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GLOBAL ATTRACTOR AND ℓ^p SOLUTIONS TO INITIAL VALUE PROBLEMS OF DISCRETE NONLINEAR SCHRÖDINGER EQUATIONS COMPLEX POTENTIAL

GUOPING ZHANG, GHDER ABURAMYAH

Abstract. In this article, we investigate the global well-posedness of initial value problems of the time-dependent discrete nonlinear Schrödinger equation with a complex potential and sufficiently general nonlinearity on a multidimensional lattice in weighted ℓ^p spaces for $1 < p < \infty$. Thanks to our improved estimates we are able to prove the existence of global attractor for ℓ^p solutions to the initial value problem.

1. Introduction

The discrete nonlinear Schrödinger equation (DNLS) describes the evolution of complex-valued wave amplitudes at discrete sites, capturing the interplay between discreteness and nonlinearity. This discreteness can arise in systems where wave propagation is confined to a lattice or network, as observed in optical waveguide arrays and certain crystalline structures.

The presence of a complex potential in the DNLS equation introduces additional complexities and nonlinearity, leading to rich dynamics and novel phenomena. The complex potential can arise from various sources, such as an external field or a spatially varying refractive index in optics. It can significantly influence the propagation characteristics of waves, including wave localization, soliton formation, and wave scattering.

Understanding the properties and dynamics of the DNLS equation is crucial for gaining insights into the behavior of discrete wave systems and exploring nonlinear effects in different physical systems. For instance, we mention nonlinear wave transmission in discrete media, propagation of localized pulses in coupled waveguides and optical fibers, and modeling Bose-Einstein condensates (see, e.g., [8, 11, 12] and references therein).

Research activity in this area mainly focuses on the so-called "breathers" which are standing waves. The profile function of such a wave solves an appropriate stationary DNLS equation. Most works in this area deal with (discrete) translation-invariant DNLS on a one-dimensional lattice and employ perturbation techniques,

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two-dimensional discrete-time dynamical systems, and numerical simulation (see, e.g., [6, 7, 8] and references therein).

On the other hand, the series of papers [3, 15, 16, 17, 18, 19, 25, 26, 29, 30, 31] applies the theory of critical points of smooth functionals to the study of breathers for DNLS with various types of nontrivial potentials. In this context, we also mention the remarkable paper [24].

The initial value problem (IVP) associated with the DNLS equation with a complex potential deals with determining the evolution of the wave function over time when its initial configuration is known. In other words, given the initial values of the wave function and its derivative at a specific time, the IVP seeks to find a solution that satisfies the DNLS equation with the given complex potential. The DNLS equation with a complex potential on a one-dimensional lattice can be written as

$$i(d\psi_n/dt) + A_n\psi_n + B_n|\psi_n|^2\psi_n + C_n\psi_{n+1} + D_n\psi_{n-1} = 0,$$

where ψ_n is the complex-valued wave function at the discrete lattice site n, and A_n, B_n, C_n, and D_n represent the coefficients associated with the linear and nonlinear interactions between adjacent lattice sites.

In [20] we investigated the weighted ℓ^2 solution of the following initial value problem for the time-dependent d-dimensional discrete nonlinear Schrödinger equation

$$i\dot{u} = -\Delta u + Wu - f(n,u) + b(t,n), \tag{1.1}$$

$$u(0,n) = u^0(n)\,, \tag{1.2}$$

where the potential $W = V + i\delta$ is a complex function of

$$n = (n_1, n_2, \dots, n_d) \in \mathbb{Z}^d\,,$$

where $\dot{u}$ stands for the time derivative and $-\Delta$ is the d-dimensional discrete Laplacian defined by

$$\Delta u(n) = \sum_{j=1}^{d}\big[u(n+e_j) + u(n-e_j)\big] - 2du(n), \quad n \in \mathbb{Z}^d, \tag{1.3}$$

where $e_j \in \mathbb{Z}^d$ has 1 at the j-th component and 0 elsewhere.

Note that if $\delta(n)$ is negative for all $n \in \mathbb{Z}^d$, the part δ of the potential represents dissipation effects. Additionally, our Assumption (iii) below allows the nonlinearity to contain a dissipative term. This DNLS (1.1) is the space discretization of the nonlinear Schrödinger equation in continuous media.

Only a few papers [9, 10, 13, 14] are devoted to equations of the form (1.1). The paper [14] focuses on the initial value problem for the DNLS with a zero potential and power nonlinearity on a one-dimensional lattice with weighted ℓ^2 initial value. The main result provides global well-posedness in weighted ℓ^2 spaces with power weights. In [9] and [10], the authors consider the DNLS with $V = 0$ and $\delta =$ const. The main results are global well-posedness in the conservative ($\delta = 0$) and dissipative ($\delta < 0$) cases, as well as the existence of attractors in weighted ℓ^2 spaces in the conservative case, on one-dimensional and multidimensional lattices, respectively. In the paper [13], the well-posedness in weighted spaces is studied for the DNLS on a one-dimensional lattice in the case when $W = V$ is a general real potential and $b = 0$.

In [20], we extended those results to the multidimensional case, allowing a sufficiently general, not necessarily bounded potential W with weighted ℓ^2 initial value. In [27], we used the integral equation defining the mild solution of the DNLS in [20] to prove the existence of global solution for the DNLS with a weighted ℓ^p initial value when $1 \leq p < 2$ by leveraging the existing ℓ^2 global solutions obtained in [20]. In [28], we investigated the global solutions in the weighted ℓ^p space when $2 < p < \infty$.

Note that the study of attractors for spatially discrete systems goes back to the paper [1] and has been continued in [32, 33, 34, 35, 20].

We observe that, in contrast to [20, Theorem 3.1], the proof of [27, Theorem 4.1] necessitates the inclusion of following additional assumptions

$$\delta(n) \leq 0, \quad \text{for all } n \in \mathbb{Z}^d \quad \text{and} \quad b \in L^1([0,\infty), \ell^2(\mathbb{Z}^d)). \tag{1.4}$$

In this paper, we aim to enhance the existing results regarding the global solution for DNLS with a weighted ℓ^p initial value, where $1 < p < \infty$. Our approach involves optimal estimates and minimal assumptions, contributing to a substantial improvement in results. Specifically, we obtain the global weighted ℓ^p solution without relying on the aforementioned assumption (1.4). Additionally, we rigorously establish the existence of a global attractor for solutions in the weighted ℓ^p space to the initial value problem.

The organization of this paper is as follows: For readers' convenience, we provide a reminder of some preliminaries on the semigroup theory of abstract differential equations in Section 2. The previous results in [27] and [28] will be reviewed in Section 3. Section 4 is devoted to the existence of weighted ℓ^p global solutions for $1 < p < \infty$. We prove the existence of the global attractor in the last section 5.

2. Semigroup theory and abstract initial value problem

The content of this section can be found in [20]. For reader's convenience, we include it here. We treat (1.1) as an abstract differential equation of the form

$$\dot{u} = Au + N(t,u) \tag{2.1}$$

in a complex Banach space. We always assume that A is a closed operator in a Banach space E with the domain $D(A)$, and $N : [0,\infty) \times E \to E$ is continuous. Let us provide a reminder of some elementary facts related to such equations.

A family $U(t)$, $t \in [0,\infty)$, of bounded linear operators in E is a strongly continuous *semigroup of operators* if

(1) $U(t)v$ is a continuous function on $[0,\infty)$ with values in E for every $v \in E$;
(2) $U(0) = I$ is the identity operator in E;
(3) $U(t+s) = U(t)U(s)$ for all $t, s \in [0,\infty)$.

If the family $U(t)$ is defined for all $t \in \mathbb{R}$ and satisfies (1)–(3) above on the whole real line, we say that $U(t)$ is a strongly continuous *group of operators*.

If $U(t)$ is a strongly continuous semigroup of operators, then its *generator* A is defined by

$$Av = \lim_{t\to 0^+} t^{-1}(U(t) - I)v, \tag{2.2}$$

where the domain $D(A)$ consists of those $v \in E$ for which the limit in (2.2) exists. The following result is well known (see [5, 21]).

Proposition 2.1. *If A is a generator of a strongly continuous semigroup in a Banach space E and B is a bounded linear operator in E, then $A+B$ is a generator of a strongly continuous semigroup.*

If A is a bounded linear operator, then it generates a one-parameter group e^{tA}. In general, if A is a generator of a strongly continuous semigroup, we still use the same exponential notation e^{tA} for the semigroup generated by A.

Now we discuss the abstract initial value problem for equation (2.1), with initial data

$$u(0) = u^0 \in E. \tag{2.3}$$

If A is a bounded operator, then it is sufficient to consider classical solutions, i.e. continuously differentiable functions with values in E that satisfy (2.1) and (2.3). In general, when the operator A is unbounded, we consider mild solutions to (2.1) and (2.3).

