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EXISTENCE OF SOLUTIONS TO STOCHASTIC $p(t,x)$-LAPLACE EQUATIONS AND APPLICATIONS

CHEN LIANG, LIXU YAN, YONGQIANG FU

Abstract. In this article, we consider a stochastic $p(t,x)$-Laplace equation. First we use the Galerkin method toobtain a unique weak solution. Then we obtain optimal controls for the corresponding stochastic optimal control problem

1. Introduction

In this article, we consider the stochastic $p(t,x)$-Laplace equation

$$\begin{gathered}
\mathrm{d}u-\operatorname{div}(|\nabla u|^{p(t,x)-2}\nabla u)\mathrm{d}t=f(u)\mathrm{d}t+g(t)\mathrm{d}t+\sigma\mathrm{d}W,\quad (t,x)\in[0,T]\times\Sigma\\
u(t,x)=0,\quad (t,x)\in[0,T]\times\partial\Sigma\\
u(0,x)=u_0,\quad x\in\Sigma
\end{gathered}\tag{1.1}$$

where $\Sigma\subset\mathbb{R}^d$ is a bounded smooth domain, $T\in(0,+\infty)$, $p(t,x)>1$, u_0, g are known functions, f is a continuous accretive operator, $u:\mathbb{R}^d\to\mathbb{R}^N$ is a vector-valued stochastic process, σ is a operator-valued function, $\{W(t)\}_{t\in[0,T]}$ is a E-valued Q-Brownian motion.

Then we consider the corresponding stochastic control system

$$\begin{gathered}
\mathrm{d}u-\operatorname{div}(|\nabla u|^{p(t,x)-2}\nabla u)\mathrm{d}t=f(u)\mathrm{d}t+g(t)\mathrm{d}t+\mathcal{A}v\mathrm{d}t+\sigma\mathrm{d}W,\quad t\in(0,T]\\
u(0,x)=u_0
\end{gathered}$$

where $\mathcal{A}v$ is a control item. The cost function is

$$J(v)=\mathbb{E}\Big\{\int_0^T\|\mathcal{H}u(v)-\mu_{\mathrm{d}}\|_{L^2}^2\mathrm{d}t+(\mathcal{K}v,v)_V\Big\}$$

where $u(v)$ is the solution of the stochastic control system, $\mathcal{H},\mathcal{K}$ are linear operators, μ_{d} is a fixed stochastic process.

The theory of partial differential equations with variable growth has a wide range of applications in solving non-standard exponential growth nonlinear problems. Stochastic partial differential equations have a wide range of applications in financial mathematics, physics, engineering technology. In recent years, with the

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development of stochastic analysis, stochastic partial differential equations have developed rapidly.

Ahemd [1] studied the case $p(t,x) = 2$,

$$\begin{gathered} \mathrm{d}u = D\Delta u \mathrm{d}t + f(x,u)\mathrm{d}W, \quad (t,x) \in (0,T] \times \Sigma, \\ \partial u/\partial v = 0, \quad (t,x) \in (0,T] \times \partial\Sigma, \\ u(0,x) = u_0(x), \quad x \in \Sigma. \end{gathered}$$

Ahemd proved that there exists a unique weak solution for a stochastic Laplace equation under suitable assumptions. Then the existence of optimal controls for the corresponding stochastic optimal control problem was obtained. Different from Laplace operator, even though $p(t,x) \equiv p \neq 2$, $p(t,x)$-Laplace is a nonlinear operator. Majee [8] studied p-Laplace equations and obtained the existence of weak solutions under multiplicative noise. Based on the variational calculus and the convexity of the costing function, the existence of optimal controls for the corresponding stochastic optimal control problems was obtained. Sapountzoglou and Zimmermann [10, 11] also discussed a stochastic p-Laplace equation and they obtained solutions for the stochastic p-Laplace equation under additive noise and multiplicative noise.

Zimmermann et al [3] discussed the stochastic $p(t,x)$-Laplace equation

$$\begin{gathered} \mathrm{d}u - \operatorname{div}\big(|\nabla u|^{p(t,x)-2}\nabla u\big)\mathrm{d}t = h(t,x,u)\mathrm{d}W, \quad (t,x) \in (0,T) \times \Sigma, \\ u(t,x) = 0, \quad (t,x) \in (0,T] \times \partial\Sigma, \\ u(0,x) = u_0(x), \quad x \in \Sigma. \end{gathered}$$

By using singular perturbation theory and a fixed point theorem, thy obtained the existence and uniqueness of solutions for the stochastic $p(t,x)$-Laplace equations under additive noise and multiplicative noise. Zimmermann and Vallet [12] used similar methods to consider stochastic $p(\omega,t,x)$-Laplace equations and got the corresponding results which are similar to [3].

2. Preliminaries

In this section, we recall some concepts of variable exponent Lebesgue spaces and Sobolev spaces and some Banach spaces which involve stochastic variables; see [9, 6] for details.

Let $\Sigma \subset \mathbb{R}^d$ be a bounded and smooth domain. $p : \Sigma \to [1,+\infty)$ is a continuous function. Let $p^+ = \sup_{x\in\overline{\Sigma}} p(x)$, $p^- = \inf_{x\in\overline{\Sigma}} p(x)$. For each function u, the modular is

$$\rho_{p(x)}(u) = \int_\Sigma |u(x)|^{p(x)}\mathrm{d}x.$$

The variable exponent Lebesgue space is defined by

$$L^{p(x)}(\Sigma) = \big\{u \text{ is a measurable function} : \rho_{p(x)}(u) < \infty\big\}$$

with the norm

$$\|u\|_{L^{p(x)}(\Sigma)} = \inf\big\{\lambda > 0 : \rho_{p(x)}\Big(\frac{u}{\lambda}\Big) \leqslant 1\big\}.$$

Then the space $L^{p(x)}(\Sigma)$ is a Banach space. Note

$$\min\big\{\|u\|^{p^-}_{L^{p(x)}(\Sigma)}, \|u\|^{p^+}_{L^{p(x)}(\Sigma)}\big\} \leqslant \rho_{p(x)}(u) \leqslant \max\big\{\|u\|^{p^-}_{L^{p(x)}(\Sigma)}, \|u\|^{p^+}_{L^{p(x)}(\Sigma)}\big\},$$

so norm convergence is equivalent to modular convergence. If the exponent p is bounded, the conjugate exponent $p^*(x) = \frac{p(x)}{p(x)-1}$; when $p(x) = 1$ the conjugate exponent is $p^*(x) = \infty$. If $1 < p^- \leqslant p^+ < +\infty$, $L^{p(x)}(\Sigma)$ is a reflexive Banach space and its dual space is $L^{p^*(x)}(\Sigma)$.

Definition 2.1. We call an exponent $p : \Sigma \to \mathbb{R}$ a globally log-Hölder continuous function if p satisfies the following conditions:

(1) There exists a positive constant α_1 such that
$$|p(x) - p(y)| \leqslant \frac{\alpha_1}{\log(\mathrm{e} + 1/|x-y|)}$$
for all points $x, y \in \Sigma$;

(2) There exists a positive constant α_2 such that
$$|p(x) - p_\infty| \leqslant \frac{\alpha_2}{\log(\mathrm{e} + 1/|x|)}$$
for all points $x \in \Sigma$.

If the exponent p is globally log-Hölder continuous, $C_0^\infty(\Sigma)$ is dense in $L^{p(x)}(\Sigma)$. The variable exponent Sobolev space is defined by
$$W^{1,p(x)}(\Sigma) = \big\{u \in L^{p(x)}(\Sigma) : \nabla u \in (L^{p(x)}(\Sigma))^d\big\}$$
with the norm
$$\|u\|_{W^{1,p(x)}(\Sigma)} = \|u\|_{L^{p(x)}(\Sigma)} + \|\nabla u\|_{(L^{p(x)}(\Sigma))^d}.$$
Note that $W^{1,p(x)}(\Sigma)$ is a Banach space. If $1 < p^- \leqslant p^+ < +\infty$, $W^{1,p(x)}(\Sigma)$ is reflexive. $W_0^{1,p(x)}(\Sigma)$ is the closure of $C_0^\infty(\Sigma)$ under the norm $\|\cdot\|_{W^{1,p(x)}(\Sigma)}$. If the exponent p is globally log-Hölder continuous, $C_0^\infty(\Sigma)$ is dense in $W^{1,p(x)}(\Sigma)$.

Definition 2.2. Let $\Sigma_T = (0,T) \times \Sigma$, and $p, m : \Sigma_T \to (1, +\infty)$ be globally log-Hölder continuous. $X(\Sigma_T)$ is defined by
$$\begin{aligned} X(\Sigma_T) = \big\{u \in L^{m(t,x)}(\Sigma_T) : \nabla u \in (L^{p(t,x)}(\Sigma_T))^d,\\ u(t,x) \in W_0^{1,p(t,x)}(\Sigma) \text{ for a.e. } t \in [0,T]\big\}. \end{aligned}$$
with the norm
$$\|u\|_{X(\Sigma_T)} = \|u\|_{L^{m(t,x)}(\Sigma_T)} + \|\nabla u\|_{(L^{p(t,x)}(\Sigma_T))^d}.$$

Note that $X(\Sigma_T)$ is a reflexive Banach space, and $C_0^\infty(\Sigma_T)$ and $C_0^\infty([0,T], C_0^\infty(\Sigma))$ are dense in $X(\Sigma_T)$.

