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NODAL SOLUTIONS FOR NONLINEAR SCHRÖDINGER SYSTEMS

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ABSTRACT. In this article we consider the nonlinear Schrödinger system

$$\begin{gathered} -\Delta u_j + \lambda_j u_j = \sum_{i=1}^{k} \beta_{ij} u_i^2 u_j, \quad \text{in } \Omega, \\ u_j(x) = 0, \quad \text{on } \partial\Omega,\ j = 1, \dots, k, \end{gathered}$$

where $\Omega \subset \mathbb{R}^N$ $(N = 2, 3)$ is a bounded smooth domain, $\lambda_j > 0$, $j = 1, \dots, k$, β_{ij} are constants satisfying $\beta_{jj} > 0$, $\beta_{ij} = \beta_{ji} \le 0$ for $1 \le i < j \le k$. The existence of sign-changing solutions is proved by the truncation method and the invariant sets of descending flow method.

1. Introduction

We consider the nonlinear Schödinger system

$$\begin{gathered} -\Delta u_j + \lambda_j u_j = \sum_{i=1}^{k} \beta_{ij} u_i^2 u_j, \quad \text{in } \Omega, \\ u_j(x) = 0, \quad \text{on } \partial\Omega,\ j = 1, \dots, k, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N = 2, 3)$ is a bounded domain with smooth boundary, and $\lambda_j > 0$, $\beta_{jj} > 0$, $1 \le j \le k$, $\beta_{ij} = \beta_{ji}$, $1 \le i < j \le k$ are constants.

This type of coupled systems, also known as Gross-Pitaevskii equations, have applications in many physical problems such as nonlinear optics and multispecies Bose-Einstein condensates [8, 18]. Physically, β_{jj}, β_{ij} $(i \ne j)$ are the intraspecies and interspecies scattering lengths respectively. In the physics literature, the signs of the coupling constants β_{ij} being positive or negative determine the nature of the system being attractive or repulsive. In the repulsive case ($\beta_{ij} < 0$, $i \ne j$, $i, j = 1, \dots, k$), the components tend to segregate with each other leading to phase separations. These phenomena have been documented in experiments as well as in numeric simulations; see [4, 17] and references therein. Mathematical work has been done extensively in recent years, refer the reader to [1, 3, 7, 9, 14, 15, 16, 19] for the existence theory and the studies of qualitative property of solutions to attractive and repulsive systems.

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Over the years there have been systematic studies on nodal solutions for scalar equations by using a combination of minimax methods and the method of invariant sets of gradient flows. We refer the reader to [2, 6, 13]. However, most of the methods in treating scalar equations are not applicable directly to systems. In [14, 15] a construction of invariant sets has been developed to locate multiple nontrivial solutions, but without giving any information about nodal property of the components of solutions. Compared with scalar equations, there are many new challenges for coupled equations in dealing with the existence of multiple solutions, in particular multiple sign-changing solutions. An attempt was made in [10, 11] for establishing an abstract framework to deal with sign-changing solutions for systems that share some of the above features. The authors in [10, 11] developed the method of multiple invariant sets of decreasing flow. In [10] for the subcritical case infinitely many sign-changing solutions were established. Specially, Chen, Lin and Zou [5] proved the existence of multiple sign-changing (i.e., both two components change sign) and semi-nodal solutions (i.e., one component changes sign and the other one is positive) for coupled Schrödinger equations for the case of $k = 2$, $\beta_{12} = \beta_{21} = \beta > 0$. Motivated by the works we mentioned above, in this paper we consider the existence of sign-changing solutions for the system (1.1) in the general case, by using the method of invariant sets of decreasing flow (see [10]) and the truncation method (see [12]).

We assume that

- (A1) $\Omega \subset \mathbb{R}^N$, $N = 2, 3$, $k \geq 2$, $\lambda_j > 0$ for $j = 1, \dots, k$.
- (A2) $\beta_{jj} > 0$, $\beta_{ij} = \beta_{ji} \leq 0$ for $1 \leq i < j \leq k$.

Solutions of (1.1) correspond to critical points of the functional

$$I(U) = \frac{1}{2} \int_\Omega \sum_{j=1}^{k} (|\nabla u_j|^2 + \lambda_j u_j^2)\, dx - \frac{1}{4} \int_\Omega \sum_{i,j=1}^{k} \beta_{ij} u_i^2 u_j^2 \, dx$$

for $U = (u_1, \dots, u_k) \in X = H_0^1(\Omega) \times \cdots \times H_0^1(\Omega)$, the k-fold product of $(H_0^1(\Omega))^k$. We shall use the equivalent inner products

$$(u, v)_j = \int_\Omega (\nabla u \nabla v + \lambda_j uv) dx, \quad j = 1, \dots, k$$

and the induced norm $\| \cdot \|_j$. The inner product

$$(U, V) = \sum_{j=1}^{k} (u, v)_j, \quad U = (u_1, \dots, u_k),\ V = (v_1, \dots, v_k),$$

gives rise to a norm $\| \cdot \|$ on X.

Firstly, we introduce the following perturbation problem. We assume $U = (u_1, \dots, u_k)$, $\varepsilon \in \mathbb{R}$ is a small parameter, $F(U, \varepsilon)$, $\frac{\partial F}{\partial u_j}(U, \varepsilon)$ are continuous functions, and $F(U, \varepsilon) = F(-U, \varepsilon)$. For $\varepsilon = 0$, we understand

$$F(U, 0) = 0, \quad \frac{\partial F}{\partial u_j}(U, 0) = 0.$$

Then we consider the perturbed problem

$$\begin{gathered} -\Delta u_j + \lambda_j u_j = \sum_{i=1}^{k} \beta_{ij} u_i^2 u_j + \frac{\partial F}{\partial u_j}(U, \varepsilon), \quad \text{in } \Omega, \\ u_j(x) = 0, \quad \text{on } \partial\Omega,\ j = 1, \dots, k. \end{gathered} \tag{1.2}$$

Here are our main results.

