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Explicit construction, uniqueness, and bifurcation curves of solutions for a nonlinear Dirichlet problem in a ball *

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Dedicated to Alan Lazer on his 60th birthday

Abstract

This paper presents a method for the explicit construction of radially symmetric solutions to the semilinear elliptic problem

$$\begin{aligned} \Delta v + f(v) = 0 \quad \text{in } B \\ v = 0 \quad \text{on } \partial B\,, \end{aligned}$$

where B is a ball in $\mathbb{R}^N$ and f is a continuous piecewise linear function. Our construction method is inspired on a result by E. Deumens and H. Warchall [8], and uses spline of Bessel's functions. We prove uniqueness of solutions for this problem, with a given number of nodal regions and different sign at the origin. In addition, we give a bifurcation diagram when f is multiplied by a parameter.

1 Introduction

The purpose of this paper is to explicitly construct radially symmetric solutions $v : B \to \mathbb{R}$ to the nonlinear Dirichlet problem

$$\begin{aligned} \Delta v + f(v) = 0 \quad \text{in } B \\ v = 0 \quad \text{on } \partial B, \end{aligned} \tag{1.1}$$

where B is the ball in $\mathbb{R}^N$ centered at the origin with radius π, Δ is the Laplacian operator, and $f : \mathbb{R} \to \mathbb{R}$ is a continous piecewise-linear function such that $f(0) = 0$, f has a positive zero, and $f'(0) = f'(\infty)$.

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We construct solutions to (1.1) with a given number of zeros in their radial profiles. Our method provides an explicit calculation rather than the existence result presented in [3, 4, 5, 6, 9, 10, 11, 12]. Our constructions further develops the authors' work in [2] and the paper by E. Deumens and H. Warchall [8].

Let $\lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$ be the eigenvalues of $-\Delta$ acting on radial functions of $H_0^1(B)$ (see [1]) and $\{\varphi_1, \varphi_2, \cdots, \varphi_k, \cdots\}$ be the corresponding complete set of eigenfunctions.

Let $\lambda_{j+1} > \alpha^2 > \lambda_j$, $\beta > 0$, and

$$f(t) = \begin{cases} \alpha^2\, t & \text{if } t \le \frac{\beta}{2}, \\ -\alpha^2\, t + \alpha^2\, \beta & \text{if } \frac{\beta}{2} \le t \le \beta, \\ \alpha^2\, t - \alpha^2\, \beta & \text{if } t \ge \beta. \end{cases} \tag{1.2}$$

In Section 2, we shall construct radially symmetric solutions to (1.1) with the above nonlinear function f.

We recall that the radial solutions to (1.1) are the solutions to the ordinary differential equation

$$\begin{gathered} v'' + \frac{N-1}{r} v' + f(v) = 0 \quad (0 < r \le \pi) \\ v'(0) = 0\,, \quad v(\pi) = 0\,. \end{gathered} \tag{1.3}$$

We give a method for finding the initial data $v(0)$ corresponding to a radially symmetric solution with i nodes in $(0, \pi)$, $0 \le i \le j-1$, (see Table 1).

We observe that Deumens and Warchall [8] studied a nonlinear wave equation in $\mathbb{R}^{N+1}$. Derrick et al [7] studied problem (1.1) in unbounded domains. It is worth remarking here that our construction is made in a bounded domain. Castro and Cossio [4] dealt with a type of nonlinearity similar to the nonlinearity found in (1.2). They use bifurcation theory to show the existence of solutions but they do not give a method for the explicit construction of solutions.

In Section 3, we prove uniqueness of the solution constructed in Section 2. More precisely, we show the following theorem.

