1) provided $R > 0$ is sufficiently large. Consequently, this will establish Theorem 2.3.

Proof of Theorem 2.3. Theorem 3.2 ensures that the solution set of problem (1.1) is uniformly bounded, thus there is a constant $C_0 > 0$ such that $\|u\|_{L^\infty} \le C_0$ for all weak solutions $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ to (1.1). As explicitly mentioned in the statement of Theorem 3.2, the constant C_0 does not depend on the function G_i entering problem (1.1) except on the lower bound a_i for $i = 1, \dots, N$. As seen from (4.1), a_i is a lower bound for each truncation G_{iR}, so the solution set of each auxiliary problem (4.2) is uniformly bound by the constant C_0, which is independent of R. In particular, the solution $u_R \in W_0^{1,\overrightarrow{p}}(\Omega)$ to problem (4.2) given by Theorem 5.1 satisfies $u_R \in L^\infty(\Omega)$ with $\|u_R\|_{L^\infty} \le C_0$, where $C_0 > 0$ is the uniform bound in Theorem 3.2.

The preceding reasoning shows that it is allowed to choose $R \ge C_0$ because C_0 is independent of $R > 0$. With such a choice, there holds $|u_R(x)| \le R$ almost everywhere on Ω. In view of (4.1) we obtain

$$G_{iR}(u_R(x)) = G_i(u_R(x)) \quad \text{for a.e. } x \in \Omega, \text{ and } i = 1, \dots, N.$$

A simple comparison regarding the statements of problems (1.1) and (4.2) confirms that $u_R \in W_0^{1,\overrightarrow{p}}(\Omega)$ is a bounded weak solution for the original problem (1.1). The proof is thus complete. $\square$

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