

ENTIRE SOLUTIONS TO FERMAT-TYPE DIFFERENCE AND PARTIAL DIFFERENTIAL-DIFFERENCE EQUATIONS IN \mathbb{C}^n

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ABSTRACT. In this article, we study the existence and the form of finite order transcendental entire solutions of systems of Fermat-type difference and partial differential-difference equations in several complex variables. Our results extend previous theorems given by Xu-Cao [49], Xu et al [52], and Zheng-Xu [55]. We give some examples to illustrate the content of this article.

1. INTRODUCTION AND MAIN RESULTS

It is well known that Nevanlinna theory is an important tool to study value distribution of entire and meromorphic solutions of complex differential equations (see [15, 21]). Initially, Fermat-type functional equations were investigated by Montel [35], Gross [6, 7], and Iyer [18], independently. In fact, Iyer [18] considered the Fermat type functional equation

$$f(z)^2 + g(z)^2 = 1, \quad (1.1)$$

and proved that the entire solutions of (1.1) are of the form $f(z) = \cos \alpha(z)$, $g(z) = \sin \alpha(z)$, where $\alpha(z)$ is entire function.

Many researchers pay considerable attention to study the existence of entire and meromorphic solutions of complex difference as well as complex differential-difference equations, and obtained a number of important results; see [29, 30, 32, 42]. Mainly utilizing difference analogues of Nevanlinna theory, which was developed by Halburd and Korhonen [8, 9], and Chiang and Feng [3], independently.

In 2012, Liu et al. [30] proved that the Fermat-type difference equation $f(z)^2 + f(z+c)^2 = 1$ has the solutions of the form $f(z) = \sin(az+b)$, where $c(\neq 0)$, $a, b \in \mathbb{C}$, $a = (4k+1)\pi/2c$, and k is an integer. In 2013, Liu and Yang [27] extended this result by considering the Fermat-type difference equation $f(z)^2 + P(z)^2 f(z+c)^2 = Q(z)$ where $P(z)$ and $Q(z)$ are two non-zero polynomials.

After that Liu [28], Liu and Dong [31] considered some variations of Fermat-type equations

$$f(z)^2 + [f(z+c) - f(z)]^2 = 1, \quad (1.2)$$

$$a_1^2 f(z)^2 + [a_2 f(z+c) + a_3 f(z)]^2 = 1, \quad (1.3)$$

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$$[a_1 f(z+c) + a_2 f(z)]^2 + [a_3 f(z+c) + a_4 f(z)]^2 = 1, \quad (1.4)$$

and obtained some remarkable results.

Hereafter, we denote $z+w = (z_1+w_1, z_2+w_2, \dots, z_n+w_n)$ for any $z = (z_1, z_2, \dots, z_n)$, $w = (w_1, w_2, \dots, w_n)$ and $c = (c_1, c_2, \dots, c_n)$, where $z, w, c \in \mathbb{C}^n$ except otherwise stated.

Considering equations (1.2)–(1.4), in 2021, Zheng and Xu [55] extended the results due to Liu [28], Liu and Dong [31] from one complex variable to several complex variables, and obtained the following results.

Theorem 1.1. [55] *Let $c = (c_1, c_2) \in \mathbb{C}^2 \setminus \{(0, 0)\}$. Then there are no transcendental entire solutions $f : \mathbb{C}^2 \rightarrow \mathbb{P}^1(\mathbb{C})$ with finite order for equation (1.2).*

Theorem 1.2. [55] *Let $c = (c_1, c_2) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ and a_1, a_2, a_3 be nonzero constants in \mathbb{C} . If (1.3) has a transcendental entire solution $f : \mathbb{C}^2 \rightarrow \mathbb{P}^1(\mathbb{C})$ with finite order, then $a_1^2 + a_3^2 = a_2^2$ and*

$$f(z) = \frac{1}{a_1} \sin(L(z) + \Phi(t) + A),$$

where $L(z) = \alpha_1 z_1 + \alpha_2 z_2$, $\alpha_1, \alpha_2, A \in \mathbb{C}$, $\Phi(t)$ is a polynomial in $t := c_2 z_1 - c_1 z_2$ in \mathbb{C} , and $L(z)$ satisfies

$$L(c) = \alpha_1 c_1 + \alpha_2 c_2 = \theta + k\pi \pm \frac{\pi}{2}, \quad \tan \theta = \frac{a_3}{a_1}.$$

Theorem 1.3. [55] *Let $c = (c_1, c_2) \in \mathbb{C}^2 \setminus \{(0, 0)\}$, a_1, a_2, a_3, a_4 be nonzero constants in \mathbb{C} , and let $D := a_1 a_4 - a_2 a_3 \neq 0$. If (1.4) has a transcendental entire solution $f : \mathbb{C}^2 \rightarrow \mathbb{P}^1(\mathbb{C})$ with finite order, then $a_1^2 + a_3^2 = a_2^2 + a_4^2$ and*

$$f(z) = \frac{1}{2D} \left[- (a_3 + ia_1) e^{L(z) + \Phi(t) + A} - (a_3 - ia_1) e^{-(L(z) + \Phi(t) + A)} \right],$$

where $L(z) = \alpha_1 z_1 + \alpha_2 z_2$, $\alpha_1, \alpha_2, A \in \mathbb{C}$, $\Phi(t)$ is a polynomial in $t := c_2 z_1 - c_1 z_2$ in \mathbb{C} , and $L(z)$ satisfies

$$e^{L(c)} = e^{\alpha_1 c_1 + \alpha_2 c_2} = -\frac{a_3 - ia_1}{a_4 - ia_2} = -\frac{a_4 + ia_2}{a_3 + ia_1}.$$

As far as our knowledge is concerned, although there are some remarkable results about the existence and forms of transcendental entire solutions of Fermat-type difference and partial differential-difference equations in several complex variables (see [17, 49, 50, 51, 55, 11, 12, 14, 45, 13, 47, 48, 46]), the number of results about the solutions of the system of Fermat-type equations in the literature (see [52]) are scanty. We would like to discuss some of these results which are relevant to the content of this article.

Theorem 1.4. [52] *Let $c = (c_1, c_2)$ be constants in \mathbb{C}^2 . Then any pair of transcendental entire solutions with finite order for the system of Fermat-type difference equations*

$$\begin{aligned} f_1(z_1, z_2)^2 + (f_2(z_1 + c_1, z_2 + c_2))^2 &= 1 \\ f_2(z_1, z_2)^2 + (f_1(z_1 + c_1, z_2 + c_2))^2 &= 1, \end{aligned}$$

has the following form

$$(f_1(z), f_2(z)) = \left(\frac{e^{L(z)+B_1} + e^{-(L(z)+B_1)}}{2}, \frac{A_{21} e^{L(z)+B_1} + A_{22} e^{-(L(z)+B_1)}}{2} \right),$$

where $L(z) = \alpha_1 z_1 + \alpha_2 z_2$, B_1 is a constant in \mathbb{C} , and c, A_{21}, A_{22} satisfy one of the following cases

- (i) $L(c) = 2k\pi i$, $A_{21} = -i$ and $A_{22} = i$, or $L(c) = (2k + 1)\pi i$, $A_{21} = i$ and $A_{22} = -i$, here and below k is an integer;
- (ii) $L(c) = (2k + 1/2)\pi i$, $A_{21} = -1$ and $A_{22} = -1$, or $L(c) = (2k - 1/2)\pi i$, $A_{21} = 1$ and $A_{22} = 1$.

Now, we consider the following systems Fermat-type functional equations on \mathbb{C}^n .

$$\begin{aligned} f_1(z_1, \dots, z_n)^2 + (\Delta_c f_2(z_1, \dots, z_n))^2 &= 1 \\ f_2(z_1, \dots, z_n)^2 + (\Delta_c f_1(z_1, \dots, z_n))^2 &= 1, \end{aligned} \quad (1.5)$$

where $c = (c_1, c_2, \dots, c_n)$ are constant in \mathbb{C}^n .

$$\begin{aligned} a_1^2 f_1(z)^2 + (a_2 f_2(z+c) + a_3 f_2(z))^2 &= 1 \\ a_1^2 f_2(z)^2 + (a_2 f_1(z+c) + a_3 f_1(z))^2 &= 1, \end{aligned} \quad (1.6)$$

$$\begin{aligned} (a_1 f_1(z+c) + a_2 f_1(z))^2 + (a_3 f_2(z+c) + a_4 f_2(z))^2 &= 1 \\ (a_1 f_2(z+c) + a_2 f_2(z))^2 + (a_3 f_1(z+c) + a_4 f_1(z))^2 &= 1, \end{aligned} \quad (1.7)$$

where $f_j : \mathbb{C}^n \rightarrow \mathbb{P}^1(\mathbb{C})$, $j = 1, 2$, $c = (c_1, c_2, \dots, c_n)$ are constants in $\mathbb{C}^n \setminus \{0\}$, a_1, a_2, a_3, a_4 are nonzero constants in \mathbb{C} , and $\Delta_c f(z) = f(z_1 + c_1, \dots, z_n + c_n) - f(z_1, \dots, z_n)$ as defined in [20].

Inspired by Theorems 1.1–1.4, one may ask the following questions.

What can be said about the existence and the forms of finite order transcendental entire solutions for the system of the Fermat-type functional equations (1.5)–(1.7)?

Can we extend all the results stated above from \mathbb{C}^2 to \mathbb{C}^n ?

