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An elliptic problem with arbitrarily small positive solutions *

Pierpaolo Omari & Fabio Zanolin

Dedicated to Alan Lazer on his 60th birthday

Abstract

We show that, for each $\lambda > 0$, the problem

$$-\Delta_p u = \lambda f(u) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega$$

has a sequence of positive solutions $(u_n)_n$ with $\max_{\bar\Omega} u_n$ decreasing to zero. We assume that $\liminf_{s\to 0^+} \dfrac{F(s)}{s^p} = 0$ and that $\limsup_{s\to 0^+} \dfrac{F(s)}{s^p} = +\infty$, where $F' = f$. We stress that no condition on the sign of f is imposed.

1 Introduction

Let us consider the quasilinear elliptic problem

$$\begin{aligned} -\Delta_p u &= \lambda\, f(u) \quad \text{in } \Omega\,, \\ u &= 0 \quad \text{on } \partial\Omega\,, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N)$ is a bounded domain, with a smooth boundary $\partial\Omega$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, with $p > 1$, $f : [0, +\infty[\to \mathbb{R}$ is a continuous function and $\lambda > 0$ is a real parameter.

Here, we are concerned with the existence and multiplicity of positive solutions of (1.1), where by a positive solution we mean a function $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, with $u \ge 0$ and $u \not\equiv 0$ in Ω, such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u\, \nabla w = \lambda \int_\Omega f(u)\, w\,,$$

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for every $w \in W_0^{1,p}(\Omega)$. Standard regularity results imply that $u \in C^{1+\sigma}(\bar{\Omega})$, for some $\sigma > 0$.

This problem has been investigated in a quite large number of papers, both in the case where $p = 2$ and in the case where $p \neq 2$, often placing conditions on the behaviour of $f(s)/s^{p-1}$ near 0 and near $+\infty$ of the following types:

$$\lim_{s \to 0^+} \frac{f(s)}{s^{p-1}} = +\infty, \tag{1.2}$$

$$\lim_{s \to +\infty} \frac{f(s)}{s^{p-1}} = 0, \tag{1.3}$$

$$\lim_{s \to 0^+} \frac{f(s)}{s^{p-1}} = 0, \tag{1.4}$$

$$\lim_{s \to +\infty} \frac{f(s)}{s^{p-1}} = +\infty. \tag{1.5}$$

When $p = 2$, assumptions (1.2) and (1.3) are usually referred to as sublinearity conditions, whereas (1.4) and (1.5) as superlinearity conditions at 0 and at $+\infty$, respectively. Just as a convention, we keep this terminology even when $p \neq 2$. Note also that conditions (1.2) and (1.4) both imply, in particular,

$$f(0) \geq 0. \tag{1.6}$$

The existence of (sometimes multiple) positive solutions was proved in the following cases:

- f is sublinear at 0 and at $+\infty$;
- f is superlinear at 0 and at $+\infty$ and has subcritical growth at $+\infty$;
- f is sublinear at 0, superlinear at $+\infty$, has subcritical growth at $+\infty$ and there exists a positive strict upper solution;
- f is superlinear at 0, sublinear at $+\infty$ and there exists a positive strict lower solution.

Classical references in this context are, for example, [1, 2, 3, 4, 7, 8, 9, 10, 11, 16, 17]. More recently, in [15] it was discussed the situation where f is eventually neither sublinear nor superlinear at $+\infty$, in the sense that

$$\liminf_{s \to +\infty} \frac{f(s)}{s^{p-1}} = 0 \quad \text{and} \quad \limsup_{s \to +\infty} \frac{f(s)}{s^{p-1}} = +\infty. \tag{1.7}$$

Yet, a counterexample given in [13] shows that, generally speaking, assumptions (1.6) and (1.7) are not sufficient to guarantee the existence of positive solutions of (1.1). Accordingly, in [15] condition (1.7) was strenghthened to

$$\liminf_{s \to +\infty} \frac{F(s)}{s^{p}} = 0 \quad \text{and} \quad \limsup_{s \to +\infty} \frac{F(s)}{s^{p}} = +\infty, \tag{1.8}$$

where $F : [0, +\infty[\to \mathbb{R}$ is such that $F' = f$. Then, it was proved that, under (1.6) and (1.8), problem (1.1) has, for each $\lambda > 0$, a sequence of $(u_n)_n$ of positive solutions, with $\max_{\bar{\Omega}} u_n \to +\infty$.