Theorem 1.1 *Let f be as in (1.2). For each $0 \le i \le j-1$ there exist unique solutions v_i and u_i to (1.1) with i nodes in $(0, \pi)$ such that $v_i(0) > \beta > 0$ and $0 < u_i(0) < \beta$.*

In Section 4, we obtain a description of the graph of the set of radial solutions to

$$\begin{gathered} \Delta v + \lambda\, f(v) = 0 \quad \text{in } B \\ v = 0 \quad \text{on } \partial B, \end{gathered} \tag{1.4}$$

where $\lambda \in \mathbb{R}$ is a parameter (see Figures 4 and 5). Figures 6, 7, and 8 were generated with software, written by the authors, following the method of construction given in Section 2.

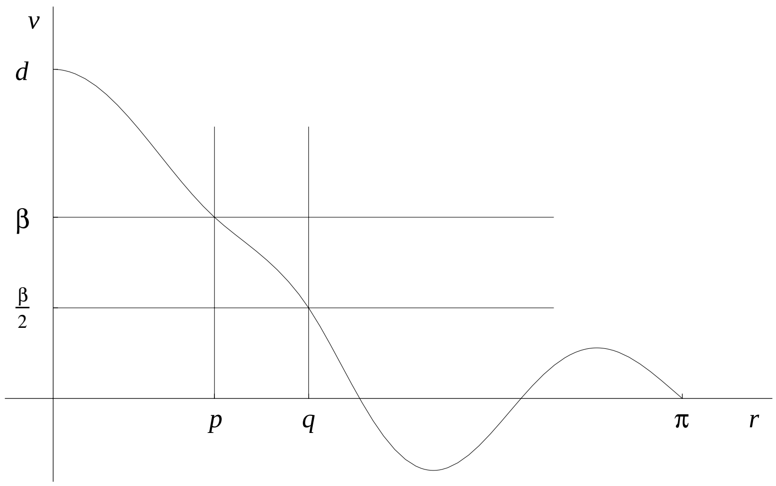

Figure 1: Radial profile of a solution of (1.3) with 2 nodes

2 Explicit construction of radially symmetric solutions

In each r-interval where $v(r)$ lies between $-\infty$ and $\frac{\beta}{2}$, or between $\frac{\beta}{2}$ and β, or between β and $+\infty$, the equation (1.3) has the form

$$v'' + \frac{N-1}{r}\, v' + K_1\, v + K_2 = 0, \tag{2.1}$$

with K_1 and K_2 constants depending only on $f'(0) = \alpha^2$ and β. The solution to this equation is

$$N \geq 2: \qquad v(r) = Ar^{-\nu} J_\nu(kr) + Br^{-\nu} N_\nu(kr) - \frac{K_2}{K_1},$$

where $k^2 = K_1$, $\nu = \frac{N-2}{2}$, and J_ν and N_ν are the Bessel and Neumann functions (see [2]).

To build solutions to (1.3) we put together several of the above pieces, subject to continuity conditions for v and its first two derivatives, and subject to the boundary conditions $v'(0) = 0$ and $v(\pi) = 0$. For the sake of clarity and easy of manipulations, we henceforth deal with the three-dimensional case.

We discuss the construction of a solution v to problem (1.3) under assumption (1.2) with i nodes in $(0, \pi)$ $(0 \leq i \leq j-1)$ and $v(0) = d > \beta$. The construction of a solution u with i nodes in $(0, \pi)$ $(0 \leq i \leq j-1)$ and $0 < u(0) = d < \beta$ follows a similar pattern.

For $0 \leq r \leq p$ we take $v(r) \geq \beta$. Thus $f(v) = \alpha^2\, v - \alpha^2\, \beta$, and the solution to (1.3) is

$$v_1(r) = \beta + \frac{p_1}{r}\, \sin \alpha\, (r - P_1).$$

For $p \le r \le q$, $\frac{\beta}{2} \le v(r) \le \beta$. Thus $f(v) = -\alpha^2 v + \alpha^2 \beta$, and the solution is

$$v_2(r) = \beta + \frac{p_2}{r} \sinh \alpha (r - P_2).$$

For $q \le r \le \pi, \quad 0 \le v(r) \le \frac{\beta}{2}$. Thus $f(v) = \alpha^2 v$, and the solution is

$$v_3(r) = \frac{p_3}{r} \sin \alpha (r - P_3).$$

This ansatz specifies the solution in terms of 3 coefficients p_1, p_2, p_3 and 2 welding points p and q, and 3 unknowns P_1, P_2, P_3. These 8 unknowns are to be found from the equations stating that v, v', and v'' are continuous at the 2 welding points and the boundary conditions.