Theorem 1.1. *Assume* (A1), (A2) *hold. Then system* (1.1) *has infinitely many solutions with each component being sign-changing.*

Theorem 1.2. *Assume* (A1), (A2) *hold and let* $l \in \mathbb{N}^+$. *Then there exists* $\varepsilon_l > 0$ *such that for* $|\varepsilon| \le \varepsilon_l$, *the system* (1.2) *has* l *pairs of sign-changing solutions.*

Corollary 1.3. *For each* $l \in \mathbb{N}^+$, *there exists* $\beta_l > 0$ *such that for* $\beta_{ij} = \beta_{ji} \le \beta_l$ *with* $1 \le i < j \le k$, *system* (1.1) *has at least* l *pairs of sign-changing solutions.*

Note that we do not assume any growth conditions for the perturbation function F. To apply critical point theorem [10, 11], we firstly have the following truncated function; the idea comes from [12]. For $M > 0$, we define

$$F_M(U,\varepsilon) = F\big(f_M(|U|)\frac{U}{|U|}\big),$$

where f_M is a monotonic smooth function, satisfying $f_M(t) = t$ if $t \le M$, $f_M(t) = M + \frac{1}{2}$ if $t \ge M$. Then we consider the truncated system

$$\begin{gathered} -\Delta u_j + \lambda_j u_j = \sum_{i=1}^k \beta_{ij} u_i^2 u_j + \frac{\partial F_M}{\partial u_j}(U,\varepsilon), \quad \text{in } \Omega, \\ u_j(x) = 0, \quad \text{on } \partial\Omega,\ j = 1,\dots,k. \end{gathered} \tag{1.3}$$

If $U = (u_1,\dots,u_k)$ is a solution of (1.3), and there exists $M > 0$ such that $|U(x)| < M$ for all $x \in \overline{\Omega}$, then U is also a solution of the perturbed problem (1.2). System (1.3) has a variational structure given by the functional

$$\begin{aligned} I_M(U) &= I(U) - \int_\Omega F_M(U,\varepsilon)\,dx \\ &= \frac{1}{2}\int_\Omega \sum_{j=1}^k (|\nabla u_j|^2 + \lambda_j u_j^2)\,dx - \frac{1}{4}\int_\Omega \sum_{i,j=1}^k \beta_{ij} u_i^2 u_j^2\,dx - \int_\Omega F_M(U,\varepsilon)\,dx\,. \end{aligned} \tag{1.4}$$

This article organized as follows. In Section 2, we study the truncated functional I_M, and construct a sequence of critical values for I_M by using the method of multiple invariant sets of descending flow. In Section 3, we obtain the sign-changing solutions of the perturbed problem (1.2), then we obtain the main result.

Throughout this article, we use $\|\cdot\|_{L^p}$ and $\|\cdot\|$ to denote the norms of L^p and X, respectively. $c, c_1, \dots$ denote constants that are independent of the sequences in the arguments but maybe different from line to line, and $c(\cdot)$ will be used to indicate the dependency of the constant c on the relevant quantity.

2. Critical points of the truncated functional I_M

To obtain sign-changing critical points of I_M, we apply an abstract critical point theorem (Theorem 2.1) to the truncated functional I_M.

Theorem 2.1. *Let* X *be a Banach space,* f *be an even* C^1*-functional on* X, A *be an odd, continuous mapping from* X *to* X, *and* P_j, Q_j, $j = 1,\dots,k$ *be open convex subsets of* X *with* $Q_j = -P_j$. *Denote* $W = \cup_{j=1}^k (P_j \cup Q_j)$, $\Sigma = \cap_{j=1}^k (\partial P_j \cap \partial Q_j)$. *Assume*

(A3) f *satisfies the Palais-Smale condition.*

(A4) $c^* = \inf_{x \in \Sigma} f(x) > 0$.
(A5) *For each* $b_0 > 0$ *and* $c_0 > 0$, *there exists* $b = b(b_0, c_0)$, *such that if* $|f(x)| \leq c_0$, $\|Df(x)\| \geq b_0$, *then*

$$\langle Df(x), x - Ax \rangle \geq b\|x - Ax\| > 0.$$

(A6) $A(\partial P_j) \subset P_j,\ A(\partial Q_j) \subset Q_j,\ j = 1, \dots, k$.

We define

$$\begin{gathered}\Gamma_j = \{E \subset X : E \textit{ is compact}, -E = E,\ \gamma(E \cap \sigma^{-1}(\Sigma)) \geq j \textit{ for } \sigma \in \Lambda\}, \\ \Lambda = \big\{\sigma \in C(X, X) : \sigma \textit{ is odd},\ \sigma(P_j) \subset P_j,\ \sigma(Q_j) \subset Q_j,\ j = 1, \dots, k, \\ \sigma(x) = x \textit{ if } f(x) < 0\big\}\end{gathered}$$

where $\gamma = \gamma(E)$ *denotes the genus of a symmetric set* E

$$\gamma = \min\{n : \textit{there is an odd map } \varphi^{(j)} : E \to \mathbb{R}^n \setminus \{0\}\}.$$

We ssume that

(A7) Γ_j *is nonempty for* $j = 1, 2, \dots$.

