

SOLUTIONS OF LINEAR AND NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS BY MEANS OF TENSOR PRODUCT THEORY OF BANACH SPACES

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ABSTRACT. In this article, we introduce an analytical method for solving both non-separable linear and non-linear partial differential equations, for which separation of variables method does not work. This method is based on the theory of tensor product in Banach spaces coupled with some properties of atoms operators. We provide some illustrative examples.

1. INTRODUCTION

Converting a real-world problem into ordinary or partial differential equations is the essential task for mathematical modeling [4, 5, 7]. Differential equations, ordinary or partial, are equations relating unknown functions and some of their derivatives. There are plenty of examples of their use in economics, physics, chemistry, and biology. In general, to find analytical or numerical solutions of partial differential equations is not straightforward. There are no generally applicable methods or approaches to solve all partial differential equations of a given order, even the classes for which we have general analytic methods of solution are quite limited [8]. Thus, we have to study fairly small classes of partial differential equations individually.

One of the most popular methods for solving specific types of partial differential equations is the method of separation of variables. This method is based on the idea that the solution of the equation is separable, that is, the final solution can be represented as the product of several functions, each of which only depends on one independent variable. However, the procedure of separation of variables does have some limitations, it works in very special cases involving equations with high degree of symmetry. Also, to be able to apply this method, many constraints need to be imposed on the coefficients expressions as well as the companion initial and boundary conditions. Some other analytical and numerical techniques of solving partial differential equations are Fourier transform, Laplace transform, Green's functions,

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finite element method (FEM), finite volume methods (FVM), and finite difference methods (FDM)[12].

In 2010, Khalil [10] introduced a novel approach to handle differential equations for both ordinary and fractional orders. The new technique is based on the theory of tensor product of Banach spaces, and it can be utilized for obtaining the so-called atomic solutions of the differential equation [1, 2, 3]. Before we introduce our main result for atomic solutions of partial differential equations with ordinary orders, we commence with some required materials that are reported in [6, 11, 13, 14, 15].

2. ATOMS OPERATORS

In this section, we introduce some preliminaries related to the main result.

Definition 2.1. Let X and Y be two Banach spaces and X^* is the dual space of X . For $x \in X$ and $y \in Y$, the operator $T : X^* \rightarrow Y$, defined by $T(x^*) = x^*(x)y = \langle x, x^* \rangle y$, is bounded one rank linear operator since the range of T is the span of y . We write $x \otimes y$ for T and such operators are called atoms.

For example, let X and Y be two Banach spaces such that $X = Y = C[0, 1]$, with the norm $\|f\| = \sup_{t \in [0, 1]} |f(t)|$ for any $f \in C[0, 1]$. Define X^* to be the space of

all regular Borel measures on $[0, 1]$. So, if $\mu \in X^*$, then $\mu(f) = \int_0^1 f d\mu$. Now, let $f \in X$, $g \in Y$, and $\mu \in X^*$. Then $(f \otimes g)\mu = \mu(f)g = g \int_0^1 f(t) d\mu(t)$. Further, $f \otimes g$ is a bounded linear operator. Also $\|(f \otimes g)(\mu)\| = |\mu(f)|\|g\| \leq \|\mu\|\|f\|\|g\|$ for all $\mu \in X^*$.

Atoms are used in theory of best approximation in Banach spaces and they are considered among the fundamental ingredients in the theory of tensor product. One of the known results that we need in our paper can be presented in theorem [9] below, which guarantees that if the sum of two atoms is an atom, then either the first components are dependent or the second ones are dependent.

Theorem 2.2 ([9]). *Let $x_1 \otimes y_1$ and $x_2 \otimes y_2$ be two nonzero atoms in $X \otimes Y$ such that $x_1 \otimes y_1 + x_2 \otimes y_2 = x_3 \otimes y_3$. Then either $x_1 = x_2 = x_3$ or $y_1 = y_2 = y_3$.*

This leads us to the following interesting theorem that lies at the heart of functional analysis as well as approximation theory and guarantee that any continuous function of several variables can be written as a sum of products of continuous separated functions [6].

