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On Properties of Nonlinear Second Order Systems under Nonlinear Impulse Perturbations *

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Abstract

In this paper, we consider the impulsive second order system

$$\ddot{x} + f(x) = 0 \quad (t \neq t_n); \quad \dot{x}(t_n + 0) = b_n \dot{x}(t_n) \quad (t = t_n)$$

where $t_n = t_0 + n\,p$ $(p > 0, n = 1, 2 \ldots)$. In a previous paper, the authors proved that if $f(x)$ is strictly nonlinear, then this system has infinitely many periodic solutions. The impulses account for the main differences in the attractivity properties of the zero solution. Here, we prove that these periodic solutions are attractive in some sense, and we give good estimates for the attractivity region.

1 Introduction

Investigations of asymptotic stability problems for the intermittently damped second order differential equation

$$\ddot{x} + g(t)\dot{x} + f(x) = 0 \tag{1}$$

have led to asymptotic stability investigations of the impulsive system

$$\begin{aligned} \ddot{x} + f(x) = 0, &\quad (t \neq t_n) \\ x(t_n + 0) = x(t_n), & \\ \dot{x}(t_n + 0) = b_n \dot{x}(t_n), & \end{aligned} \tag{2}$$

where $t_n \to \infty$ $(n \to \infty)$, $xf(x) > 0$ $(x \neq 0)$, and f is continuous $(x \in \mathbb{R})$ (see [3, 7, 8]). Although there are analogies between the systems (1) and (2) in the case $0 \leq b_n \leq 1$, system (2) has unexpected properties due to the instantaneous effects. In addition, if $b_n < 0$ ([3, 5]), there are some new beating phenomena, and the beating impulses can stabilize the oscillatory behavior of the system (see [5]). In particular, in both the positive and negative impulse

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cases, if $t_n = t_0 + n\,p$ $(p > 0)$, there can exist nonzero periodic solutions, which are small or large depending on the nonlinearity of the function $f(x)$. The existence of such solutions can destroy the global nature of the attractivity, or the attractivity itself, of the zero solution. In this paper, we investigate the attractivity properties of these periodic solutions. We show that the periodic solutions are attractive in some sense, and we describe the attractivity regions as well.

2 Definitions and Preliminaries

For the system (2), we use the following assumptions: $t_n = t_0 + n\,p$ $(p > 0)$, $f(x)$ is continuous, $xf(x) > 0$ $(x \neq 0)$, and for the sake of simplicity, we assume that f is an odd function, i.e., $f(-x) = -f(x)$.

We say that the zero solution of (2) or (1) is *stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|x(0)| + |\dot{x}(0)| < \delta$ implies $|x(t)| + |\dot{x}(t)| < \varepsilon$ $(t \geq 0)$. The zero solution is *asymptotically stable* (a.s.) if it is stable and there exists $\delta > 0$ such that $|x(0)|+|\dot{x}(0)| < \delta$ implies $\lim_{t\to\infty}(x(t), \dot{x}(t)) = (0, 0)$. The asymptotic stability is *global* (g.a.s.) if $\delta = \infty$.

We investigate the solutions of the equations with the aid of the energy function

$$V(x, y) = y^2 + 2\int_0^x f =: y^2 + F(x), \tag{3}$$

and often use the notation $V(t) = V(x(t), \dot{x}(t))$ for the solutions of system (2). Furthermore, without future reference, we assume that

$$\lim_{x\to\pm\infty} F(x) = \infty.$$

This condition allows us to obtain boundedness of the solutions from the boundedness of the energy.

Let us consider the undamped equation

$$\ddot{u} + f(u) = 0. \tag{4}$$

All solutions are periodic, and the energy is constant along each solution. The distance between the extremal points is given by (see [9])

$$\Delta(r) = \int_{-F^{-1}(r)}^{F^{-1}(r)} \frac{dx}{\sqrt{r - F(x)}}, \tag{5}$$

where F^{-1} is the inverse of the positive part of $F(x)$. Calculations yield the following expressions for $\Delta(r)$ in the case where $f(x) = |x|^\alpha \mathrm{sgn}\,(x)$:

$$\text{a)} \quad \alpha = 1, \quad \Delta(r) = \pi, \tag{6}$$

b) $\alpha \neq 1$, $\Delta(r) = Ar^{\beta}$ with $\beta = \dfrac{1-\alpha}{2(\alpha+1)}$ and

$$A = 2\left(\frac{\alpha+1}{2}\right)^{\frac{1}{\alpha+1}} \frac{\sqrt{\pi}\Gamma\left(\frac{1}{\alpha+1}\right)}{(\alpha+1)\Gamma\left(\frac{3+\alpha}{2(1+\alpha)}\right)},$$

where $\Gamma(\cdot)$ denotes Euler's Γ function. In the linear case, we obtain the known value π provided $\beta < 0$ for $\alpha > 1$ or $0 < \beta < 1/2$ for $0 < \alpha < 1$.