Then we define

$$\begin{gathered}c_j = \inf_{E \in \Gamma_j} \sup_{x \in E \setminus W} f(x),\ j = 1, 2, \dots, \\ K_c = \{x \in X : Df(x) = 0,\ f(x) = c\}, \quad K_c^* = K_c \setminus W.\end{gathered}$$

Then

(1) $c_j \geq c_*$, $K_{c_j}^* \neq \emptyset$ *for* $j = 1, 2, \dots$.
(2) $c_j \to +\infty$, *as* $j \to \infty$.
(3) *If* $c_j = c_{j+1} = \dots = c_{j+l-1} = c$, *then* $\gamma(K_c^*) \geq l$.

Lemma 2.2. *I_M is a C^1-functional on X, and satisfies the Palais-Smale condition.*

Proof. It is easy to verify that I_M is a C^1-functional. Also, for $\Phi = (\varphi_1, \dots, \varphi_k) \in X$, we have

$$\begin{aligned}\langle DI_M(U),\, \Phi\rangle &= \int_\Omega \sum_{j=1}^k (\nabla u_j \nabla \varphi_j + \lambda_j u_j \varphi_j)\, dx - \int_\Omega \sum_{i,j=1}^k \beta_{ij} u_i^2 u_j \varphi_j \, dx \\ &\quad - \int_\Omega \sum_{j=1}^k \frac{\partial F_M}{\partial u_j}(U, \varepsilon) \varphi_j \, dx,\end{aligned} \tag{2.1}$$

there exists an arbitrary small constant ε_M, such that for $|\varepsilon| \leq \varepsilon_M$, we have

$$\begin{aligned}&I_M(U) - \frac{1}{4}\langle DI_M(U),\, U\rangle \\ &= \frac{1}{4}\int_\Omega \sum_{j=1}^k (|\nabla u_j|^2 + \lambda_j u_j^2)\, dx - \int_\Omega \big(F_M(U, \varepsilon) - \frac{1}{4}\sum_{j=1}^k \frac{\partial F_M}{\partial u_j}(U, \varepsilon) u_j\big)\, dx \\ &\geq \frac{1}{4}\|U\|^2 - c.\end{aligned} \tag{2.2}$$

Then any Palais-Smale sequence of I_M is bounded in X. Let $U_n = (u_{n,1}, \dots, u_{n,k}) \in X$ be a Palais-Smale sequence of the functional I_M. Notice that the imbedding

$H_0^1(\Omega) \hookrightarrow L^4(\Omega)$ is compact and we can assume that $U_n \to U$ in $L^4(\Omega)$. Then we have

$$
\begin{aligned}
&\int_\Omega \sum_{j=1}^k (|\nabla(u_{n,j}-u_{m,j})|^2 + \lambda_j(u_{n,j}-u_{m,j})^2)\,dx \\
&= \langle DI_M(U_n) - DI_M(U_m), U_n - U_m\rangle + \int_\Omega \sum_{i,j=1}^k \beta_{ij} u_{n,i}^2 u_{n,j}(u_{n,j}-u_{m,j})\,dx \\
&\quad - \int_\Omega \sum_{i,j=1}^k \beta_{ij} u_{m,i}^2 u_{m,j}(u_{n,j}-u_{m,j})\,dx \\
&\quad + \int_\Omega \sum_{j=1}^k \Big(\frac{\partial F_M}{\partial u_j}(U_n,\varepsilon) - \frac{\partial F_M}{\partial u_j}(U_m,\varepsilon)\Big)(u_{n,j}-u_{m,j})\,dx \\
&\le o(1) + c\|U_n\|_{L^4(\Omega)}^3 \Big(\int_\Omega \sum_{j=1}^k (u_{n,j}-u_{m,j})^4\,dx\Big)^{1/4} \\
&\quad + c\|U_m\|_{L^4(\Omega)}^3 \Big(\int_\Omega \sum_{j=1}^k (u_{n,j}-u_{m,j})^4\,dx\Big)^{1/4} \\
&\quad + \int_\Omega \sum_{j=1}^k \Big|\frac{\partial F_M}{\partial u_j}(U_n,\varepsilon) - \frac{\partial F_M}{\partial u_j}(U_m,\varepsilon)\Big|\,|u_{n,j}-u_{m,j}|\,dx \\
&\le o(1) + c\|U_n - U_m\|_{L^4(\Omega)} \to 0, \quad \text{as } n,m\to\infty.
\end{aligned}
$$

Therefore, we conclude that up to a subsequence a Palais-Smale sequence U_n is a Cauchy sequence in X, hence a convergent sequence. □

Definition 2.3. An odd and continuous operator $A : U = (u_1,\dots,u_k) \in X \mapsto V = (v_1,\dots,v_k) = AU \in X$ is defined by the system

$$
\begin{aligned}
&\int_\Omega (\nabla v_j \nabla \varphi_j + \lambda_j v_j \varphi_j)\,dx - \int_\Omega \sum_{i=1, i\neq j}^k \beta_{ij} u_i^2 v_j \varphi_j\,dx \\
&= \int_\Omega \beta_{jj} u_j^3 \varphi_j\,dx + \int_\Omega \frac{\partial F_M}{\partial u_j}(U,\varepsilon)\varphi_j\,dx,
\end{aligned} \tag{2.3}
$$

For $j = 1,\dots,k$ and $\Phi = (\varphi_1,\dots,\varphi_k) \in X$.