Our main goal is to investigate the existence and form of finite order transcendental entire solutions of system (1.5)–(1.7) with the help of Nevanlinna theory and the difference logarithmic lemma in several complex variables (see [2, 20]).

We extend Theorems 1.1–1.4 from the complex Fermat-type difference equations to the Fermat-type system of difference equations. Now we list our main results.

Theorem 1.5. *There is no pair of transcendental entire solutions with finite order for the system of Fermat-type difference equation (1.5).*

Theorem 1.6. *Let a_1, a_2, a_3 be three non-zero complex constants in one variable and $c = (c_1, \dots, c_n) \in \mathbb{C}^n \setminus \{(0, 0, \dots, 0)\}$. If (f_1, f_2) is a pair of transcendental entire solution with finite order of simultaneous Fermat-type difference equation (1.6), then (f_1, f_2) takes one of the following forms*

$$(i) \quad (f_1, f_2) = \left(\frac{1}{a_1} \cos(L(z) + \Phi(z) + A), \frac{1}{a_1} \cos(L(z) + \Phi(z) + A + k) \right), \text{ where } a_2^2 = a_1^2 + a_3^2, e^{2ik} = 1,$$

$$e^{2iL(c)} = -\frac{a_1 - ia_3 e^{-ik}}{a_1 + ia_3 e^{ik}},$$

where $L(z) = \sum_{j=1}^n \alpha_j z_j$, $\alpha_j, A \in \mathbb{C}$, $j = 1, 2, \dots, n$, and

$$\Phi(z) = \sum_{i_1, i_2=1}^n (i_1 < i_2) H_{i_1, i_2}(c_{i_2} z_{i_1} - c_{i_1} z_{i_2})$$

$$\begin{aligned}
& + \sum_{i_1, i_2, i_3=1}^n (i_1 < i_2 < i_3) H_{i_1, i_2, i_3}(c_{i_2} c_{i_2} z_{i_1} - 2c_{i_3} c_{i_1} z_{i_2} + c_{i_1} c_{i_2} z_{i_3}) + \cdots \\
& + H(c_2 c_3 \cdots c_n z_1 - (n-1)c_1 c_3 \cdots c_n z_2 + \cdots + c_1 c_2 \cdots c_{n-1} z_n),
\end{aligned}$$

where H_{i_1, i_2} is a polynomial in $c_{i_2} z_{i_1} - c_{i_1} z_{i_2}$, $1 \leq i_1 < i_2 \leq n$, H_{i_1, i_2, i_3} is a polynomial in $c_{i_2} c_{i_2} z_{i_1} - 2c_{i_3} c_{i_1} z_{i_2} + c_{i_1} c_{i_2} z_{i_3}$, $1 \leq i_1 < i_2 < i_3 \leq n$, ..., and H is a polynomial in $c_2 c_3 \cdots c_n z_1 - (n-1)c_1 c_3 \cdots c_n z_2 + \cdots + c_1 c_2 \cdots c_{n-1} z_n$;

(ii) $(f_1, f_2) = (\frac{1}{a_1} \cos(-L(z) + \Phi(z) + A) + k, \frac{1}{a_1} \cos(L(z) + \Phi(z) + A))$, where $L(z)$ and $\Phi(z)$ are defined as in (i), and $L(z)$, k , a_1, a_2, a_3 satisfy one of the following conditions:

- (a) $e^{iL(c)} = 1$, $e^{ik} = \pm i$ and $a_1 = \pm(a_2 + a_3)$.
- (b) $e^{iL(c)} = -1$, $e^{ik} = \pm i$ and $a_1 = \pm(a_2 - a_3)$.

The following examples show the existence of transcendental entire solutions with finite order for system (1.6).

Example 1.7. Let $a_1 = 3$, $a_2 = 5$, $a_3 = 4$ and $L(z) = 7z_1 - 5z_2$. Choose $k \in \mathbb{C}$ such that $e^{ik} = 1$. Also, let $c = (c_1, c_2, c_3) \in \mathbb{C}^3$ such that $e^{iL(c)} = \cos \frac{2\pi + \alpha}{2} + i \sin \frac{2\pi + \alpha}{2}$, where $\tan \alpha = 24/7$. Let

$$\begin{aligned}
f_1 &= \frac{\cos(7z_1 - 5z_2 + z_3 + i(c_2 c_3 z_1 - 2c_3 c_1 z_2 + c_1 c_2 z_3)^3 + 3)}{3}, \\
f_2 &= \frac{\cos(7z_1 - 5z_2 + z_3 + i(c_2 c_3 z_1 - 2c_3 c_1 z_2 + c_1 c_2 z_3)^3 + 3 + k)}{3}.
\end{aligned}$$

Then, it can be easily verified that (f_1, f_2) is a solution of (1.6).

Example 1.8. Let $a_1 = 1$, $a_2 = -2$, $a_3 = \sqrt{3}$, and $L(z) = z_1 + 2z_2$. Choose $k \in \mathbb{C}$ such that $e^{ik} = 1$. Also, let $c = (c_1, c_2) \in \mathbb{C}^2$ such that $e^{iL(c)} = (1 + i\sqrt{3})/2$. Then, it can be easily verified that

$$(f_1(z), f_2(z)) = (\cos(z_1 + 2z_2 + 3), \cos(z_1 + 2z_2 + 3 + k))$$

is a solution of (1.6).

Example 1.9. Let $a_1 = 1$, $a_2 = -2$, $a_3 = \sqrt{3}$, and $L(z) = 5z_1 - 2z_2$. Choose $k \in \mathbb{C}$ such that $e^{ik} = -1$. Also, let $c = (c_1, c_2) \in \mathbb{C}^2$ such that $e^{iL(c)} = (1 - i\sqrt{3})/2$. Then, it can be easily verified that

$$(f_1(z), f_2(z)) = (\cos(5z_1 - 2z_2 + 10i), \cos(5z_1 - 2z_2 + 10i + k))$$

is a solution of (1.6).

Example 1.10. Let $a_1 = 12$, $a_2 = 7$, $a_3 = 5$ and $L(z) = z_1 + iz_2$. Choose $k \in \mathbb{C}$ such that $e^{ik} = i$. Also, let $c = (c_1, c_2) \in \mathbb{C}^2$ such that $e^{iL(c)} = 1$. Then, it can be easily verified that

$$(f_1(z), f_2(z)) = \left(\frac{1}{12} \cos(-(z_1 + iz_2 + 17i) + k), \frac{1}{12} \cos(z_1 + iz_2 + 17i) \right)$$

is a solution of (1.6).

Theorem 1.11. Let a_1, a_2, a_3, a_4 be four non-zero constants in \mathbb{C} such that $a_1^2 + a_3^2 = a_2^2 + a_4^2$ and $D := a_1 a_4 - a_2 a_3 \neq 0$. Let $c = (c_1, \dots, c_n) \in \mathbb{C}^n \setminus \{(0, 0, \dots, 0)\}$. If (f_1, f_2) is a pair of transcendental entire solution with finite order of Fermat-type simultaneous difference equation (1.7), then one of the following cases must occur.

(i)

$$f_1(z) = \frac{-1}{2D} [(a_3 + ia_1 e^k) e^{L(z)+A+\Phi(z)} + (a_3 - ia_1 e^{-k}) e^{-(L(z)+A+\Phi(z))}] \quad (1.8)$$

$$f_2(z) = \frac{-1}{2D} [(a_3 e^k + ia_1) e^{L(z)+A+\Phi(z)} + (a_3 e^{-k} - ia_1) e^{-(L(z)+A+\Phi(z))}], \quad (1.9)$$

where $L(z)$ and $\Phi(z)$ are defined the conclusion (i) in Theorem 1.6 such that

$$e^{L(c)} = \frac{a_3 e^k - ia_1}{ia_2 - a_4 e^k} = \frac{a_3 e^{-k} - ia_1}{ia_2 - a_4 e^{-k}} = \frac{-(a_4 e^k + ia_2)}{a_3 + ia_1 e^k} = \frac{-(a_4 e^k + ia_2)}{a_3 e^k + ia_1},$$

and $e^{2k} = 1, k \in \mathbb{C}$;

(ii)

$$f_1(z_1, z_2) = \frac{-1}{2D} [(a_3 - ia_1 e^{-k}) e^{L(z)+\Phi(z)+\xi} + (a_3 + ia_1 e^k) e^{-(L(z)+\Phi(z)+\xi)}],$$

$$f_2(z_1, z_2) = \frac{-1}{2D} [(ia_2 - a_4 e^k) e^{-(L(z)+\Phi(z)+\xi)} + (ia_2 + a_4 e^{-k}) e^{L(z)+\Phi(z)+\xi}],$$

where $L(z)$ and $\Phi(z)$ are defined the conclusion (i) in Theorem 1.6 such that $e^{2L(c)} = 1, e^{2k} = -1$ and $a_1 a_3 = a_2 a_4$.

The following examples show the existence of transcendental entire solutions with finite order of the system (1.7).