Consider the system (2). Since $\lim_{n\to\infty} t_n = \infty$, every solution can be continued to ∞. In addition, the solutions are differentiable, and $\dot{x}(t)$ is piecewise continuous and continuous from the left at every $t > 0$.

The variation of the energy along the solutions of (2) is given by

$$\begin{aligned} V(t_{n+1}) - V(t_n) &= V(t_n+0) - V(t_n) \\ &= \dot{x}^2(t_n+0) + F(x(t_n+0)) - \dot{x}^2(t_n) - F(x(t_n)) \qquad (7) \\ &= b_n^2\dot{x}^2(t_n) - \dot{x}^2(t_n) = -\dot{x}^2(t_n)(1-b_n^2) = -a_n\dot{x}^2(t_n), \end{aligned}$$

where $a_n = 1 - b_n^2$ is the n-th energy-quantum. The energy is nonincreasing if $b_n^2 \leq 1$, independent of the sign of b_n, and it is constant between any t_n and t_{n+1}. In case $b_n = 0$, the solutions of initial value problems are not unique in the backwards direction. In this case, there can be solutions which are identically zero on $[t_n, \infty)$.

Consider the energy along the solutions of (2) on the interval $[0, t]$. Using (7) repeatedly easily yields

$$V(t) = V(0) - \sum_{t_n < t} a_n\dot{x}^2(t_n) \qquad (8)$$

along solutions of (2). From inequality (8), it is easy to prove the following lemma.

Lemma 1 *If $|b_n| \leq 1$ for every $n = 1, 2, \ldots$, then $V(t)$ is nonincreasing along every solution. Moreover, every solution is bounded, and the zero solution of* (2) *is stable.*

3 Asymptotic Stability and Existence of Periodic Solutions

The following theorem guarantees the existence of periodic solutions; it is based on Theorem 20 in [2].

Theorem 1 *Suppose $0 \leq b_n \leq 1$, $p > 0$, and $t_n = t_0 + n\,p$, and let $D_0 = \{r : \Delta(r) = p/k,\ k = 1, 2 \ldots\}$. The solutions of* (2) *with initial conditions satisfying* $F(x(t_i)) = r \in D_0$, $\dot{x}(t_i) = 0$ $(i = 0, 1, \ldots)$ *are periodic and satisfy equation* (4).

This theorem assures the existence of periodic solutions in both the superlinear and sublinear cases. If $\limsup_{r\to\infty} \Delta(r) = 0$, then the set D_0 is unbounded, i.e., there are infinitely many periodic solutions in some set $\{V(x,y) > r_0\}$. This is the case if $f(x) = |x|^\alpha \operatorname{sgn} x \ \ (\alpha > 1)$. On the other hand, if $\liminf_{r\to 0} \Delta(r) = 0$, then D_0 has no positive infimum, i.e., there are arbitrarily small periodic solutions (e.g., $f(x) = |x|^\alpha \operatorname{sgn} x \ \ (0 < \alpha < 1)$). If $\lim_{r\to 0} \Delta(r) = \infty$, D_0 has a positive minimal element.

As consequence of the above statements, we obtain that the zero solution of the system (2) cannot be globally asymptotically stable in the strictly superlinear case ($\limsup_{r\to\infty} \Delta(r) = 0$), and it cannot be asymptotically stable in the strictly sublinear case ($\liminf_{r\to 0} \Delta(r) = 0$). More precisely, a simplified version of Corollary 3.3 in [5] is the following.

Theorem 2 *Let $t_n = t_0 + n\,p$ and $|b_n| \le 1$, and assume that there exists a sequence of integers $\{n_k\}$ such that*

$$\sum_k \min(a_{n_k}, a_{n_k+1}) = \infty. \tag{9}$$

Then:
Case (a): $0 < \inf_{r>0} \Delta(r)$. *If $p < \inf_{r>0} \Delta(r)$, the zero solution is globally asymptotically stable. If $p > \inf_{r>0} \Delta(r)$, the behavior depends on the shape of $\Delta(r)$.*
Case (b): $\lim_{r\to 0} \Delta(r) = \infty$ *and* $\lim_{r\to\infty} \Delta(r) = 0$. *The zero solution is asymptotically (but not globally) stable.*
Case (c): $\lim_{r\to 0} \Delta(r) = 0$. *The zero solution is not asymptotically stable.*

4 Attractivity of the Periodic Solutions

Following Theorem 1, let u_p denote the periodic solution of equation (4) (it is also a solution of system (2)) such that $r_p = V(u_p(t_0), \dot u_p(t_0)) = \Delta^{-1}(p)$ and $\dot u_p(t_0) = 0$ (i.e., $F(u_p(t_0)) = \Delta^{-1}(p)$). First, we consider which attractivity properties might reasonably be expected. Let $x(t)$ be another solution of (2) for which $r = V(x(t_0), \dot x(t_0)) < r_p$. Since $V(u_p(t), \dot u_p(t))$ is constant and $V(x(t), \dot x(t))$ is nonincreasing, $x(t) - u_p(t)$ cannot tend to zero as $t \to \infty$. Consequently, the relation $\lim_{t\to\infty}(x(t) - u_p(t)) = 0$ can hold only if $V(x(t_0), \dot x(t_0)) \ge r_p$.