A continuous function u on $[0, T]$ with values in E is a *mild solution* of the initial value problem (2.1) and (2.3) if it satisfies the integral equation

$$u(t) = e^{tA}u^0 + \int_0^t e^{(t-s)A} N(s, u(s))ds. \tag{2.4}$$

In the case when the operator A is bounded, these are classical solutions. We need the following well-known result (see [2, 21]).

Proposition 2.2. *Let A be a generator of a strongly continuous semigroup in a Banach space E, and $N(t, u) : [0, \infty) \times E \to E$ be continuous in t and locally Lipschitz continuous in u with the Lipschitz constant being bounded on bounded intervals of t. That is, for any $T > 0$ and $R > 0$, there exists $C = C(T, R) > 0$ such that*

$$\max_{0 \le t \le T} \|N(t, w) - N(t, w')\| \le C\|w - w'\| \tag{2.5}$$

whenever $\|w\| \le R$ and $\|w'\| \le R$.

(a) *For every $u^0 \in E$, there exists a unique local mild solution of the initial value problem* (2.1) *and* (2.3) *defined on the maximal interval* $[0, \tau_{\max})$.
(b) *If $\tau_{\max} < \infty$, then* $\lim_{t \nearrow \tau_{\max}} \|u(t)\| = \infty$.
(c) *The solution $u(t)$ depends continuously on u^0 in the topology of uniform convergence on bounded closed subintervals of* $[0, \tau_{\max})$.
(d) *Assume, in addition, that the map $N : [0, \infty) \times E \to E$ is locally Lipschitz continuous,* i.e., *for any $T > 0$ and $R > 0$, there exists $C = C(T, R) > 0$ such that*

$$\|N(t, w) - N(t', w')\| \le C(|t - t'| + \|w - w'\|) \tag{2.6}$$

whenever $t \in [0, T]$, $t' \in [0, T]$, $\|w\| \le R$ and $\|w'\| \le R$. If $u^0 \in D(A)$, then the mild solution of the initial value problem (2.1) *and* (2.3) *is a classical solution.*

Note that (2.5) implies automatically that N is bounded on bounded sets.

Remark 2.3. If $N(t, u)$ is globally Lipschitz continuous in u, i.e. there exists a constant $C = C(T) > 0$ such that

$$\max_{0 \le t \le T} \|N(t, w) - N(t, w')\| \le C\|w - w'\|, \quad \forall w, w' \in E, \tag{2.7}$$

then the initial value problem (2.1) and (2.3) possesses a unique global mild solution defined on $[0,\infty)$. Moreover, the solution $u(t)$ depends continuously on u^0 in the topology of uniform convergence on bounded closed subintervals of $[0,\infty)$.

Remark 2.4. Let

$$N(t,u) = M(u) + f(t)\,.$$

Then assumption (2.6) holds if and only if M and f are locally Lipschitz continuous on E and $[0,\infty)$, respectively.

3. Assumptions and review

We review local solution to the equation (1.1) under the following assumptions:

(i) The complex potential $W = V + i\delta$ is such that both V and δ are real-valued functions on $\mathbb{Z}^d$, and

$$\overline{\delta} = \sup\{\delta(n) | n \in \mathbb{Z}^d\} < \infty\,.$$

(ii) The nonlinearity $f : \mathbb{Z}^d \times \mathbb{C} \to \mathbb{C}$ satisfies the following conditions:

  (1) $f(n,0) = 0$,

  (2) $f(n,z) = o(z)$ as $z \to 0$ uniformly with respect to $n \in \mathbb{Z}^d$,

  (3) f is uniformly locally Lipschitz continuous, that is, for every $R > 0$, there exists a constant $C = C(R)$ independent of $n \in \mathbb{Z}^d$ such that

$$|f(n,z) - f(n,z')| \le C|z - z'|$$

  for all $n \in \mathbb{Z}^d$ whenever $|z| \le R$ and $|z'| \le R$.

(iii) The nonlinearity $f(n,z)$ is of the form $f(n,z) = g(n,|z|)z$, where $g(n,r)$ is a function and its imaginary part is nonnegative.

Let $\Theta = (\theta_n)_{n\in\mathbb{Z}^d}$ be a sequence of positive numbers (weights). The space $\ell^p_\Theta(\mathbb{Z}^d)$ consists of all two-sided sequences of complex numbers such that the norm

$$\|u\|_{\ell^p_\Theta} = \Big(\sum_{n\in\mathbb{Z}^d} |u(n)\theta_n|^p\Big)^{1/p}$$

is finite, where $1 \le p < \infty$. We notice that $u \in \ell^p_\Theta(\mathbb{Z}^d)$ if and only if $u\Theta \in \ell^p(\mathbb{Z}^d)$ and

$$\|u\|_{\ell^p_\Theta} = \|u\Theta\|_{\ell^p}.$$

Therefore for $1 \le p < q \le \infty$ we have

$$\|u\|_{\ell^q_\Theta} \le \|u\|_{\ell^p_\Theta} \quad \text{and} \quad \ell^p_\Theta(\mathbb{Z}^d) \subset \ell^q_\Theta(\mathbb{Z}^d).$$

We always assume that the weight Θ is *regular* in the sense that

(iv) The sequence Θ is bounded below by a positive constant, and there exists a constant $c_0 \ge 1$ such that

$$c_0^{-1} \le \frac{\theta_{n+e_i}}{\theta_n} \le c_0$$

for all $n \in \mathbb{Z}^d$ and $i = 1,\dots,d$, where $e_i \in \mathbb{Z}^d$ has 1 at the i-th component and 0 elsewhere.

From Assumption (iv), we obtain

$$\|u\|_{l^p(\mathbb{Z}^d)} \leq C_0\|u\|_{l^p_\Theta}, \tag{3.1}$$

which implies that $\ell^p_\Theta(\mathbb{Z}^d)$ is densely and continuously embedded into $\ell^p(\mathbb{Z}^d)$. Setting Θ_0 as the constant weight with unit components, we have that

$$\ell^p_{\Theta_0}(\mathbb{Z}^d) = \ell^p(\mathbb{Z}^d)$$

From the perspective of functional analysis, Assumption (iv) means that the space $\ell^p_\Theta(\mathbb{Z}^d)$ is translation invariant. More precisely, let S_i and T_i be the operators defined by

$$(S_i w)(n) = w(n - e_i), \quad (T_i w)(n) = w(n + e_i), \quad i = 1, \dots, d.$$

To understand the equation (1.1) in the framework of evolution equations, we interpret it as an evolution equation of the form (2.1), where $A = -iH$ and H is the Schrödinger operator defined as

$$H = -\Delta + W \tag{3.2}$$

and the operator N is defined as

$$N(t,u)(n) = if(n, u(t,n)) - ib(t,n). \tag{3.3}$$

To establish a precise interpretation, we need to analyze certain properties of the Schrödinger operator H in the space $\ell^p_\Theta(\mathbb{Z}^d)$. First, we observe that the operator (of multiplication by) $-iW = -iV + \delta$ is a diagonal operator. Since V is real and $\delta(n) \leq \overline{\delta}$ for all $n \in \mathbb{Z}^d$, the operator $-iW$ generates a strongly continuous semigroup in $\ell^p_\Theta(\mathbb{Z}^d)$ given by

$$(e^{-itW}u)(n) = e^{-iV(n)t}e^{\delta(n)t}u(n), \;\; n \in \mathbb{Z}^d\,.$$

The domain of this operator in $\ell^p_\Theta(\mathbb{Z}^d)$ is defined as

$$D_\Theta = \{u \in \ell^p_\Theta(\mathbb{Z}^d) : Wu \in \ell^p_\Theta(\mathbb{Z}^d)\}. \tag{3.4}$$

where we use the notation D to represent the domain of the operator W in $\ell^p(\mathbb{Z}^d)$. It is clear that $D_\Theta \subset D$.

Based on Proposition 2.1, we derived the following lemma in [27].

Lemma 3.1. *The operator $A = -iH$ is a generator of strongly continuous group e^{tA} in the space $\ell^p_\Theta(\mathbb{Z}^d)$, where $1 \leq p < \infty$. Moreover, there exist two constants $M \geq 1$ and ω such that for all $t \geq 0$*

$$\|e^{tA}\| \leq Me^{\omega t}. \tag{3.5}$$

We define the operator

$$N(t,u)(n) = if(n, u(n)) - ib(t,n).$$

Then equation (1.1) can be expressed in the form of equation (2.1). The following local well-posedness result is proved in [27].