Lemma 2.4. *The operator A is well-defined and continuous.*

Proof. Note that $V = AU$ can be obtained by solving the minimization problem

$$\inf\{G(V) : V \in X\}$$

where

$$
\begin{aligned}
G(V) = \frac{1}{2}\int_\Omega \sum_{j=1}^k (|\nabla v_j|^2 + \lambda_j v_j^2)\,dx - \frac{1}{2}\int_\Omega \sum_{i,j=1, i\neq j}^k \beta_{ij} u_i^2 v_j^2\,dx \\
- \int_\Omega \sum_{j=1}^k \beta_{jj} u_j^3 v_j\,dx - \int_\Omega \sum_{j=1}^k \frac{\partial F_M}{\partial u_j}(U,\varepsilon) v_j\,dx.
\end{aligned}
$$

Let $V = AU$, $\bar{V} = A\bar{U}$, $\bar{V} = (\bar{v}_1, \dots, \bar{v}_k)$, $\bar{U} = (\bar{u}_1, \dots, \bar{u}_k)$. By (2.3), we have

$$
\begin{aligned}
&\|V - \bar{V}\|^2 \\
&= \int_\Omega \sum_{j=1}^k (|\nabla(v_j - \bar{v}_j)|^2 + \lambda_j (v_j - \bar{v}_j)^2)\, dx \\
&= \int_\Omega \sum_{i,j=1,i\neq j}^k \beta_{ij}(u_i^2 v_j - \bar{u}_i^2 \bar{v}_j)(v_j - \bar{v}_j)\, dx + \int_\Omega \sum_{j=1}^k \beta_{jj}(u_j^3 - \bar{u}_j^3)(v_j - \bar{v}_j)\, dx \\
&\quad + \int_\Omega \sum_{j=1}^k \Big(\frac{\partial F_M}{\partial u_j}(U,\varepsilon) - \frac{\partial F_M}{\partial u_j}(\bar{U},\varepsilon)\Big)(v_j - \bar{v}_j)\, dx \\
&\le c \int_\Omega \sum_{i,j=1,i\neq j}^k |u_i^2 - \bar{u}_i^2|\, |v_j|\, |v_j - \bar{v}_j|\, dx + c\int_\Omega \sum_{j=1}^k |u_j^3 - \bar{u}_j^3|\, |v_j - \bar{v}_j|\, dx \\
&\quad + \int_\Omega \sum_{j=1}^k \Big|\frac{\partial F_M}{\partial u_j}(U,\varepsilon) - \frac{\partial F_M}{\partial u_j}(\bar{U},\varepsilon)\Big|\, |v_j - \bar{v}_j|\, dx \\
&\le c(\|U - \bar{U}\|\, \|V - \bar{V}\| + \|\frac{\partial F_M}{\partial u_j}(U,\varepsilon) - \frac{\partial F_M}{\partial u_j}(\bar{U},\varepsilon)\|\, \|V - \bar{V}\|),
\end{aligned}
$$

hence $AU - A\bar{U} = V - \bar{V} \to 0$ as $U \to \bar{U}$ in X. □

Lemma 2.5. *For each $b_0, c_0 > 0$, then the following property holds: if $|I_M(U)| \le c_0$ and $\|DI_M(U)\| \ge b_0$, then there exists $b = b(b_0, c_0)$ such that*

$$
\langle DI_M(U), U - AU\rangle \ge b\|U - AU\| > 0.
$$

Proof. We have

$$
\begin{aligned}
&\langle DI_M(U), \Phi\rangle \\
&= \int_\Omega \sum_{j=1}^k (\nabla(u_j - v_j)\nabla\varphi_j + \lambda_j(u_j - v_j)\varphi_j)\, dx - \int_\Omega \sum_{i,j=1,i\neq j}^k \beta_{ij} u_i^2 (u_j - v_j)\varphi_j\, dx \\
&= \langle U - V, \Phi\rangle - \int_\Omega \sum_{i,j=1,i\neq j}^k \beta_{ij} u_i^2 (u_j - v_j)\varphi_j\, dx
\end{aligned}
\tag{2.4}
$$

for $\Phi = (\varphi_1, \dots, \varphi_k) \in X$. By using $\Phi = U - V$ in (2.4), we obtain

$$
\langle DI_M(U), U - V\rangle = \|U - V\|^2 - \int_\Omega \sum_{i,j=1,i\neq j}^k \beta_{ij} u_i^2 (u_j - v_j)^2\, dx.
$$

Notice that if $\beta_{ij} = \beta_{ji} \le 0$ for $1 \le i < j \le k$, then

$$
\langle DI_M(U), U - V\rangle \ge \|U - V\|^2 \tag{2.5}
$$

and

$$
\langle DI_M(U), U - V\rangle \ge -\int_\Omega \sum_{i,j=1,i\neq j}^k \beta_{ij} u_i^2 (u_j - v_j)^2\, dx. \tag{2.6}
$$

It follows from (2.4) and (2.6) that

$$\begin{aligned}|\langle DI_M(U),\Phi\rangle| &= \Big|\langle U-V,\,\Phi\rangle - \int_\Omega \sum_{i,j=1,i\neq j}^k \beta_{ij}u_i^2(u_j-v_j)\varphi_j\,dx\Big|\\ &\le \|U-V\|\,\|\Phi\| + \Big(-\int_\Omega \sum_{i,j=1,i\neq j}^k \beta_{ij}u_i^2(u_j-v_j)^2\,dx\Big)^{1/2}\\ &\quad\times \Big(-\int_\Omega \sum_{i,j=1,i\neq j}^k \beta_{ij}u_i^2\varphi_j^2\,dx\Big)^{1/2}\\ &\le \|U-V\|\,\|\Phi\| + c\|U\|_{L^4(\Omega)}\|\Phi\|_{L^4(\Omega)}\langle DI_M(U),U-V\rangle^{1/2}\end{aligned}$$

which implies that

$$\|DI_M(U)\| \le \|U-V\| + c\|U\|\langle DI_M(U),\,U-V\rangle^{1/2}. \tag{2.7}$$