To formulate our results, we need some additional concepts. We assume that $\Delta(r)$ is monotone on some interval I_p of form $[r_p, r_p + \varepsilon]$ or in $[r_p, \infty)$. This is a reasonable assumption since it is satisfied, for example, if $f(x)$ is an odd power of x. More generally, if $f(x)/x$ is monotone, then $\Delta(F(x))$ is monotone in the opposite direction ([15; Theorem 3.1.6]).

For a solution $u(t)$ of (4), let $U_0 = (u(0), \dot u(0)) \in \mathbb{R}^2$ and $U(t; U_0) = (u(t), \dot u(t))$, and for a solution $x(t)$ of the impulsive system (2), let $X_0 = (x(0{+}0), \dot x(0{+}0)) \in \mathbb{R}^2$ and $X(t; X_0) = (x(t), \dot x(t))$. To simplify the notation, let $u_p > 0$ be such that $F(u_p) = r_p$, and define $\tau(p, r) = \Delta(r) - \Delta(r_p) = \Delta(r) - p$. If

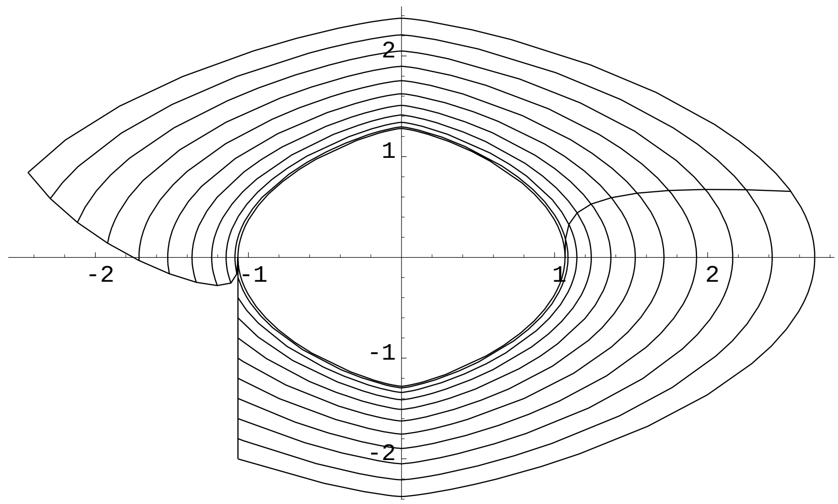

Figure 1: The mapping $\delta_- \to \gamma_+ \to \{(-x_p, y)\}$

$\Delta(r)$ is increasing on the interval I_p, then $\tau(p,r) > 0$, and if $\Delta(r)$ is decreasing, then $\tau(p,r) < 0$; moreover, $|\tau(p,r)|$ is increasing for $x \geq u_0$.

Let γ_+ and γ_- be the curves which are mapped to the sets $\{(-u_p, y) : y \in \mathbb{R}\}$ and $\{(u_p, y) : y \in \mathbb{R}\}$, respectively, by the mapping $U(p;\cdot)$. It is easy to see that this is equivalent to $\gamma_+ = \{U(\tau(p,r); U_0) : U_0 = (u_p, y)\}$ and $\gamma_- = \{U(\tau(p,r); U_0) : U_0 = (-u_p, y)\}$ where $r = V(u_p, y)$ is the energy of $U(t; U_0)$. From the symmetry of $f(x)$, we see that $\gamma_- = \{(x,y) : (-x,-y) \in \gamma_+\}$. The mapping $\delta_- \to \gamma_+ \to \{(-x_p, y)\}$ is shown in Figure 1.

The monotonicity of $\Delta(r)$ and the continuous dependence of solutions on initial conditions imply that the curve γ_+ is a graph of a continuous function of one variable, and so it can be written in the form $\dot{u}(\tau(p,r); U_0) = \gamma_+(u(\tau(p,r); U_0))$. To see this, assume the contrary. Let $(x, y_1), (x, y_2) \in \gamma_+$, $|y_1| < |y_2|$, and let $r_1 = V(x, y_1) < r_2 = V(x, y_2)$. Then

$$|\tau(p, r_1)| = \int_{u_p}^{x} \frac{ds}{\sqrt{r_1 - F(s)}} > \int_{u_p}^{x} \frac{ds}{\sqrt{r_2 - F(s)}} = |\tau(p, r_2)|,$$

which contradicts the monotonicity of $|\tau(r,p)|$. The case $y_1 = -y_2$ cannot happen because of the uniqueness of solutions to initial value problems for equation (4).