Example 1.12. Let us choose $a_1 = a_2 = a_3 = 1, a_4 = -1, L(z) = \sum_{j=1}^n j z_j, c = (c_1, c_2, \dots, c_n) \in \mathbb{C}^n$ such that $\sum_{j=1}^n j c_j = (2m + 1/2)\pi i, m$ being an integer, and $\Phi(t) = (\sum_{j=1}^n d_j z_j)^{10}$, where $d_j = \prod_{i=1, i \neq j}^n c_1 c_2 \dots c_{j-1} c_{j+1} \dots c_n$. Let

$$f_1(z) = \frac{1}{4} \left((1 + i) e^{L(z)+\Phi(z)+3} + (1 - i) e^{-(L(z)+\Phi(z)+3)} \right),$$

$$f_2(z) = \frac{1}{4} \left((1 + i) e^{L(z)+\Phi(z)+3} - (i - 1) e^{-(L(z)+\Phi(z)+3)} \right).$$

Then one can easily verify that (f_1, f_2) is a solution of (1.7)

Example 1.13. Let us choose $a_1 = a_2 = a_3 = 1, a_4 = -1, L(z) = i(z_1 - z_2), \Phi(t) = i(c_2 z_1 - c_1 z_2)^5, A = 3,$ and $c = (c_1, c_2) \in \mathbb{C}^2$ such that $c_1 + 2c_2 = (2m - 1/2)\pi, m$ being an integer. Let

$$f_1(z) = \frac{\cos(z_1 - z_2 + (c_2 z_1 - c_1 z_2)^5 - 3i) + \sin(z_1 - z_2 + (c_2 z_1 - c_1 z_2)^5 - 3i)}{2},$$

$$f_2(z) = -\frac{\cos(z_1 - z_2 + (c_2 z_1 - c_1 z_2)^5 - 3i) + \sin(z_1 - z_2 + (c_2 z_1 - c_1 z_2)^5 - 3i)}{2}.$$

Then one can easily verify that $(f_1(z), f_2(z))$ is a solution of (1.7)

2. SOLUTIONS OF FERMAT-TYPE PARTIAL DIFFERENTIAL-DIFFERENCE EQUATIONS IN SEVERAL COMPLEX VARIABLES

The study partial differential equations, which is a generalizations of the well-known eikonal equation in real variable case has a long history [4, 5, 37]. Recently, many mathematicians have paid considerable attention to the study of entire and meromorphic solutions of Fermat type partial differential equations in several complex variables; see [25, 24, 39, 40, 23, 34, 33]). In 1995, Khavinson [19] pointed out that in \mathbb{C}^2 , the entire solution of the Fermat type partial differential equation

$f_{z_1}^2 + f_{z_2}^2 = 1$ must necessarily be linear. In view of the result of Khavinson, Xu and Cao [49] investigated the entire solutions of Fermat-type partial differential-difference equation

$$\left(\frac{\partial f(z_1, z_2)}{\partial z_1}\right)^n + f^m(z_1 + c_1, z_2 + c_2) = 1, \quad (2.1)$$

and proved that in \mathbb{C}^2 , Equation (2.1) does not have any transcendental entire solution with finite order, where m and n are two distinct positive integers. In 2020, Xu and Wang [51] generalized the result by considering the Fermat-type partial differential-difference equation

$$\left(\frac{\partial f(z_1, z_2)}{\partial z_1} + \frac{\partial f(z_1, z_2)}{\partial z_2}\right)^n + f^m(z_1 + c_1, z_2 + c_2) = 1, \quad (2.2)$$

and proved that if (2.2) satisfies one of the conditions: (i) $m > n$ and (ii) $n > m \geq 2$, then (2.2) does not have any finite order transcendental entire solutions.

In 2020, Xu et al. [52] considered the system of partial differential-difference equations and obtained the following result.

Theorem 2.1. [52] *Let $c = (c_1, c_2)$ be a constant in \mathbb{C}^2 , and m_j, n_j ($j = 1, 2$) be positive integers. If the system of Fermat-type partial differential-difference equations*

$$\begin{aligned} \left(\frac{\partial f_1(z_1, z_2)}{\partial z_1}\right)^{n_1} + f_2(z_1 + c_1, z_2 + c_2)^{m_1} &= 1 \\ \left(\frac{\partial f_2(z_1, z_2)}{\partial z_1}\right)^{n_2} + f_1(z_1 + c_1, z_2 + c_2)^{m_2} &= 1, \end{aligned}$$

satisfies one of the conditions (i) $m_1 m_2 > n_1 n_2$ or (ii) $m_j > \frac{n_j}{n_j - 1}$, $j = 1, 2$, then the above system does not have any pair of transcendental entire solution with finite order.

As far as we know, the Fermat-type mixed partial differential-difference equations in several complex variables has not been addressed in the literature before. To generalize Theorem 1.5, we consider the partial differential-difference equation

$$\begin{aligned} (a\partial^I f_1(z_1, z_2) + b\partial^J f_1(z_1, z_2))^{n_1} + f_2(z_1 + c_1, z_2 + c_2)^{m_1} &= 1, \\ (a\partial^I f_2(z_1, z_2) + b\partial^J f_2(z_1, z_2))^{n_2} + f_1(z_1 + c_1, z_2 + c_2)^{m_2} &= 1, \end{aligned} \quad (2.3)$$

where

$$\partial^I f_j(z_1, z_2) = \frac{\partial^{|I|} f_j(z_1, z_2)}{\partial z_1^{\alpha_1} \partial z_2^{\alpha_2}} \quad \text{and} \quad \partial^J f_j(z_1, z_2) = \frac{\partial^{|J|} f_j(z_1, z_2)}{\partial z_1^{\beta_1} \partial z_2^{\beta_2}}$$

with $I = (\alpha_1, \alpha_2)$ and $J = (\beta_1, \beta_2)$ are multi-index with $I \neq J$, where $\alpha_1, \alpha_2, \beta_1$, and β_2 are non-negative integers and $a, b \in \mathbb{C}$, not both zero. We denote by $|I|$ to denote the length of I , that is, $|I| = \alpha_1 + \alpha_2$. Similarly, for J also.

As a matter of fact, we prove the next result for any order Fermat-type partial differential-difference equation (2.3).

Theorem 2.2. *Let $c = (c_1, \dots, c_n)$ be a constant in \mathbb{C}^n and m_j, n_j be positive integers with $j = 1, 2$. If the Fermat-type simultaneous partial differential-difference equation (2.3) satisfies one of the following conditions:*

- (i) $m_1 m_2 > n_1 n_2$;
- (ii) $m_j > \frac{n_j}{n_j - 1}$, for $n_j \geq 2$, $j = 1, 2$,

then (2.3) does not have any pair of finite order transcendental entire solutions of the form (f_1, f_2) .

The following example shows the existence of finite order transcendental entire solutions of system (2.3) when $n_1 = n_2 = 2$ and $m_1 = m_2 = 1$.

Example 2.3. Let us consider the following particular type of system of equation of (2.3).

$$\begin{aligned} \left(\frac{\partial^2 f_1}{\partial z_1^2}\right)^2 + f_2(z+c) &= 1 \\ \left(\frac{\partial^2 f_2}{\partial z_1^2}\right)^2 + f_1(z+c) &= 1. \end{aligned} \tag{2.4}$$

Let $c = (c_1, c_2) \in \mathbb{C}^2$ be such that $c_1 = 0$ and $e^{c_2} = \frac{1}{3}$. Let

$$f_1(z_1, z_2) = f_2(z_1, z_2) = -\frac{1}{144}z_1^4 + \frac{1}{2}z_1^2e^{z_2} + 1 - 9e^{2z_2}.$$

Then one can easily verify that (f_1, f_2) is a solution of (2.4).

Example 2.4. Let $c = (c_1, c_2) \in \mathbb{C}^2$ be such that $c_1 = 0$ and $e^{c_2} = -\frac{1}{3}$. Let

$$f_1(z_1, z_2) = -\frac{1}{144}z_1^4 - \frac{1}{2}z_1^2e^{z_2} - 9e^{2z_2} + 1, \quad f_2(z_1, z_2) = -\frac{1}{144}z_1^4 + \frac{1}{2}z_1^2e^{z_2} - 9e^{2z_2} + 1.$$

Then one can easily verify that (f_1, f_2) is a solution of (2.4).

3. PROOF OF MAIN RESULTS

First, we present here some lemmas which play key role to prove the main results.

Lemma 3.1 ([16]). *Let $f_j \not\equiv 0$ ($j = 1, 2, \dots, m$; $m \geq 3$) be meromorphic functions on \mathbb{C}^n such that f_1, \dots, f_{m-1} are not constant, $f_1 + f_2 + \dots + f_m = 1$ and such that*

$$\sum_{j=1}^m \left\{ N_{n-1}\left(r, \frac{1}{f_j}\right) + (m-1)\overline{N}(r, f_j) \right\} < \lambda T(r, f_j) + O(\log^+ T(r, f_j))$$

holds for $j = 1, \dots, m-1$ and all r outside possibly a set with finite logarithmic measure, where $\lambda < 1$ is a positive number. Then $f_m = 1$.

Lemma 3.2 ([22, 38, 41]). *For an entire function F on \mathbb{C}^n , $F(0) \not\equiv 0$ and put $\rho(n_F) = \rho < \infty$. Then there exist a canonical function f_F and a function $g_F \in \mathbb{C}^n$ such that $F(z) = f_F(z)e^{g_F(z)}$. For the special case $n = 1$, f_F is the canonical product of Weierstrass.*

Lemma 3.3 ([36]). *If g and h are entire functions on the complex plane \mathbb{C} and $g(h)$ is an entire function of finite order, then there are only two possible cases: either*

- (i) *the internal function h is a polynomial and the external function g is of finite order; or*
- (ii) *the internal function h is not a polynomial but a function of finite order, and the external function g is of zero order.*

Lemma 3.4 ([1, 54]). *Let f be a non-constant meromorphic function on \mathbb{C}^n and let $I = (\alpha_1, \dots, \alpha_n)$ be a multi-index with length $|I| = \sum_{j=1}^n \alpha_j$. Assume that $T(r_0, f) \geq e$ for some r_0 . Then*

$$m\left(r, \frac{\partial^I f}{f}\right) = S(r, f)$$

for all $r \geq r_0$ outside a set $E \subset (0, +\infty)$ of finite logarithmic measure, $\int_E \frac{dt}{t} < \infty$, where $\partial^I f = \frac{\partial^{|\alpha_1|} f}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}}$.