Theorem 3.2.
1. *Under Assumptions (i), (ii), (iv), if $b \in C([0,\infty), \ell^p_\Theta(\mathbb{Z}^d))$, where $1 \leq p < \infty$, then for every $u^0 \in \ell^p_\Theta(\mathbb{Z}^d)$, problem* (1.1) *and* (1.2) *has a unique local mild solution $u \in C([0,\tau_{\max}), \ell^p_\Theta(\mathbb{Z}^d))$ which is defined on the maximal interval $[0,\tau_{\max})$.*
2. *If $\tau_{\max} < \infty$, then $\lim_{t \nearrow \tau_{\max}} \|u(t)\| = \infty$. The solution $u(t)$ depends continuously on u^0 in the topology of uniform convergence on bounded closed subintervals of $[0,\tau_{\max})$.*

(3) *The mild solution $u(t) \in C([0,\tau_{\max}), \ell^p_\Theta(\mathbb{Z}^d))$ of problem (1.1) and (1.2) obtained in part (1) is a classical solution if one of the following conditions holds*
 (a) *$u^0 \in \ell^p_\Theta(\mathbb{Z}^d)$ and W is bounded;*
 (b) *$u^0 \in D(A) = D_\Theta$ and $b : [0,\infty) \to \ell^p_\Theta(\mathbb{Z}^d)$ is locally Lipschitz continuous.*

4. Global solutions for $1 < p < \infty$

In [27] we utilized the fact $\ell^p_\Theta(\mathbb{Z}^d) \subset \ell^2_\Theta(\mathbb{Z}^d)$ for the case $1 \le p < 2$, and we relied on the established global existence of ℓ^2 solutions in [20] to prove the following theorem.

Theorem 4.1. (1) *Let assumptions (i)–(iv) be satisfied, and $1 \le p \le 2$, if $\overline{\delta} \le 0$ and $b \in C([0,\infty), \ell^p_\Theta(\mathbb{Z}^d)) \cap L^1([0,\infty), \ell^2(\mathbb{Z}^d))$, then for every $u^0 \in \ell^p_\Theta(\mathbb{Z}^d)$, problem (1.1) and (1.2) has a unique global mild solution $u \in C([0,\infty), \ell^p_\Theta(\mathbb{Z}^d))$ which continuously depends on u^0 in the topology of uniform convergence on bounded closed subintervals of $[0,\infty)$. Moreover, for any $t \ge 0$*

$$\|u(t)\|_{\ell^p_\Theta} \le (\|u^0\|_{\ell^p_\Theta} + B(\omega + CM, t)) e^{(\omega+CM)t}, \tag{4.1}$$

where

$$B(\omega + CM, t) = \int_0^t e^{-(\omega+CM)s} \|b(s)\|_{\ell^p_\Theta}\, ds,$$

C is the Lipschitz constant independent of t, ω and M are the constants in Lemma 3.1.

(2) *The global mild solution $u(t) \in C([0,\infty), \ell^p_\Theta(\mathbb{Z}^d))$ of problem (1.1) and (1.2) obtained in (1) is a classical solution if one of the following conditions holds*
 (a) *$u^0 \in \ell^p_\Theta(\mathbb{Z}^d)$ and W is bounded;*
 (b) *$u^0 \in D(A) = D_\Theta$ and $b : [0,\infty) \to \ell^p_\Theta(\mathbb{Z}^d)$ is locally Lipschitz continuous.*

We observed that, when comparing with Theorem 3.1 in [20], the proof of Theorem 4.1 required the following additional assumption:

$$\overline{\delta} \le 0, \quad \text{and} \quad b \in L^1([0,\infty), \ell^2(\mathbb{Z}^d)).$$

However, using a different method, we proved the following theorem in the case $2 \le p < \infty$ without the aforementioned additional assumption (refer to [28]).

Theorem 4.2. *(1) Assume that assumptions (i)–(iv) are satisfied and $2 \le p < \infty$, if $b \in C([0,\infty), \ell^p_\Theta(\mathbb{Z}^d))$, then for every $u^0 \in \ell^p_\Theta(\mathbb{Z}^d)$, problem (1.1) and (1.2) has a unique global mild solution $u \in C([0,\infty), \ell^p_\Theta(\mathbb{Z}^d))$ which continuously depends on u^0 in the topology of uniform convergence on bounded closed subintervals of $[0,\infty)$. Moreover, for any $t \ge 0$,*

$$\|u(t)\|_{\ell^p_\Theta} \le (\|u^0\|_{\ell^p_\Theta} + B(\overline{\delta} + 2dc_0, t)) e^{(\overline{\delta}+2dc_0)t}, \tag{4.2}$$

where

$$B(\overline{\delta} + 2dc_0, t) = \int_0^t e^{-(\overline{\delta}+2dc_0)s} \|b(s)\|_{\ell^p_\Theta} ds,$$

$\overline{\delta}$ and c_0 are constants in assumption (i) and (iv) respectively.

(2) The global mild solution $u(t) \in C([0,\infty), \ell^p_\Theta(\mathbb{Z}^d))$ of problem (1.1) and (1.2) obtained in (1) is a classical solution if one of the following conditions holds

(a) $u^0 \in \ell^p_\Theta(\mathbb{Z}^d)$ *and* W *is bounded;*
(b) $u^0 \in D(A) = D_\Theta$ *and* $b : [0,\infty) \to \ell^p_\Theta(\mathbb{Z}^d)$ *is locally Lipschitz continuous.*

In this section, we reconsider about the global solution in the case of $1 < p < 2$ without those additional assumptions. To do so, we need some preparations.

Lemma 4.3. *If* $u \in C^1([0,T], l^p_\Theta(\mathbb{Z}^d))$, $1 < p < \infty$, *then for each* $n \in \mathbb{Z}^d$ *and* $t \in [0,T]$,

$$\frac{d}{dt}|u(t,n)|^p = \mathrm{Re}\ \ Big(p|u(t,n)|^{p-2}\bar{u}(t,n)\frac{d}{dt}u(t,n)\Big) \tag{4.3}$$

Proof. Fix $t_0 \in (0,T)$. If $u(t_0,n) \neq 0$, then there exists $\delta > 0$ such that

$$|u(t,n)| > \frac{|u(t_0,n)|}{2} > 0, \quad \text{whenever } |t - t_0| < \delta,$$

by the chain rule, we obtain

$$\begin{aligned}
\frac{d}{dt}|u(t,n)|^p\big|_{t=t_0} &= \frac{d}{dt}[u(t,n)\bar{u}(t,n)]^{p/2}\big|_{t=t_0} \\
&= p|u(t_0,n)|^{p-2}\,\mathrm{Re}\Big(\bar{u}(t_0,n)\frac{d}{dt}u(t_0,n)\Big).
\end{aligned}$$

If $u(t_0,n) = 0$, by definition

$$\begin{aligned}
\frac{d}{dt}|u(t,n)|^p\big|_{t=t_0} &= \lim_{t\to t_0}\frac{|u(t,n)|^p - |u(t_0,n)|^p}{t-t_0} = \lim_{t\to t_0}\frac{|u(t,n)|^p}{t-t_0} \\
&= \big|\lim_{t\to t_0}\frac{u(t,n)-u(t_0,n)}{t-t_0}\big|^p \lim_{t\to t_0}(t-t_0)^{p-1} \\
&= \big|\frac{d}{dt}u(t,n)\big|_{t=t_0}\big|^p \lim_{t\to t_0}(t-t_0)^{p-1} = 0.
\end{aligned}$$

Notice that

$$|p|u(t_0,n)|^{p-2}\,\mathrm{Re}\Big(\bar{u}(t_0,n)\frac{d}{dt}u(t_0,n)\Big)| \le p|u(t_0,n)|^{p-1}|\frac{d}{dt}u(t_0,n)|,$$

we obtain (4.3) for $0 < t < T$. Similarly we can prove (4.3) for $t = 0$ or T if we replace two-sided limit by one-sided limit. □

For each positive integer k we define

$$\chi_k(n) = \begin{cases} 1, & \text{when } |n| \le k \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 4.4. *If* $u \in C^1([0,T], l^p_\Theta(\mathbb{Z}^d))$, $1 < p < \infty$, *then*

$$w(t) \equiv \|u(t)\|^p_{l^p_\Theta} = \sum_{n\in\mathbb{Z}^d}|u(t,n)\theta_n|^p$$

is differentiable and

$$\begin{aligned}
\frac{dw}{dt} &= \frac{d}{dt}\sum_{n\in\mathbb{Z}^d}|u(t,n)\theta_n|^p = \sum_{n\in\mathbb{Z}^d}\frac{d}{dt}|u(t,n)|^p\theta_n^p \\
&= \mathrm{Re}\Big(\sum_{n\in\mathbb{Z}^d}p\theta_n^p|u(t,n)|^{p-2}\bar{u}(t,n)\frac{d}{dt}u(t,n)\Big).
\end{aligned}$$