The aim of this paper is to show that the above considered conditions at $+\infty$ can be replaced by similar ones at 0, in order to produce arbitrarily small positive solutions of (1.1). Namely, the following holds.

Theorem *Assume*

$$\liminf_{s\to 0^+} \frac{F(s)}{s^p} = 0 \quad \textit{and} \quad \limsup_{s\to 0^+} \frac{F(s)}{s^p} = +\infty. \tag{1.9}$$

Then, problem (1.1) *has, for each* $\lambda > 0$, *a sequence* $(u_n)_n$ *of positive solutions, satisfying* $\max_{\bar{\Omega}} u_n \searrow 0$ *and* $\frac{1}{p}\int_\Omega |\nabla u_n|^p - \lambda \int_\Omega F(u_n) \nearrow 0$.

Remark 1 The assumptions

$$\limsup_{s\to 0^+} \frac{F(s)}{s^p} = +\infty \quad \text{and} \quad \liminf_{s\to +\infty} \frac{F(s)}{s^p} = 0$$

and, respectively,

$$\liminf_{s\to 0^+} \frac{F(s)}{s^p} = 0 \quad \text{and} \quad \limsup_{s\to +\infty} \frac{F(s)}{s^p} = +\infty,$$

together with some other technical conditions, have been also considered in [14], [12], [6] and [5], for proving the existence of at least one positive radial solution of (1.1), in the case where Ω is an annular domain.

Remark 2 No condition on the sign of f is required in our result; yet, if $f(s) \geq 0$ in a neighbourhood of 0, the strong maximum principle implies that every (small) positive solution u of (1.1) is actually strictly positive, i.e. $u(\mathrm{x}) > 0$ in Ω and $\frac{\partial u}{\partial \nu}(\mathrm{x}) < 0$ on $\partial\Omega$. The same conclusion still holds in the case where f changes sign near 0, provided that the nondecreasing regularization $\widehat{f^-}$ of f^-, defined by $\widehat{f^-}(s) = \max_{t\in[0,s]} f^-(t)$, satisfies $\int_0^1 (s\widehat{f^-}(s))^{-1/p} ds = +\infty$. To verify this, it is sufficient to observe that $-\Delta_p u \geq -\lambda \widehat{f^-}(u)$ in Ω and to apply Theorem 5 in [18].

Remark 3 It is quite easy to find continuous functions $f : [0, +\infty[\to \mathbb{R}$ which change sign in any neighbourhood of 0 and for which condition (1.9) is fulfilled. For instance, one can take $f = F'$, with

$$F(s) = s^q \sin(s^{-\gamma}) + s^r \cos(s^{-\gamma}) \;\text{ for } s > 0 \quad \text{and} \quad F(0) = 0,$$

where q, r, γ satisfy $q > p > r > 1 + \gamma$ and $\gamma > 0$. On the contrary, it seems less immediate to exhibit positive functions f, for which (1.9) holds. We produce here the example of a continuous (even nondecreasing) function $f : [0, +\infty[\to \mathbb{R}$, with $f(s) > 0$ in $]0, +\infty[$, such that F satisfies (1.9). Let $(s_n)_n$, $(t_n)_n$ and $(\delta_n)_n$ be sequences defined by

$$s_n = 2^{-\frac{1}{2}n!}, \quad t_n = 2^{-2n!} \quad \text{and} \quad \delta_n = 2^{-(n!)^2}.$$

Observe that, for all large n,

$$s_{n+1} < t_n < s_n - \delta_n.$$

Fix $p > 1$. Let $f : [0, +\infty[\to \mathbb{R}$ be a continuous nondecreasing function such that $f(0) = 0$, $f(s) > 0$ for $s > 0$ and, for all large n,

$$f(s) = 2^{-(p-1)n!} \quad \text{for } s \in [s_{n+1}, s_n - \delta_n].$$

Let us set $F(s) = \int_0^s f(t)dt$ for $s \geq 0$. Then, it is not difficult to verify that

$$F(s_n)/{s_n}^p \leq \left(f(s_{n+1})s_n + f(s_n)\delta_n\right)/{s_n}^p \to 0$$

and

$$F(t_n)/{t_n}^p \geq \left(f(s_{n+1})(t_n - s_{n+1})\right)/{t_n}^p \to +\infty,$$

as $n \to +\infty$. Since $F(s) > 0$ for $s > 0$, we can conclude that condition (1.9) holds.