Lemma 3.5 ([2, 20]). *Let f be a non-constant meromorphic function with finite order on \mathbb{C}^n such that $f(0) \neq 0, \infty$, and let $\epsilon > 0$. Then for $c \in \mathbb{C}^n$,*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = S(r, f)$$

for all $r \geq r_0$ outside a set $E \subset (0, +\infty)$ of finite logarithmic measure, $\int_E \frac{dt}{t} < \infty$.

Lemma 3.6 ([16]). *Let $f_j (\neq 0)$, $j = 1, 2, 3$ be meromorphic functions on \mathbb{C}^m such that f_1 is not constant. If $f_1 + f_2 + f_3 = 1$, and if*

$$\sum_{j=1}^m \left\{ N_2\left(r, \frac{1}{f_j}\right) + 2\overline{N}(r, f_j) \right\} < \lambda T(r, f_j) + O(\log^+ T(r, f_j)),$$

for all r outside possibly a set with finite logarithmic measure, where $\lambda < 1$ is a positive number, then either $f_2 = 1$ or $f_3 = 1$.

Proof of Theorem 1.5. Suppose that (f_1, f_2) is a pair of transcendental entire functions with finite order satisfying system (1.5). We write (1.5) as follows:

$$\begin{aligned} (f_1(z) + i(f_2(z+c) - f_2(z)))(f_1(z) - i(f_2(z+c) - f_2(z))) &= 1 \\ (f_2(z) + i(f_1(z+c) - f_1(z)))(f_2(z) - i(f_1(z+c) - f_1(z))) &= 1. \end{aligned} \quad (3.1)$$

Since f_1, f_2 are transcendental entire functions with finite order, there exist polynomials $p_1(z), p_2(z)$ in \mathbb{C}^n such that

$$\begin{aligned} f_1(z) + i(f_2(z+c) - f_2(z)) &= e^{p_1(z)} \\ f_1(z) - i(f_2(z+c) - f_2(z)) &= e^{-p_1(z)} \\ f_2(z) + i(f_1(z+c) - f_1(z)) &= e^{p_2(z)} \\ f_2(z) - i(f_1(z+c) - f_1(z)) &= e^{-p_2(z)}. \end{aligned} \quad (3.2)$$

In view of (3.2), we obtain

$$\begin{aligned} f_1(z) &= \frac{1}{2}(e^{p_1(z)} + e^{-p_1(z)}) \\ f_2(z+c) - f_2(z) &= \frac{1}{2i}(e^{p_1(z)} - e^{-p_1(z)}) \\ f_2(z) &= \frac{1}{2}(e^{p_2(z)} + e^{-p_2(z)}) \\ f_1(z+c) - f_1(z) &= \frac{1}{2i}(e^{p_2(z)} - e^{-p_2(z)}). \end{aligned} \quad (3.3)$$

After simple calculations, it follows from (3.3) that

$$-ie^{p_2(z)+p_1(z+c)} - ie^{p_2(z)-p_1(z+c)} + ie^{p_2(z)+p_1(z)} + ie^{p_2(z)-p_1(z)} + e^{2p_2(z)} = 1 \quad (3.4)$$

and

$$-ie^{p_1(z)+p_2(z+c)} - ie^{p_1(z)-p_2(z+c)} + ie^{p_1(z)+p_2(z)} + ie^{p_1(z)-p_2(z)} + e^{2p_1(z)} = 1. \tag{3.5}$$

Now, we consider the following two possible cases.

Case 1: Let $p_2(z) - p_1(z) = k$, where k is a constant in \mathbb{C} . Then it follows from (3.4) and (3.5) that

$$\begin{aligned} -ie^{p_1(z)+p_1(z+c)+k} - ie^{p_1(z)-p_1(z+c)+k} + ie^{2p_1(z)+k} + e^{2p_1(z)+2k} &= 1 - ie^k \\ -ie^{p_1(z)+p_1(z+c)+k} - ie^{p_1(z)-p_1(z+c)-k} + ie^{2p_1(z)+k} + e^{2p_1(z)} &= 1 - ie^{-k}. \end{aligned} \tag{3.6}$$

First we consider that $e^k \neq \pm i$. Then, we obtain from (3.6) that

$$\begin{aligned} \frac{-ie^k}{1 - ie^k} e^{p_1(z)+p_1(z+c)} + \frac{-ie^k}{1 - ie^k} e^{p_1(z)-p_1(z+c)} + \frac{(i + e^k)e^k}{1 - ie^k} e^{2p_1(z)} &= 1 \\ \frac{-ie^k}{1 - ie^{-k}} e^{p_1(z)+p_1(z+c)} + \frac{-ie^{-k}}{1 - ie^{-k}} e^{p_1(z)-p_1(z+c)} + \frac{1 + ie^k}{1 - ie^{-k}} e^{2p_1(z)} &= 1. \end{aligned} \tag{3.7}$$

By Lemma 3.1, we obtain from (3.7) that

$$\begin{aligned} \frac{-ie^k}{1 - ie^k} e^{p_1(z)-p_1(z+c)} &= 1, \\ \frac{-ie^{-k}}{1 - ie^{-k}} e^{p_1(z)-p_1(z+c)} &= 1. \end{aligned} \tag{3.8}$$

It follows from (3.7) and (3.8) that

$$\begin{aligned} e^{-p_1(z)+p_1(z+c)} &= 1 - ie^k \\ e^{-p_1(z)+p_1(z+c)} &= 1 - ie^{-k}. \end{aligned} \tag{3.9}$$

It follows from (3.8) and (3.9) that $-ie^{-k} = (1 - ie^{-k})(1 - ie^k)$, which yields that $ie^k = 0$, a contradiction.

Next, suppose that $e^k = -i$. Then from (3.6), we obtain that

$$e^{-p_1(z)+p_1(z+c)} + e^{-(p_1(z)+p_1(z+c))} = -2.$$

This implies that

$$T(r, e^{-(p_1(z)+p_1(z+c))}) = T(r, e^{-p_1(z)+p_1(z+c)}) + S(r, e^{-p_1(z)+p_1(z+c)}).$$

By the second fundamental theorem of Nevanlinna for several complex variables, we have

$$\begin{aligned} T(r, e^{-(p_1(z)+p_1(z+c))}) &\leq \bar{N}(r, e^{-(p_1(z)+p_1(z+c))}) + \bar{N}\left(r, \frac{1}{e^{-(p_1(z)+p_1(z+c))}}\right) \\ &\quad + \bar{N}\left(r, \frac{1}{e^{-(p_1(z)+p_1(z+c))} + 2}\right) + S(r, e^{-(p_1(z)+p_1(z+c))}) \\ &\leq \bar{N}\left(r, \frac{1}{e^{-p_1(z)+p_1(z+c)}}\right) + S(r, e^{-p_1(z)+p_1(z+c)}) \\ &\leq S(r, e^{-(p_1(z)+p_1(z+c))}) + S(r, e^{-p_1(z)+p_1(z+c)}), \end{aligned}$$

which implies that $p_1(z)$ is a constant, which is a contradiction. Similarly, we can get a contradiction for the cases $e^k = i$.

Case 2: Let $p_2(z) - p_1(z)$ be non-constant. We consider the following two possible subcases:

Subcase 2.1 Let $p_2(z) + p_1(z) = k$, where $k \in \mathbb{C}$. Then, from (3.4) and (3.5), we obtain

$$\begin{aligned} ie^k e^{p_1(z+c)-p_1(z)} + ie^k e^{-(p_1(z)+p_1(z+c))} - (i + e^k)e^k e^{-2p_1(z)} &= ie^k - 1 \\ ie^k e^{p_1(z)-p_1(z+c)} + ie^{-k} e^{p_1(z)+p_1(z+c)} - (ie^{-k} + 1)e^{2p_1(z)} &= ie^k - 1. \end{aligned} \quad (3.10)$$

Observe that $e^k \neq -i$. Otherwise, it follows from (3.10) that $e^{2p_1(z+c)} = -1$, which implies that $p_1(z)$ is constant, a contradiction. By Lemma 3.1, we obtain from (3.10) that

$$\begin{aligned} ie^k e^{p_1(z+c)-p_1(z)} &= ie^k - 1 \\ ie^k e^{p_1(z)-p_1(z+c)} &= ie^k - 1. \end{aligned} \quad (3.11)$$

In view of (3.10) and (3.11), we obtain that

$$\begin{aligned} ie^{-p_1(z+c)+p_1(z)} &= i + e^k \\ ie^{-k} e^{-p_1(z)+p_1(z+c)} &= ie^{-k} + 1. \end{aligned} \quad (3.12)$$

It follows from (3.11) and (3.12) that $e^k = -i/2 = -2i$, which is not possible.

Subcase 2.2 Suppose $p_2(z) + p_1(z)$ is non-constant.