There exists a small constant ε_M, so that for $|\varepsilon|\le\varepsilon_M$, by (1.4) and (2.4), we have

$$\begin{aligned}&I_M(U) - \frac14\langle U-V,\,U\rangle\\ &= I_M(U) - \frac14\langle DI_M(U),\,U\rangle - \frac14\int_\Omega \sum_{i,j=1,i\neq j}^k \beta_{ij}u_i^2u_j(u_j-v_j)\,dx\\ &= \frac14\|U\|^2 + \int_\Omega \Big(\frac14\sum_{j=1}^k \frac{\partial F_M}{\partial u_j}(U,\varepsilon)u_j - F_M(U,\varepsilon)\Big)\,dx\\ &\quad - \frac14\int_\Omega \sum_{i,j=1,i\neq j}^k \beta_{ij}u_i^2u_j(u_j-v_j)\,dx\\ &\ge \frac14\|U\|^2 - \frac14\int_\Omega \sum_{i,j=1,i\neq j}^k \beta_{ij}u_i^2u_j(u_j-v_j)\,dx - c.\end{aligned} \tag{2.8}$$

So by (2.8), we obtain

$$\begin{aligned}&\|U\|^2\\ &\le c(1+|I_M(U)|) + c|\langle U-V,U\rangle| + c\Big|\int_\Omega \sum_{i,j=1,i\neq j}^k \beta_{ij}u_i^2u_j(u_j-v_j)\,dx\Big|\\ &\le c(1+|I_M(U)|) + c\|U-V\|^2 + \frac14\|U\|^2 + c\|U\|^2_{L^4(\Omega)}\langle DI_M(U),U-V\rangle^{1/2}.\end{aligned} \tag{2.9}$$

Given a positive constant a, if

$$\langle DI_M(U),\,U-V\rangle \ge a^2,$$

then by (2.5) we can easily obtain

$$\langle DI_M(U),\,U-V\rangle \ge a\|U-V\| > 0.$$

The conclusion holds; if not, let

$$\langle DI_M(U),\,U-V\rangle \le a^2, \tag{2.10}$$

by (2.9) and (2.10), we have

$$\|U\|^2 \le c(1+|I_M(U)|+\|U-V\|^2) + c_0a\|U\|^2. \tag{2.11}$$

Hence, taking a such that $c_0 a \leq 1/2$, then we have

$$\|U\|^2 \leq c(1 + |I_M(U)| + \|U - V\|^2). \tag{2.12}$$

Substituting (2.12) into (2.7), we obtain

$$\begin{aligned} &\|DI_M(U)\| \\ &\leq \|U - V\| + c(1 + |I_M(U)| + \|U - V\|^2)^{1/2}\langle DI_M(U), U - V\rangle^{1/2} \\ &\leq \|U - V\| + \frac{1}{2}\|DI_M(U)\| + c(1 + |I_M(U)| + \|U - V\|^2)\|U - V\|. \end{aligned} \tag{2.13}$$

Therefore

$$\|DI_M(U)\| \leq c(1 + |I_M(U)| + \|U - V\|^2)\|U - V\|.$$

If $|I_M(U)| \leq c_0$ and $\|DI_M(U)\| \geq b_0 > 0$, we deduce that there exists $b = b(b_0, c_0)$ such that $\|U - V\| > b$. So it follows from (2.5) that

$$\langle DI_M(U),\, U - AU\rangle \geq b\|U - AU\| > 0. \qquad \square$$

Let P_j, Q_j for $j = 1, \dots, k$ be open convex subsets of X, defined by

$$\begin{gathered} P_j = P_j(\delta) = \{U = (u_1, \dots, u_k) \in X : \|u_j^-\|_{L^4(\Omega)} < \delta\}, \\ Q_j = Q_j(\delta) = \{U = (u_1, \dots, u_k) \in X : \|u_j^+\|_{L^4(\Omega)} < \delta\}. \end{gathered}$$

Lemma 2.6. *There exist $\delta > 0$ and $\varepsilon_M > 0$ such that for $|\varepsilon| \leq \varepsilon_M$, it holds that*

$$A(\partial P_j) \subset P_j, \quad A(\partial Q_j) \subset Q_j, \quad \text{for } j = 1, \dots, k.$$

Proof. Choose $\Phi = V^+ = (v_1^+, \dots, v_k^+)$ as test function in (2.3), we have

$$\begin{aligned} &\int_\Omega \big(|\nabla v_j^+|^2 + \lambda_j (v_j^+)^2\big)\, dx - \int_\Omega \sum_{i=1, i\neq j}^k \beta_{ij} u_i^2 (v_j^+)^2\, dx \\ &= \int_\Omega \beta_{jj} u_j^3 v_j^+\, dx + \int_\Omega \frac{\partial F_M}{\partial u_j}(U, \varepsilon) v_j^+\, dx \\ &\leq c\Big(\int_\Omega (u_j^+)^3 v_j^+\, dx + \int_\Omega \big|\frac{\partial F_M}{\partial u_j}(U, \varepsilon)\big| v_j^+\, dx\Big). \end{aligned}$$

Then

$$\|v_j^+\|_{L^4(\Omega)}^2 \leq c_1 \|u_j^+\|_{L^4(\Omega)}^3 \|v_j^+\|_{L^4(\Omega)} + c_2 \|\frac{\partial F_M}{\partial u_j}(U, \varepsilon)\|_{L^\infty(\Omega)} \|v_j^+\|_{L^4(\Omega)}. \tag{2.14}$$