Theorem 2.3 ([6]). *Let I, J be two compact intervals, and $C(I), C(J)$, and $C(I \times J)$ be the spaces of continuous functions on I, J , and $I \times J$, respectively. Then every $f \in C(I \times J)$ can be written in the form $f(x, y) = \sum_{i=1}^{\infty} u_i(x)v_i(y)$, where $u_i(x) \in C(I)$ and $v_i(y) \in C(J)$.*

3. GENERAL SCHEME FOR THE ATOMIC SOLUTION METHOD

Consider the second order non-linear partial differential equation

$$u_{xy}(x, y) + u(x, y)u_y(x, y) = u_x(x, y) + u^2(x, y), \quad (3.1)$$

where $u(x, y)$ is an unknown function and subjected to the following conditions:

$$u(0, 0) = 1, \quad u_x(0, 0) = 1, \quad u_y(0, 0) = 1. \quad (3.2)$$

Clearly, equation (3.1) is non-separable as we cannot move the x -terms to one side and the y -terms to the other. Hence, the method of separation of variables does not work. From (3.2), we can assume without loss of generality that

$$P(0) = Q(0) = P'(0) = Q'(0) = 1. \quad (3.3)$$

According to Theorem 2.3, we start our approach by assuming that

$$u(x, y) = P(x)Q(y). \quad (3.4)$$

Now, we substitute (3.4) into the main partial differential equation (3.1). Hence,

$$P'(x)Q'(y) + P(x)Q(y)P(x)Q'(y) = P'(x)Q(y) + P^2(x)Q^2(y). \quad (3.5)$$

Equivalently, (3.5) can be written as

$$P'(x)Q'(y) + P^2(x)Q(y)[Q'(y) - Q(y)] = P'(x)Q(y) \quad (3.6)$$

Clearly, each term of (3.6) is just a product of two functions one of them only in x and the other one only in y . Therefore, in tensor product form, (3.6) can be written as

$$P'(x) \otimes Q'(y) + P^2(x) \otimes [Q(y)Q'(y) - Q^2(y)] = P'(x) \otimes Q(y). \quad (3.7)$$

This implies that the sum of two atoms is an atom. By Theorem 2.2, we have the following two cases:

- (i) $P'(x) = P^2(x) = P'(x)$,
- (ii) $Q'(y) = [Q(y)Q'(y) - Q^2(y)] = Q(y)$.

Case (i): This case has only the situation $P'(x) = P^2(x)$ which can be solved for $P(x)$, by taking into account (3.3), as

$$P(x) = \frac{1}{1-x}. \quad (3.8)$$

Our next step is to substitute (3.8) into (3.5) taking into account that $P'(x) = P^2(x)$. This yields

$$Q'(y) + [Q(y)Q'(y) - Q^2(y)] = Q(y), \quad (3.9)$$

or equivalently,

$$(Q'(y) - Q(y))(1 + Q(y)) = 0. \quad (3.10)$$

So, by referring to (3.3), case (i) gives two values for $Q(y)$, namely, $Q(y) = -1$ and $Q(y) = e^y$. Hence, by taking into account these two values of $Q(y)$ together with both (3.4) and (3.8), case (i) generates two atomic solutions:

$$u_{11}(x, y) = \frac{1}{x-1}, \quad u_{12}(x, y) = \frac{e^y}{1-x}. \quad (3.11)$$

Case (ii): This case has the following three situations:

- (a) $Q'(y) = Q(y)Q'(y) - Q^2(y)$,
- (b) $Q'(y) = Q(y)$,
- (c) $Q(y) = Q(y)Q'(y) - Q^2(y)$.

Again by (3.3) the last two situations (b) and (c) yield, respectively, $Q(y) = e^y$ and $Q(y) = 2e^y - 1$. Therefore, according to Theorem 2.2 there is no atomic solution for case (ii) and hence, this case does not admit an atomic solution.

The two atomic solutions $u_{11}(x, y)$ and $u_{12}(x, y)$ in (3.11) of problem (3.1) are displayed, in Figures 1 and 2.