Proof. For each $k \geq 1$ we define the partial sum sequence

$$w_k(t) = \sum_{|n|\leq k} |u(t,n)\theta_n|^p = \sum_{n\in\mathbb{Z}^d} |u(t,n)\chi_k(n)\theta_n|^p.$$

By Lemma 4.3 we obtain

$$\begin{aligned}\frac{d}{dt}w_k(t) &= \mathrm{Re}\Big(\sum_{|n|\leq k} p\theta_n^p|u(t,n)|^{p-2}\bar{u}(t,n)\frac{d}{dt}u(t,n)\Big)\\ &= \mathrm{Re}\Big(\sum_{n\in\mathbb{Z}^d} p\theta_n^p|u(t,n)\chi_k(n)|^{p-2}\bar{u}(t,n)\chi_k(n)\frac{d}{dt}u(t,n)\Big).\end{aligned}$$

$u \in C^1([0,T], l^p_\Theta(\mathbb{Z}^d))$ implies $w_k(t)$ converges uniformly to $w(t)$ on $[0,T]$, and $\sup_{0\leq t\leq T}\|u'(t)\|_{l^p_\Theta} < \infty$. We denote

$$\begin{aligned}s(t) &= \mathrm{Re}\Big(\sum_{n\in\mathbb{Z}^d} p\theta_n^p|u(t,n)|^{p-2}\bar{u}(t,n)\frac{d}{dt}u(t,n)\Big)\\ s_k(t) &= \mathrm{Re}\Big(\sum_{n\in\mathbb{Z}^d} p\theta_n^p|u(t,n)\chi_k(n)|^{p-2}\bar{u}(t,n)\chi_k(n)\frac{d}{dt}u(t,n)\Big).\end{aligned}$$

By Hölder's inequality,

$$\begin{aligned}\sup_{0\leq t\leq T}|s_k(t)-s(t)| &\leq \sup_{0\leq t\leq T}\sum_{n\in\mathbb{Z}^d} p\theta_n^p|u(t,n)(1-\chi_k(n))|^{p-1}|\frac{d}{dt}u(t,n)|\\ &\leq p\sup_{0\leq t\leq T}\|u(t)(1-\chi_k)\|^{p-1}_{l^p_\Theta}\sup_{0\leq t\leq T}\|u'(t)\|_{l^p_\Theta},\end{aligned}$$

which implies that $s_k(t) = \frac{d}{dt}w_k(t)$ converges uniformly to $s(t)$ on $[0,T]$ and $s(t) \in C([0,T],\mathbb{R})$. By [22, Theorem 316] we obtain

$$s(t) = \lim_{k\to\infty}\frac{d}{dt}w_k(t) = \frac{d}{dt}\lim_{k\to\infty}w_k(t) = \frac{d}{dt}w(t). \qquad \square$$

Lemma 4.5. *Let $1 < p < \infty$ and $u \in \ell^p_\Theta(\mathbb{Z}^d))$. If Θ satisfies the assumption (iv), then*

$$\sum_{n\in\mathbb{Z}^d}\mathrm{Im}[-\Delta u(n)\overline{u(n)}]|u(n)|^{p-2}\theta_n^p \leq 2dc_0\|u\|^p_{\ell^p_\Theta}.$$

Proof. Since for all $n \in \mathbb{Z}^d$ we have

$$-\Delta u(n)\overline{u(n)} = -\sum_{j=1}^d\big[u(t,n+e_j)+u(t,n-e_j)\big]\overline{u(t,n)} + 2d|u(n)|^2 \tag{4.4}$$

which implies

$$\begin{aligned}&\sum_{n\in\mathbb{Z}^d}\mathrm{Im}[-\Delta u(n)\overline{u(n)}]|u(n)|^{p-2}\theta_n^p\\ &= -\sum_{j=1}^d\sum_{n\in\mathbb{Z}^d} Im\big[[u(t,n+e_j)+u(t,n-e_j)]\overline{u(n)}\big]|u(n)|^{p-2}\theta_n^p\\ &\leq \sum_{j=1}^d\sum_{n\in\mathbb{Z}^d}|u(n)\theta_n|^{p-1}|u(t,n+e_j)\theta_{n+e_j}|\frac{\theta_n}{\theta_{n+e_j}}\end{aligned}$$

$$+\sum_{j=1}^{d}\sum_{n\in\mathbb{Z}^d}|u(n)\theta_n|^{p-1}|u(t,n-e_j)\theta_{n-e_j}|\frac{\theta_n}{\theta_{n-e_j}}.$$

Since Θ is a regular weight by assumption (iv), then by Hölder's inequality we obtain

$$\sum_{n\in\mathbb{Z}^d}\operatorname{Im}[-\Delta u(n)\overline{u(n)}]|u(n)|^{p-2}\theta_n^p\le 2dc_0\|u\|^p_{\ell^p_\Theta}. \qquad \square$$

For $1<p<\infty$ and Θ satisfying assumption (iv), we define

$$\rho_\Theta(p)\equiv\sup_{\|u\|_{\ell^p_\Theta}=1}\sum_{n\in\mathbb{Z}^d}\operatorname{Im}[-\Delta u(n)\overline{u(n)}]|u(n)|^{p-2}\theta_n^p. \tag{4.5}$$

By Lemma 4.5 we know that $\rho_\Theta(p)\le 2dc_0$.

Let

$$N(t,u)(n)=if(n,u(t,n))-ib(t,n),\quad A=-iH=-i(-\Delta+W).$$

For $u^0\in\ell^p_\Theta$, $1<p<\infty$, by Theorem 3.2, there is a unique mild solution $u(t)$ defined on the maximal interval $[0,\tau_{\max})$ and if $u^0\in D(A)$ and $b(t)$ is locally Lipschitz in $[0,\tau_{\max})$, then $u\in C^1([0,\tau_{\max}),\ell^p_\Theta)$ is a classical solution satisfying

$$\frac{d}{dt}u(t,n)=Au(t,n)+N(t,u)(n),\quad\forall n\in\mathbb{Z}^d. \tag{4.6}$$

Lemma 4.6. *Let assumptions (i)–(iv) be satisfied and $b\in C([0,\infty),\ell^p_\Theta(\mathbb{Z}^d))$ with $1<p<\infty$. If $u^0\in\ell^p_\Theta(\mathbb{Z}^d)$, then there exists a constant $\bar C=\bar\delta+\rho_\Theta(p)$, such that the mild solution $u\in C([0,\tau_{\max}),\ell^p_\Theta(\mathbb{Z}^d))$ satisfies*

$$\|u(t)\|_{\ell^p_\Theta}\le e^{\bar Ct}\Big(\|u^0\|_{\ell^p_\Theta}+B(\bar C,t)\Big) \tag{4.7}$$

for all $0\le t<\tau_{\max}$.

Proof. First of all we assume that $u^0\in D(A)$ and $b(t)$ is locally Lipschitz in $[0,\tau_{\max})$, then $u\in C^1([0,\tau_{\max}),l^p_\Theta)$ is a classical solution by Theorem 3.2. By Lemma 4.3 we obtain

$$\frac{d}{dt}|u(t,n)|^p=\operatorname{Re}\big[p\big|u(t,n)\big|^{p-2}\overline{u(t,n)}\frac{d}{dt}u(t,n).\big]$$

Multiplying (4.6) by $p\big|u(t,n)\big|^{p-2}\overline{u(t,n)}$, and taking the real part we obtain

$$\begin{aligned}
&\frac{d}{dt}|u(t,n)|^p\\
&=\operatorname{Re}\big[p|u(t,n)|^{p-2}\overline{u(t,n)}[(-iH)u(t,n)]+p|u(t,n)|^{p-2}\overline{u(t,n)}N(t,u)(n)\big]\\
&=\operatorname{Re}\big[p|u(t,n)|^{p-2}\overline{u(t,n)}[(i\Delta u(t,n)-iWu(t,n)]\\
&\quad+p|u(t,n)|^{p-2}\overline{u(t,n)}\big(if(n,u(t,n)\big)-p|u(t,n)|^{p-2}\overline{u(t,n)}ib(t,n).\big]
\end{aligned}$$