Subcase 2.2.1 Let $p_2(z) - p_1(z + c) = k_1$, a constant in \mathbb{C} . Then (3.4) reduces to

$$-ie^{p_2(z)+p_1(z+c)} + ie^{p_2(z)+p_1(z)} + ie^{p_2(z)-p_1(z)} + e^{2p_2(z)} = 1 + ie^{k_1}. \quad (3.13)$$

If $1 + ie^{k_1} = 0$, then it follows from (3.13) that

$$e^{p_1(z+c)-p_1(z)} - e^{-2p_1(z)} - ie^{p_2(z)-p_1(z)} = 1. \quad (3.14)$$

By Lemma 3.1, it follows from (3.14) that $e^{p_1(z+c)-p_1(z)} = 1$, which implies that $p_1(z + c) - p_1(z) = \text{constant}$. Then, we can assume that $p_1(z) = L(z) + \Phi(z) + \xi$, where $L(z) = \sum_{j=1}^n \alpha_j z_j$, $\Phi(z)$ is a polynomial defined in (i) in Theorem 1.6. Hence, $p_2(z) = L(z) + \Phi(z) + L(c) + \xi + k_1$. But, then $p_2(z) - p_1(z) = L(c) + k_1 = \text{constant}$, which is a contradiction. Hence, $1 + ie^{k_1} \neq 0$.

In view of Lemma 3.1, it follows from (3.13) that $-ie^{p_2(z)+p_1(z+c)} = 1 + ie^{k_1}$. This implies that $p_2(z) + p_1(z + c) = k_2$, a constant in \mathbb{C} , say. But then $p_2(z) = (k_1 + k_2)/2 = \text{constant}$, which is a contradiction.

Subcase 2.2.2 Let $p_2(z) - p_1(z + c)$ be non-constant. Then, by Lemma 3.1, it follows from (3.4) that

$$-ie^{p_2(z)+p_1(z+c)} = 1, \quad (3.15)$$

which yields that $p_2(z) + p_1(z + c)$ is constant, say k_2 . In view of (3.4) and (3.15) that

$$-ie^{p_1(z)-p_1(z+c)} + ie^{2p_1(z)} + e^{p_2(z)+p_1(z)} = -i. \quad (3.16)$$

By Lemma 3.1, it follows from (3.16) that $-ie^{p_1(z)-p_1(z+c)} = -i$, which yields that $p_1(z) - p_1(z + c) = \text{constant}$, say k_3 . But then, $p_2(z) + p_1(z) = k_1 + k_3$ is constant, which is a contradiction. This completes the proof of the theorem. \square

Proof of Theorem 1.6. As we know the entire solutions of the functional equation $f^2 + g^2 = 1$ are $f = \cos \alpha(z)$ and $g = \sin \alpha(z)$, where $\alpha(z)$ is an entire function. If f

and g are finite order entire functions, then $p(z)$ must be a non-constant polynomial (see [6, 7, 35]). In view of the above fact, we easily obtain from (1.6) that

$$\begin{aligned} a_1 f_1(z) &= \frac{1}{2} \left(e^{ip_1(z)} + e^{-ip_1(z)} \right) \\ a_2 f_2(z+c) + a_3 f_2(z) &= \frac{1}{2i} \left(e^{ip_1(z)} - e^{-ip_1(z)} \right) \\ a_1 f_2(z) &= \frac{1}{2} \left(e^{ip_2(z)} + e^{-ip_2(z)} \right) \\ a_2 f_1(z+c) + a_3 f_1(z) &= \frac{1}{2i} \left(e^{ip_2(z)} - e^{-ip_2(z)} \right), \end{aligned} \tag{3.17}$$

where $p_1(z), p_2(z)$ are two non-constant polynomials in \mathbb{C}^n .

After some simple calculations, from (3.17) we obtain

$$\begin{aligned} -\frac{ia_2}{a_1} e^{i(p_2(z)+p_1(z+c))} - \frac{ia_2}{a_1} e^{i(p_2(z)-p_1(z+c))} - \frac{ia_3}{a_1} e^{i(p_2(z)+p_1(z))} \\ - \frac{ia_3}{a_1} e^{i(p_2(z)-p_1(z))} + e^{2ip_2(z)} = 1 \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} -\frac{ia_2}{a_1} e^{i(p_1(z)+p_2(z+c))} - \frac{ia_2}{a_1} e^{i(p_1(z)-p_2(z+c))} - \frac{ia_3}{a_1} e^{i(p_1(z)+p_2(z))} \\ - \frac{ia_3}{a_1} e^{i(p_1(z)-p_2(z))} + e^{2ip_1(z)} = 1. \end{aligned} \tag{3.19}$$

Now, we consider the following possible two cases.

Case 1. Let $p_2(z) - p_1(z) = k$, a constant in \mathbb{C} . It follows from (3.18) and (3.19) that

$$\begin{aligned} -ia_2 e^{i(p_1(z)+p_1(z+c))} - ia_2 e^{i(p_1(z)-p_1(z+c))} - (ia_3 - a_1 e^{ik}) e^{2ip_1(z)} \\ = a_1 e^{-ik} + ia_3 \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} -ia_2 e^{i(p_1(z)+p_1(z+c))} - ia_2 e^{-2ik} e^{i(p_1(z)-p_1(z+c))} - (ia_3 - a_1 e^{-ik}) e^{2ip_1(z)} \\ = (a_1 + ia_3 e^{-ik}) e^{-ik}. \end{aligned} \tag{3.21}$$

If $e^{ik} \neq ia_1/a_3, -ia_3/a_1$, then by Lemma 3.1, it follows from (3.20) and (3.21) that

$$\begin{aligned} -ia_2 e^{i(p_1(z)-p_1(z+c))} &= a_1 e^{-ik} + ia_3 \\ -ia_2 e^{i(p_1(z)-p_1(z+c))} &= a_1 e^{ik} + ia_3. \end{aligned} \tag{3.22}$$

In view of (3.20), (3.21) and (3.22), it follows that

$$\begin{aligned} -ia_2 e^{i(-p_1(z)+p_1(z+c))} &= ia_3 - a_1 e^{ik} \\ -ia_2 e^{i(-p_1(z)+p_1(z+c))} &= ia_3 - a_1 e^{-ik}. \end{aligned} \tag{3.23}$$

In view of (3.23), we conclude that $p_1(z+c) - p_1(z)$ is constant, and hence we can assume that $p_1(z) = L(z) + \Phi(z) + \xi$, where $L(z) = \sum_{j=1}^n \alpha_j z_j$ with $\alpha_j, \xi \in \mathbb{C}$, $j = 1, 2, \dots, n$, and $\Phi(z)$ is a polynomial defined in **(i)** in Theorem 1.6.

Therefore, from (3.22) and (3.23), we obtain that

$$\begin{aligned} -ia_2e^{-iL(c)} &= a_1e^{-ik} + ia_3, \\ -ia_2e^{-iL(c)} &= a_1e^{ik} + ia_3, \\ -ia_2e^{iL(c)} &= ia_3 - a_1e^{ik}, \\ -ia_2e^{iL(c)} &= ia_3 - a_1e^{-ik}. \end{aligned} \tag{3.24}$$

It follows from (3.24) that

$$e^{2ik} = 1, \quad a_2^2 = a_1^2 + a_3^2, \quad e^{2iL(c)} = -\frac{a_1 - ia_3e^{-ik}}{a_1 + ia_3e^{ik}}.$$

It follows from (3.17) that the solution of the system (1.5) is

$$(f_1(z), f_2(z)) = \left(\frac{1}{a_1} \cos(L(z) + \Phi(z) + \xi), \frac{1}{a_1} \cos(L(z) + \Phi(z) + \xi + k) \right).$$

Case 2. Let $p_2(z) - p_1(z)$ be non-constant. We discuss the following two possible subcases:

Subcase 2.1 Let $p_2(z) + p_1(z) = k$, a constant, $k \in \mathbb{C}$. Therefore, from (3.18) and (3.19), we obtain

$$\begin{aligned} -ia_2e^{i(-p_1(z)+p_1(z+c))} - ia_2e^{-i(p_1(z)+p_1(z+c))} - (ia_3 - a_1e^{ik})e^{-2ip_1(z)} \\ = a_1e^{-ik} + ia_3 \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} -ia_2e^{i(p_1(z)-p_1(z+c))} - ia_2e^{-2ik}e^{i(p_1(z)+p_1(z+c))} \\ - (ia_3e^{-ik} - a_1)e^{-ik}e^{2ip_1(z)} = a_1e^{-ik} + ia_3. \end{aligned} \tag{3.26}$$

If $a_1e^{-ik} + ia_3 = 0$, then it follows from (3.25) that

$$e^{i(p_1(z)+p_1(z+c))} + e^{i(p_1(z)-p_1(z+c))} = w, \tag{3.27}$$

where $w = \frac{a_1e^{ik} - ia_3}{ia_2}$. Observe from (3.27) that

$$\overline{N}\left(r, \frac{1}{e^{i(p_1(z)+p_1(z+c))} - w}\right) = \overline{N}\left(r, \frac{1}{e^{i(p_1(z)-p_1(z+c))}}\right) = S\left(r, e^{i(p_1(z)-p_1(z+c))}\right).$$

By the second fundamental theorem of Nevanlinna for several complex variables, we have

$$\begin{aligned} T\left(r, e^{i(p_1(z)+p_1(z+c))}\right) &\leq \overline{N}\left(r, e^{i(p_1(z)+p_1(z+c))}\right) + \overline{N}\left(r, \frac{1}{e^{i(p_1(z)+p_1(z+c))}}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{e^{i(p_1(z)+p_1(z+c))} - w}\right) + S\left(r, e^{i(p_1(z)+p_1(z+c))}\right) \\ &\leq S\left(r, e^{i(p_1(z)+p_1(z+c))}\right) + S\left(r, e^{i(p_1(z)-p_1(z+c))}\right). \end{aligned}$$