Note that $\gamma_+(x)$ is positive (negative) if $\Delta(r)$ is increasing (decreasing), $x - u_p$ is small enough, and $x > u_p$. Let $\hat{x}$ be the first zero of γ_+ if $F(\hat{x}) \in I_p$, let $\bar{x} = \min\{\hat{x}, \sup I_p\}$, and let $\bar{r} = F(\bar{x})$. We also use the notation $\gamma_+^- = \{(x,y) : (x,-y) \in \gamma_+\}$ and $\gamma_-^- = \{(x,y) : (x,-y) \in \gamma_-\}$. Now, we can define the following closed sets:

$$G_+ := \begin{cases} \{(x,y) : x \geq x_p,\ V(x,y) \leq \bar{r},\ y \leq \gamma_+(x)\}, & \text{if } \Delta(r) \text{ is increasing,} \\ \{(x,y) : x \geq x_p,\ V(x,y) \leq \bar{r},\ y \geq \gamma_+(x)\}, & \text{if } \Delta(r) \text{ is decreasing,} \end{cases}$$

$$G_- := \{(x,y) : (-x,-y) \in G_+\},$$
$$G_+^- := \{(x,y) : (x,-y) \in G_+\},$$
$$G_-^- := \{(x,y) : (x,-y) \in G_-\}.$$

Next, we consider the impulsive system (2), and assume that $X(t_{n-1}+0;X_0) \in G_+ \ \ (G_-)$ and $-1 \leq b_n \leq 0$ for an impulse at t_{n-1}. Then $X(t_n - 0;X_0) \in G_-^- \ \ (G_+^-)$ and $X(t_n+0;X_0) \in G_- \ \ (G_+)$. The first relation follows immediately from the properties of the sets G_i^j $(i,j \in \{+,-\})$. For the second one, we only have to prove that $(x,y) \in G_+$ implies $(x,b\,y) \in G_+$ for every $0 \leq b \leq 1$. But this immediately follows from the fact that γ_+ is a function of x.

Applying the above arguments, we can formulate the basic attractivity theorem for the beating impulses.

Theorem 3 *Assume that $\Delta(r)$ is monotone on an interval $[r_p, r_p+\varepsilon]$ or $[r_p,\infty)$, and $-1 < b_n \leq 0$ for every $n = 1,2,\ldots$. If a solution $x(t)$ of (2) satisfies $(x(t_0+0),\dot{x}(t_0+0)) = X_0 \in G_+ \ \ (G_-)$, then $(x(t_0+2np+0),\dot{x}(t_0+2np+0)) = X_0 \in G_+ \ \ (G_-)$ for every $n = 1,2,\ldots$. Since $V(x(t),\dot{x}(t))$ is nonincreasing, the solution $u_p(t)$ is conditionally stable with respect to the set G_+. If, in addition, condition (9) holds, then $\lim_{t\to\infty} V(x(t),\dot{x}(t)) = r_p$, i.e., $\lim_{t\to\infty}(x(t)-u_p(t)) = 0$.*

Proof We only have to prove the second part. This proof is analogous to the proof of Theorem 19 in [2]. Consider the case where $\Delta(r)$ increasing; the decreasing case can be proved analogously. To the contrary, suppose $x(t)$ is a solution of (2) such that $(x(t_0+2np+0),\dot{x}(t_0+np+0)) \in G_+ \cup G_-$ and $\lim_{t\to\infty} V(x(t),\dot{x}(t)) = r > r_p$. Then $\Delta(V(x(t_n),\dot{x}(t_n))) > p_1 > p$ for $n > N$. Now, it is easy to see that there exist a positive number μ, independent of n, such that

$$\max(\dot{x}^2(t_n),\dot{x}^2(t_{n+1})) > \mu$$

for $n > N$. Consequently,

$$\begin{aligned} V(t) &= V(t_0) - \sum_{t_n<t} a_n \dot{x}^2(t_n) \\ &\leq V(0) - \sum_{t_{n_k+1}<t} \min(a_{n_k} + a_{n_k+1})(\dot{x}^2(t_{n_k}) + \dot{x}^2(t_{n_k+1}) \\ &\leq V(0) - \sum_{t_{n_k+1}<t} \mu \min(a_{n_k} + a_{n_k+1}), \end{aligned}$$

which tends to $-\infty$ as $k \to \infty$. This contradiction completes the proof of the theorem. ◊