Remark 4 It will be clear from the proof that condition (1.9) can be replaced by

$$-\infty < \lambda \liminf_{s \to 0^+} \frac{F(s)}{s^p} < \mu_* \leq \mu^* < \lambda \limsup_{s \to 0^+} \frac{F(s)}{s^p},$$

where μ_*, μ^* are suitable positive constants, depending only on Ω and p.

Remark 5 Our result extends to equations involving a more general class of quasilinear operators of the type $\operatorname{div} A(\mathrm{x}, \nabla u)$, where A satisfies suitable ellipticity and growth conditions of Leray-Lions type, and nonlinearities f also depending on the x-variable. The existence of positive periodic solutions for some classes of quasilinear parabolic equations can be proved along the same lines too.

2 Proof

We will exploit some arguments similar to those introduced in [15], therefore only the main steps of the proof will be produced.

At first, we notice that condition (1.9) implies that $F(0) = 0$ and $f(0) = 0$. Hence, we have, in particular, that the function 0 is a (lower) solution of problem (1.1). It is also convenient for the sequel to extend f and F to the whole of $\mathbb{R}$, as an odd and as an even function, respectively. Throughout this proof, we further suppose that the coefficient $\lambda > 0$ is fixed.

Then, using the former condition in (1.9), we prove the existence of a sequence $(\beta_n)_n \subset C^1(\bar{\Omega})$ of upper solutions of (1.1), satisfying

$$\min_{\bar{\Omega}} \beta_n > 0 \quad \text{and} \quad \max_{\bar{\Omega}} \beta_n \to 0. \tag{2.1}$$

It is obvious that, if $\inf\{s > 0 \,|\, f(s) \le 0\} = 0$, then there exists a sequence $(\beta_n)_n$ of constant upper solutions satisfying (2.1). Therefore, let us suppose that there is a number $s_0 > 0$ such that

$$f(s) > 0 \quad \text{for } s \in]0, s_0] \tag{2.2}$$

and therefore

$$F(s) > 0 \quad \text{for } s \in]0, s_0]. \tag{2.3}$$

By the former condition in (1.9), we can find a sequence $(c_n)_n \subset]0, s_0[$ such that $c_n \searrow 0$ and

$$\frac{F(c_n)}{{c_n}^p} \to 0. \tag{2.4}$$

Let $]a, b[$ be the projection of Ω onto, say, the x_1-axis and consider, for each n, the initial-value problem

$$\begin{aligned} -(|v'|^{p-2}v')' &= \lambda f(v) \quad \text{in } [a, b[, \\ v(a) &= c_n, \\ v'(a) &= 0\,. \end{aligned} \tag{2.5}$$

By a local solution of (2.5) we mean a function v defined on some interval $I \subset [a, b[$, with $a \in I$, which is of class C^1 in I, together with $|v'|^{p-2}v'$, and satisfies the equation in I and the initial conditions. It is known that (2.5) admits local solutions, which can be extended to a right maximal interval of existence $[a, \omega[\subset [a, b[$. Let v be a noncontinuable solution of (2.5) and define

$$\sigma = \sup\{t \in]a, \omega[\;|\; \frac{1}{2}c_n < v(s) < s_0 \text{ in } [a, t]\}.$$

We want to prove that $\sigma = b$. By (2.2), we immediately realize that $|v'|^{p-2}v'$ and, hence, v' are decreasing in $[a, \sigma[$. Hence, we have $v'(t) < 0$ in $]a, \sigma[$. Multiplying the equation in (2.5) by v' and integrating between a and t, with $t \in]a, \sigma[$, we obtain

$$\frac{p-1}{p}|v'(t)|^p = \lambda(F(c_n) - F(v(t)))$$

and then, by (2.3),

$$-v'(t) \le \left(\frac{p}{p-1}\right)^{1/p} (\lambda F(c_n))^{1/p}. \tag{2.6}$$