This implies that p_1 is a constant, which is a contradiction. Therefore, $a_1e^{-ik} + ia_3 \neq 0$. By Lemma 3.1, we obtain from (3.25) and (3.26) that

$$\begin{aligned} -ia_2e^{i(-p_1(z)+p_1(z+c))} &= a_1e^{-ik} + ia_3, \\ -ia_2e^{i(p_1(z)-p_1(z+c))} &= a_1e^{-ik} + ia_3. \end{aligned} \tag{3.28}$$

In view of (3.25), (3.26) and (3.29), we have

$$\begin{aligned} -ia_2e^{i(p_1(z)-p_1(z+c))} &= ia_3 - a_1e^{ik}, \\ -ia_2e^{i(-p_1(z)+p_1(z+c))} &= ia_3 - a_1e^{ik}. \end{aligned} \tag{3.29}$$

In view of (3.28), we conclude that $p_1(z + c) - p_1(z)$ is constant, and hence we can assume that $p_1(z) = L(z) + \Phi(z) + \xi$, where $L(z), \Phi(z)$ are defined in Case 1. Therefore, $p_2(z) = -(L(z) + \Phi(z) + \xi) + k$. It follows from (3.28) and (3.29) that

$$\begin{aligned} -ia_2e^{iL(c)} &= a_1e^{-ik} + ia_3, \\ -ia_2e^{-iL(c)} &= a_1e^{-ik} + ia_3, \\ -ia_2e^{-iL(c)} &= ia_3 - a_1e^{ik}, \\ -ia_2e^{iL(c)} &= ia_3 - a_1e^{ik}. \end{aligned} \tag{3.30}$$

In view of (3.30), it follows that

$$e^{2iL(c)} = 1, \quad e^{2ik} = 1, \quad a_2^2 = (a_1 \pm a_3)^2.$$

In view of (3.17), the solution of the system (1.5) is

$$(f_1, f_2) = \left(\frac{1}{a_1} \cos[L(z) + \Phi(z) + \xi], \frac{1}{a_1} \cos(-[L(z) + \Phi(z) + \xi] + k) \right).$$

Subcase 2.2 Let $p_2(z) + p_1(z)$ be non-constant. Now, if $p_2(z) - p_1(z + c)$ is non-constant, then by Lemma 3.1, we obtain from (3.18) that $a_2e^{i(p_2(z)+p_1(z+c))} = ia_1$, which yields that $p_2(z) + p_1(z + c)$ is constant, say $k \in \mathbb{C}$. It follows from (3.18) that

$$e^{i(p_1(z)-p_2(z))} + e^{-i(p_1(z)+p_2(z))} = \frac{a_1 - ia_2e^{-ik}}{ia_3}. \tag{3.31}$$

Clearly, $a_1 - ia_2e^{-ik} \neq 0$. Otherwise, it follows from (3.31) that $p_1(z)$ is constant, which is a contradiction. In view of (3.31), we observe that

$$N\left(r, \frac{1}{e^{-i(p_1(z)+p_2(z))} - \frac{a_1 - ia_2e^{-ik}}{ia_3}}\right) = S\left(r, e^{-i(p_1(z)+p_2(z))}\right).$$

By the second fundamental theorem of Nevanlinna for several complex variables, we have that

$$\begin{aligned} &[T\left(r, e^{-i(p_1(z)+p_2(z))}\right)] \\ &\leq N\left(r, e^{-i(p_1(z)+p_2(z))}\right) + N\left(r, \frac{1}{e^{-i(p_1(z)+p_2(z))}}\right) \\ &\quad + N\left(r, \frac{1}{e^{-i(p_1(z)+p_2(z))} - \frac{a_1 - ia_2e^{-ik}}{ia_3}}\right) + S\left(r, e^{-i(p_1(z)+p_2(z))}\right) \\ &\leq S\left(r, e^{-i(p_1(z)+p_2(z))}\right) + S\left(r, e^{i(p_1(z)-p_2(z))}\right). \end{aligned}$$

This implies that $p_1(z) + p_2(z)$ is a constant in \mathbb{C} , which contradicts our assumption. Thus, $p_2(z) - p_1(z + c)$ is a constant in \mathbb{C} .

Let $p_2(z) - p_1(z + c) = k, k \in \mathbb{C}$. Then (3.18) reduces to

$$\begin{aligned} -ia_2e^{i(p_2(z)+p_1(z+c))} - ia_3e^{i(p_2(z)+p_1(z))} - ia_3e^{i(p_2(z)-p_1(z))} + a_1e^{2ip_2(z)} \\ = a_1 + ia_2e^{ik}. \end{aligned} \tag{3.32}$$

If $a_1 + ia_2e^{ik} \neq 0$, then using Lemma 3.1, we obtain from (3.32) that $a_1 + ia_2e^{ik} = -ia_2e^{i(p_2(z)+p_1(z+c))}$, which implies that $p_2(z) + p_1(z+c) = k_1 \in \mathbb{C}$. But, then $2p_2(z) = k + k_1$, a constant in \mathbb{C} , which contradicts to the fact that $p_2(z)$ is non-constant. Thus, $a_1 + ia_2e^{ik} = 0$. Then after simple computation, equation (3.32) reduces to equation (3.31). Then, by similar argument, we can get a contradiction. \square

Proof of Theorem 1.11. Assume that (f_1, f_2) is a pair of transcendental entire solution of (1.7) with each f_j of finite order for $j = 1, 2$. Then, by an argument similar to the one in the proof of Theorem 1.6, we obtain

$$\begin{aligned} a_1f_1(z+c) + a_2f_1(z) &= \frac{1}{2} \left(e^{p_1(z)} + e^{-p_1(z)} \right), \\ a_3f_2(z+c) + a_4f_2(z) &= \frac{1}{2i} \left(e^{p_1(z)} - e^{-p_1(z)} \right), \\ a_1f_2(z+c) + a_2f_2(z) &= \frac{1}{2} \left(e^{p_2(z)} + e^{-p_2(z)} \right), \\ a_3f_1(z+c) + a_4f_1(z) &= \frac{1}{2i} \left(e^{p_2(z)} - e^{-p_2(z)} \right), \end{aligned}$$

where $p_1(z), p_2(z)$ are two non-constant polynomials. Since $D := a_1a_4 - a_2a_3 \neq 0$, solving the above system of equations, we obtain

$$f_1(z+c) = \frac{1}{2D} \left(a_4(e^{p_1(z)} + e^{-p_1(z)}) + ia_2(e^{p_2(z)} - e^{-p_2(z)}) \right), \quad (3.33)$$

$$f_1(z) = \frac{1}{-2D} \left(a_3(e^{p_1(z)} + e^{-p_1(z)}) + ia_1(e^{p_2(z)} - e^{-p_2(z)}) \right), \quad (3.34)$$

$$f_2(z+c) = \frac{1}{2D} \left(a_4(e^{p_2(z)} + e^{-p_2(z)}) + ia_2(e^{p_1(z)} - e^{-p_1(z)}) \right), \quad (3.35)$$

$$f_2(z) = \frac{1}{-2D} \left(a_3(e^{p_2(z)} + e^{-p_2(z)}) + ia_1(e^{p_1(z)} - e^{-p_1(z)}) \right). \quad (3.36)$$

It follows from (3.33) and (3.34) that

$$\begin{aligned} a_3e^{p_1(z+c)+p_2(z)} + a_3e^{-p_1(z+c)+p_2(z)} + ia_1e^{p_2(z+c)+p_2(z)} \\ - ia_1e^{-p_2(z+c)+p_2(z)} + a_4e^{p_2(z)+p_1(z)} + a_4e^{-p_1(z)+p_2(z)} + ia_2e^{2p_2(z)} = ia_2. \end{aligned} \quad (3.37)$$

From (3.35) and (3.36), we obtain that

$$\begin{aligned} a_3e^{p_2(z+c)+p_1(z)} + a_3e^{-p_2(z+c)+p_1(z)} + ia_1e^{p_1(z+c)+p_1(z)} \\ - ia_1e^{-p_1(z+c)+p_1(z)} + a_4e^{p_1(z)+p_2(z)} + a_4e^{p_1(z)-p_2(z)} + ia_2e^{2p_1(z)} = ia_2. \end{aligned} \quad (3.38)$$

Now, we consider the following two possible cases.

Case 1. Suppose $p_2(z) - p_1(z) = k$, where k is a constant in \mathbb{C} . Then (3.37) and (3.38), respectively, yield

$$\begin{aligned} e^k (a_3 + ia_1e^k) e^{p_1(z)+p_1(z+c)} + (a_3e^k - ia_1) e^{p_1(z)-p_1(z+c)} \\ + e^k (a_4 + ia_2e^k) e^{2p_1(z)} = (ia_2 - a_4e^k) \end{aligned} \quad (3.39)$$

and

$$\begin{aligned} (a_3e^k + ia_1) e^{p_1(z)+p_1(z+c)} + (a_3e^{-k} - ia_1) e^{p_1(z)-p_1(z+c)} \\ + (a_4e^k + ia_2) e^{2p_1(z)} = (ia_2 - a_4e^{-k}). \end{aligned} \quad (3.40)$$

Now, we show that all of $a_3 + ia_1e^k$, $a_3e^k - ia_1$, $a_4 + ia_2e^k$, $ia_2 - a_4e^k$, $a_3e^k + ia_1$, $a_3e^{-k} - ia_1$, $a_4e^k + ia_2$, and $ia_2 - a_4e^{-k}$ are non-zero. Suppose $a_3 + ia_1e^k = 0$. Then, clearly, $a_4 + ia_2e^k \neq 0$. It follows from (3.39) that

$$(a_3e^k - ia_1) e^{p_1(z)-p_1(z+c)} + e^k (a_4 + ia_2e^k) e^{2p_1(z)} = ia_2 - a_4e^k. \tag{3.41}$$

Since $a_4 + ia_2e^k \neq 0$, it follows from (3.41) that $a_3e^k - ia_1 \neq 0$. Otherwise, $p_1(z)$ would be constant, which is not possible.