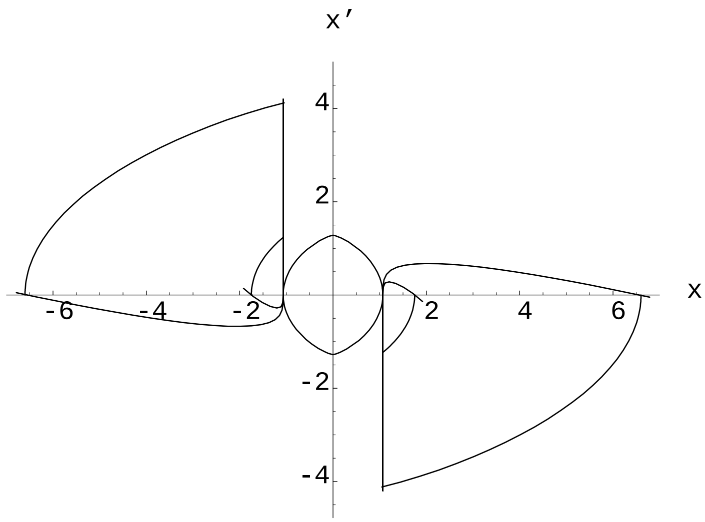

Figure 2: The sets G_+, G_-, H_+, and H_-, and the trajectory of $u_p(t)$

The left hand side of Figure 3 shows the attractivity of u_p for negative impulses.

Remark. As was the case with the asymptotic stability theorems in [8], condition (9) can not be replaced by the weaker condition $\sum_n a_n = \infty$.

To see this, let $p > 0$, $t_n = np$. $b_{2n} = 0$, and $b_{2n-1} = -1$, $(n = 1, 2, \ldots)$. Let $x(t)$ be any solution with $\dot{x}(t_2) = 0$. This solution is periodic but does not satisfy the equation $\ddot{x} + f(x) = 0$.

We can observe that the conditional stability of the solution u_p is satisfied independently of the specific values of the impulse constants b_n. The negative sign guarantees that the sets G_+ and G_- map into each other by the mapping $X(t_{n-1} + p + 0; \cdot)$. The case of positive impulses is different. If b_n is positive, such invariance will hold for much narrower sets under stronger conditions on b_n.

To formulate results for the case $0 \le b_n \le 1$, we need some additional definitions. Let δ_+ and δ_- be the curves that are mapped respectively to γ_- and γ_+ by the mapping $U(p; \cdot)$. Obviously, these curves can be defined analogously to the γ curves above, that is, $\delta_+ = \{U(2\tau(p, r); U_0) : U_0 = (u_p, y)\}$ and $\delta_- = \{U(2\tau(p, r); U_0) : U_0 = (-u_p, y)\} = \{(x, y) : (-x, -y) \in \delta_+\}$.

The curves δ_+ and δ_- also represent graphs of continuous functions of one variable, and can be written in the form $\delta_+(x)$ and $\delta_-(x)$. Similarly, $\delta_+(x)$ is positive (negative) if $\Delta(r)$ is increasing (decreasing), $x - u_p$ is small enough, and $x > u_p$. Let $\check{x}$ be the first zero of δ_+ if $F(\check{x}) \in I_p$, let $\tilde{x} = \min\{\check{x}, \sup I_p\}$,

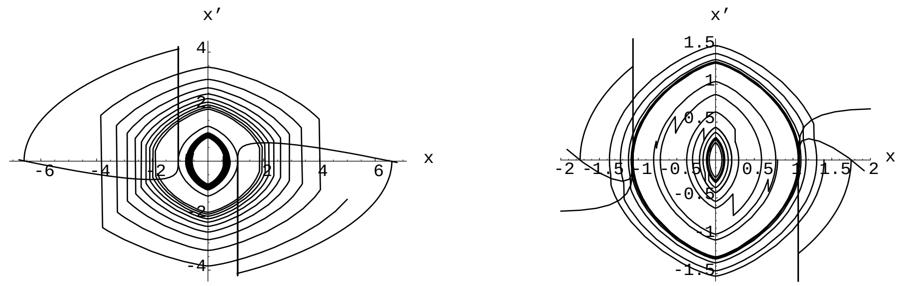

Figure 3: The attractivity of $u_p(t)$, $f(x) = x^{1/3}$, $b_n = 0.7$ (l.h.s) and $b_n = 0.6$ (r.h.s)

and let $\tilde{r} = F(\tilde{x})$. We can then define the following closed sets:

$$H_+ := \begin{cases} \{(x,y) : x \geq x_p,\ V(x,y) \leq \tilde{r},\ y \leq \delta_+(x)\}, & \text{if } \Delta(r) \text{ is increasing,} \\ \{(x,y) : x \geq x_p,\ V(x,y) \leq \tilde{r},\ y \geq \delta_+(x)\}, & \text{if } \Delta(r) \text{ is decreasing,} \end{cases}$$

$$H_- := \{(x,y) : (-x,-y) \in H_+\}.$$

The sets G_+, G_-, H_+, and H_- are shown in Figure 2.