From assumption (i) we have $-iW=-iV+\delta$ and from assumption (iii) we have

$$f(n,u)=g(n,|u|)u(t,n)=(\operatorname{Re}\ g+i\operatorname{Im}g)u(t,n),$$

$$\begin{aligned}
\frac{d}{dt}|u(t,n)|^p=\operatorname{Re}\Big[&p|u(t,n)|^{p-2}\overline{u(t,n)}[(i\Delta u(t,n)-iV(n)u(t,n)+\delta(n)u(t,n)]\\
&+p|u(t,n)|^{p-2}|u(t,n)|^2\big(i\operatorname{Re}g-\operatorname{Im}g)\big)\\
&-p|u(t,n)|^{p-2}\overline{u(t,n)}\ ib(t,n).\Big]
\end{aligned}$$

$$
\begin{gathered}
\Delta u(t,n)=\sum_{j=1}^{d}\big[u(t,n+e_j)+u(t,n-e_j)\big]-2du(t,n),\\
i\Delta u(t,n)\overline{u(t,n)}=i\sum_{j=1}^{d}\big[u(t,n+e_j)+u(t,n-e_j)\big]\overline{u(t,n)}-i2d|u(t,n)|^2, \quad (4.8)\\
\delta(n)u(t,n)\overline{u(t,n)}=\delta(n)|u(t,n)|^2. \quad (4.9)
\end{gathered}
$$

Using (4.4) and (4.9), we have

$$
\begin{aligned}
\frac{d}{dt}|u(t,n)|^p = p\delta(n)|u(t,n)|^p &- p|u(t,n)|^{p-2}\operatorname{Img}\sum_{j=1}^{d}\big[u(t,n+e_j)\overline{u(t,n)}\\
&+u(t,n-e_j)\overline{u(t,n)}\big]-p|u(t,n)|^p\big(\operatorname{Img}\big)\\
&+p|u(t,n)|^{p-2}\operatorname{Im}(\overline{u(t,n)}b).\big]
\end{aligned}
$$

Multiplying both sides by θ_n^p, then by assumption (iii), $\operatorname{Im} g \geq 0$ implies that $-p|u(t,n)|^p(\operatorname{Im} g)\leq 0$. Thus by assumption (i) we have

$$
\begin{aligned}
\frac{d}{dt}|u(t,n)|^p\theta_n^p \leq p\bar{\delta}\ |u(t,n)|^p\theta_n^p &- p|u(t,n)|^{p-2}\operatorname{Im}\sum_{j=1}^{d}\big[u(t,n+e_j)\theta_n^p\overline{u(t,n)}\\
&+u(t,n-e_j)\theta_n^p\overline{u(t,n)}\big]+p|u(t,n)|^{p-2}\operatorname{Im}(\overline{u(t,n)}b)\theta_n^p.
\end{aligned}
$$

Taking summation over $n\in\mathbb{Z}^d$, by Lemma 4.4 we obtain

$$
\begin{aligned}
&\frac{d}{dt}\Big(\sum_{n\in\mathbb{Z}^d}|u(t,n)\theta_n|^p\Big)\\
&\leq p\bar{\delta}\sum_{n\in\mathbb{Z}^d}|u(t,n)\theta_n|^p - p\operatorname{Im}\sum_{j=1}^{d}\sum_{n\in\mathbb{Z}^d}|u(t,n)|^{p-2}\overline{u(t,n)}\big[u(t,n+e_j)\theta_n^p\\
&\quad+u(t,n-e_j)\theta_n^p\big]+p|u(t,n)|^{p-2}\operatorname{Im}(\overline{u(t,n)}b(t,n)\theta_n^p).
\end{aligned}
$$

Since

$$
\operatorname{Im}(\overline{u(t,n)}b(t,n))\leq|\overline{u(t,n)}b(t,n)|=|u(t,n)||b(t,n)|,
$$

by Lemma 4.5 we have

$$
\frac{d}{dt}\|u\|_{\ell^p_\Theta}^p\leq p\bar{\delta}\|u(t)\|_{\ell^p_\Theta}^p+p\rho_\Theta(p)\|u(t)\|_{\ell^p_\Theta}^p+p\sum_{n\in\mathbb{Z}^d}|u(t,n)\theta_n|^{p-1}|b(t,n)||\theta_n|.
$$

Then by Hölder's inequality we obtain

$$
\frac{d}{dt}\|u(t)\|_{\ell^p_\Theta}^p\leq p(\overline{\delta}+\rho_\Theta(p))\|u(t)\|_{\ell^p_\Theta}^p+p\|u(t)\|_{\ell^p_\Theta}^{p-1}\|b(t)\|_{\ell^p_\Theta}.
$$

Let $\overline{C}=\overline{\delta}+\rho_\Theta(p)$ and $\alpha=\frac{p-1}{p}\in(0,1)$, and by the definition of $w(t)=\|u(t)\|_{\ell^p_\Theta}^p$ we obtain

$$
\frac{d}{dt}w(t)\leq p\overline{C}w(t)+p\|b(t)\|_{\ell^p_\Theta}w(t)^\alpha. \quad (4.10)
$$

Since the inequality (4.7) holds if $w(t)=0$, we only need to consider the case $w(t)>0$. Then $Z(t)=w(t)^{1-\alpha}$ is differentiable and by the chain rule we obtain

$$
\frac{d}{dt}Z(t)=(1-\alpha)w(t)^{-\alpha}\frac{d}{dt}w(t)\leq(1-\alpha)p\overline{C}Z(t)+(1-\alpha)p\|b(t)\|_{\ell^p_\Theta}.
$$

Notice that $(1-\alpha)p = 1$ and $Z(t) = \|u(t)\|_{\ell^p_\Theta}$ and by using Grönwall's inequality we obtain

$$\|u(t)\|_{\ell^p_\Theta} \le e^{\overline{C}t}\big(\|u^0\|_{\ell^p_\Theta} + B(\overline{C}, t)\big).$$

Now we assume that $u^0 \in \ell^p_\Theta(\mathbb{Z}^d)$ and b is a continuous function with values in $\ell^p_\Theta(\mathbb{Z}^d)$. Let u be a solution of problem (1.1)-(1.2) with maximal interval of existence $[0, \tau_{\max})$. Choose $u^{0,k} \in D(A)$ and $b^k \in C^1([0, \tau_{\max}), \ell^p_\Theta(\mathbb{Z}^d))$ such that $u^{0,k} \to u^0$ in $\ell^p_\Theta(\mathbb{Z}^d)$ and $b^k \to b$ uniformly on compact intervals. Let u^k be a solution of problem (1.1)- (1.2). By Theorem 3.2, for any $T \in (0, \tau_{\max})$ the solution u^k is defined on $[0, T]$ for all k large enough and $u^k \to u$ uniformly on $[0, T]$. Applying inequality (4.7) to u^k and passing to the limit, we extend these inequalities to the general case. □

Now, we are ready to prove the following theorem regarding the existence of global weighted ℓ^p solution to the initial value problem(1.1) and (1.2) for $1 < p < \infty$.

Theorem 4.7. *(1) Assume that assumptions (i)–(iv) are satisfied and $1 < p < \infty$, if $b \in C([0, \infty), \ell^p_\Theta(\mathbb{Z}^d))$, then for every $u^0 \in \ell^p_\Theta(\mathbb{Z}^d)$, problem (1.1) and (1.2) has a unique global mild solution $u \in C([0, \infty), \ell^p_\Theta(\mathbb{Z}^d))$ which continuously depends on u^0 in the topology of uniform convergence on bounded closed subintervals of $[0, \infty)$. Moreover, for any $t \ge 0$*

$$\|u(t)\|_{\ell^p_\Theta} \le (\|u^0\|_{\ell^p_\Theta} + B(\overline{\delta} + \rho_\Theta(p), t))e^{(\overline{\delta}+\rho_\Theta(p))t}\,, \tag{4.11}$$

where $\rho_\Theta(p) \le 2dc_0$ defined in (4.5) and

$$B(\overline{\delta} + \rho_\Theta(p), t) = \int_0^t e^{-(\overline{\delta}+\rho_\Theta(p))s}\|b(s)\|_{\ell^p_\Theta} ds,$$

$\overline{\delta}$ and c_0 are constants in assumption (i) and (iv) respectively.