Also, it is clear that $ia_2 - a_4e^k$ is non-zero. Otherwise, then we must have from (3.41) that

$$-\left(\frac{a_3e^k - ia_1}{a_4 + ia_2e^k}\right) e^{-(p_1(z)+p_1(z+c))} = 1,$$

which implies that $p_1(z) + p_1(z + c)$, and hence $p_1(z)$ is constant, which is a contradiction.

In view of (3.41),

$$T(r, e^{p_1(z)-p_1(z+c)}) = T(r, e^{2p_1(z)}) + S(r, e^{2p_1(z)}).$$

Since $p_1(z)$ is a polynomial, it is easy to see that

$$N\left(r, \frac{1}{e^{p_1(z)-p_1(z+c)}}\right) = N(r, e^{p_1(z)-p_1(z+c)}) = N\left(r, \frac{1}{e^{2p_1(z)}}\right) = S(r, e^{p_1(z)}).$$

Then, in view of (3.41) and using the second fundamental theorem of Nevanlinna in several complex variables, we obtain

$$\begin{aligned} T(r, e^{p_1(z)-p_1(z+c)}) &\leq N\left(r, \frac{1}{e^{p_1(z)-p_1(z+c)}}\right) + N(r, e^{p_1(z)-p_1(z+c)}) \\ &\quad + N\left(r, \frac{1}{e^{p_1(z)-p_1(z+c)} - \alpha}\right) + S(r, e^{p_1(z)-p_1(z+c)}) \\ &\leq N\left(r, \frac{1}{e^{2p_1(z)}}\right) + S(r, e^{p_1(z)-p_1(z+c)}) \\ &\leq S\left(r, e^{p_1(z)-p_1(z+c)} + S(r, e^{2p_1(z)})\right) \end{aligned}$$

where $\alpha = (ia_2 - a_4e^k)/(a_3e^k - ia_1)$. This implies $T(r, e^{2p_1(z)}) = o(T(r, e^{2p_1(z)}))$, which is not possible as $e^{p_1(z)}$ is transcendental entire. We conclude that $a_3 + ia_1e^k \neq 0$. Similarly, we can prove that the others are also non-zero.

In view of Lemma 3.1, from (3.39) we obtain that

$$\left(\frac{a_3e^k - ia_1}{ia_2 - a_4e^k}\right) e^{p_1(z)-p_1(z+c)} = 1. \tag{3.42}$$

In view of (3.42) and (3.39), we have

$$-\left(\frac{a_3 + ia_1e^k}{a_4 + ia_2e^k}\right) e^{-p_1(z)+p_1(z+c)} = 1. \tag{3.43}$$

Again, in view of Lemma 3.1, from (3.40) we obtain that

$$\left(\frac{a_3e^{-k} - ia_1}{ia_2 - a_4e^{-k}}\right) e^{p_1(z)-p_1(z+c)} = 1. \tag{3.44}$$

Using (3.44) in (3.40), we obtain that

$$-\left(\frac{a_3e^k + ia_1}{a_4e^k + ia_2}\right) e^{-p_1(z)+p_1(z+c)} = 1. \tag{3.45}$$

In view of the fact that $D \neq 0$, from (3.42) and (3.44), we obtain $e^{2k} = 1$. Multiplying (3.42) and (3.43), we obtain

$$((a_2^2 + a_4^2) - (a_1^2 + a_3^2))e^k + i(a_2a_4 - a_1a_3)(e^{2k} - 1) = 0.$$

As $e^{2k} = 1$, the above equation yields

$$a_2^2 + a_4^2 = a_1^2 + a_3^2.$$

In view of (3.42), we conclude that $p_1(z) - p_1(z+c)$ is constant. Thus, we can write

$$p_1(z) = L(z) + \Phi(z) + A,$$

where $L(z) = \sum_{j=1}^n \alpha_j z_j$, $\alpha_j, A \in \mathbb{C}$, $j = 1, 2, \dots, n$, and $\Phi(z)$ is a polynomial defined in in Theorem 1.6(i). Therefore, from (3.42), (3.43), (3.44). and (3.45), we obtain

$$e^{L(c)} = \frac{a_3e^k - ia_1}{ia_2 - a_4e^k} = \frac{a_3e^{-k} - ia_1}{ia_2 - a_4e^{-k}} = \frac{-(a_4e^k + ia_2)}{a_3 + ia_1e^k} = \frac{-(a_4e^k + ia_2)}{a_3e^k + ia_1}.$$

Case 2. Let $p_2(z) - p_1(z)$ be non-constant. We consider the following subcases:

Subcase 2.1 Suppose $p_2(z) + p_1(z) = k$, where k is a constant in \mathbb{C} . Then from (3.37) and (3.38), we obtain

$$\begin{aligned} & (a_3e^k - ia_1) e^{-p_1(z)+p_1(z+c)} + e^k (a_3 + ia_1e^k) e^{-(p_1(z)+p_1(z+c))} \\ & + e^k (a_4 + ia_2e^k) e^{-2p_1(z)} \\ & = (ia_2 - a_4e^k) \end{aligned} \quad (3.46)$$

and

$$\begin{aligned} & (a_3e^k - ia_1) e^{p_1(z)-p_1(z+c)} + (a_3e^{-k} + ia_1) e^{p_1(z)+p_1(z+c)} \\ & + (ia_2 + a_4e^{-k}) e^{2p_1(z)} \\ & = (ia_2 - a_4e^k). \end{aligned} \quad (3.47)$$

In a similar manner as in Case 1, we can prove that $a_3e^k - ia_1$, $a_3 + ia_1e^k$, $a_4 + ia_2e^k$, $ia_2 - a_4e^k$, $a_3e^{-k} + ia_1$, and $ia_2 + a_4e^{-k}$ are non-constant.

As $p_1(z)$ is a non-constant polynomial in \mathbb{C}^2 , in view of Lemma 3.1, and equations (3.46) and (3.47), we obtain that

$$\begin{aligned} & (a_3e^k - ia_1) e^{-p_1(z)+p_1(z+c)} = ia_2 - a_4e^k, \\ & (a_3e^k - ia_1) e^{p_1(z)-p_1(z+c)} = ia_2 - a_4e^k. \end{aligned} \quad (3.48)$$

In view of (3.46), (3.47), and (3.48), we obtain that

$$\begin{aligned} & -(a_3 + ia_1e^k) e^{p_1(z)-p_1(z+c)} = a_4 + ia_2e^k, \\ & -(a_3e^{-k} + ia_1) e^{p_1(z+c)-p_1(z)} = ia_2 + ia_4e^{-k}. \end{aligned} \quad (3.49)$$

Since $p_1(z)$ is a non-constant polynomial in \mathbb{C}^2 , it follows from (3.48) that $p_1(z+c) - p_1(z) = k$, a constant in \mathbb{C} . This implies that $p_1(z) = L(z) + \Phi(z) + \xi$, where $L(z) = \sum_{j=1}^n \alpha_j z_j$, $\Phi(z)$ is a polynomial defined in (i) of Theorem 1.6, ξ, α_j are in

\mathbb{C} , $j = 1, 2, \dots, n$. In view of the form of $p_1(z)$, we obtain from (3.48) and (3.49) that

$$\begin{aligned} (a_3e^k - ia_1) e^{L(c)} &= ia_2 - a_4e^k, \\ (a_3e^k - ia_1)e^{-L(c)} &= ia_2 - a_4e^k, \\ -(a_3 + ia_1e^k) e^{-L(c)} &= a_4 + ia_2e^k, \\ -(a_3e^{-k} + ia_1) e^{L(c)} &= ia_2 + ia_4e^{-k}. \end{aligned} \tag{3.50}$$

Now, in view of equations of (3.50), we can easily obtain that

$$e^{2L(c)} = 1, \quad e^{2k} = -1, \quad a_1^2 + a_3^2 = a_2^2 + a_4^2, \quad a_1a_3 = a_2a_4.$$

Thus, in view of (3.34) and (3.36), it follows that

$$f_1(z_1, z_2) = \frac{-1}{2D} [(a_3 - ia_1e^{-k})e^{L(z)+\Phi(z)+\xi} + (a_3 + ia_1e^k)e^{-(L(z)+\Phi(z)+\xi)}]$$

and

$$f_2(z_1, z_2) = \frac{-1}{2D} [(ia_2 - a_4e^k)e^{-(L(z)+\Phi(z)+\xi)} + (ia_2 + a_4e^{-k})e^{L(z)+\Phi(z)+\xi}].$$

Subcase 2.2 Let $p_2(z) + p_1(z)$ be non-constant.

