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Quantization and irreducible representations of infinite-dimensional transformation groups and Lie algebras *

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Abstract

We present an analytic version of a theorem of Burnside and apply it to the study of irreducible representations of doubly-transitive groups and Lie algebras. Application to the Dirac quantization problem is given.

1 Group actions

Let G be a group and let M be a set. An *action* of G on M is a map from G to the permutations of M such that, for $g, h \in G$ and $x \in M$

$$g \cdot (h \cdot x) = (gh) \cdot x$$
$$e \cdot x = x \quad (e = \text{ identity}).$$

Example. Let G be a group and let H be a subgroup of G. Let $M = G/H$, the space of left cosets. Define

$$g \cdot (aH) = (ga)H,$$

the obvious "translation" action of G on the coset space. (If $H = \{e\}$ we have $M = G$ and the action is simply G acting on itself by left translation.)

Often M has additional structure; for example, M may be a manifold. Then we want G to act by diffeomorphisms (smooth mappings) of M. Or, if M carries a smooth measure, we may want G to act via measure-preserving diffeomorphisms.

2 Burnside's theorem

This is basically a nineteenth century theorem. See [2].

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Theorem. *Let G be a discrete group acting on a discrete set M. Suppose that the action of G is doubly transitive: that is, if x, y, x', y' are in M, there exists $g \in G$ with $g \cdot x = x'$, $g \cdot y = y'$.*

Then the natural unitary representation U of G on $l^2(M)$ is (essentially) irreducible. That is, U is irreducible if $|M| = \infty$, while if $|M| < \infty$ there are just two irreducible components, viz. (scalars) and $l^2(M)\ominus$ (scalars), the orthogonal complement.

Note. The "natural representation" U is just given by left translation:

$$(U_a f)(x) = f(a^{-1} \cdot x),$$

$a \in G$, $x \in M$, $f \in l^2(M)$.

Proof . Let $T : l^2(M) \to l^2(M)$ be an intertwining operator for U; that is, for all $a \in G$, $TU_a = U_a T$. Since M is discrete, the operator T has a matrix kernel K such that, for $f \in l^2$,

$$(Tf)(x) = \sum_{y \in M} K(x, y) f(y).$$

The intertwining condition readily implies the identity $K(a \cdot x, a \cdot y) = K(x, y)$, which means that K is constant on the G-orbits in $M \times M$. But there are just two such orbits, namely the diagonal Δ and its complement.

Hence the space of intertwining operators is at most two-dimensional, generated by the identity I and projection onto the scalars P. But the operator P is 0 if $|M|$ is infinite, so in the latter case the representation is irreducible. □

3 Main results

Our main results are analogues of Burnside's theorem, but the analytic details are more involved. For example, we use the Schwartz kernel theorem to study the intertwining operators.

Transitive and doubly-transitive actions of Lie algebras

Le M be a smooth manifold, $\mathrm{Vect}(M)$ the Lie algebra of smooth vector fields on M, and $\mathfrak{G}$ any Lie algebra. An action of $\mathfrak{G}$ on M is just a homomorphism

$$A : \mathfrak{G} \to \mathrm{Vect}(M)$$

$X \in \mathfrak{G} \mapsto A(X)$, a vector field on M which is linear and such that $A([X, Y]) = [A(X), A(Y)]$. (This is simply the "infinitesimal analogue" of a group action.)

Definition. 1. $\mathfrak{G}$ acts transitively on M provided that, for each point $p \in M$, $\{A(X)_p : X \in \mathfrak{G}\} = T_p(M)$, the tangent space of M at the point p.

2. $\mathfrak{G}$ acts doubly-transitively on M provided $\mathfrak{G}$ acts transitively on $M \times M \backslash \Delta$. That is, given $p \neq q \in M$, $v \in T_p(M)$, $w \in T_q(M)$, there exists $X \in \mathfrak{G}$ with $A(X)_p = v$ and $A(X)_q = w$.

3. n-fold transitivity may be similarly defined.

Examples

A. Let (M, μ) be a smooth manifold with a smooth measure μ. Let $\mathfrak{G} = \mathrm{Vect}_\mu(M)$, the Lie algebra of divergence-free vector fields on M. If $\dim M \geq 2$, $\mathfrak{G}$ acts n-fold transitively on M for all $n \geq 1$. (This is easy to see.)

B. Let ω be a closed 2-form on M, so that (M, ω) is a symplectic manifold (= a "phase space"). From ω we define a Poisson bracket structure on $C^\infty(M)$:

$$\{f, g\} = \omega(\xi_f, \xi_g)$$

where ξ_f is the Hamiltonian vector field corresponding to $f \in C^\infty(M)$.

For example, take $M = \mathbb{R}^{2n}$ with canonical coordinates q's and p's;

$$\begin{aligned}
\omega &= \sum_i dq_i \wedge dp_i \\
\{f, g\} &= \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \\
\xi_f &= \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q_i} \right).
\end{aligned}$$

ξ_f may be viewed as a vector field, or as a first-order skew-symmetric differential operator.

Then $A : f \mapsto \xi_f$ is m-fold transitive for all $m \geq 1$. (This is via an easy "patching" argument using partitions of unity.)