(2) The global mild solution $u(t) \in C([0, \infty), \ell^p_\Theta(\mathbb{Z}^d))$ of problem (1.1) and (1.2) obtained in (1) is a classical solution if one of the following conditions holds

(a) *$u^0 \in \ell^p_\Theta(\mathbb{Z}^d)$ and W is bounded;*
(b) *$u^0 \in D(A) = D_\Theta$ and $b : [0, \infty) \to \ell^p_\Theta(\mathbb{Z}^d)$ is locally Lipschitz continuous.*

Proof of Theorem 4.7. (1) From Theorem 3.2 we know u is the unique mild solution to initial value problem(1.1) and (1.2) on $[0, \tau_{\max})$. By Lemma 4.6, for all $0 \le t < \tau_{\max}$ we have

$$\|u(t)\|_{\ell^p_\Theta} \le (\|u^0\|_{\ell^p_\Theta} + B(\overline{\delta} + \rho_\Theta(p), t))e^{(\overline{\delta}+\rho_\Theta(p))t}.$$

If $\tau_{\max} < \infty$, the limit of the right-hand side of the inequality is finite as $t \to \tau_{\max}$ since all functions of t involved are continuous on $[0, \infty)$ which implies

$$\lim_{t\to\tau_{\max}} \|u(t)\|_{\ell^p_\Theta} \le (\|u^0\|_{\ell^p_\Theta} + B(\overline{\delta} + \rho_\Theta(p), \tau_{\max}))e^{(\overline{\delta}+\rho_\Theta(p))\tau_{\max}} < \infty,$$

which contradicts with Theorem 3.2 (2). Therefore we must have $\tau_{\max} = \infty$ and the mild solution $u(t)$ is global.

(2) The proof follows from a similar reasoning in [20]. □

Remark 4.8. (1) Theorem 4.7 holds for all $1 < p < \infty$, which represents an improvement over Theorem 4.1 and 4.2. However, its proof is not valid for the case $p = 1$, which was covered in Theorem 4.1.

(2) There are no results concerning global solutions in ℓ^∞_Θ.

(3) When $p = 2$, the constant $\rho_\Theta(p)$ is similar to ρ_Θ defined in [20]. We can easily prove that $\rho_{\Theta_0}(p) = 0$ if and only if $p = 2$. Therefore, with a slight different definition of the weighted l^p space, our results generalize the theorems in [20].

If we set $f = b = 0$ in equation (1.1), we obtain the following corollary, which is an enhanced version of Lemma 3.1 for the case $1 < p < \infty$.

Corollary 4.9. *Let W satisfy assumption (i) and Θ satisfy assumption (iv) and $H = -\Delta + W$, then the operator $-iH$ generates a strongly continuous semigroup e^{-itH} in the space $\ell^p_\Theta(\mathbb{Z}^d)$ for $1 \le p < \infty$. Moreover if $1 < p < \infty$, then for each $u^0 \in \ell^p_\Theta(\mathbb{Z}^d)$ we have the following estimate for all $t \ge 0$*

$$\|e^{-itH}u^0\|_{\ell^p_\Theta} \le e^{(\overline{\delta}+\rho_\Theta(p))t}\|u^0\|_{\ell^p_\Theta}, \tag{4.12}$$

where $\rho_\Theta(p) \le 2dc_0$ defined in (4.5) and $\overline{\delta}$ and c_0 are constants in assumption (i) and (iv) respectively.

5. Global attractor for $1 < p < \infty$

Now, we investigate the long-term behavior of global solutions to the initial value problem for the autonomous DNLS equation. In equation (1.1), we introduce the additional assumption that

(v) The forcing term $b(t, n) \equiv b(n)$ is independent of t.

Let us denote by $S(t) = S_\Theta(t)$ the solution operator to

$$\begin{array}{ccc} \ell^p_\Theta(\mathbb{Z}^d) & \overset{S(t)}{\to} & \ell^p_\Theta(\mathbb{Z}^d) \\ u^0 & \mapsto & S(t)u^0 = u(t) \end{array}, \tag{5.1}$$

where $t \to u(t)$ represents the solution to the initial value problem (1.1), (1.2). It is important to note that under assumptions (i)–(v), Theorem 4.2 is applicable, and $u(t)$ is a globally defined classical solution. In this scenario, the path $t \to S(t)$ constitutes a nonlinear semigroup of operators as described in [21], operating within the space $\ell^p_\Theta(\mathbb{Z}^d)$.

Theorem 5.1. *Assume that assumptions (i)–(v) hold, and that $1 < p < \infty$. If $\overline{\delta} + \rho_\Theta(p) < 0$ and $b \in \ell^p_\Theta(\mathbb{Z}^d)$, then the semigroup of operators $S(t)$ in (5.1) has a compact, and connected, global attractor set A_Θ that is maximal among all functional invariant sets in $\ell^p_\Theta(\mathbb{Z}^d)$.*

For the proof of the existence of this global attractor in $\ell^p_\Theta(\mathbb{Z}^d)$ we follow the strategy outlined in [20], which is in turn quite similar to that in [23, Theorem 1.1.1] (see also [4, Chapter 2, Theorem 3.1]). We proceed in steps, starting with the following Lemma on the existence of an absorbing set for $S(t)$.

Lemma 5.2. *Assume that assumptions (i)–(v) hold, and that $1 < p < \infty$. If $\overline{\delta} + \rho_\Theta(p) < 0$ and $b \in l^p_\Theta$, then the solution operator $S(t)$ in (5.1) possesses a bounded absorbing set B_0 in $\ell^p_\Theta(\mathbb{Z}^d)$, that is to say, for every bounded set $B \subset \ell^p_\Theta(\mathbb{Z}^d)$ there exists $t_0(B)$ such that $S(t)B \subset B_0$ for all $t \ge t_0(B)$.*

Proof. If $u^0 \in \ell^p_\Theta$, then $u(t) \in \ell^p_\Theta$ satisfies (4.2). For convenience, we set $\kappa = -(\overline{\delta} + \rho_\Theta(p))$, which is positive by hypothesis. Since (v) holds, we obtain that

$$\|u(t)\|_{\ell^p_\Theta} \le \|u^0\|_{\ell^p_\Theta} e^{-\kappa t} + (1 - e^{-\kappa t})\frac{\|b\|_{\ell^p_\Theta}}{\kappa} \to \frac{\|b\|_{\ell^p_\Theta}}{k}, \tag{5.2}$$

and so

$$\lim_{t\to\infty} \sup \|u(t)\|_{\ell^p_\Theta} \le \frac{\|b\|_{\ell^p_\Theta}}{\kappa} := r\,,$$

where we use the expression on the right side to define r.

Let B be any bounded subset in $\ell^p_\Theta(\mathbb{Z}^d)$. Then, B is contained within a ball with its center at the origin of radius $2r+l$. To demonstrate its absorption in finite time, we can assume, without loss of generality, that B is the ball of radius $R \ge 2r+1$ centered at the origin.

We choose R_0 in the interval $(r, 2r+1)$ and let B_0 be the ball of radius R_0 centered at $(0,0)$. Then, for all $u^0 \in B$, the solution $u(t) = S(t)u^0$ satisfies

$$\|u(t)\|_{\ell^p_\Theta} \le \|u^0\|_{\ell^p_\Theta} e^{-\kappa t} + \frac{1-e^{-\kappa t}}{\kappa}\|b\|_{\ell^p_\Theta} \le (\|u^0\|_{\ell^p_\Theta} - r)e^{-\kappa t} + r \le (R-r)e^{-\kappa t} + r\,.$$

It follows that if

$$t_0 = t_0(B) = \frac{1}{\kappa}\ln\frac{R-r}{R_0-r}\,,$$

for each $t \ge t_0$ we have that $\|u(t)\|_{\ell^p_\Theta} \le R_0$ and so $u(t) \subset B_0$. Thus, B_0 is an absorbing set. $\square$

As in [20], we introduce the following piecewise line Lipschitz continuous function

$$\varphi(r) = \begin{cases} 0 & \text{if } 0 \le r \le 1,\\ r-1 & \text{if } 1 \le r < 2,\\ 1 & \text{if } r \ge 2, \end{cases}$$

with Lipschitz constant 1. When $R > 0$, we use the notation $\phi(n) = \varphi(|n|/R)$ for all $n \in \mathbb{Z}^d$. We suppress the explicit dependency of ϕ on R in the notation for convenience. It is worth noting that for any $n \in \mathbb{Z}^d$, we have

$$0 \le \phi(n) \le 1, \quad |\phi(n \pm e_j) - \phi(n)| \le \frac{1}{R}, \quad 1 \le j \le d\,.$$

We continue our study of the semigroup $S(t)$ by establishing the following lemma, which provides a tail estimate.

Lemma 5.3. *Assume that assumptions (i)–(v) hold, and that $1 < p < \infty$. If $\overline{\delta} + \rho_\Theta(p) < 0$, $b \in \ell^p_\Theta$, and B is a bounded subset of $\ell^p_\Theta(\mathbb{Z}^d)$, then for any $\varepsilon > 0$, there exist positive constants $R_0 = R_0(\varepsilon)$, and $T = T(\varepsilon)$ such that for all $u^0 \in B$ the solution $u(t) = S(t)u^0$ of (1.1)-(1.2) is such that*

$$\sum_{|n|>2R} |u(t,n)|^p \theta_n^p \le \Big(\frac{\varepsilon}{|\overline{\delta} + \rho_\Theta(p)|}\Big)^p, \tag{5.3}$$

for all $t \ge T$ and $R \ge R_0$.