The weld point p and the 3 unknowns P_1, P_2, and P_3 are determined by the conditions $v_1'(0) = 0, v_1(p) = v_2(p) = \beta$, and $v_3(\pi) = 0$, and we find

$$p = \frac{\pi}{\alpha}, \quad P_1 = 0, \quad P_2 = \frac{\pi}{\alpha}, \quad P_3 = \frac{\pi}{\alpha}(\alpha - k),$$

where $k \in \mathbb{Z} - \{0\}$.

Remark 1: Note that all solutions of (1.3) with $v(0) > \beta$ satisfy $v(\frac{\pi}{\alpha}) = \beta$.

Since $v_3(r)$ has $(k-1)$ nodes in $(\frac{\pi}{\alpha}(\alpha-k), \pi)$, in order to construct a solution with i nodes in $(0, \pi)$ to problem (1.3) we take $k = i + 1$. Let $z = \alpha q$. Since $v_2'(q) = v_3'(q)$ it follows that z must be a solution of the equation

$$g(z) := \frac{z}{\tan(z - \pi \alpha)} + \frac{z}{\tanh(z - \pi)} - 2 = 0. \tag{2.2}$$

Equation (2.2) has a unique solution z over the interval $(\pi(\alpha - k), \pi(\alpha - k + 1))$ (see Figure 2), which can be found by using Newton's method with initial condition $z_0 \in (\pi(\alpha - k), \pi(\alpha - k + 1))$ and $z_0 \simeq \pi(\alpha - k + 1)$. Using the solution z we get the weld point $q = \frac{z}{\alpha}$.

The remaining continuity conditions yield

$$p_1 = -p_2 = \frac{\beta z}{2\alpha \sinh(z - \pi)}$$

and

$$p_3 = \frac{\beta z}{2\alpha \sin(z - \pi(\alpha - i - 1))}.$$

Since $\lim_{r \to 0^+} v_1(r) = d$, it follows that

$$d = \beta + \frac{\beta z}{2 \sinh(z - \pi)} \quad (z > \pi). \tag{2.3}$$

Thus, we have constructed a solution with i nodes in $(0, \pi)$ and initial condition $d = v(0) > \beta$.

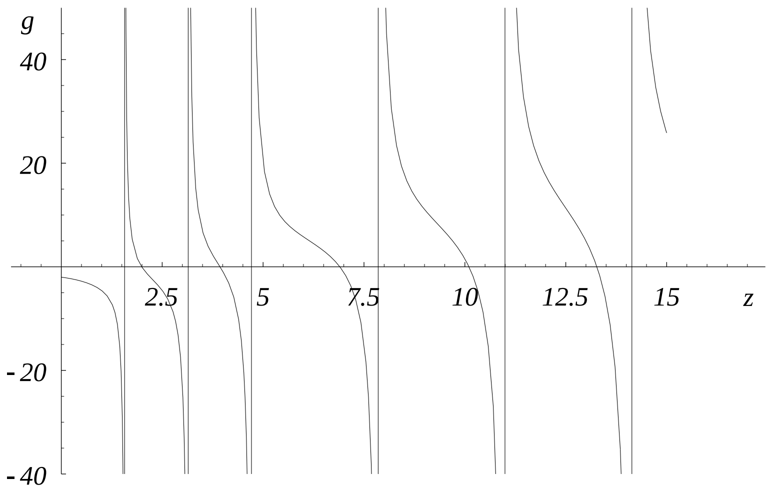

Figure 2: Solutions to (2.2) with $\alpha = 4.9$

Remark 2: For each positive integer m with $1 \leq m \leq j$, let $\alpha_m = \alpha - j + m$. Since $j < \alpha < j + 1$, it follows that

$$m < \alpha_m < m + 1.$$

Therefore, using our method of construction we can obtain solutions with i nodes in $(0, \pi)$ $(0 \leq i \leq m - 1)$ to (1.3) with nonlinearity f given by (1.2) with $\alpha = \alpha_m$. Let us call d_{mi} the initial data corresponding to this solution, which can be found by using (2.3).