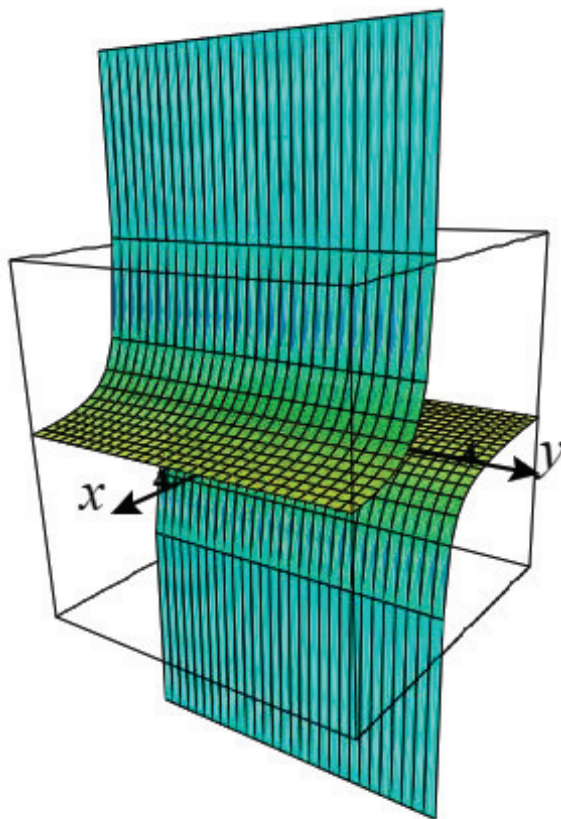


FIGURE 1. First atomic solution $u_{11}(x, y)$ of problem (3.1).

4. APPLICATIONS

In this section, we provide two examples for solving non-separable linear partial differential equations utilizing the atomic method.

Example 4.1. Consider the initial value problem

$$f(x)u_x + u_y = g(y)u, \quad (4.1)$$

where $u(x, y)$ is the unknown function, and $f(x)$ and $g(y)$ are given functions.

By substituting $u(x, y) = P(x)Q(y)$ into (4.1) we obtain

$$f(x)P'(x)Q(y) + P(x)Q'(y) = g(y)P(x)Q(y). \quad (4.2)$$

Therefore, in tensor product form, (4.2) becomes

$$f(x)P'(x) \otimes Q(y) + P(x) \otimes Q'(y) = P(x) \otimes g(y)Q(y), \quad (4.3)$$

By Theorem 2.2, we have the following two cases:

- (i) $f(x)P'(x) = P(x) = P(x)$,
- (ii) $Q(y) = Q'(y) = g(y)Q(y)$.

For case (i) we have only one situation, $f(x)P'(x) = P(x)$ which yields

$$P(x) = c_1 e^{\int 1/f(x) dx} \quad (4.4)$$

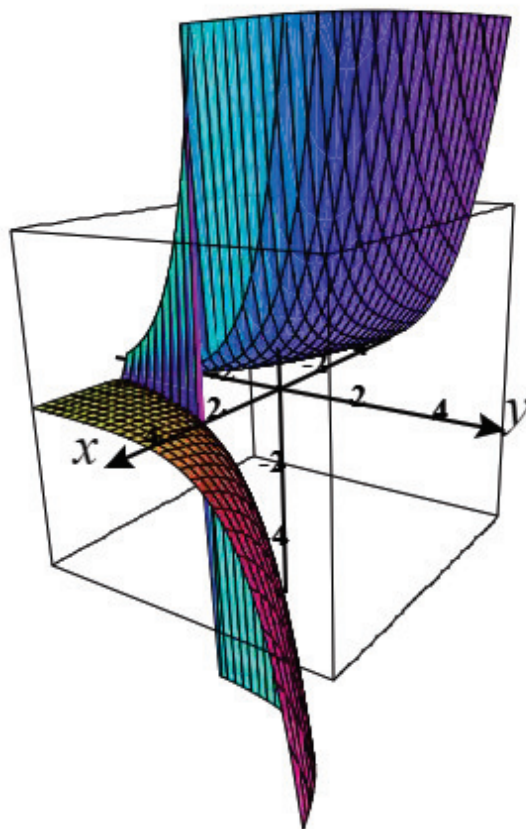


FIGURE 2. Second atomic solution $u_{12}(x, y)$ of problem (3.1).

where c_1 is a constant. Therefore, an atomic solution exists for case (i), and it can be obtained by substituting (4.4) into (4.2),

$$Q'(y) + [1 - g(y)]Q(y) = 0. \quad (4.5)$$

This is a linear ordinary differential equation which has solution

$$Q(y) = c_2 e^{\int (g(y)-1) dy}. \quad (4.6)$$

Hence, referring to (4.4) and (4.6), the first atomic solution for case (i) is

$$u_1(x, y) = c_* e^{\int 1/f(x) dx + \int (g(y)-1) dy}, \quad (4.7)$$

where $c_* = c_1 c_2$.

For case (ii), we have three situations:

- (a) $Q(y) = Q'(y)$,
- (b) $Q(y) = g(y)Q(y)$,
- (c) $Q'(y) = g(y)Q(y)$.