For a vector-valued function $u = (u_1, u_2, \dots, u_N)^{\mathrm{T}}$, we can define the space
$$(L^{p(x)}(\Sigma))^N = \big\{u : \sum_{i=1}^N \|u_i\|_{L^{p(x)}(\Sigma)} < \infty\big\}$$
with the norm
$$\|u\|_{(L^{p(x)}(\Sigma))^N} = \sum_{i=1}^N \|u_i\|_{L^{p(x)}(\Sigma)}.$$

Similarly, we define the space
$$\big(W^{1,p(x)}(\Sigma)\big)^N = \big\{u \in (L^{p(x)}(\Sigma))^N : \nabla u \in (L^{p(x)}(\Sigma))^{d\times N}\big\}$$

with the norm

$$\|u\|_{(W^{1,p(x)}(\Sigma))^N} = \|u\|_{(L^{p(x)}(\Sigma))^N} + \|\nabla u\|_{(L^{p(x)}(\Sigma))^{d\times N}}\,.$$

Then we have the vector-valued function space

$$\begin{gathered} X(\Sigma_T) = \big\{u \in \big(L^{m(t,x)}(\Sigma_T)\big)^N : \nabla u \in \big(L^{p(t,x)}(\Sigma_T)\big)^{d\times N}, \\ u(t,x) \in \big(W_0^{1,p(t,x)}(\Sigma)\big)^N \text{ a.e. } t \in [0,T]\big\} \end{gathered}$$

with the norm

$$\|u\|_{X(\Sigma_T)} = \|u\|_{(L^{m(t,x)}(\Sigma_T))^N} + \|\nabla u\|_{(L^{p(t,x)}(\Sigma_T))^{d\times N}}$$

Next we will recall some Banach spaces which involve stochastic variables. Let $(\Omega, \mathscr{F}, \mathscr{P})$ be a complete probability space with a filtration $\mathscr{F}_{t\in[0,T]}$. Let

$$\begin{gathered} L_2^{\mathscr{F}_0}(\Omega, X) = \big\{u \text{ is } \mathscr{F}_0 \text{ adapted } X\text{-valued stochastic variable } : \mathbb{E}\|u\|_X^2 < \infty\big\}, \\ L_2^{\mathscr{F}_T}(\Omega, X) = \big\{u \text{ is } \mathscr{F}_T \text{ adapted } X\text{-valued stochastic variable} : \mathbb{E}\|u\|_X^2 < \infty\big\}, \\ L^\infty(\Omega) = \Big\{\xi \text{ is measurable } \mathbb{R}^N\text{-valued stochastic variable} : \\ \inf\{M : \mathscr{P}(|\xi| > M) < \infty\}\Big\}, \\ C^1([0,T], (C_0^\infty(\Sigma))^N) = \big\{\varphi \text{ is a continuous function on } \Sigma_T : \varphi(t), \\ \frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} \in (C_0^\infty(\Sigma))^N\big\}. \end{gathered}$$

For each positive constant $p \in [1, +\infty)$, let

$$\begin{aligned} &L_p^{\mathscr{F}}([0,T], X) \\ &= \big\{u \text{ is a } \mathscr{F}_{t\in[0,T]} \text{ adapted stochastic process } : \mathbb{E}\int_0^T \|u(t)\|_X^p \mathrm{d}t < \infty\big\} \end{aligned}$$

with the norm

$$\|u\|_{L_p^{\mathscr{F}}([0,T],X)} = \Big(\mathbb{E}\int_0^T \|u(t)\|_X^p \mathrm{d}t\Big)^{1/p}.$$

When $p = +\infty$, we let

$$\begin{aligned} &L_\infty^{\mathscr{F}}([0,T], X) \\ &= \big\{u \text{ is a } \mathscr{F}_{t\in[0,T]} \text{ adapted stochastic process} : \operatorname{ess\,sup}_{t\in[0,T]} \mathbb{E}\|u(t)\|_X^2 < \infty\big\} \end{aligned}$$

with the norm

$$\|u\|_{L_\infty^{\mathscr{F}}([0,T],X)} = \operatorname{ess\,sup}_{t\in[0,T]} \big(\mathbb{E}\|u(t)\|_X^2\big)^{1/2}.$$

For any $p \in [1, +\infty]$, $L_p^{\mathscr{F}}([0,T], X)$ is a Banach space. When $p \in (1, +\infty)$, $L_p^{\mathscr{F}}([0,T], X)$ is a reflective Banach space. When $p < +\infty$, $L_p^{\mathscr{F}}([0,T], X)$ is a separable Banach space.

Next we define the space $L^{p(x)}(\Omega \times \Sigma)$

$$L^{p(x)}(\Omega\times\Sigma) = \Big\{u : \mathbb{E}\big\{\int_\Sigma |u|^{p(x)}\mathrm{d}x\big\} < +\infty\Big\}$$

with the norm

$$\|u\|_{L^{p(x)}(\Omega\times\Sigma)} = \inf\Big\{\lambda > 0 : \mathbb{E}\big\{\int_\Sigma |\frac{u}{\lambda}|^{p(x)}\mathrm{d}x\big\} < +\infty\Big\}.$$

Note that $L^{p(x)}(\Omega \times \Sigma)$ is a reflective Banach space. Now we define the space

$$\begin{aligned} &L^{\mathscr{F}}_{p(t,x)}(\Omega \times \Sigma_T) \\ &= \Big\{u \text{ is a } \mathscr{F}_{t\in[0,T]} \text{ adapted stochastic process} : \mathbb{E}\big\{\int_{\Sigma_T} |u|^{p(t,x)} \mathrm{d}x\mathrm{d}t\big\} < +\infty\Big\} \end{aligned}$$

with the norm

$$\|u\|_{L^{\mathscr{F}}_{p(t,x)}(\Omega\times\Sigma_T)} = \inf\Big\{\lambda > 0 : \ \mathbb{E}\big\{\int_{\Sigma_T} |\frac{u}{\lambda}|^{p(t,x)} \mathrm{d}x\mathrm{d}t\big\} < +\infty\Big\}.$$

for $1 < p_- \leqslant p^+ < +\infty$, $L^{\mathscr{F}}_{p(t,x)}(\Omega \times \Sigma_T)$ is a reflective Banach space. The following theorem gives a relation between almost everywhere convergence and weak convergence in $L^{\mathscr{F}}_{p(t,x)}(\Omega \times \Sigma_T)$.

Theorem 2.3 ([5]). *Let p be a bounded globally log-Hölder continuous function with $p(t,x) > 1$. If $\{u_n\}$ is bounded in $L^{\mathscr{F}}_{p(t,x)}(\Omega \times \Sigma_T)$ and $u_n \to u$ a.e. $(\omega,t,x) \in \Omega \times \Sigma_T$, then there exist a subsequence $\{u_n\}$, such that $u_n \to u$ weakly in $L^{\mathscr{F}}_{p(t,x)}(\Omega\times\Sigma_T)$.*

Similarly, the above spaces can be extended to the case of vector-valued function spaces. Hence we introduce the space $L^{\mathscr{F}}(\Omega, X(\Sigma_T))$.

Definition 2.4.

$$\begin{aligned} L^{\mathscr{F}}(\Omega, X(\Sigma_T)) = \big\{&u \in (L^{\mathscr{F}}_{m(t,x)}(\Omega \times \Sigma_T))^N, \nabla u \in (L^{\mathscr{F}}_{p(t,x)}(\Omega \times \Sigma_T))^{d\times N}, \\ &u(\omega,t,x) \in X(\Sigma_T), \text{a.e. } \omega \in \Omega\big\} \end{aligned}$$

with the norm

$$\|u\|_{L^{\mathscr{F}}(\Omega,X(\Sigma_T))} = \|u\|_{(L^{\mathscr{F}}_{m(t,x)}(\Omega\times\Sigma_T))^N} + \|\nabla u\|_{(L^{\mathscr{F}}_{p(t,x)}(\Omega\times\Sigma_T))^{d\times N}}$$

Note that $L^{\mathscr{F}}(\Omega, X(\Sigma_T))$ is a reflective Banach space. In this article we set $m(t,x) = 2$. Let E be a separable Hilbert space.