Let l be a positive integer less than or equal to i. Since

$$(\pi\,(\alpha_m - (i+1)), \pi\,(\alpha_m - i)) = (\pi\,(\alpha_m - l - (i - l + 1)), \pi\,(\alpha_m - l - (i - l))),$$

we see that finding a solution of (2.3) on $(\pi\,(\alpha_m - (i+1)), \pi\,(\alpha_m - i))$ it is equivalent to find a solution of (2.3) over the interval $(\pi\,(\alpha_m - l - (i - l + 1)),$ $\pi\,(\alpha_m - l - (i - l)))$. Therefore,

$$d_{mi} = d_{(m-l)(i-l)}, \quad (1 \leq m \leq j,\ 0 \leq i \leq m - 1,\ 0 \leq l \leq i).$$

We summarize the above discussion in Table 2 which will be useful for constructing bifurcation diagrams in Section 4.

3 Proof of Theorem 1.1

In this section, we prove uniqueness for the solution to (1.3) with i nodes in $(0, \pi)$ and initial data $v(0) > \beta$.

As we mentioned in Remark 1, solutions to (1.3) satisfy the equation $v(p) = v(\frac{\pi}{\alpha}) = \beta$. Next we derive a basic lemma about the solutions of (1.3).

$\alpha \setminus$ nodes	0	1	2	...	m-1
$1 < \alpha_1 < 2$	d_{10}				
$2 < \alpha_2 < 3$	d_{20}	$d_{21} = d_{10}$			
$3 < \alpha_3 < 4$	d_{30}	$d_{31} = d_{20}$	$d_{32} = d_{10}$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$		
$m < \alpha_m < m+1$	d_{m0}	$d_{m1} = d_{m-1,0}$	$d_{m2} = d_{m-2,0}$	...	$d_{m,m1} = d_{10}$

Table 1: Initial data $v(0) = d$ corresponding to solutions of (1.3)

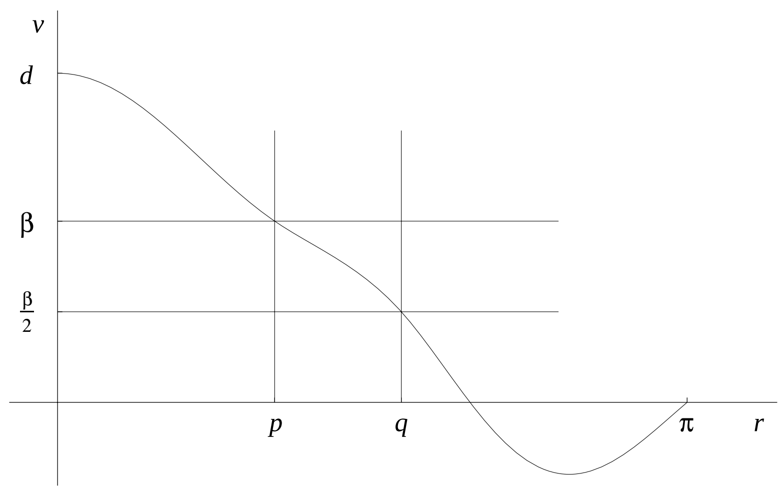

Figure 3: Radial profile of a solution $v(r)$ to problem (1.3)