Proof. By Theorem 4.2, if $u^0 \in u^0 \in D_\Theta \cap B$, then $u(t) = S(t)u^0$ is a global classical solution of the initial value problem (1.1)-(1.2). With the help of Lemma 4.4, we can then obtain

$$\begin{aligned} \frac{d}{dt}\|\phi u(t)\|^p_{\ell^p_\Theta} &= \frac{d}{dt}\sum_{n\in\mathbb{Z}^d} |u(t,n)\phi(n)\theta_n|^p \\ &= \sum_{n\in\mathbb{Z}^d} \frac{d}{dt}|u(t,n)\phi(n)|^p \theta_n^p \end{aligned}$$

$$= \mathrm{Re}\Big(\sum_{n\in\mathbb{Z}^d} p\theta_n^p |u(t,n)\phi(n)|^{p-2}\phi(n)^2\overline{u}(t,n)\frac{d}{dt}u(t,n)\Big),$$

and if we multiply both sides of (1.1) by $p\theta_n^p\phi(n)^p|u(t,n)|^{p-2}\overline{u}(t,n)$, and then use assumptions (i) and (iii) to find the real parts of both sides both sides of the resulting expression, we obtain that

$$\begin{aligned}
&\frac{d}{dt}\|\phi u(t)\|_{\ell^p_\Theta}^p\\
&\le p\overline{\delta}\sum_{n\in\mathbb{Z}^d}|(\phi u)(n)|^p\theta_n^p + p\sum_{n\in\mathbb{Z}^d}\mathrm{Im}[-\Delta u(n)\overline{u(n)}]|u(n)|^{p-2}\theta_n^p\phi(n)^p\\
&\quad + p\sum_{n\in\mathbb{Z}^d}|(\phi u)(n)|^{p-2}\,\mathrm{Im}(\overline{\phi u}(n)b(n)\phi(n))\theta_n^p\\
&\le p\overline{\delta}\|\phi u\|_{\ell^p_\Theta}^p + p\|\phi u\|_{\ell^p_\Theta}^{p-1}\|\phi b\|_{\ell^p_\Theta} + p\sum_{n\in\mathbb{Z}^d}\mathrm{Im}[-\Delta u(n)\overline{u(n)}]|u(n)|^{p-2}\theta_n^p\phi(n)^p.
\end{aligned}$$

By the Lipschitz property of ϕ, condition (iv), Lemma 4.5, and Hölder's inequality, we obtain that

$$\begin{aligned}
&p\sum_{n\in\mathbb{Z}^d}\mathrm{Im}[-\Delta u(n)\overline{u(n)}]|u(n)|^{p-2}\theta_n^p\phi(n)^p\\
&= -p\sum_{i=1}^d\sum_{n\in\mathbb{Z}^d}|(\theta\phi u)(n)|^{p-2}\,\mathrm{Im}\big[(u(n+e_j)\phi(n)\theta_n + u(n-e_j)\phi(n)\theta_n)\overline{\theta\phi u}(n)\big]\\
&= -p\sum_{i=1}^d\sum_{n\in\mathbb{Z}^d}|(\theta\phi u)(n)|^{p-2}\,\mathrm{Im}\big[u(n+e_j)\phi(n+e_j)\theta_n\overline{\theta\phi u}(n)\big]\\
&\quad - p\sum_{i=1}^d\sum_{n\in\mathbb{Z}^d}|(\theta\phi u)(n)|^{p-2}\,\mathrm{Im}\big[u(n-e_j)\phi(n-e_j)\theta_n\overline{\theta\phi u}(n)\big]\\
&\quad + p\sum_{i=1}^d\sum_{n\in\mathbb{Z}^d}|(\theta\phi u)(n)|^{p-2}\,\mathrm{Im}\big[u(n+e_j)(\phi(n+e_j)-\phi(n))\theta_n\overline{\theta\phi u}(n)\big]\\
&\quad + p\sum_{i=1}^d\sum_{n\in\mathbb{Z}^d}|(\theta\phi u)(n)|^{p-2}\,\mathrm{Im}\big[u(n-e_j)(\phi(n-e_j)-\phi(n))\theta_n\overline{\theta\phi u}(n)\big]\\
&= p\sum_{n\in\mathbb{Z}^d}\mathrm{Im}[-\Delta(\phi u)(n)\overline{\phi u}(n)]|(\phi u)(n)|^{p-2}\theta_n^p\\
&\quad + p\sum_{i=1}^d\sum_{n\in\mathbb{Z}^d}|(\theta\phi u)(n)|^{p-2}\,\mathrm{Im}\big[u(n+e_j)(\phi(n+e_j)-\phi(n))\theta_n\overline{\theta\phi u}(n)\big]\\
&\quad + p\sum_{i=1}^d\sum_{n\in\mathbb{Z}^d}|(\theta\phi u)(n)|^{p-2}\,\mathrm{Im}\big[u(n-e_j)(\phi(n-e_j)-\phi(n))\theta_n\overline{\theta\phi u}(n)\big]\\
&\le p\rho_\Theta(p)\|\phi u(t)\|_{\ell^p_\Theta}^p + p(2d/R)c_0\|u(t)\|_{\ell^p_\Theta}\|\phi u(t)\|_{\ell^p_\Theta}^{p-1},
\end{aligned}$$

which implies that

$$\frac{d}{dt}\|\phi u(t)\|_{\ell^p_\Theta}^p \le p(\overline{\delta}+\rho_\Theta(p))\|\phi u\|_{\ell^p_\Theta}^p + p(2d/R)c_0\|u(t)\|_{\ell^p_\Theta}\|\phi u(t)\|_{\ell^p_\Theta}^{p-1}.$$

Since $\kappa = \overline{\delta} + \rho_\Theta(p) < 0$, by (5.2) we obtain

$$\sup_{t\geq 0} \|u(t)\|_{\ell^p_\Theta} \leq \max\{\|u^0\|_{\ell^p_\Theta}, \ \frac{\|b\|_{\ell^p_\Theta}}{|\overline{\delta} + \rho_\Theta(p)|}\} := M\,. \tag{5.4}$$

Therefore,

$$\begin{aligned}\frac{d}{dt}\|\phi u(t)\|^p_{\ell^p_\Theta} &\leq p(\overline{\delta} + \rho_\Theta(p))\|\phi u\|^p_{\ell^p_\Theta} + \frac{2pdc_0M}{R}\|\phi u\|^{p-1}_{\ell^p_\Theta} + p\|\phi u\|^{p-1}_{\ell^p_\Theta}\|\phi b\|_{\ell^p_\Theta}\\ &= p(\overline{\delta} + \rho_\Theta(p))\|\phi u(t)\|^p_{\ell^p_\Theta} + p\Big(\frac{2dc_0M}{R} + \|\phi b\|_{\ell^p_\Theta}\Big)\|\phi u\|^{p-1}_{\ell^p_\Theta}\,.\end{aligned}$$

By Grönwall's inequality, we conclude that

$$\|\phi u(t)\|_{\ell^p_\Theta} \leq \|\phi u^0\|_{\ell^p_\Theta} e^{(\overline{\delta}+\rho_\Theta(p))t} + \frac{\|\phi b\|_{\ell^p_\Theta} + \frac{2dc_0M}{R}}{|\overline{\delta} + \rho_\Theta(p)|}(1 - e^{(\overline{\delta}+\rho_\Theta(p))t})\,. \tag{5.5}$$

If we now consider an initial condition such that $u_0 \in B$, we take any sequence $u^0_k \in D_\Theta \cap B$ such that $u^0_k \to u^0$ in $\ell^p_\Theta(\mathbb{Z}^d)$. Then, as functions with values in $\ell^p_\Theta(\mathbb{Z}^d)$, the solutions $u_k(t) = S(t)u^0_k$ converge to $u(t) = S(t)u^0$ uniformly on bounded intervals, and applying inequality (5.5) to $u_k(t)$ and passing to the limit, we can see (5.5) holds also if $u(t)$ is the solution corresponding to the initial condition u^0 also.