Theorem 2.5 ([7]). *Let $Q \in \mathscr{L}(E)$ be a symmetric nonnegative operator, $\mathrm{Tr}Q < \infty$. B is an E-valued Q-Wiener process. For each $t \in [0,T]$, $y \in E$ we have:*

1. *B is E-valued Gauss process and*
$$\mathbb{E}(B(t), y)_E = 0, \quad \mathbb{E}(B(t), y)^2_E = t(Qy, y).$$
2. *B has the expression*
$$B(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\beta_j(t)\overline{e_j}, \tag{2.1}$$
where $\{\overline{e_i}\}_{i=1}^{\infty}$ is an orthonormal basis of E, $\{\lambda_i\}_{i=1}^{\infty}$ is the sequence of eigenvalues of Q. $\beta_i(t)$ is a sequence of Brownian motions which are independent from each other on probability space $\big(\Omega, \mathscr{F}, \mathscr{P}, \mathscr{F}_{t\in[0,T]}\big)$. The series converges strongly to B in $L^{\mathscr{F}}_2(\Omega, C([0,T], E))$.
3. *Let O be a separable Hilbert space. If $\sigma(t) \in \mathscr{L}(E, O)$ $(t \in [0,T])$, then*
$$\int_0^T \sigma(s)\mathrm{d}B(s) = \sum_{j=1}^{\infty} \sqrt{\lambda_j} \int_0^T \sigma(s)(\overline{e_j})\mathrm{d}\beta_j(s). \tag{2.2}$$
The series converges strongly to $\int_0^T \sigma(s)\mathrm{d}B(s)$ in $L^{\mathscr{F}}_2(\Omega, C([0,T], O))$.

Finally we recall the Crandal-Liggett theorem.

Theorem 2.6 ([2]). *Let $\mathscr{Y}$ be a Banach space, L is a m-accretive operator, Δ_n is a partition of $[0,T]$, and $|\Delta_n| \to 0$ as $n \to \infty$. If $u_0 \in \overline{D(L)}$, then there exists a $u \in C([0,T],\mathscr{Y})$ and a nonlinear operator semigroup $\{T(t)\}_{t\geqslant 0}$, such that*

$$u(t) = T(t)u_0.$$

If u_n is the implicit interpolation approximation of u, then

$$\|u_n(t) - u(t)\|_{\mathscr{Y}} \to 0, \quad \text{as } n \to \infty$$

uniformly on $[0,T]$.

We denote by $\|\cdot\|_{L^2}$ the norm $(L^2(\Sigma))^N$; denote by $\|\cdot\|_{L^{2q(x)}}$ the norm of $(L^{2q(x)}(\Sigma))^N$; denote by $\|\cdot\|_{L^2(\Omega\times\Sigma_T)}$ the norm of $(L_2^{\mathscr{F}}(\Omega\times\Sigma_T))^N$; denote by $\|\cdot\|_{L^{p(t,x)}(\Omega\times\Sigma_T)}$ the norm of $(L^{\mathscr{F}}_{p(t,x)}(\Omega\times\Sigma_T))^{d\times N}$; denote by $(\cdot,\cdot)_{L^2}$ the product of $(L^2(\Sigma))^N$.

3. Existence and uniqueness of weak solutions

Let E be a separable Hilbert space, $\sigma(t)$ be a bounded linear operator from E to $(L^2(\Sigma))^N$ and

$$\|\sigma(t)\| \leqslant M, \quad \forall t \in [0,T]. \tag{3.1}$$

Let $Q \in \mathscr{L}(E)$ be a symmetric nonnegative operator, $\{W(t)\}_{t\in[0,T]}$ be a E-valued Q-Brown motion defined on $(\Omega,\mathscr{F},\mathscr{P},\mathscr{F}_{t\in[0,T]})$.

Fix $\omega \in \Omega$, $t \in [0,T]$, f is a continuous accretive operator from $\left(L^{2q(x)}(\Sigma)\right)^N$ to $\left((L^{2q(x)}(\Sigma))^*\right)^N$, where $q(x)$ is continuous and $q(x) \geqslant 1$. Additionally, f satisfies the following conditions:

(H1) There exist $c_1 \in [0,+\infty)$ and $c_2 \in (0,+\infty)$, such that
$$\langle f(u), u\rangle_{(L^{2q(x)})^*,L^{2q(x)}} \leqslant c_1\|u\|_{L^2}^2 - c_2\|u\|_{L^{2q(x)}}^2.$$

(H2) $f(u)$ with respect to u is a completely continuous operator from the space $L_2^{\mathscr{F}}\left([0,T],(L^{2q(x)}(\Sigma))^N\right)$ to $\left(L_2^{\mathscr{F}}\left([0,T],(L^{2q(x)}(\Sigma))^N\right)\right)^*$.

(H3) For each $u,v \in (L^{2q(x)}(\Sigma))^N$,
$$\langle f(u)-f(v), u-v\rangle_{(L^{2q(x)})^*,L^{2q(x)}} \leqslant 0.$$

Next we give the concept of weak solutions for system (1.1).

Definition 3.1. An $\mathbb{R}^N$-valued stochastic process

$$u \in L_\infty^{\mathscr{F}}\left([0,T],(L^2(\Sigma))^N\right) \cap L^{\mathscr{F}}(\Omega, X(\Sigma_T)) \cap L_2^{\mathscr{F}}\left([0,T],(L^{2q(x)}(\Sigma))^N\right)$$

is a weak solution of (1.1), if for each $\varphi \in C^1([0,T],(C_0^\infty(\Sigma))^N)$, u satisfies

$$\begin{aligned}
&(u(T),\varphi(T))_{L^2} - (u_0,\varphi(0))_{L^2} - \int_0^T \big(u(t),\frac{\mathrm{d}\varphi}{\mathrm{d}t}\big)_{L^2}\mathrm{d}t \\
&+ \int_0^T\int_\Sigma |\nabla u|^{p(t,x)-2}\nabla u\nabla\varphi \mathrm{d}x\mathrm{d}t \\
&= \int_0^T \langle f(u(t)),\varphi\rangle_{(L^{2q(x)})^*,L^{2q(x)}}\mathrm{d}t + \int_0^T (g(t),\varphi(t))_{L^2}\mathrm{d}t \\
&\quad + \int_0^T (\varphi(t),\sigma(t)\mathrm{d}W(t))_{L^2}
\end{aligned} \tag{3.2}$$

Under the above conditions, we use Galërkin's method to prove that equation (1.1) admits a unique weak solution. The main result of this section reads as follows.

Theorem 3.2. *Let $p(t,x)$ be a bounded globally log-Hölder continuous function and $p(t,x) > 1$. If* (H1)–(H3) *and* (3.1) *hold, then equation* (1.1) *has a unique weak solution.*

$$u \in L_\infty^{\mathscr{F}}\left([0,T],(L^2(\Sigma))^N\right) \cap L^{\mathscr{F}}(\Omega, X(\Sigma_T)) \cap L_2^{\mathscr{F}}\left([0,T],(L^{2q(x)}(\Sigma))^N\right)$$

for any $u_0 \in L_2^{\mathscr{F}_0}\left(\Omega,(L^2(\Sigma))^N\right)$ *and* $g \in L_2^{\mathscr{F}}\left([0,T],(L^2(\Sigma))^N\right)$.

Proof. This proof is divided into four steps.

Step 1: Uniqueness of a weak solution. Assume that two solutions satisfy

$$u,v \in L_\infty^{\mathscr{F}}\left([0,T],(L^2(\Sigma))^N\right) \cap L^{\mathscr{F}}(\Omega, X(\Sigma_T)) \cap L_2^{\mathscr{F}}\left([0,T],(L^{2q(x)}(\Sigma))^N\right)$$

with initial states u_0, v_0 and g_1, g_2. Since u,v satisfy system (1.1) in the weak sense, by integrating by parts, we deduce that

$$\begin{aligned}
&\frac{1}{2}\|u-v\|_{L^2}^2 + \int_0^t\int_\Sigma \left(|\nabla u|^{p(s,x)-2}\nabla u - |\nabla v|^{p(s,x)-2}\nabla v\right)(\nabla u - \nabla v)\mathrm{d}x\mathrm{d}s\\
&= \frac{1}{2}\|u_0-v_0\|_{L^2}^2 + \int_0^t \langle f(u)-f(v), u-v\rangle_{(L^{2q(x)})^*, L^{2q(x)}}\mathrm{d}s\\
&\quad + \int_0^t (g_1-g_2, u-v)_{L^2}\,\mathrm{d}s + \int_0^t (u-v, \sigma\mathrm{d}W)_{L^2}\,.
\end{aligned}$$

As

$$\int_0^t\int_\Sigma \left(|\nabla u|^{p(s,x)-2}\nabla u - \nabla v|^{p(s,x)-2}\nabla v\right)(\nabla u - \nabla v)\mathrm{d}x\mathrm{d}s \geqslant 0,$$

by (H2), we have

$$\frac{1}{2}\|u-v\|_{L^2}^2 \leqslant \frac{1}{2}\|u_0-v_0\|_{L^2}^2 + \int_0^t (g_1-g_2, u-v)_{L^2}\,\mathrm{d}s + \int_0^t (u-v,\sigma\mathrm{d}W)_{L^2}$$

and further after taking the expectation we have

$$\frac{1}{2}\mathbb{E}\|u-v\|_{L^2}^2 \leqslant \frac{1}{2}\mathbb{E}\|u_0-v_0\|_{L^2}^2 + \mathbb{E}\int_0^t (g_1-g_2, u-v)_{L^2}\,\mathrm{d}s.$$

When $u_0 = v_0$ and $g_1 = g_2$, we deduce $u = v$.