Lemma 3.1 *Let v_1 and v_2 be two solutions of (1.3) such that $v_1(q) = v_2(q)$. Then*

$$v_1 = v_2 \quad \text{on } [p, q]\,.$$

Proof. Let

$$w(r) = v_1(r) - v_2(r), \quad r \in [p, q].$$

Because v_1 and v_2 are solutions of (1.3), w satisfies

$$w'' + \frac{2}{r} w' + f(v_1) - f(v_2) = 0 \quad p \le r \le q$$
$$w(p) = w(q) = 0\,.$$

Using the Mean Value Theorem, we see that there exists ξ such that

$$w'' + \frac{2}{r}\, w' + f'(\xi)\, w(r) = 0 \quad r \in [p, q]. \tag{3.1}$$

We multiply (3.1) by r^2. This yields

$$(r^2\, w')' + r^2\, f'(\xi)\, w = 0, \quad r \in [p, q].$$

Now we multiply by w and integrate by parts over $[p, q]$, we obtain

$$-\int_p^q r^2\, (w')^2 + \int_p^q r^2\, f'(\xi)\, w^2 = 0\,. \tag{3.2}$$

To prove the lemma we proceed by contradiction. Suppose $w \neq 0$ on $[p, q]$. Since $r \in (p, q)$ we know that $v, \xi \in (\frac{\beta}{2}, \beta)$ so that $f'(\xi) < 0$ on $[p, q]$, we see that

$$-\int_p^q r^2\, (w')^2 + \int_p^q r^2\, f'(\xi)\, w^2 < 0\,. \tag{3.3}$$

This contradicts (3.2). The contradiction shows that $w \equiv 0$ on $[p, q]$. The proof of the lemma follows.

Proof of Theorem 1.1. Let v_1 and v_2 be solutions to (1.3), with $v_1(0) = d_1$ and $v_2(0) = d_2$. Since $v_1(p) = v_2(p) = \beta$, by uniqueness of the initial value problem for ordinary differential equations applied to (1.3) on $[0, p]$, we see that

$$d_1 \neq d_2 \Longrightarrow v_1'(p) \neq v_2'(p).$$

Using Lemma 3.1 we obtain

$$v_1'(p) \neq v_2'(p) \Longrightarrow v_1(q) \neq v_2(q).$$

Finally, using again the uniqueness of the initial value problem for ordinary differential equations, we obtain

$$v_1(q) \neq v_2(q) \Longrightarrow v_1(\pi) \neq v_2(\pi).$$

Therefore, if $d_1 \neq d_2$ we infer that

$$v_1(\pi) \neq v_2(\pi),$$

which is a contradiction because $v_1(\pi) = 0 = v_2(\pi)$. Hence $d_1 = d_2$. This proves uniqueness of solutions to (1.3). Thus, we have proved Theorem 1.1.

4 Construction of bifurcation curves and graphs of solutions

In this section we give a description of the graph of the set of radial solutions to

$$\begin{aligned} \Delta v + \lambda\, f(v) = 0 \quad \text{in } B \\ v = 0 \quad \text{on } \partial B, \end{aligned} \tag{4.1}$$

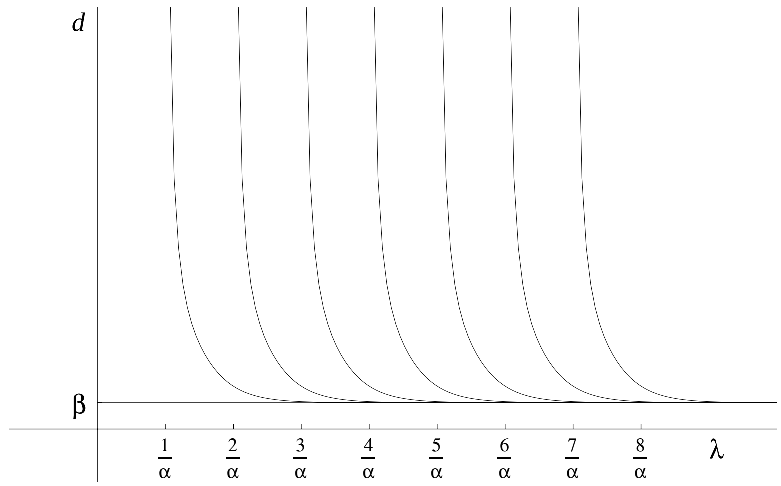

Figure 4: Bifurcation diagram for (4.1) with initial data $d_{mi} > \beta$

where $\lambda \in \mathbb{R}^+$ is a parameter.