According to Theorem 2.2, to obtain an atomic solution for case (ii), we require

$$Q(y) = Q'(y) = c_3 e^y \quad \text{and} \quad g(y) = 1, \quad (4.8)$$

where c_3 is a constant. Now, on substituting (4.8) into (4.2), we have

$$f(x)P'(x) = 0. \quad (4.9)$$

Therefore, $P(x) = c_4$ (a constant) and hence, the second atomic solution associated with case (ii), can be achieved by compiling both the value of $P(x)$ from (4.9) and the obtained value of $Q(x)$ from (4.8) as follows

$$u_2(x, y) = c_{**}e^y, \quad (4.10)$$

where $c_{**} = c_3c_4$.

Example 4.2. Consider the partial differential equation

$$u_{xxy} + u_{xyy} = yu, \quad (4.11)$$

where $u(x, y)$ is the unknown function. The following conditions are imposed on u :

$$u(0, 0) = 1, \quad u_x(0, 0) = 1, \quad u_y(0, 0) = 1. \quad (4.12)$$

By substituting $u(x, y) = P(x)Q(y)$ into (4.11) we obtain

$$P''(x)Q'(y) + P'(x)Q''(y) = yP(x)Q(y). \quad (4.13)$$

Therefore, in tensor product form, (4.13) becomes

$$P''(x) \otimes Q'(y) + P'(x) \otimes Q''(y) = P(x) \otimes yQ(y). \quad (4.14)$$

Using Theorem 2.2, we have one of the following two cases:

- (i) $P''(x) = P'(x) = P(x)$,
- (ii) $Q''(y) = Q'(y) = yQ(y)$.

For case (i), we have the following three situations:

- (a) $P''(x) = P'(x)$,
- (b) $P''(x) = P(x)$,
- (c) $P'(x) = P(x)$.

From (4.12), we can assume, without loss of generality, that

$$P(0) = Q(0) = P'(x) = Q'(y) = 1. \quad (4.15)$$

Thus, all three situations (a), (b), and (c) yield the same result, that is

$$P(x) = e^x. \quad (4.16)$$

By substituting (4.16) into (4.13) we obtain $Q''(y) + Q'(y) - yQ(y) = 0$ which is a linear ordinary differential equation with variable coefficients and can be solved by implementing the power series method such that $Q(y) = \sum_{n=0}^{\infty} k_n y^n$ as follows:

$$\begin{aligned} Q(y) &= k_0 \left[1 + \frac{y^3}{2 \cdot 3} - \frac{y^4}{2 \cdot 3 \cdot 4} + \frac{y^5}{2 \cdot 3 \cdot 4 \cdot 5} + \dots \right] \\ &\quad + k_1 \left[y - \frac{y^2}{2} + \frac{y^3}{2 \cdot 3} + \frac{y^4}{2 \cdot 3 \cdot 4} - \dots \right] \\ &=: k_0 Q_1 + k_1 Q_2. \end{aligned} \quad (4.17)$$

Hence, from (4.16) and (4.17), the first atomic solution with respect to case (i) is

$$u_1(x, y) = e^x(k_0 Q_1 + k_1 Q_2), \quad (4.18)$$

where Q_1 and Q_2 are given in (4.17).

For case (ii), we have three situations,

- (a) $Q''(y) = Q'(y)$,
- (b) $Q''(y) = yQ(y)$,

$$(c) \quad Q'(y) = yQ(y).$$

By assuming both situations (a) and (b), we have $Q'(y) = yQ(y)$ which together with conditions (4.15) yields, $Q(y) = e^{y^2/2}$. But situation (a) gives $Q(y) = e^y$. Therefore, according to Theorem 2.2 there is no atomic solution for case (ii) and hence, this case does not admit an atomic solution. Therefore, (4.11) has only one atomic solution, that is (4.18).

5. CONCLUSIONS

This article has introduced a new analytical method for handling non-separable, linear and non-linear partial differential equations via atomic solutions method. The theory of tensor product of Banach spaces coupled with some properties of atoms operators have been utilized for achieving such a notion. For atomic solution method, we emphasize the following points:

1. In most cases, the atomic solution approach can provide an exact solutions to non-separable, non-homogeneous, and non-linear partial differential equations when the method of separation of variables does not work.
2. It is not necessary that each case reported in Theorem 2.3 admits an atomic solution. This means that the three situations of each case have to provide the same result; otherwise, there is no atomic solution.

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