Step 2: Existence of solutions for finite dimensional truncated systems. We choose an orthonormal basis $\{e_i\}_{i=1}^\infty$, such that $\{e_i\}_{i=1}^\infty \subset (C_0^\infty(\Sigma))^N \subset \overline{\bigcup_{n=1}^\infty V_n}^{(C^1(\Sigma))^N}$, where $V_n = \operatorname{span}\{e_1, e_2, \dots, e_n\}$. Let $\{\overline{e}_i\}$ be an orthonormal basis of E, W_n is an n-dimensional Brown motion. We consider the truncation of system (1.1):

$$\begin{gathered}
\mathrm{d}u_n - \operatorname{div}\left(|\nabla u_n|^{p(t,x)-2}\nabla u_n\right)\mathrm{d}t = f(u_n)\mathrm{d}t + g_n(t)\mathrm{d}t + \sigma\mathrm{d}W_n, \quad (t,x)\in(0,T]\times\Sigma,\\
u_n(t,x) = 0, \quad (t,x)\in[0,T]\times\partial\Sigma,\\
u_n(0,x) = \sum_{j=1}^n (u_0, e_j)_{L^2} e_j, \quad x \in \Sigma,
\end{gathered} \tag{3.3}$$

where

$$u_n(t)=\sum_{j=1}^n\theta_n^j(t)e_j,\quad u_n(0)=\sum_{j=1}^n(u_0,e_j)_{L^2}e_j=\sum_{j=1}^n\theta_n^j(0)e_j,$$

$$g_n(t)=\sum_{j=1}^n(g(t),e_j)_{L^2}e_j,\quad W_n(t)=\sum_{j=1}^n(W(t),\overline{e}_j)_E\overline{e}_j.$$

$\{\theta_n^j(t)\}$ are unknown functions. Let L, F and G be n-dimensional vectors, A is a $n\times n$ matrix, whose entries are

$$L_i(\theta)\triangleq\int_\Sigma\Big|\sum_{j=1}^n\theta_n^j\nabla e_j\Big|^{p(t,x)-2}\Big(\sum_{j=1}^n\theta_n^j\nabla e_j\Big)\nabla e_i\mathrm{d}x,$$

$$F_i(\theta_n)\triangleq\big\langle f(\sum_{j=1}^n\theta_n^je_j),e_i\big\rangle_{(L^{2q(x)})^*,L^{2q(x)}},$$

$$G_i(t)\triangleq\sum_{j=1}^n\left(g(t),e_j\right)_{L^2}e_i,\quad a_{ij}(t)\triangleq\sqrt{\lambda}_j(e_i,\sigma\overline{e}_j)_{L^2},$$

where $1\leqslant i,j\leqslant n$. We consider the n dimensional stochastic system

$$\mathrm{d}\theta_n=L\theta_n\mathrm{d}t+F(\theta_n)\mathrm{d}t+G\mathrm{d}t+A\mathrm{d}W_n,\quad t\in[0,T]\tag{3.4}$$

We claim F is a m-accretive operator on $\mathbb{R}^N$. On one hand, because f is accretive, we obtain F is a accretive operator. On the other hand, F is continuous, according to [4, Appendix D Corollary D.10], we can obtain F is m-accretive. Let $\Delta_k=\big\{0=t_k^0<t_k^1<,\dots,<t_k^k=T\big\}$ be the kth uniform partition of $[0,T]$, denote $\delta_k\triangleq|\Pi_k|$, the sequence of approximate solutions $\{\theta_{n,k}(t)\}$ is given by

$$\begin{aligned}\theta_{n,k}\left(t_k^i\right)=&\left(I-\delta_kF\right)^{-1}\big[\theta_{n,k}\left(t_k^{i-1}\right)+\delta_kL\theta_{n,k}\left(t_k^{i-1}\right)+\delta_kG\left(t_k^{i-1}\right)\\&+A\left(t_k^{i-1}\right)\left(B_n\left(t_k^i\right)-B_n\left(t_k^{i-1}\right)\right)\big],\end{aligned}\tag{3.5}$$

where $i=1,2,\dots,k$. By Theorem 2.6, there exists $\theta_n\in C^{\mathscr{F}}\left([0,T],\mathbb{R}^n\right)$, such that

$$\theta_{n,k}(t)\to\theta_n(t)\quad\text{strongly in }\mathbb{R}^n,$$

uniformly on $[0,T]$. Thus

$$\theta_n=\left(\theta_n^1,\theta_n^2,\dots,\theta_n^n\right)^{\mathrm{T}}$$

is a solution of (3.4), so $u_n=\sum_{j=1}^n\theta_n^je_j$ is a solution of the system (3.3).

Step 3: A priori estimate. Integrating by parts on (3.3) we obtain

$$\begin{aligned}&\mathbb{E}\|u_n(t)\|_{L^2}^2+\mathbb{E}\int_0^t\int_\Sigma|\nabla u_n|^{p(s,x)}\mathrm{d}x\mathrm{d}s+2C_2\mathbb{E}\int_0^t\|u_n(s)\|_{L^{2q(x)}}^2\mathrm{d}s\\&\leqslant\mathbb{E}\|u_n(0)\|_{L^2}^2+C\mathbb{E}\int_0^t\|u_n(s)\|_{L^2}^2\mathrm{d}s+2C_\varepsilon\mathbb{E}\int_0^t\|g_n(s)\|_{L^2}^2\mathrm{d}s\end{aligned}.\tag{3.6}$$

Since $u_0\in L_2^{\mathscr{F}_0}\left(\Omega,(L^2(\Sigma))^N\right)$, $g\in L_2^{\mathscr{F}}\left([0,T],(L^2(\Sigma))^N\right)$,

$$\begin{aligned}\mathbb{E}\|u_n(0)\|_{L^2}^2&=\mathbb{E}\Big\|\sum_{j=1}^n(u_0,e_j)_{L^2}e_j\Big\|_{L^2}^2\\&=\mathbb{E}\Big(\sum_{j=1}^n|(u_0,e_j)_{L^2}|^2\Big)\end{aligned}$$

$$
\begin{aligned}
&\leqslant \mathbb{E}\Big(\sum_{j=1}^{\infty}|(u_0,e_j)_{L^2}|^2\Big)\\
&=\mathbb{E}\|u_0\|_{L^2}^2=\|u_0\|_{L_2^{\mathscr{F}_0}(\Omega,(L^2(\Sigma))^N)},
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\int_0^T\|g_n(t)\|_{L^2}^2\mathrm{d}t&=\mathbb{E}\int_0^T\Big\|\sum_{j=1}^{n}(g(t),e_j)_{L^2}e_j\Big\|_{L^2}^2\mathrm{d}t\\
&=\mathbb{E}\int_0^T\Big(\sum_{j=1}^{n}|(g(t),e_j)_{L^2}|^2\Big)\mathrm{d}t\\
&\leqslant\int_0^T\mathbb{E}\Big(\sum_{j=1}^{\infty}|(g(t),e_j)_{L^2}|^2\Big)\mathrm{d}t\\
&=\mathbb{E}\int_0^T\|g(t)\|_{L^2}^2\mathrm{d}t\\
&=\|g\|_{L_2^{\mathscr{F}}([0,T],(L^2(\Sigma))^N)}^2,
\end{aligned}
$$

It follows that the first term and third term on the right=hand side of (3.3) are bounded. Then by Gronwall's inequality, we obtain

$$
\mathbb{E}\|u_n(t)\|_{L^2}^2\leqslant C, \tag{3.7}
$$

where $C=C\big(\|u_0\|_{L_2^{\mathscr{F}_0}(\Omega,L^2)},\|g\|_{L_2^{\mathscr{F}}([0,T],L^2)},T,\varepsilon,c_1,c_2\big)$. As T is a fixed positive number,

$$
\mathbb{E}\int_0^T\|u_n(t)\|_{L^2}^2\mathrm{d}t\leqslant C.
$$

By (3.6), we arrive at

$$
\|\nabla u_n\|_{L^{p(t,x)}(\Omega\times\Sigma_T)}\leqslant C,\quad \mathbb{E}\int_0^T\|u_n(t)\|_{L^{2q(x)}}^2\mathrm{d}t\leqslant C.
$$