It follows immediately from the definition of the sets that the mapping $U(p;\cdot)$ maps the sets H_+ and H_- into $G_- \cap \{V(x,y) \leq \tilde{r}\}$ and $G_+ \cap \{V(x,y) \leq \tilde{r}\}$, respectively. Now let $x(t)$ be a solution of (2) such that $X(t_{n-1}+0;X_0) \in H_+$ (H_-) and $0 \leq b_n \leq 1$. Then $X(t_n - 0; X_0) \in G_- \cap \{V(x,y) \leq \tilde{r}\}$ $(G_+ \cap \{V(x,y) \leq \tilde{r}\})$. To guarantee that $X(t_n+0;X_0) \in H_-$ $(X(t_n+0;X_0) \in H_+)$, we need an additional condition on b_n, such as $b_n \leq \sup\{\delta_+(x)/\gamma_+(x) : x \in (u_p, x(t_n-0))\}$. The following theorem then holds.

Theorem 4 *Assume that $\Delta(r)$ is monotonic on an interval $[r_p, r_p+\varepsilon]$ or $[r_p,\infty)$. Let $r_0 \leq \tilde{r}$, and assume that*

$$0 \leq b_n \leq \sup\{\delta_+(x)/\gamma_+(x) : x \in (u_p, F^{-1}(r_0))\}, \quad n = 1,2,\ldots \tag{10}$$

If a solution $x(t)$ of (2) satisfies $(x(t_0+0), \dot{x}(t_0+0)) = X_0 \in H_+ \cap \{V(x,y) \leq r_0\}$ $(H_- \cap \{V(x,y) \leq r_0\})$, then $(x(t_0+2np+0), \dot{x}(t_0+2np+0)) \in H_+$ (H_-). In addition, $\lim_{t\to\infty} V(x(t),\dot{x}(t)) = r_p$, i.e., $\lim_{t\to\infty}(x(t)-u_p(t)) = 0$.

For the proof of the last statement, we have only to note that condition (10) is stronger than (9) since the supremum in (10) is smaller than 1. The right hand side of Figure 3 shows the attractivity of u_p for positive impulses.

To illustrate our theorem, in Figure 4 we show the values of the mappings $X(t_0+np;\cdot)$ for $b_n = 1, 0.9, 0.8, 0.7, 0.6, 0.5$. Later, we will return to the question of the sharpness of the estimates of the fraction $\delta_+(x)/\gamma_+(x)$.

Combining the arguments for negative and positive impulses, we can formulate the following more general theorem.

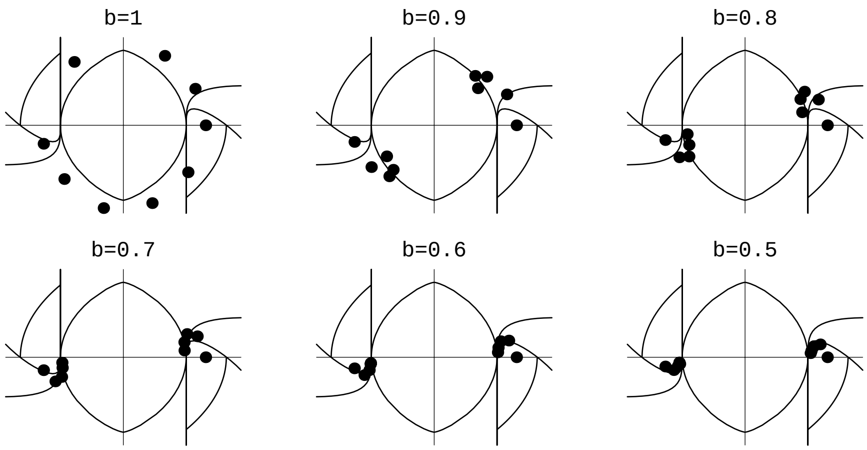

Figure 4: Mappings $X(t_0+np;\cdot)$, $f(x)=x^{1/3}$, $p=3$, $b_n=1,0.9,0.8,0.7,0.6,0.5$

Theorem 5 *Assume that* $\Delta(r)$ *is monotone on an interval* $[r_p, r_p+\varepsilon]$ *or* $[r_p,\infty)$, $r_0 \le \tilde{r}$, *and assume that conditions* (9) *and*

$$-1 \le b_n \le \sup\{\delta_+(x)/\gamma_+(x) : x \in (u_p, F^{-1}(r_0))\} \quad n = 1,2,\ldots \tag{11}$$

hold. If a solution $x(t)$ *of* (2) *satisfies* $(x(t_0+0),\dot{x}(t_0+0)) = X_0 \in H_+ \cap \{V(x,y) \le r_0 \le \tilde{r}\}$ $(H_- \cap \{V(x,y) \le r_0 \le \tilde{r}\})$, *then* $\lim_{t\to\infty} V(x(t),\dot{x}(t)) = r_p$, *i.e.,* $\lim_{t\to\infty}(x(t)-u_p(t)) = 0$.