Let $\lambda \in \mathbb{R}^+$, $m \in \mathbb{N}$ be such that $m < \lambda\,\alpha < m+1$, and $i = 0, 1, \cdots, m-1$. Now, as we have seen in Section 2, we can find a unique solution $z = z(\lambda)$ to the equation

$$\frac{z}{\tan(z - \pi\,(\lambda\,\alpha))} + \frac{z}{\tanh(z-\pi)} - 2 = 0, \quad \text{on } (\pi\,(\lambda\,\alpha - (i+1)), \pi\,(\lambda\,\alpha - i)).$$

With this solution and (2.3) we find the initial data $d_{mi} > \beta$ corresponding to the solution with i nodes in $(0, \pi)$. Since

$$d = \beta + \frac{\beta\,z}{2\sinh(z-\pi)} \quad (z > \pi),$$

we see that

$$\begin{aligned} d'(z) &< 0 \quad (z > \pi), \\ \lim_{z\to\infty} d(z) &= \beta, \quad \text{and} \\ d'(\lambda) &< 0. \end{aligned}$$

The sequence $\{d_{mi}\}_{i=m-1}^{0} = \{d_{j0}\}_{j=1}^{m}$ is decreasing. Thus, using Table 1 and the previous information, we obtain the following bifurcation diagram

Similarly, we can construct the bifurcation diagram for solutions with initial data $0 < d_{mi} < \beta$ (see Figure 5). In this case, since

$$d = \beta - \frac{\beta\,z}{2\sinh(z)} \quad (z > 0),$$

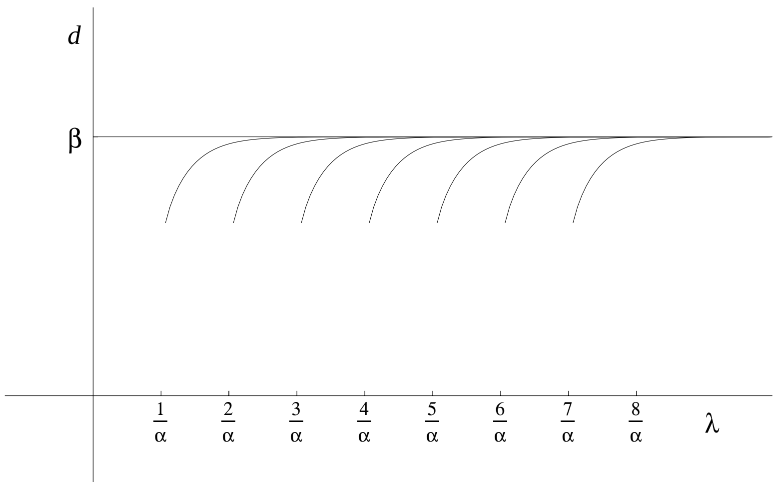

Figure 5: Bifurcation diagram for (4.1) with initial data $0 < d_{mi} < \beta$

we see that

$$\begin{aligned} &d'(z) > 0 \quad (z > 0), \\ &\lim_{z\to\infty} d(z) = \beta, \quad \text{and} \\ &d'(\lambda) > 0\,. \end{aligned}$$

The sequence $\{d_{mi}\}_{i=m-1}^{0} = \{d_{j0}\}_{j=1}^{m}$ is increasing.

Figures 6-8 of radially symmetric solutions to problem (1.1) were generated with software, written by the authors, following the method of construction given in Section 2.