Hence, $\{u_n\}$ is bounded in $L_\infty^{\mathscr{F}}\left([0,T],(L^2(\Sigma))^N\right)\cap L^{\mathscr{F}}\left(\Omega,X(\Sigma_T)\right)\cap L_2^{\mathscr{F}}\left([0,T],(L^{2q(x)}(\Sigma))^N\right)$. By Eberlein-Smulian theorem and Alaoglu theorem, there exists a subsequence (still denoted by $\{u_n\}$) and a stochastic process u such that

$$
u_n\to u\quad\text{weakly}* \text{ in } L_\infty^{\mathscr{F}}\left([0,T],(L^2(\Sigma))^N\right), \tag{3.8}
$$

$$
u_n\to u\quad\text{weakly in } L^{\mathscr{F}}\left(\Omega,X(\Sigma_T)\right), \tag{3.9}
$$

$$
u_n\to u\quad\text{weakly in } L_2^{\mathscr{F}}\big([0,T],(L^{2q(x)}(\Sigma))^N\big), \tag{3.10}
$$

Step 4: Limit process. We prove u is a weak solution of (1.1) by showing u satisfies (3.2). For any $\varphi\in C^1\left([0,T],(C_0^\infty(\Sigma))^N\right)$ and $\xi\in L^\infty(\Omega)$, from (3.3) we obtain

$$
\begin{aligned}
0&=\mathbb{E}\{\xi(u_n(0),\varphi(0))_{L^2}\}-\mathbb{E}\{\xi(u_n(T),\varphi(T))_{L^2}\}+\mathbb{E}\Big\{\xi\int_0^T\big(u_n(t),\frac{\mathrm{d}\varphi}{\mathrm{d}t}\big)_{L^2}\mathrm{d}t\Big\}\\
&\quad-\mathbb{E}\Big\{\xi\int_0^T\int_\Sigma|\nabla u_n|^{p(t,x)-2}\nabla u_n\nabla\varphi\mathrm{d}x\mathrm{d}t\Big\}\\
&\quad+\mathbb{E}\Big\{\xi\int_0^T\langle f(u_n(t)),\varphi\rangle_{(L^{2q(x)})^*,L^{2q(x)}}\mathrm{d}t\Big\}+\mathbb{E}\Big\{\xi\int_0^T(g_n(t),\varphi)_{L^2}\mathrm{d}t\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+\mathbb{E}\big\{\xi\int_0^T(\varphi,\sigma(t)\mathrm{d}W_n(t))_{L^2}\big\}\\
&=I_1-I_2+I_3-I_4+I_5+I_6+I_7.
\end{aligned}
$$

Next we analyze the limits of $I_1,\dots,I_7$.

(1) Consider I_1. Noting $u_n(0)$ is the n-dimensional truncation of $u(0)$, we obtain

$$
\begin{aligned}
\|u_0-u_n(0)\|_{L^2}^2&=\big\|\sum_{j=n+1}^{\infty}(u_0,e_j)_{L^2}e_j\big\|_{L^2}^2\\
&=\Big(\sum_{j=n+1}^{\infty}|(u_0,e_j)_{L^2}|^2\Big)\\
&\leqslant\Big(\sum_{j=1}^{\infty}|(u_0,e_j)_{L^2}|^2\Big)\\
&=\|u_0\|_{L^2}^2.
\end{aligned}
$$

Since

$$
\|u_n(0)-u(0)\|_{L^2}^2\to 0\quad\text{as } n\to\infty,
$$

by using dominated convergence Theorem, we obtain

$$
\mathbb{E}\|u_n(0)-u(0)\|_{L^2}^2\to 0\quad\text{as } n\to\infty,
$$

that is

$$
u_n(0)\to u_0\quad\text{strongly in } L_2^{\mathscr{F}_0}\left(\Omega,(L^2(\Sigma))^N\right).
$$

Since $\varphi(0)\in(C^1(\Sigma)^N)\subset L_2^{\mathscr{F}_0}\left(\Omega,(L^2(\Sigma))^N\right)$, we derive that

$$
\mathbb{E}\big\{\xi(u_n(0),\varphi(0))_{L^2}\big\}\to\mathbb{E}\big\{\xi(u_0,\varphi(0))_{L^2}\big\}\text{ as } n\to\infty. \tag{3.11}
$$

(2) Similarly, for I_6, we obtain

$$
g_n\to g\quad\text{strongly in } L_2^{\mathscr{F}}\left([0,T],(L^2(\Sigma))^N\right).
$$

In view of $\varphi\in C^1\left([0,T],(C_0^\infty(\Sigma))^N\right)\subset L_2^{\mathscr{F}}([0,T],\left(L^2(\Sigma)\right)^N)$, we obtain

$$
\mathbb{E}\big\{\xi\int_0^T(g_n(t),\varphi(t))_{L^2}\mathrm{d}t\big\}\to\mathbb{E}\big\{\xi\int_0^T(g(t),\varphi(t))_{L^2}\mathrm{d}t\big\}\quad\text{as } n\to\infty. \tag{3.12}
$$

(3) Consider I_3. By

$$
u_n\to u\quad\text{weakly}*\text{ in } L_\infty^{\mathscr{F}}\left([0,T],(L^2(\Sigma))^N\right)
$$

and $\frac{\mathrm{d}\varphi}{\mathrm{d}t}\in C\left([0,T],(C_0^\infty(\Sigma))^N\right)\subset L_1^{\mathscr{F}}\left([0,T],(L^2(\Sigma))^N\right)$, we obtain

$$
\mathbb{E}\big\{\xi\int_0^T\big(u_n(t),\frac{\mathrm{d}\varphi}{\mathrm{d}t}\big)_{L^2}\mathrm{d}t\big\}\to\mathbb{E}\big\{\xi\int_0^T\big(u(t),\frac{\mathrm{d}\varphi}{\mathrm{d}t}\big)_{L^2}\mathrm{d}t\big\} \tag{3.13}
$$

as $n\to\infty$.

(4) Consider I_5. Because

$$
u_n\to u\quad\text{weakly in } L^{\mathscr{F}}\left(\Omega,X(\Sigma_T)\right),
$$

By H(2), we know that

$$
f(u_n)\to f(u)\quad\text{strongly in }\left(L_2^{\mathscr{F}}\left([0,T],(L^{2q(x)}(\Sigma))^N\right)\right)^*.
$$

Since $\varphi \in C^1\left([0,T],(C_0^\infty(\Sigma))^N\right) \subset L_2^{\mathscr{F}}\left([0,T],(L^{2q(x)}(\Sigma))^N\right)$, thus we obtain

$$\mathbb{E}\big\{\xi\int_0^T \langle f(u_n(t)),\varphi\rangle_{(L^{2q(x)})^*,L^{2q(x)}}\mathrm{d}t\big\} \to \mathbb{E}\big\{\xi\int_0^T \langle f(u(t)),\varphi\rangle_{(L^{2q(x)})^*,L^{2q(x)}}\mathrm{d}t\big\} \tag{3.14}$$

as $n\to\infty$.

(5) Consider I_7. According to Theorem 2.5, we know that

$$W_n \to W \quad \text{strongly in } L_2^{\mathscr{F}}\left(\Omega, C([0,T],E)\right).$$

As

$$\mathbb{E}\int_0^T (\varphi(t),\sigma \mathrm{d}W_n(t))_{L^2} = \mathbb{E}\int_0^T \Big(\varphi(t),\sum_{i=1}^n \sqrt{\lambda_i}\sigma(\bar{e}_i)\mathrm{d}\beta_i(t)\Big)_{L^2}$$

and $\varphi \in C^1\left([0,T],(C_0^\infty(\Sigma))^N\right)$ is a deterministic function, by Theorem 2.5(3), we obtain

$$\mathbb{E}\big\{\xi\int_0^T (\varphi(t),\sigma\mathrm{d}W_n(t))\big\} \to \mathbb{E}\big\{\xi\int_0^T (\varphi(t),\sigma\mathrm{d}W(t))\big\} \quad \text{as } n\to\infty. \tag{3.15}$$

(6) Consider I_4 and I_2. Since

$$\mathbb{E}\big\{\int_{\Sigma_T} ||\nabla u_n|^{p(t,x)-2}\nabla u_n|^{p^*(t,x)}\mathrm{d}x\mathrm{d}t\big\} = \mathbb{E}\big\{\int_{\Sigma_T}|\nabla u_n|^{p(t,x)}\mathrm{d}x\mathrm{d}t\big\} \leqslant C,$$

there exists a subsequence (still denoted by $\{u_n\}$) and a stochastic process η, such that

$$|\nabla u_n|^{p(t,x)-2}\nabla u_n \to \eta \quad \text{weakly in } \left(L^{\mathscr{F}}_{L^{p^*(t,x)}}(\Omega\times\Sigma_T)\right)^{d\times N},$$

and further

$$\mathbb{E}\big\{\xi\int_0^T\int_\Sigma |\nabla u_n|^{p(t,x)-2}\nabla u_n\nabla\varphi\mathrm{d}x\mathrm{d}t\big\} \to \mathbb{E}\big\{\xi\int_0^T\int_\Sigma \eta\nabla\varphi\mathrm{d}x\mathrm{d}t\big\} \tag{3.16}$$

as $n\to\infty$. In view of (3.7), we obtain

$$\mathbb{E}\big\{\|u_n(T)\|_{L_2}^2\big\} \leqslant C,$$

therefore there exists a function $\hat{u} \in L_2^{\mathscr{F}_T}\left(\Omega,(L^2(\Sigma))^N\right)$ such that

$$u_n(T) \to \hat{u} \quad \text{weakly in } L_2^{\mathscr{F}_T}\left(\Omega,(L^2(\Sigma))^N\right).$$