Cocycles for a Lie algebra action

Let $A : \mathfrak{G} \to \mathrm{Vect}(M)$ be an action of $\mathfrak{G}$ on M, by divergence-free vector fields for simplicity. Consider a 0th order perturbation of A:

$$B(X) = A(X) + i\rho(X).$$

Here $\rho(X) \in C^\infty(M)$ depends linearly on $X \in \mathfrak{G}$. $B(X)$ is a skew-symmetric first-order differential operator. We want the mapping $X \mapsto B(X)$ to be a Lie algebra homomorphism:

$$B([X, Y]) = [B(X), B(Y)].$$

This leads to the following *cocycle identity*:

$$\rho([X, Y]) = A(X) \cdot \rho(Y) - A(Y) \cdot \rho(X).$$

Example. (L. van Hove, 1951). Let $M = \mathbb{R}^{2n}$, $\mathcal{F} = C^\infty(M)$ = the Poisson bracket Lie algebra over M. Let $A(f) = \xi_f$, the Hamiltonian vector field corresponding to $f \in \mathcal{F}$. Set

$$B(f) = \xi_f + i\theta(f)$$

where $\theta : \mathcal{F} \to \mathcal{F}$ is linear and θ satisfies the cocycle identity (expressed in terms of Poisson brackets):

$$\theta(\{f, g\}) = \{f, \theta(g)\} + \{\theta(f), g\}.$$

But this just says that θ is a derivation of the Lie algebra $\mathcal{F}$.

Thinking of $B(f)$, acting on $L^2(M)$, as a quantum operator corresponding to the classical observable (= function) f, we impose the non-triviality condition

$$\theta(1) = 1$$

so that $B(1) = I$ = the identity operator on $L^2(M)$.

The derivations of $C^\infty(M, \omega) = \mathcal{F}$ have been completely determined for general symplectic manifolds (M, ω). For $M = \mathbb{R}^{2n}$, van Hove discovered the formula

$$\theta(f) = f - \sum_{i=1}^{n} p_i \partial f / \partial p_i.$$

Then, as required, $\theta(1) = 1$. Moreover θ is unique up to an inner derivation.

Irreducibility theorems

These are analogues of Burnside's theorem and theorems of Mackey and Shoda.

Theorem 1. *Let (M, μ) be a connected manifold with a smooth measure μ. Let $A : \mathfrak{G} \to Vect_\mu(M)$ be an action of the Lie algebra $\mathfrak{G}$ via divergence-free skew-adjoint vector fields on M. Assume that the action is doubly transitive.*

Let ρ be a cocycle for the action A. The representation B is defined by

$$B(X) = A(X) + i\rho(X).$$

Also, assume that the dimension of M is ≥ 2 or that $M = S^1$, so that $M \times M \backslash \Delta$ is connected.

Then the representation B on $L^2(M, \mu)$ has at most two irreducible components.

Sketch of Proof. Consider $T : L^2 \to L^2$ an intertwining operator for B with kernel K a distribution on $M \times M$. (Here we use the Schwartz kernel theorem.) The intertwining condition leads to a family of partial differential equations satisfied by the kernel K. Moreover this family is elliptic. Hence K is smooth off the diagonal Δ, and the double transitivity of A may be used to show that there exists at most a two-dimensional family of intertwining operators. $\square$

Theorem 2. *Let (M, μ) be a connected manifold with smooth measure μ. Let the action A of $\mathfrak{G}$ and the cocycle ρ satisfy the hypotheses of Theorem 1. In particular, A is assumed to be doubly transitive.*

Also assume that the cocycle ρ satisfies the following condition: Given a point $p \in M$ denote by ρ_p the character of the stabilizer algebra $\mathfrak{G}_p = \{X \in \mathfrak{G} : A(X)_p = 0\}$, determined by restricting the character ρ to $\mathfrak{G}_p$.

Finally, assume that there are two points $p, q \in M$ such that ρ_p and ρ_q restrict to distinct characters of $\mathfrak{G}_p \cap \mathfrak{G}_q$.

Conclusion. The representation $B = A + \rho$ is irreducible on $L^2(M, \mu)$.

(N.B. In this theorem we do not need to assume that $M \times M \backslash \Delta$ is connected. So the theorem holds for $M = \mathbb{R}$, e.g.)

Proof. Theorem 2 is basically an application of Theorem 1. The condition on the character ρ is used to show that the intertwining kernel $K(x, y)$ must vanish off the diagonal, from which it follows that the intertwining operators are just scalar multiples of the identity I. □

Applications

1. Van Hove's prequantization representations are irreducible:

Here $M = \mathbb{R}^{2n}$, $\mathfrak{G} = \mathcal{F} = C^\infty_{\text{comp}}(\mathbb{R}^n, \omega)$, the Poisson bracket Lie algebra; $A(f) = \xi_f =$ The Hamiltonian vector field generated by $f \in \mathcal{F}$; $\rho = \lambda\theta$, where λ is a real non-0 scalar; $\theta(f) =$ van Hove's derivation $= f - \sum_{i=1}^n p_i \theta f / \partial p_i$. $\mathcal{F}_q \cap \mathcal{F}_b = \{f \in \mathcal{F} : \nabla f$ (or ξ_f) vanishes at the points a and $b\}$.

If $f \in \mathcal{F}_a \cap \mathcal{F}_b$, $\rho_a(f) = \lambda f(a)$ and $\rho_b(f) = \lambda f(b)$. But $f(a)$ and $f(b)$ can be anything at all, so $\rho_a \neq \rho_b$. Therefore Theorem 2 applies to show that the representation on $L^2(\mathbb{R}^n)$ given by

$$B_\lambda(f) = \xi_f + i\lambda\rho(f)$$

is irreducible.

2. The above generalizes to the case of any non-compact symplectic manifold (M, ω) with ω exact.

3. For compact (M, ω), A. Avez defines

$$\theta(f) = \text{mean value of } f \text{ on } M.$$

Then $B_\lambda(f) = \xi_f + i\lambda\theta(f)$ has two irreducible components, namely the scalars and their orthogonal complement.

4. The prequantization representations of Souriau, Kostant, and Urwin are all (essentially) irreducible.

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