Take $\delta > 0$ such that $c_1 \delta^2 \leq 1/4$ and choose $\varepsilon_M > 0$, such that for $|\varepsilon| \leq \varepsilon_M$, $c_2 \|\frac{\partial F_M}{\partial u_j}(U, \varepsilon)\|_{L^\infty(\Omega)} \leq \delta/4$. Then for $U \in \partial Q_j$, $\|u_j^+\|_{L^4(\Omega)} = \delta$, we have

$$\|v_j^+\|_{L^4(\Omega)}^2 \leq \frac{1}{4}\delta \|v_j^+\|_{L^4(\Omega)} + \frac{1}{4}\delta \|v_j^+\|_{L^4(\Omega)},$$

hence

$$\|v_j^+\|_{L^4(\Omega)} \leq \frac{1}{2}\delta.$$

That is for $U \in \partial Q_j$, we have $V = AU \in Q_j$ and $A(\partial Q_j) \subset Q_j$, $j = 1, \dots, k$. Similarly, $A(\partial P_j) \subset P_j$, $j = 1, \dots, k$. $\square$

Lemma 2.7. *There exist $\delta > 0$ and $c^* > 0$, such that if $U \in \Sigma$ and $|\varepsilon| \leq \varepsilon_M$, then $I_M(U) \geq c^*$.*

Proof. Note that

$$\begin{aligned} I_M(U) &= \frac{1}{2}\int_\Omega \sum_{j=1}^{k}(|\nabla u_j|^2 + \lambda_j u_j^2)\,dx - \frac{1}{4}\int_\Omega \sum_{i,j=1}^{k}\beta_{ij}u_i^2u_j^2\,dx - \int_\Omega F_M(U,\varepsilon)\,dx \\ &\geq \frac{1}{2}\|U\|^2 - \frac{1}{4}\int_\Omega \sum_{j=1}^{k}\beta_{jj}u_j^4\,dx - \int_\Omega F_M(U,\varepsilon)\,dx \\ &\geq c_1\|U\|^2_{L^4(\Omega)} - c_2\|U\|^4_{L^4(\Omega)} - \|F_M(U,\varepsilon)\|_{L^\infty(\Omega)}. \end{aligned}$$

For $U \in \Sigma = \cap_{j=1}^k(\partial P_j \cap \partial Q_j)$, we have

$$\|U\|^4_{L^4(\Omega)} = \int_\Omega \sum_{j=1}^{k}\big((u_j^+)^4 + (u_j^-)^4\big)\,dx = 2k\|u_j^+\|^4_{L^4(\Omega)} = 2k\delta^4.$$

By Lemma 2.6, taking $\delta > 0$ such that $c_2\delta^2 \leq \frac{1}{4}c_1$, and choosing ε_M such that for $|\varepsilon| \leq \varepsilon_M$, we have $\|F_M(U,\varepsilon)\|_{L^\infty(\Omega)} \leq \frac{1}{4}c_1\delta^2$. Therefore,

$$I_M(U) \geq c_1\delta^2 - c_2\delta^4 - \frac{1}{4}c_1\delta^2 \geq \frac{1}{2}c_1\delta^2 := c^* > 0\,. \qquad \square$$

Let

$$\begin{aligned} \Gamma_j &= \{E \subset X : E \text{ is compact}, -E = E,\ \gamma(E \cap \sigma^{-1}(\Sigma)) \geq j \text{ for } \sigma \in \Lambda\}, \\ \Lambda &= \big\{\sigma \in C(X,X) : \sigma \text{ odd},\ \sigma(P_j) \subset P_j,\ \sigma(Q_j) \subset Q_j,\ j = 1,\dots,k, \\ &\quad \sigma(U) = U \text{ if } I_M(U) < 0\big\}, \end{aligned}$$

and $\gamma = \gamma(E)$ is the genus of E,

$$\gamma = \min\{n : \text{there is an odd map } \varphi^{(j)} : E \to \mathbb{R}^n \setminus \{0\}\}.$$

Now we define a sequence of critical values of the truncated functional I_M,

$$c_j(M,\varepsilon) = \inf_{E \in \Gamma_j}\sup_{U \in E \setminus W} I_M(U), \quad j = 1,2,\dots$$

where $W = \cup_{j=1}^k(P_j \cup Q_j)$.

Lemma 2.8. *The set Γ_j is nonempty, and there exist $d_j > 0$ independent of M, ε and $\varepsilon_M^{(j)} > 0$, such that if $|\varepsilon| \leq \varepsilon_M^{(j)}$, then $c_j(M,\varepsilon) \leq d_j$.*

Proof. Let B^{nk} be the unit closed ball of $\mathbb{R}^{nk}$. Assume $n = j + k$. Denote $t \in \mathbb{R}^{nk}$ by $t = (t_1,\dots,t_k)$ and $t_m = (t_{1m}, t_{2m},\dots,t_{nm}) \in \mathbb{R}^n$ for $m = 1,\dots,k$. Let $v_{im} \in C_0^\infty(\Omega)$, $i = 1,\dots,n$, $m = 1,\dots,k$ be nk functions in X with disjoint supports. Define $\varphi^{(j)} : B^{nk} \to X$ by

$$\varphi^{(j)}(t) = R\big(\sum_{i=1}^{n} t_{i1}v_{i1},\dots,\sum_{i=1}^{n} t_{ik}v_{ik}\big) \in X$$

where R is large enough such that $I(\varphi^{(j)}(t)) < -10$ for $t \in \partial B^{nk}$. Then there exists $\varepsilon_M > 0$, so that if $|\varepsilon| \leq \varepsilon_M$, then we have

$$I_M(\varphi^{(j)}(t)) \leq I(\varphi^{(j)}(t)) + 1 < 0$$

for $t \in \partial B^{nk}$. By [11, Lemma 5.6], we have $E_j := \varphi^{(j)}(B^{nk}) \in \Gamma_j$. Then Γ_j is nonempty.