Now we prove $u(T)=\hat{u}$. For any $\psi \in (C_0^\infty(\Sigma))^N$ and any $\phi \in \left(C^1[0,T]\right)^N$, we have

$$\begin{aligned}0 = &-(u_n(T),\psi\phi(T))_{L^2} + (u_n(0),\psi\phi(0))_{L^2} + \int_0^T \Big(u_n(t),\psi\frac{\mathrm{d}\phi}{\mathrm{d}t}\Big)_{L^2}\mathrm{d}t \\ &- \int_0^T\int_\Sigma |\nabla u_n|^{p(t,x)-2}\nabla u_n\nabla\phi\psi\mathrm{d}x\mathrm{d}t + \int_0^T \langle f(u_n(t)),\psi\phi\rangle_{(L^{2q(x)})^*,L^{2q(x)})}\mathrm{d}t \\ &+ \int_0^T (g_n(t),\psi\phi)_{L^2}\mathrm{d}t + \int_0^T (\phi\psi,\sigma\mathrm{d}W_n(t))_{L^2}.\end{aligned}$$

Letting $n \to \infty$, we obtain

$$\begin{aligned} 0 = &-(\hat{u}, \psi\phi(T))_{L^2} + (u_0, \psi\phi(0))_{L^2} + \int_0^T \big(u(t), \psi\frac{\mathrm{d}\phi}{\mathrm{d}t}\big)_{L^2}\mathrm{d}t \\ &- \int_0^T \int_\Sigma \eta\nabla\phi\psi\mathrm{d}x\mathrm{d}t + \int_0^T \langle f(u(t)), \psi\phi\rangle_{(L^{2q(x)})^*, L^{2q(x)}}\mathrm{d}t \\ &+ \int_0^T (g(t), \psi\phi)_{L^2}\mathrm{d}t + \int_0^T (\phi\psi, \sigma\mathrm{d}W(t))_{L^2} \end{aligned} \tag{3.17}$$

for any $\phi \in \big(C_0^1([0,T])\big)^N \subset \big(C^1[0,T]\big)^N$ and further

$$\begin{aligned} 0 = &\int_0^T \big(u(t), \psi\frac{\mathrm{d}\phi}{\mathrm{d}t}\big)_{L^2}\mathrm{d}t - \int_0^T \int_\Sigma \eta\nabla\phi\psi\mathrm{d}x\mathrm{d}t \\ &+ \int_0^T \langle f(u(t)), \psi\phi\rangle_{(L^{2q(x)})^*, L^{2q(x)}}\mathrm{d}t + \int_0^T (g(t), \psi\phi)_{L^2}\mathrm{d}t + \int_0^T (\phi\psi, \sigma\mathrm{d}W(t))_{L^2}. \end{aligned}$$

A density argument and the definition of derivatives with respect to time in the distributional sense imply

$$\begin{aligned} 0 = &\int_0^T \Big(u(t), \frac{\mathrm{d}\varphi}{\mathrm{d}t}\Big)_{L^2}\mathrm{d}t - \int_0^T \int_\Sigma \eta\nabla\varphi\mathrm{d}x\mathrm{d}t \\ &+ \int_0^T \langle f(u(t)), \varphi\rangle_{(L^{2q(x)})^*, L^{2q(x)}}\mathrm{d}t + \int_0^T (g(t), \varphi)_{L^2}\mathrm{d}t + \int_0^T (\varphi, \sigma\mathrm{d}W(t))_{L^2}. \end{aligned}$$

For each $\varphi \in (C_0^\infty(\Sigma_T))^N$, $\frac{\mathrm{d}u}{\mathrm{d}t}$ satisfies

$$\begin{aligned} 0 = &-\int_0^T \big(\frac{\mathrm{d}u}{\mathrm{d}t}, \varphi\big)_{L^2}\mathrm{d}t - \int_0^T \int_\Sigma \eta\nabla\varphi\mathrm{d}x\mathrm{d}t \\ &+ \int_0^T \langle f(u(t)), \varphi\rangle_{(L^{2q(x)})^*, L^{2q(x)}}\mathrm{d}t + \int_0^T (g(t), \varphi)_{L^2}\mathrm{d}t \\ &+ \int_0^T (\varphi, \sigma\mathrm{d}W(t))_{L^2}. \end{aligned}$$

Then we obtain

$$\begin{aligned} \int_0^T \Big(\frac{\mathrm{d}u}{\mathrm{d}t}, \varphi\Big)_{L^2}\mathrm{d}t &= \int_0^T \int_\Sigma \operatorname{div}\eta\varphi\mathrm{d}x\mathrm{d}t + \int_0^T \langle f(u(t)), \varphi\rangle_{(L^{2q(x)})^*, L^{2q(x)}}\mathrm{d}t \\ &\quad + \int_0^T (g(t), \varphi)_{L^2}\mathrm{d}t + \int_0^T (\varphi, \sigma\mathrm{d}W(t))_{L^2} \\ &\triangleq \langle S, \varphi\rangle. \end{aligned}$$

Furthermore, for any $\psi \in (C_0^\infty(\Sigma))^N$ and any $\phi \in \big(C^1[0,T]\big)^N$, we obtain

$$\begin{aligned} &-\int_0^T \Big(u(t), \psi\frac{\mathrm{d}\phi}{\mathrm{d}t}\Big)_{L^2}\mathrm{d}t + \int_0^T \int_\Sigma \eta\nabla\phi\psi\mathrm{d}x\mathrm{d}t - \int_0^T \langle f(u(t)), \psi\phi\rangle_{(L^{2q(x)})^*, L^{2q(x)}}\mathrm{d}t \\ &- \int_0^T (g(t), \psi\phi)_{L^2}\mathrm{d}t - \int_0^T (\phi\psi, \sigma\mathrm{d}W(t))_{L^2} \\ &= -\int_0^T \big(u(t), \psi\frac{\mathrm{d}\phi}{\mathrm{d}t}\big)_{L^2}\mathrm{d}t - \langle S, \phi\psi\rangle \end{aligned}$$

$$\begin{aligned}&= -\int_0^T \big(u(t), \psi\frac{\mathrm{d}\phi}{\mathrm{d}t}\big)_{L^2}\mathrm{d}t - \int_0^T \big(\frac{\mathrm{d}u}{\mathrm{d}t}, \psi\phi\big)_{L^2}\mathrm{d}t\\
&= (u(0), \phi\psi(0))_{L^2} - (u(T), \phi\psi(T))_{L^2}.\end{aligned}$$

In view of (3.17), we obtain $u(T) = \hat{u}$ and

$$u_n(T) \to u(T) \quad \text{weakly in} \quad L_2^{\mathscr{F}_T}\left(\Omega, (L^2(\Sigma))^N\right).$$

By the weak lower semi-continuity of the norm, we obtain

$$\liminf_{n\to\infty} \left(\mathbb{E}\|u_n(T)\|_{L^2}^2\right) \leqslant \mathbb{E}\|u(T)\|_{L^2}^2. \tag{3.18}$$

Since $\varphi(T) \in \left(C^1(\Sigma)^N\right) \subset L_2^{\mathscr{F}_T}\left(\Omega, (L^2(\Sigma))^N\right)$,

$$\mathbb{E}\big\{\xi(u_n(T), \varphi(T))_{L^2}\big\} \to \mathbb{E}\big\{\xi(u(T), \varphi(T))_{L^2}\big\} \text{ as } n \to \infty. \tag{3.19}$$

Combining (3.11), (3.13), (3.14), (3.12), (3.15), (3.16), and (3.19), we have

$$\begin{aligned}0 &= (u_0, \varphi(0))_{L^2} - (u(T), \varphi(T))_{L^2} + \int_0^T \big(u(t), \frac{\mathrm{d}\varphi}{\mathrm{d}t}\big)_{L^2}\mathrm{d}t\\
&\quad - \int_0^T\int_\Sigma \eta\nabla\varphi \mathrm{d}x\mathrm{d}t + \int_0^T \langle f(u(t)), \varphi\rangle_{(L^{2q(x)})^*, L^{2q(x)}}\mathrm{d}t\\
&\quad + \int_0^T (g_n(t), \varphi)_{L^2}\mathrm{d}t + \int_0^T (\varphi, \sigma(t)\mathrm{d}W_n(t))_{L^2}.\end{aligned} \tag{3.20}$$