Subcase 2.2.1 Let $p_1(z + c) + p_2(z) = k \in \mathbb{C}$. Then, clearly $-p_1(z + c) + p_2(z)$ is non-constant. Otherwise, we obtain that $p_2(z)$ is constant, a contradiction. It follows from (3.37) that

$$\begin{aligned} a_3e^{-p_1(z+c)+p_2(z)} + ia_1e^{p_2(z+c)+p_2(z)} - ia_1e^{p_2(z)-p_2(z+c)} + a_4e^{p_1(z)+p_2(z)} \\ + a_4e^{p_2(z)-p_1(z)} + ia_2e^{2p_2(z)} = ia_2 - a_3e^k. \end{aligned} \tag{3.51}$$

Subcase 2.2.1.1 Let $ia_2 - a_3e^k = 0$. Then equation (3.51) reduces to

$$\begin{aligned} ia_1e^{p_2(z+c)-p_2(z)} - ia_1e^{-[p_2(z+c)+p_2(z)]} + a_4e^{p_1(z)-p_2(z)} + a_4e^{-[p_1(z)+p_2(z)]} \\ = -(ia_2 + a_3e^{-k}). \end{aligned} \tag{3.52}$$

Subcase 2.2.1.1.1 Let $ia_2 + a_3e^{-k} = 0$. Then (3.52) becomes

$$ia_1e^{p_2(z+c)+p_1(z)} - ia_1e^{p_1(z)-p_2(z+c)} + a_4e^{2p_1(z)} = -a_4. \tag{3.53}$$

In view of Lemma 3.6, it follows from (3.53) that either

$$ia_1e^{p_2(z+c)+p_1(z)} = -a_4, \tag{3.54}$$

or

$$-ia_1e^{-p_2(z+c)+p_1(z)} = -a_4. \tag{3.55}$$

First, we assume that (3.54) holds. Since $p_1(z), p_2(z)$ are non-constant polynomials in \mathbb{C}^2 , it follows from (3.54) that $p_2(z + c) + p_1(z) = k_1$, a constant in \mathbb{C} . As $p_1(z + c) + p_2(z) = k$, it follows that $p_1(z + 2c) - p_1(z) = k - k_1$. Then, we may assume that $p_1(z) = L(z) + \Phi(z) + \xi$, where $L(z), \Phi(z)$ are defined in the Theorem 1.6(i). Hence, $p_2(z) = -[L(z) + \Phi(z) + \xi] + k - L(c)$. Thus, $p_1(z) + p_2(z) = k - L(c)$, a constant in \mathbb{C} , which contradicts to our assumption. In a similar manner we can obtain a contradiction for the case (3.55).

Subcase 2.2.1.1.2 Let $ia_2 + a_3e^{-k} \neq 0$. Then in view of Lemma 3.1, it follows from (3.32) that

$$ia_1e^{p_2(z+c)-p_2(z)} = -(ia_2 + a_3e^{-k}). \tag{3.56}$$

Therefore, in view of (3.32) and (3.56), we obtain that

$$-ia_1e^{-p_2(z+c)+p_1(z)} + a_4e^{2p_1(z)} = -a_4. \quad (3.57)$$

In view of (3.57), we observe that

$$N\left(r, \frac{1}{e^{2p_1(z)} + 1}\right) = N\left(r, \frac{1}{e^{-p_2(z+c)+p_1(z)}}\right) = S(r, e^{-p_2(z+c)+p_1(z)}).$$

Now, by the second fundamental theorem of Nevanlinna for several complex variables, we obtain that

$$\begin{aligned} T(r, e^{2p_1(z)}) \\ \leq N(r, e^{2p_1(z)}) + N\left(r, \frac{1}{e^{2p_1(z)}}\right) + N\left(r, \frac{1}{e^{2p_1(z)} + 1}\right) + S(r, e^{2p_1(z)}) \\ \leq S(r, e^{-p_2(z+c)+p_1(z)}) + S(r, e^{2p_1(z)}). \end{aligned}$$

This implies that $p_1(z)$ is constant in \mathbb{C} , which is a contradiction.

Subcase 2.2.1.2 Let $ia_2 - a_3e^k \neq 0$. Then, in view of Lemma 3.1, we obtain from (3.51) that $-ia_1e^{-p_2(z+c)+p_2(z)} = ia_2 - a_3e^k$. This implies that $-p_2(z+c) + p_2(z) = k_1$, a constant in \mathbb{C} . As $p_1(z+c) + p_2(z) = k$, it follows that $p_1(z+c) - p_2(z) = p_1(z) - p_2(z) = k + k_1$, which is a contradiction.

Subcase 2.2.2 Let $p_1(z+c) + p_2(z)$ be non-constant.

Subcase 2.2.2.1 Let $-p_1(z+c) + p_2(z) = k$, a constant in \mathbb{C} . Then, from (3.37), we obtain that

$$\begin{aligned} a_3e^{p_1(z+c)+p_2(z)} + ia_1e^{p_2(z+c)+p_2(z)} - ia_1e^{p_2(z)-p_2(z+c)} + a_4e^{p_1(z)+p_2(z)} \\ + a_4e^{p_2(z)-p_1(z)} + ia_2e^{2p_2(z)} = ia_2 - a_3e^k. \end{aligned}$$

Then by an argument similar one used in Subcase 2.2.1.1 and Subcase 2.2.1.2, we can easily obtain a contradiction.

Subcase 2.2.2.2 Let $-p_1(z+c) + p_2(z)$ be non-constant. Then, in view of Lemma 3.1, it follows from (3.37) that

$$-ia_1e^{p_2(z)-p_2(z+c)} = ia_2. \quad (3.58)$$

As $p_2(z)$ is a non-constant polynomials in \mathbb{C}^2 , it follows from (3.58) that $p_2(z) - p_2(z+c)$ is a constant in \mathbb{C} . Thus, $p_2(z+c) + p_1(z)$ and $-p_2(z+c) + p_1(z)$ both are non-constant.

In view of Lemma 3.1, it follows from (3.38) that

$$-ia_1e^{-p_1(z+c)+p_1(z)} = ia_2. \quad (3.59)$$

Substituting (3.58) in (3.37), we obtain

$$\begin{aligned} a_3e^{p_1(z+c)-p_2(z)} + a_3e^{-(p_1(z+c)+p_2(z))} + ia_1e^{p_2(z+c)-p_2(z)} + a_4e^{p_1(z)-p_2(z)} \\ + a_4e^{-(p_1(z)+p_2(z))} = -ia_2. \end{aligned} \quad (3.60)$$

Substituting (3.59) in (3.38), we obtain

$$\begin{aligned} a_3e^{p_2(z+c)-p_1(z)} + a_3e^{-(p_2(z+c)+p_1(z))} + ia_1e^{p_1(z+c)-p_1(z)} + a_4e^{p_2(z)-p_1(z)} \\ + a_4e^{-(p_1(z)+p_2(z))} = -ia_2. \end{aligned} \quad (3.61)$$

Again, in view of Lemma 3.1, it follows from (3.60) that

$$ia_1e^{p_2(z+c)-p_2(z)} = -ia_2. \quad (3.62)$$

Substituting (3.62) in (3.60), we obtain

$$a_3 e^{p_1(z+c)+p_1(z)} + a_3 e^{p_1(z)-p_1(z+c)} + a_4 e^{2p_1(z)} = -a_4. \quad (3.63)$$

In view of Lemma 3.1, we obtain from (3.61) that

$$ia_1 e^{p_1(z+c)-p_1(z)} = -ia_2. \quad (3.64)$$

In view of Lemma 3.1, we obtain from (3.63) that

$$a_3 e^{p_1(z)-p_1(z+c)} = -a_4. \quad (3.65)$$

Substituting (3.65) in (3.63), we obtain that

$$a_3 e^{p_1(z+c)-p_1(z)} = -a_4. \quad (3.66)$$

In view of (3.59), we conclude that $-p_1(z+c) + p_1(z)$ must be constant in \mathbb{C} . This implies that $p_1(z) = L(z) + \Phi(z) + \xi$, where $L(z), \Phi(z)$ are defined in Theorem 1.6(i). Therefore, in view of (3.59), (3.64), (3.65) and (3.66), we obtain that

$$-ia_1 e^{-L(c)} = ia_2, \quad -ia_1 e^{L(c)} = ia_2, \quad a_3 e^{-L(c)} = -a_4, \quad a_3 e^{L(c)} = -a_4.$$

From the above four equations, we can easily obtain that $D = 0$, which contradicts to our assumption. \square

Proof of Theorem 2.2. Using Lemmas 3.4 and 3.5, the proof of this theorem can be carried out with arguments similar to those in the proof of [52, Theorem 1.1]. \square

Concluding remark and an open question. Observe that if $p(z) = L(z) + \Phi(z) + \xi$, where $L(z) = \sum_{j=1}^n a_j z_j$ and $\Phi(z)$ is defined as in the conclusion (i) of Theorem 1.6, then $p(z+c) - p(z)$ must be a constant in \mathbb{C} , $c = (c_1, c_2, \dots, c_n) \in \mathbb{C}^n$, $\xi, a_j \in \mathbb{C}$, $j = 1, 2, \dots, n$. But, we are still unable to prove the converse part. Therefore, we pose the following open problem.

What will be the exact form of the polynomial $p(z) : \mathbb{C}^n \rightarrow \mathbb{P}^1(\mathbb{C})$ if it satisfies the relation $p(z+c) - p(z) = \xi$, where $c = (c_1, c_2, \dots, c_n) \in \mathbb{C}^n$ and $\xi \in \mathbb{C}$?

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