The key to the applicability of our results is to either find the curves γ_+ and δ_+ analytically or to approximate them numerically. In either case, computer algebra programs are very useful. The curves in Figures 1 and 2 are obtained from the definitions of γ_+ and δ_+, interpolating the points $\{U(p,U_0) : U_0 \in \{(u_0, i\,d), i=1,\ldots\}\}$ and $\{U(2p,U_0) : U_0 \in \{(u_0, i\,d), i=1,\ldots\}\}$, respectively, where the step size d is small enough. This approach is quite fast and good enough (although not analytically certain) to verify that a point (x,y) is in G_+ (H_+), but it is not applicable to estimate the quotient $\delta_+(x)/\gamma_+(x)$ if x is close to u_p since $\lim_{x\to u_0+0}\delta_+(x)/\gamma_+(x)$ is of form $0/0$.

First, let us give estimates for the sets G_+ and H_+; for simplicity, we assume that f is monotonic. From the definition, we have

$$\gamma_+(x) = y_p - \int\limits_0^{\tau(p,r)} f(u(s;(u_p,y_p)))\,ds.$$

Let $r_p < r_0 \le \tilde{r}$ be given, and assume that $x > u_p$ and $V(x,\gamma_+(x)) = r \le r_0$. Since $\gamma_+(x) > 0$ (< 0) and $\tau(p,r) > 0$ (< 0) for $x > u_0$, $r > r_p$, $\Delta(r)$ is increasing (decreasing), and $f(x)$ is monotonic, we have

$$\sqrt{|r-r_p|} - f(F^{-1}(r_p))|\Delta(r)-p| \ge |\gamma_+(x)| \ge \sqrt{|r-r_p|} - f(F^{-1}(r_0))|\Delta(r)-p|,$$

where $|y_p| = \sqrt{|r - r_p|} = \sqrt{|r - \Delta^{-1}(p)|}$. If, in addition, $\Delta(r)$ is differentiable and $|\tau(p, r)|$ is concave up for $r_p < r \le r_0$, then

$$\begin{aligned} &\sqrt{|r - r_p|} - f(F^{-1}(r_p))|\Delta'(r_p)||r - r_p| \\ &\geq \ |\gamma_+(x)| \\ &\geq \ \sqrt{|r - r_p|} - f(F^{-1}(r_0))|\Delta'(r_0)||r - r_p|. \end{aligned}$$

From the above estimates, we can easily give sufficient conditions for (x, y) to be in H_+. Let (x, y) be such that $r_p \le r = V(x, y) \le r_0$, $x \ge u_0$, and $y \ge 0$ if $\Delta(r)$ is increasing and $y \le 0$ if $\Delta(r)$ is decreasing. Then $(x, y) \in H_+$, if and only if $x = u(t_1; (u_p, y_p))$ and $t_1 \ge \tau(p, r)$. Obviously, for $u_p \le \bar{x}$, the inequality $t_1 \ge \tau(p, r)$ is equivalent to $|y| \le |\dot{u}(\tau(p, r))| = |\gamma_+(F^{-1}(r))|$. If the inequality

$$|y| \le \sqrt{|r - r_p|} - f(F^{-1}(r_0))|\Delta'(r_0)||r - r_p| \tag{12}$$

holds, then $(x, y) \in G_+$ since

$$\begin{aligned} |y| \le \sqrt{|r - r_p|} - f(F^{-1}(r))|\Delta(r) - p| \ &\le \ |y_p| - \int_0^{|\tau(p,r)|} f(u(s; (u_p, y_p)))\, ds \\ &= \ |\gamma_+(F^{-1}(r))|\,. \end{aligned}$$

Similarly, for δ_+, we obtain

$$\begin{aligned} &\sqrt{|r - r_p|} - 2f(F^{-1}(r_p))|\Delta'(r_p)||r - r_p| \\ &\geq \ |\gamma_+(x)| \\ &\geq \ \sqrt{|r - r_p|} - 2f(F^{-1}(r_0))|\Delta'(r_0)||r - r_p|. \end{aligned}$$