Next we prove that $\eta = |\nabla u|^{p(t,x)-2}\nabla u$. By (3.20), we know that u is a weak solution of the problem

$$\begin{gathered}\mathrm{d}u - \operatorname{div}\eta\mathrm{d}t = f(u)\mathrm{d}t + g(t)\mathrm{d}t + \sigma\mathrm{d}W, \quad (t,x) \in [0,T] \times \Sigma,\\
u(t,x) = 0, \quad (t,x) \in [0,T] \times \partial\Sigma,\\
u(0,x) = u_0, \quad x \in \Sigma.\end{gathered} \tag{3.21}$$

Integrating by parts, on (3.21) we obtain

$$\begin{aligned}0 &= \frac{1}{2}\|u_0\|_{L^2}^2 - \frac{1}{2}\|u(T)\|_{L^2}^2 - \int_0^T\int_\Sigma \eta\nabla u\mathrm{d}x\mathrm{d}t\\
&\quad + \int_0^T \langle f(u(t)), u\rangle_{(L^{2q(x)})^*, L^{2q(x)}}\mathrm{d}t + \int_0^T (g(t), u)_{L^2}\mathrm{d}t\\
&\quad + \int_0^T (u, \sigma(t)\mathrm{d}W(t))_{L^2}.\end{aligned} \tag{3.22}$$

From

$$\begin{aligned}0 &\leqslant \mathbb{E}\Big\{\int_{\Sigma_T} \big(|\nabla u_n|^{p(t,x)-2}\nabla u_n - |\nabla u|^{p(t,x)-2}\nabla u\big)(\nabla u_n - \nabla u)\mathrm{d}x\mathrm{d}t\Big\}\\
&= \frac{1}{2}\mathbb{E}\big\{\|u_n(0)\|_{L^2}^2\big\} - \frac{1}{2}\mathbb{E}\big\{\|u_n(T)\|_{L^2}^2\big\} + \mathbb{E}\Big\{\int_0^T \langle f(u_n(t)), u_n(t)\rangle_{(L^{2q(x)})^*, L^{2q(x)}}\mathrm{d}t\Big\}\\
&\quad + \mathbb{E}\Big\{\int_0^T (g_n(t), u_n(t))_{L^2}\,\mathrm{d}t\Big\} + \mathbb{E}\Big\{\int_0^T (u_n(t), \sigma(t)\mathrm{d}W_n(t))_{L^2}\Big\}\\
&\quad + \mathbb{E}\Big\{\int_{\Sigma_T} |\nabla u_n|^{p(t,x)-2}\nabla u_n\nabla u - |\nabla u|^{p(t,x)-2}\nabla u(\nabla u_n - \nabla u)\mathrm{d}x\mathrm{d}t\Big\},\end{aligned}$$

and (3.16) and (3.20), we have

$$\begin{aligned}0 &\leqslant \frac{1}{2}\mathbb{E}\big\{\|u_0\|_{L^2}^2\big\} - \frac{1}{2}\mathbb{E}\big\{\|u(T)\|_{L^2}^2\big\} + \mathbb{E}\big\{\int_0^T \langle f(u(t)), u(t)\rangle_{(L^{2q(x)})^*, L^{2q(x)}}\mathrm{d}t\big\}\\ &\quad + \mathbb{E}\big\{\int_0^T (g(t), u(t))_{L^2}\,\mathrm{d}t\big\} + \mathbb{E}\big\{\int_0^T (u(t), \sigma(t)\mathrm{d}W(t))_{L^2}\big\}\\ &\quad + \mathbb{E}\big\{\int_{\Sigma_T} \eta\nabla u\mathrm{d}x\mathrm{d}t\big\} = 0.\end{aligned}$$

Furthermore,

$$\mathbb{E}\big\{\int_{\Sigma_T}\Big(|\nabla u_n|^{p(t,x)-2}\nabla u_n - |\nabla u|^{p(t,x)-2}\nabla u\Big)(\nabla u_n - \nabla u)\,\mathrm{d}x\mathrm{d}t\big\} \to 0$$

as $n \to +\infty$,

$$\begin{aligned}&\mathbb{E}\big\{\int_{\Sigma_T} |\nabla u_n - \nabla u|^{p(t,x)}\mathrm{d}x\mathrm{d}t\big\}\\ &\leqslant C\mathbb{E}\big\{\int_{\Sigma_T}\Big(|\nabla u_n|^{p(t,x)-2}\nabla u_n - |\nabla u|^{p(t,x)-2}\nabla u\Big)(\nabla u_n - \nabla u)\,\mathrm{d}x\mathrm{d}t\big\}\\ &\to 0 \quad \text{as } n \to +\infty.\end{aligned}$$

Therefore,

$$\nabla u_n \to \nabla u \quad \text{strongly in } \big(L^{\mathscr{F}}_{L^{p(t,x)}}(\Omega \times \Sigma_T)\big)^{d\times N}.$$

Thus there exists a subsequence (still denoted by $\{u_n\}$) such that

$$\nabla u_n \to \nabla u, \quad \text{a.e.}(\omega, t, x) \in \Omega \times \Sigma_T.$$

Furthermore,

$$|\nabla u_n|^{p(t,x)-2}\nabla u_n \to |\nabla u|^{p(t,x)-2}\nabla u, \quad \text{a.e. } (\omega, t, x) \in \Omega \times \Sigma_T$$

By Theorem 2.5, we obtain $\eta = |\nabla u|^{p(t,x)-2}\nabla u$.

In summary, for each $\varphi \in C^1\big([0,T], (C_0^\infty(\Sigma))^N\big)$, we obtain

$$\begin{aligned}&(u(T)), \varphi(T))_{L^2} - (u_0, \varphi(0))_{L^2} - \int_0^T \big(u, \frac{\mathrm{d}\varphi}{\mathrm{d}t}\big)_{L^2}\mathrm{d}t\\ &+ \int_0^T\int_\Sigma |\nabla u|^{p(t,x)-2}\nabla u\nabla\varphi\mathrm{d}x\mathrm{d}t\\ &= \int_0^T \langle f(u), \varphi\rangle_{(L^{2q(x)})^*, L^{2q(x)}}\mathrm{d}t + \int_0^T (g(t), \varphi)_{L^2}\mathrm{d}t + \int_0^T (\varphi, \sigma(t)\mathrm{d}W(t))_{L^2},\end{aligned}$$

i.e., u is a solution of (1.1). □

4. Existence of optimal controls

For a real Hilbert space V, the set $\mathscr{V} = L^{\mathscr{F}}_\infty([0,T], V)$ is the control function space, and the linear operator $\mathcal{A} \in \mathscr{L}\big(\mathscr{V}, L_2^{\mathscr{F}}\big([0,T], (L^2(\Sigma))^N\big)\big)$ is the control item. We consider the stochastic control problem

$$\begin{gathered}\mathrm{d}u - \operatorname{div}\Big(|\nabla u|^{p(t,x)-2}\nabla u\Big)\mathrm{d}t = f(u)\mathrm{d}t + g(t)\mathrm{d}t + \mathcal{A}v\mathrm{d}t + \sigma\mathrm{d}W, \quad t \in (0,T],\\ u(0,x) = u_0.\end{gathered} \tag{4.1}$$

Define the solution map as follows:

$$\Phi : v \to u(v)$$

$$L^{\mathscr{F}}_{\infty}([0,T],V) \to L^{\mathscr{F}}_{\infty}\left([0,T],(L^2(\Sigma))^N\right) \cap L^{\mathscr{F}}(\Omega, X(\Sigma_T)) \cap L^{\mathscr{F}}_2\left([0,T],(L^{2q(x)}(\Sigma))^N\right),$$

where $u(v)$ is the solution of (4.1) called the state of the control problem (4.1). The observed state is denoted by $\mathcal{H}(u(v))$ where

$$\mathcal{H} \in \mathscr{L}\left(L^{\mathscr{F}}(\Omega, X(\Sigma_T)), L^{\mathscr{F}}_2\left([0,T],(L^2(\Sigma))^N\right)\right)$$

is a linear operator. A fixed stochastic process $\mu_{\rm d} \in L^{\mathscr{F}}_2\left([0,T],(L^2(\Sigma))^N\right)$ is called the desired state. The cost function is defined as

$$J(v) = \mathbb{E}\Big\{\int_0^T \|\mathcal{H}u(v) - \mu_{\rm d}\|^2_{L^2}{\rm d}t + (\mathcal{K}v, v)_V\Big\}, \tag{4.2}$$

where the operator $\mathcal{K} \in \mathscr{L}(V,V)$ satisfies

$$(\mathcal{K}v(t), v(t))_V = (v(t), \mathcal{K}v(t))_V \geqslant k\|v(t)\|^2_V$$

for $k \in [0,+\infty)$. Let $V_{\rm ad} \subset \mathscr{V}$ be an admissible set. We call $v_0 \in V_{\rm ad}$ the optimal control if

$$J(v_0) = \min_{v \in V_{\rm ad}} J(v).$$

Thus, we have the following result.