Next we estimate $c_j(M, \varepsilon)$ for $|\varepsilon| \le \varepsilon_M$. We have

$$c_j(M,\varepsilon) = \inf_{E\in\Gamma_j} \sup_{U\in E\setminus W} I_M(U) \le \sup_{U\in E_j} I_M(U) \le \sup_{U\in E_j} \big(I(U)+1\big) := d_j\,. \qquad \square$$

3. Proof of main results

In this section, we complete the proof of Theorem 1.1 and Theorem 1.2. For fixed $M > 0$ and $\varepsilon = 0$, we will obtain the critical point U of I.

Lemma 3.1. *Assume $DI_M(U) = 0$, $I_M(U) \le L$. Then there exist $\varepsilon_M > 0$ and $K = K(L)$ independent of M, ε, such that for $|\varepsilon| \le \varepsilon_M$,*

$$\|U(x)\|_{L^\infty(\Omega)} \le K.$$

Proof. Denote $U = (u_1, \dots, u_k)$. By (2.2), for $|\varepsilon| \le \varepsilon_M$, we have

$$\begin{aligned}
L &\ge I_M(U) - \frac{1}{4}\langle DI_M(U), U\rangle \\
&= \frac{1}{4}\int_\Omega \sum_{j=1}^k (|\nabla u_j|^2 + \lambda_j u_j^2)\,dx - \int_\Omega \Big(F_M(U,\varepsilon) - \frac{1}{4}\sum_{j=1}^k \frac{\partial F_M}{\partial u_j}(U,\varepsilon)u_j\Big)\,dx \qquad (3.1)\\
&\ge \frac{1}{4}\|U\|^2 - c.
\end{aligned}$$

We know that there exists $C(L) > 0$, such that $\|U\| \le C(L)$. Choose $\phi = u_{jT}|u_{jT}|^{2r-2}$ as the test function in $\langle DI_M(u_j), \phi\rangle = 0$, where $r \ge 1$, $T > 1$, and $u_{jT}(x) = \pm T$ if $\pm u_j(x) \ge T$, $u_{jT}(x) = u_j(x)$ if $|u_j(x)| \le T$. We have

$$\int_\Omega (\nabla u_j \nabla\phi + \lambda_j u_j\phi)\,dx = \int_\Omega \sum_{i=1}^k \beta_{ij} u_i^2 u_j \phi\,dx + \int_\Omega \frac{\partial F_M}{\partial u_j}(U,\varepsilon)\phi\,dx. \qquad (3.2)$$

By (3.2), it is easy to obtain the inequality

$$\int_\Omega \nabla u_j\nabla\phi\,dx \le \int_\Omega \beta_{jj} u_j^3\phi\,dx + \int_\Omega |\frac{\partial F_M}{\partial u_j}(U,\varepsilon)\phi|\,dx. \qquad (3.3)$$

Firstly, we estimate the left-hand side of (3.3),

$$\begin{aligned}
\int_\Omega \nabla u_j\nabla\phi\,dx &\ge (2r-1)\int_\Omega |\nabla u_{jT}|^2|u_{jT}|^{2r-2}\,dx \\
&\ge \frac{2r-1}{r^2}\int_\Omega |\nabla|u_{jT}|^r|^2\,dx \qquad (3.4)\\
&\ge \frac{c(2r-1)}{r^2}\Big(\int_\Omega \big(|u_{jT}|^r\big)^{2^*}\,dx\Big)^{2/2^*}.
\end{aligned}$$

Let $M > 0$, there exists ε_M such that for $|\varepsilon| \le \varepsilon_M$, we have $\|\frac{\partial F_M}{\partial u_j}(U,\varepsilon)\|_{L^\infty(\Omega)} < 1$. Then the right-hand side of (3.3) satisfies

$$
\begin{aligned}
&\int_\Omega \beta_{jj} u_j^3 \phi\,dx + \int_\Omega \big|\frac{\partial F_M}{\partial u_j}(U,\varepsilon)\phi\big|\,dx \\
&\le c\Big(\int_\Omega |u_j|^3 |u_{jT}|^{2r-1}\,dx + \int_\Omega 1\cdot |u_{jT}|^{2r-1}\,dx\Big) \\
&\le c\Big(\int_\Omega (1+|u_j|^3)|u_j|^{2r-1}\,dx\Big) \\
&\le c\Big(1 + \int_\Omega |u_j|^3 |u_j|^{2r-1}\,dx\Big) \\
&\le c\Big(1 + \Big(\int_\Omega |u_j|^{2^*}\,dx\Big)^{\frac{2}{2^*}} \Big(\int_\Omega (|u_j|^r)^{\frac{2\cdot 2^*}{2^*-2}}\,dx\Big)^{\frac{2^*-2}{2^*}}\Big) \\
&\le c\Big(1 + \Big(\int_\Omega (|u_j|^r)^{\frac{2\cdot 2^*}{2^*-2}}\,dx\Big)^{\frac{2^*-2}{2^*}}\Big) \\
&\le c\max\Big\{1, \Big(\int_\Omega (|u_j|^r)^{\frac{2\cdot 2^*}{2^*-2}}\,dx\Big)^{\frac{2^*-2}{2^*}}\Big\}.
\end{aligned}
\tag{3.5}
$$