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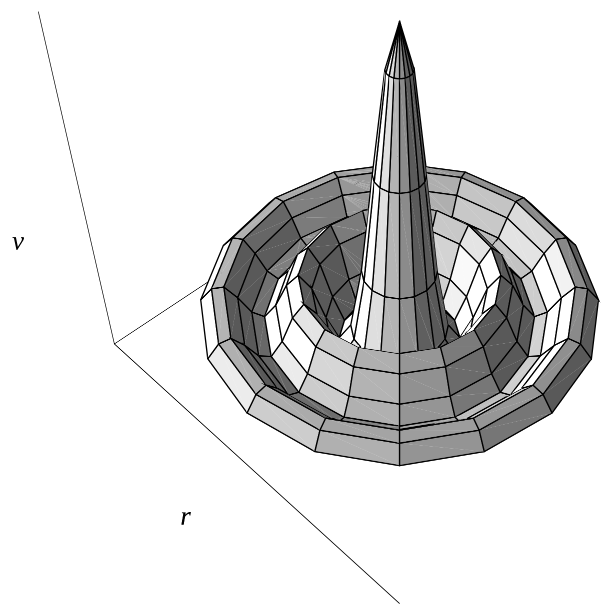

Figure 6: Radial solution in three dimensions with $\alpha = 5.1$, $\beta = 2.0$, and $i = 4$

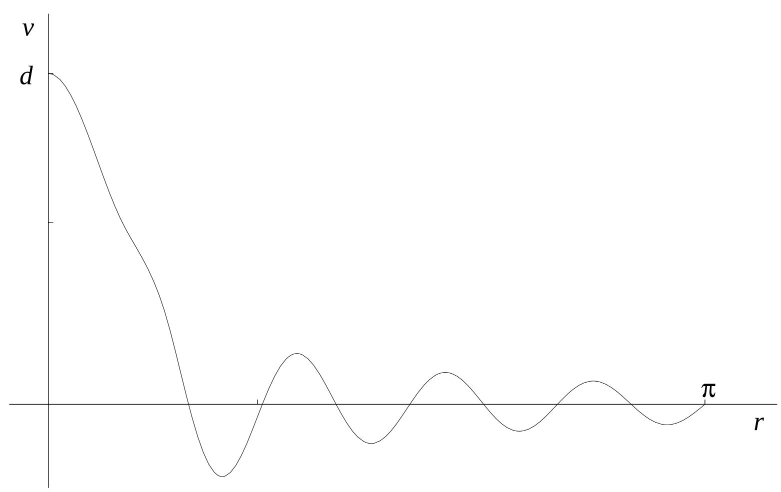

Figure 7: Radial profile of the solution with $\alpha = 8.9$, $\beta = 3.0$, and $i = 7$

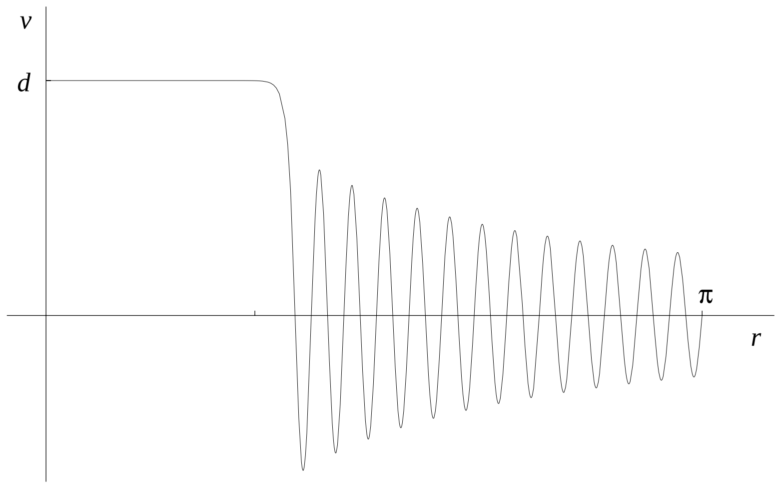

Figure 8: Radial profile of the solution with $\alpha = 40.3$, $\beta = 3.0$, and $i = 25$

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