Theorem 4.1. *Let the assumptions in Theorem 3.2 be satisfied and let $V_{\rm ad}$ is a compact subset of V. Then stochastic control problem* (4.1) *with cost function* (4.2) *has at least one optimal control $v_0 \in V_{\rm ad}$.*

Proof. Since $V_{\rm ad}$ is compact, we need only to prove that Φ is continuous and J is lower semi-continuous. Let $\{v_k\} \in V_{\rm ad}$ and

$$v_k \to \bar{v} \text{ in } V_{\rm ad}.$$

Step 1: Φ is continuous. Suppose that $\{u_k\}$ and $\bar{u}$ are weak solutions of (4.1). Then $u_k - \bar{u}$ satisfies

$$\begin{aligned} &{\rm d}u_k - {\rm d}\bar{u} + {\rm div}\left(|\nabla \bar{u}|^{p(t,x)-2}\nabla \bar{u}\right){\rm d}t - {\rm div}\left(|\nabla u_k|^{p(t,x)-2}\nabla u_k\right){\rm d}t \\ &= f(u_k){\rm d}t - f(\bar{u}){\rm d}t + \mathcal{A}v_k{\rm d}t - \mathcal{A}\bar{v}{\rm d}t \end{aligned}$$

in the weak sense. After integrating by parts, we obtain

$$\begin{aligned} &\frac{1}{2}\mathbb{E}\|u_k - \bar{u}\|^2_{L^2} + \mathbb{E}\int_0^t \int_\Sigma \left(|\nabla u|^{p(t,x)-2}\nabla u - |\nabla \bar{u}|^{p(t,x)-2}\nabla \bar{u}\right)(\nabla u - \nabla \bar{u})\,{\rm d}x{\rm d}s \\ &= \mathbb{E}\int_0^t \langle f(u_k) - f(\bar{u}), u_k - \bar{u}\rangle_{(L^{2q(x)})^*, L^{2q(x)}}{\rm d}s + \mathbb{E}\int_0^t (\mathcal{A}v_k - \mathcal{A}\bar{v}, u_k - \bar{u})_{L^2}{\rm d}s \end{aligned}$$

By (H3) and Hölder's inequality, we have

$$\begin{aligned} &\frac{1}{2}\mathbb{E}\|u_k - \bar{u}\|^2_{L^2} + \mathbb{E}\int_0^t \int_\Sigma \left(|\nabla u|^{p(s,x)-2}\nabla u - |\nabla \bar{u}|^{p(s,x)-2}\nabla \bar{u}\right)(\nabla u - \nabla \bar{u})\,{\rm d}x{\rm d}s \\ &\leqslant \Big(\mathbb{E}\int_0^t \|\mathcal{A}v_k - \mathcal{A}\bar{v}\|^2_{L^2}{\rm d}s\Big)^{1/2}\Big(\mathbb{E}\int_0^t \|u_k - \bar{u}\|^2_{L^2}{\rm d}s\Big)^{1/2}. \end{aligned}$$

Since $\{u_k\}$ is bounded in $L^{\mathscr{F}}_{\infty}\left([0,T],(L^2(\Sigma))^N\right)$, there exists a constant $M>0$, such that

$$\begin{gathered}\max\big\{\|\bar{u}\|_{L^{\mathscr{F}}_{\infty}([0,T],(L^2(\Sigma))^N)},\ \|u_1\|_{L^{\mathscr{F}}_{\infty}([0,T],(L^2(\Sigma))^N)},\\ \dots,\|u_k\|_{L^{\mathscr{F}}_{\infty}([0,T],(L^2(\Sigma))^N)},\dots\big\}\leqslant M.\end{gathered}$$

Furthermore,

$$\mathbb{E}\Big(\int_0^t\|u_k-\bar{u}\|^2_{L^2}\mathrm{d}s\Big)\leqslant C.$$

Hence we have

$$\begin{aligned}&\frac{1}{2}\mathbb{E}\|u_k-\bar{u}\|^2_{L^2}+\mathbb{E}\int_0^t\int_\Sigma\Big(|\nabla u|^{p(t,x)-2}\nabla u-|\nabla\bar{u}|^{p(t,x)-2}\nabla\bar{u}\Big)\left(\nabla u-\nabla\bar{u}\right)\mathrm{d}x\mathrm{d}s\\ &\leqslant C\Big(\mathbb{E}\int_0^t\|\mathcal{A}v_k-\mathcal{A}\bar{v}\|^2_{L^2}\mathrm{d}s\Big)^{1/2}.\end{aligned}$$

Taking the limit, we obtain

$$\begin{aligned}&\lim_{k\to\infty}\sup_{t\in[0,T]}\big\{\frac{1}{2}\mathbb{E}\|u_k-\bar{u}\|^2_{L^2}\\ &+\mathbb{E}\int_0^t\int_\Sigma\Big(|\nabla u|^{p(t,x)-2}\nabla u-|\nabla\bar{u}|^{p(t,x)-2}\nabla\bar{u}\Big)\left(\nabla u-\nabla\bar{u}\right)\mathrm{d}x\mathrm{d}s\big\}=0\end{aligned}$$

which implies

$$u_k\to\bar{u}\quad\text{strongly in }L^{\mathscr{F}}_{\infty}\left([0,T],(L^2(\Sigma))^N\right)\cap L^{\mathscr{F}}(\Omega,X(\Sigma_T)).$$

Step 2: J is lower semi-continuous. We deenote

$$J(v)=\mathbb{E}\big\{\int_0^T\|\mathcal{H}u(v)-\mu_{\mathrm{d}}(t)\|^2_{L^2}\mathrm{d}t\big\}+\mathbb{E}\big\{(\mathcal{K}v(t),v(t))_V\big\}\triangleq J_1(v)+J_2(v).$$

As

$$\begin{aligned}J_1(\bar{v})=\mathbb{E}\big\{&\int_0^T\|\mathcal{H}\bar{u}(v)-\mu_{\mathrm{d}}(t)\|^2_{L^2}-\|\mathcal{H}u_k(v)-\mu_{\mathrm{d}}(t)\|^2_{L^2}\mathrm{d}t\big\}\\ &+\mathbb{E}\big\{\int_0^T\|\mathcal{H}u_k(v)-\mu_{\mathrm{d}}(t)\|^2_{L^2}\mathrm{d}t\big\}\end{aligned}$$

and $\mathcal{H}\in\mathscr{L}\left(L^{\mathscr{F}}(\Omega,X(\Sigma_T)),(L^2(\Sigma))^N\right)$, for any $\varepsilon>0$, there exists $N_\varepsilon\in\mathbb{N}$ such that

$$\big|\|\mathcal{H}\bar{u}(v)-\mu_{\mathrm{d}}(t)\|^2_{L^2}-\|\mathcal{H}u_k(v)-\mu_{\mathrm{d}}(t)\|^2_{L^2}\big|<\varepsilon$$

whenever $k>N_\varepsilon$. Furthermore,

$$J_1(\bar{v})\leqslant T\varepsilon+\mathbb{E}\big\{\int_0^T\|\mathcal{H}u_k(v)-\mu_{\mathrm{d}}(t)\|^2_{L^2}\mathrm{d}t\big\}=T\varepsilon+J_1(v_k).$$

So we arrive at

$$J_1(\bar{v})\leqslant\liminf_{k\to\infty}J_1(v_k)$$

by the arbitrariness of ε.

From the convergence $v_k\to\bar{v}$ in $\mathscr{V}$, we derive that

$$v_k(t)\to\bar{v}(t)\quad\text{a.e. }t\in[0,T]$$

so $\{v_k(t)\}$ is bounded in V, which implies

$$(\mathcal{K}v_k(t), v_k(t))_V \leqslant \|\mathcal{K}\| \, \|v_k(t)\|_V^2 < +\infty.$$

By Fatou's Lemma, we obtain

$$\begin{aligned}\liminf_{k\to\infty} \left\{\mathbb{E}\left(\mathcal{K}v_k(t), v_k(t)\right)_V\right\} &\leqslant \left\{\mathbb{E}\liminf_{k\to\infty} \left(\mathcal{K}v_k(t), v_k(t)\right)_V\right\} \\ &= \left\{\mathbb{E}\liminf_{k\to\infty} \left(\mathcal{K}\bar{v}(t), \bar{v}(t)\right)_V\right\}.\end{aligned}$$

Furthermore,

$$J_2(\bar{v}) \leqslant \liminf_{k\to\infty} J_2(v_k).$$

At last $J(\bar{v}) \leqslant \liminf_{k\to\infty} J(v_k)$. □

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Chen Liang
School of Mathematics, Harbin Institute of Technology, Harbin, 150001, Heilongjiang, China
Email address: `liangchen3515@163.com`

Lixu Yan
Department of Mathematics, Northeast Forestry University, Harbin, 150040, Heilongjiang, China
Email address: `yanlxmath@163.com`

Yongqiang Fu (corresponding author)
School of Mathematics, Harbin Institute of Technology, Harbin, 150001, Heilongjiang, China
Email address: fuyongqiang@hit.edu.cn