Let us set

$$r = \sup\{\|u^0\|_{\ell^p_\Theta} : \ u^0 \in B\}\,.$$

Given any $\varepsilon > 0$, we define

$$T(\varepsilon) = \max\Big\{0, \ \frac{\ln\frac{2r|\overline{\delta}+\rho_\Theta(p)|}{\varepsilon}}{|\overline{\delta} + \rho_\Theta(p)|}\Big\}\,.$$

Then for all $t > T(\varepsilon)$, we readily obtain that

$$\|\phi u^0\|_{\ell^p_\Theta} e^{(\overline{\delta}+\rho_\Theta(p))t} < \frac{\varepsilon}{2|\overline{\delta} + \rho_\Theta(p)|}\,.$$

On the other hand, since $\phi(n)$ is supported on $n > R$, we see easily that

$$\lim_{R\to\infty}\Big(\|\phi b\|_{\ell^p_\Theta} + \frac{2dc_0M}{R}\Big) = \lim_{R\to\infty}\|\phi b\|_{\ell^p_\Theta} = 0\,,$$

and so each $\varepsilon > 0$, there exists $R_0 > 0$ such that, for any $R > R_0$ and $t \geq 0$, we have

$$\Big(\|\phi b\|_{\ell^p_\Theta} + \frac{2dc_0M}{R}\Big)\frac{(1 - e^{(\overline{\delta}+\rho_\Theta(p))t})}{|\overline{\delta} + \rho_\Theta(p)|} \leq \frac{\|\phi b\|_{\ell^p_\Theta} + \frac{2dc_0M}{R}}{|\overline{\delta} + \rho_\Theta(p)|} < \frac{\varepsilon}{2|\overline{\delta} + \rho_\Theta(p)|}\,,$$

and the estimate (5.3) follows. □

Lemma 5.4. *Let $\Theta = \{\theta_n\}$ be a sequence of positive terms, and p and q conjugate numbers such that $1 \leq p < \infty$, $1 < q \leq \infty$. Then*

$$(\ell^p_\Theta(\mathbb{Z}^d))^* = \ell^q_\Theta(\mathbb{Z}^d)\,,$$

and if $1 < p < \infty$ then $\ell^p_\Theta(\mathbb{Z}^d)$ is a reflexive Banach space, that is to say,

$$(\ell^p_\Theta(\mathbb{Z}^d))^{**} = \ell^p_\Theta(\mathbb{Z}^d)\,.$$

Proof. The identification $(\ell^p(\mathbb{Z}^d))^* = \ell^q(\mathbb{Z}^d)$ is well known. Let $v \in \ell^q_\Theta(\mathbb{Z}^d)$. By the definition of $\ell^q_\Theta(\mathbb{Z}^d)$, the multiplier operator $\Theta : \ell^q_\Theta(\mathbb{Z}^d) \to \ell^q(\mathbb{Z}^d)$ is an isomorphism. Therefore, $\Theta v \in \ell^q(\mathbb{Z}^d)$, and this element induces the linear functional

$$\ell^p_\Theta(\mathbb{Z}^d) \ni u \to (u,\ \Theta v) = \sum_{n\in\mathbb{Z}^d} u_n \overline{(\theta_n v_n)}\,,$$

associated with the quadratic form

$$\ell^p_\Theta(\mathbb{Z}^d) \times \ell^q_\Theta(\mathbb{Z}^d) \ni u, v \to (u,\ v)_\Theta = (\Theta u,\ \Theta v) = \sum_{n\in\mathbb{Z}^d} (\theta_n u_n)\overline{(\theta_n v_n)}\,.$$

This proves that the dual of $\ell^p_\Theta(\mathbb{Z}^d)$ is $\ell^q_\Theta(\mathbb{Z}^d)$, which addresses the first part of the statement. The second part, regarding the double dual, follows by applying the first part to the dual of $\ell^q_\Theta(\mathbb{Z}^d)$. ☐

Since bounded sets in a reflexive Banach space are precompact in the weak* topology, we now obtain the following.

Lemma 5.5. *Assume that assumptions (i)–(v) hold, and that $1 < p < \infty$. If $\overline{\delta} + \rho_\Theta(p) < 0$ and $b \in \ell^p_\Theta(\mathbb{Z}^d)$, then the semigroup $S(t)$ is asymptotically compact in $\ell^p_\Theta(\mathbb{Z}^d)$, that is, if the sequence ψ_k is bounded in $\ell^p_\Theta(\mathbb{Z}^d)$, and $t_k \to \infty$ as $k \to \infty$, then $S(t_k)\psi_k$ is precompact in $\ell^p_\Theta(\mathbb{Z}^d)$.*

Proof. Let $\psi_k \in \ell^p_\Theta(\mathbb{Z}^d)$ be a bounded sequence. According to Lemma 5.2, there exist an absorbing set B_0 and a $T > 0$ such that $S(t)\psi_k \subset B_0$ for all $t \geq T$. Therefore, if t_k is a sequence converging to infinity, the sequence $w_k = S(t_k)\psi_k$ has only finitely many terms outside of B_0 and is bounded in the reflexive Banach space $\ell^p_\Theta(\mathbb{Z}^d)$ (as shown in Lemma 5.4). Consequently, we can pass to a subsequence, if necessary, and assume that for some w_0, $w_k \to w_0$ in the weak* topology of $\ell^p_\Theta(\mathbb{Z}^d)$.

For any $R > 0$, we consider the characteristic function of the R ball in $\mathbb{Z}^d$ centered at the origin,

$$\chi_R(n) = \begin{cases} 1 & \text{if } n \in \mathbb{Z}^d,\ |n| \leq R,\\ 0 & \text{otherwise.}\end{cases}$$

According to Lemma 5.3, for any $\varepsilon > 0$ there exist constants $R = R(\varepsilon)$ and $N = N(\varepsilon)$ such that

$$\|(1-\chi_R)w_k\|_{\ell^p_\Theta} \leq \varepsilon,$$

for all $k \geq N$, and enlarging R if necessary, we also have that

$$\|(1-\chi_R)w_0\|_{\ell^p_\Theta} \leq \varepsilon\,.$$

Therefore,

$$\begin{aligned}\|w_k - w_0\|_{\ell^p_\Theta} &\leq \|\chi_R(w_k - w_0)\|_{\ell^p_\Theta} + \|(1-\chi_R)(w_k - w_0)\|_{\ell^p_\Theta}\\ &\leq \|\chi_R(w_k - w_0)\|_{\ell^p_\Theta} + \|(1-\chi_R)w_k\|_{\ell^p_\Theta} + \|(1-\chi_R)w_0\|_{\ell^p_\Theta}\\ &\leq \|\chi_R(w_k - w_0)\|_{\ell^p_\Theta} + 2\varepsilon,\end{aligned}$$

for $k \geq N$.

Since all functions $\chi_R(w_k - w_0)$ belong to a finite-dimensional subspace of $\ell^p_\Theta(\mathbb{Z}^d)$, and on a finite-dimensional subspace of a reflexive Banach space, the weak* convergence implies the strong convergence. Therefore, we can conclude that $\chi_R(w_k -$

$w_0) \to 0$ strongly in $\ell^p_\Theta(\mathbb{Z}^d)$. Consequently, we can find $\overline{N}$ such that

$$\|\chi_R(w_k - w_0)\|_{\ell^p_\Theta} < \varepsilon \,,$$

which, when used in the previous estimate, yields:

$$\|w_k - w_0\|_{\ell^p_\Theta} \le 3\varepsilon$$

for any $k \ge \max\{\overline{N}, N\}$. Since ε is an arbitrary positive number, this proves that $w_k \to w_0$ strongly in $\ell^p_\Theta(\mathbb{Z}^d)$. □

Proof of Theorem 5.1. Under the stated hypothesis, the initial value problem has a global classical solution (1.1),(1.2) in $\ell^p_\Theta(\mathbb{Z}^d)$. We let $B_0 = B_{0,\Theta}$ be the absorbing set produced by Lemma 5.2, and set A_Θ to be its ω-limit,

$$A_\Theta := \omega(B_{0,\Theta}) = \cap_{s\ge 0}\overline{\cup_{t\ge s} S(t)B_{0,\Theta}},$$

where $S(t)$ is the semigroup of operators in $\ell^p_\Theta(\mathbb{Z}^d)$ in (5.1), and the closure is taken with respect to the ℓ^p_Θ-norm. According to [23, Theorem 1.1.1] (see also [4, Chapter 2, Theorem 3.1]), this set is compact and connected, and by definition, it is maximal among all invariant sets in $\ell^p_\Theta(\mathbb{Z}^d)$. □

When $\Theta = \Theta_0$, we have $\ell^p_{\Theta_0}(\mathbb{Z}^d) = \ell^p(\mathbb{Z}^d)$. Theorem 5.1 then demonstrates the existence of a global attractor in $\ell^p(\mathbb{Z}^d)$ space.

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Guoping Zhang

Department of Mathematics, Morgan State University, Baltimore, MD 21251, USA

Email address: `guouping.zhang@morgan.edu`

Ghder Aburamyah

Department of Mathematics, Morgan State University, Baltimore, MD 21251, USA

Email address: `ghabu2@morgan.edu`