Let $T \to \infty$ such that $u_{jT}(x) \to u_j(x)$. By (3.4) and (3.5), we obtain

$$
\Big(\int_\Omega \big(|u_{jT}|^r\big)^{2^*}\,dx\Big)^{\frac{2}{2^*}} \le \frac{cr^2}{2r-1} \max\Big\{1, \Big(\int_\Omega (|u_j|^r)^{\frac{2\cdot 2^*}{2^*-2}}\,dx\Big)^{\frac{2^*-2}{2^*}}\Big\}. \tag{3.6}
$$

Denote $d = \frac{2^*}{\frac{2\cdot 2^*}{2^*-2}} = \frac{2}{N-2} > 1$, $q = \frac{2\cdot 2^*}{2^*-2} = N$. By (3.6), we have

$$
\Big(\int_\Omega (|u_j|^r)^{qd}\,dx\Big)^{\frac{1}{qdr}} \le \big(\frac{cr^2}{2r-1}\big)^{\frac{1}{2r}} \max\Big\{1, \Big(\int_\Omega (|u_j|^r)^q\,dx\Big)^{\frac{1}{qr}}\Big\}. \tag{3.7}
$$

Choose r_0, such that $r_0 q = 2^*$ and $\int_\Omega |u_j|^{qr_0}\,dx < \infty$. So

$$
\Big(\int_\Omega (|u_j|^{r_0})^{qd}\,dx\Big)^{\frac{1}{qdr_0}} \le \big(\frac{cr_0^2}{2r_0-1}\big)^{\frac{1}{2r_0}} \max\Big\{1, \Big(\int_\Omega (|u_j|^{r_0})^q\,dx\Big)^{\frac{1}{qr_0}}\Big\}. \tag{3.8}
$$

Using iteration, we note that $r_0 d = r_1$ in (3.8), then

$$
\Big(\int_\Omega |u_j|^{r_1 q}\,dx\Big)^{\frac{1}{qr_1}} \le \big(\frac{cr_0^2}{2r_0-1}\big)^{\frac{1}{2r_0}} \max\Big\{1, \Big(\int_\Omega |u_j|^{r_0 q}dx\Big)^{\frac{1}{qr_0}}\Big\}. \tag{3.9}
$$

Therefore, by (3.9), we obtain

$$
\begin{aligned}
\Big(\int_\Omega |u_j|^{r_{k+1} q}\,dx\Big)^{\frac{1}{qr_{k+1}}} &\le \big(\frac{cr_k^2}{2r_k-1}\big)^{\frac{1}{2r_k}} \max\Big\{1, \Big(\int_\Omega |u_j|^{r_k q}\,dx\Big)^{\frac{1}{qr_k}}\Big\} \\
&\le \prod_{i=0}^k \big(\frac{cr_i}{2r_i-1}\big)^{\frac{1}{2r_i}} \max\Big\{1, \Big(\int_\Omega |u_j|^{r_0 q}\,dx\Big)^{\frac{1}{qr_0}}\Big\},
\end{aligned}
$$

where $r_i = d^i r_0$, we denote $C_0 = \prod_{i=0}^k \big(\frac{cr_i}{2r_i-1}\big)^{\frac{1}{2r_i}}$, then

$$
\|u_j\|_{L^{r_0 q d^{k+1}}(\Omega)} \le C_0(1 + \|u_j\|_{L^{2^*}(\Omega)}). \tag{3.10}
$$

Let $k \to \infty$ in (3.10), by (3.1), we have

$$
\|u_j\|_{L^\infty(\Omega)} \le C_0(1 + \|u_j\|_{L^{2^*}(\Omega)}) \le c = c(L)\,. \qquad \square
$$

Proof of Theorem 1.2. By Lemmas 2.2, 2.5-2.8, for a sufficiently small parameter ε, the functional I_M satisfies the conditions (A3), (A4)–(A7) of the abstract critical point theorem (Theorem 2.1). Then, $c_j(M,\varepsilon)$ is a critical value of the functional I_M, and each component of the corresponding critical point $U_j(M,\varepsilon)$ is sign-changing. That is, $U_j(M,\varepsilon)$ is a sign-changing solution of the truncated system (1.3). Moreover, given $l \in \mathbb{N}^+$, $L^* > 0$, by Lemma 2.8, there exists $\varepsilon_M^* > 0$ such that for $|\varepsilon| \le \varepsilon_M^* = \min\{\varepsilon_M^{(1)}, \dots, \varepsilon_M^{(l)}\}$,

$$c_j(M,\varepsilon) \le L^* = \max\{d_1, \dots, d_l\}, \quad j = 1, \dots, l.$$

By Lemma 3.1, there exist the constant K^* independent of M, ε, and $\varepsilon_M > 0$, such that for $|\varepsilon| \le \varepsilon_M$,

$$\|U_j(M,\varepsilon)\|_{L^\infty(\Omega)} \le K^*, \quad j = 1, \dots, l.$$

Now take $M \ge K^* + 1$, then for $|\varepsilon| \le \varepsilon_l$, $U_j(\varepsilon) := U_j(M,\varepsilon)$, $j = 1, \dots, l$ are sign-changing solutions of the perturbed system (1.2). □

Note that taking $\varepsilon = 0$, we have $F(U,0) = 0$ and $\frac{\partial F}{\partial u_j}(U,0) = 0$, then the solutions to the perturbed system (1.2) are also solutions to the original system (1.1).

In Section 2, we have obtained the sign-changing critical points of the truncated functional I_M. Therefore, by Theorem 1.2, we know that system (1.2) has l pairs of sign-changing solutions. Then, for $\varepsilon = 0$, the system (1.1) has infinitely many sign-changing solutions, and we have thus proved the main result.

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