A sufficient condition for (x, y) to be in H_+ for $x \ge u_p$ and $r_p \le V(x, y) = r \le r_0$ is

$$|y| \le \sqrt{|r - r_p|} - 2f(F^{-1}(r_0))|\Delta'(r_0)||r - r_p|. \tag{13}$$

On the basis of the above estimations, we can define the curves $\underline{\gamma}_+$ and $\underline{\delta}_+$ as the set of points (x, y) satisfying

$$y = \operatorname{sgn}(\tau(p, V(x, y)))(\sqrt{|V(x, y) - r_p|} - f(F^{-1}(r_0))|\Delta'(r_0)||V(x, y) - r_p|)$$

and

$$y = \operatorname{sgn}(\tau(p, V(x, y)))(\sqrt{|V(x, y) - r_p|} - 2f(F^{-1}(r_0))|\Delta'(r_0)||V(x, y) - r_p|),$$

respectively. The above equations can be solved for y, and the curves can be written in the form $\underline{\gamma}_+(x)$ and $\underline{\delta}_+(x)$. The expressions are very complicated, and

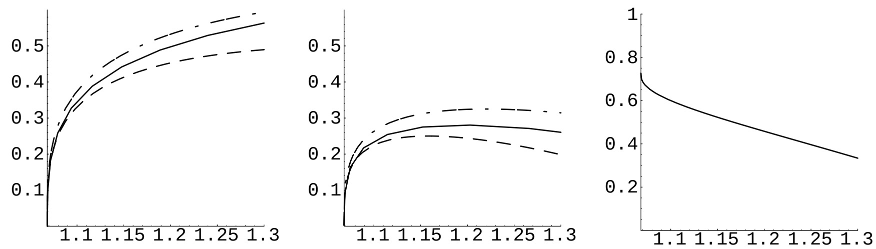

Figure 5: The curves $\underline{\delta}_+(x),\ \delta_+(x),\ \bar{\delta}_+(x),\ \underline{\gamma}_+(x),\ \gamma_+(x),\ \bar{\gamma}_+(x)$; the fraction $\frac{\underline{\delta}_+(x)}{\bar{\gamma}_+(x)}$ $(f(x) = x^{1/3},\ p = 3,\ u_p = 1.06813,\ x_0 = 1.3)$

we omit the details here. (We suggest that the reader perform the calculations with the aid of a computer algebra system such as Mathematica.) Using the curves $\underline{\gamma}_+$ and $\underline{\delta}_+$ instead of γ_+ and δ_+, we can define the sets $\underline{G}_+,\ \underline{G}_-, \underline{H}_+$ and $\underline{H}_-$ as attractivity regions.

Finally, to increase the applicability of Theorems 4 and 5, we give a lower estimate for $\delta_+(x)/\gamma_+(x)$. Let $\bar{\gamma}_+$ be the set of points (x, y) satisfying

$$y = \operatorname{sgn}(\tau(p, V(x, y)))(\sqrt{|V(x, y) - r_p|} - f(F^{-1}(r_0))|\Delta'(r_p)||V(x, y) - r_p|);$$

a similar expression holds for the curve $y = \bar{\delta}_+$. If $x \geq u_p$, $V(x, \bar{\gamma}_+(x))) \leq r_0$, and $\underline{\delta}_+(x)$ has no zero on $(u_p, F^{-1}(r_0))$, then

$$\frac{\delta_+(x)}{\gamma_+(x)} \geq \frac{\underline{\delta}_+(x)}{\bar{\gamma}_+(x)}.$$

Computer calculations with Mathematica resulted in Figure 5, which shows the curves $\underline{\delta}_+(x)$, $\delta_+(x)$, $\bar{\delta}_+(x)$, $\underline{\gamma}_+(x)$, $\gamma_+(x)$, and $\bar{\gamma}_+(x)$, and the fraction $\frac{\underline{\delta}_+(x)}{\bar{\gamma}_+(x)}$ for the case $f(x) = x^{1/3}$, $p = 3$, $u_p = 1.06813$, and $x_0 = 1.3$. This explains why, in Figure 4, the attractivity properties change somewhere around 0.7.

5 Generalizations and Open Problems

In the theorems in the previous section, we considered only the periodic solution with energy $r_p = \Delta^{-1}(p)$, but in the proofs we did not use the fact that $2p$ is the smallest period. Thus, the theorems can also be formulated for the periodic solutions $r_{p/k} = \Delta^{-1}(p/k)$ by changing r_p to $r_{p/k}$. The definition of τ is $\tau(p/k, r) = k(\Delta(r) - \Delta(r_{p/k})) = k\Delta(r) - p$.

If $\lim_{r\to 0} \Delta(r) = 0$, then there are infinitely many periodic solutions, and they accumulate at the origin (this is the case for $f(x) = x^{1/3}$). The following question arises: are all the solutions trapped at some t_n and by some periodic trajectory? The answer is closely related to the classical problem on the existence of solutions that tend to zero as $t \to \infty$. We note that this problem is still unsolved for the equation

$$\ddot{x} + g(t)\dot{x} + f(x) = 0$$

if $f(x)$ is a nonlinear function, e.g., a power function. For the linear case, see [6].

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