Now, assume, by contradiction, that $\sigma < b$ and set $v(\sigma) = \lim_{t\to\sigma^-} v(t) = \frac{1}{2}c_n$. Integrating (2.6) between a and σ, we get

$$\frac{1}{2}c_n = \int_a^\sigma -v'(t) \le \left(\frac{p}{p-1}\right)^{1/p} (\lambda F(c_n))^{1/p} \operatorname{diam}(\Omega).$$

Dividing by c_n and passing to the limit, condition (2.4) yields a contradiction. Hence, we can conclude that there is a sequence $(v_n)_n$ of solutions of (2.5), defined on $[a, b]$ and satisfying $\frac{1}{2}c_n \le v_n(t) \le c_n$ in $[a, b]$. Therefore, setting, for each n,

$$\beta_n(x_1, \ldots, x_n) = v_n(x_1) \quad \text{for } (x_1, \ldots, x_n) = \mathrm{x} \in \bar{\Omega},$$

we define a sequence $(\beta_n)_n \subset C^1(\bar{\Omega})$ of upper solutions of problem (1.1), such that, for every n,

$$\frac{1}{2}c_n \le \beta_n(\mathrm{x}) \le c_n \quad \text{in } \bar{\Omega}. \tag{2.7}$$

Now, let us introduce the functional $\phi : W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \to \mathbb{R}$, defined by $\phi(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \lambda \int_\Omega F(u)$.

Let $\zeta \in C^1(\bar{\Omega})$ be a function such that $\zeta(\mathrm{x}) = 1$ in some closed ball $B \subset \Omega$, $\zeta(\mathrm{x}) > 0$ in Ω and $\zeta(\mathrm{x}) = 0$ on $\partial\Omega$. By the former condition in (1.9), we can find a number $s_\lambda > 0$ such that $F(s) \ge \frac{-s^p}{\lambda}$ in $[0, s_\lambda]$. On the other hand, the latter condition in (1.9) yields the existence of a sequence $(d_n)_n \subset]0, s_\lambda[$, such that $d_n \searrow 0$ and $\frac{F(d_n)}{{d_n}^p} \to +\infty$. Hence, we get

$$\begin{aligned}
\phi(d_n\zeta) &= \frac{1}{p}{d_n}^p \int_\Omega |\nabla\zeta|^p - \lambda \int_{\Omega\setminus B} F(d_n\zeta) - \lambda \int_B F(d_n) \\
&\le {d_n}^p \left(\frac{1}{p} \int_\Omega |\nabla\zeta|^p + \int_\Omega \zeta^p - \lambda \,\mathrm{meas}(B)\, \frac{F(d_n)}{{d_n}^p} \right) \;<\; 0,
\end{aligned}$$

for all n large enough.

Now, we are in position of constructing a sequence $(u_n)_n$ of positive solutions of problem (1.1), with $\max_{\bar{\Omega}} u_n \to 0$. Since 0 is a lower solution and β_1 is an upper solution of (1.1), with $\min_{\bar{\Omega}} \beta_1 > 0$, there exists a solution u_1 of (1.1), satisfying $0 \le u_1 \le \beta_1$ in Ω and $\phi(u_1) = \min\{\phi(u) \,|\, u \in W_0^{1,p}(\Omega), 0 \le u \le \beta_1\}$. Since we can find a positive number, say d_{n_1}, such that $d_{n_1}\zeta \le \min_{\bar{\Omega}} \beta_1$ in Ω and $\phi(d_{n_1}\zeta) < 0$, it follows that $\phi(u_1) < 0$ and therefore $u_1 \not\equiv 0$. Hence, u_1 is a positive solution of (1.1), which, by (2.7), satisfies $\max_{\bar{\Omega}} u_1 \le c_1$. Next, we pick an upper solution, say β_2, such that $\max_{\bar{\Omega}} \beta_2 < \max_{\bar{\Omega}} u_1$. Proceeding as above, we find a solution u_2 of (1.1) such that $0 \le u_2 \le \beta_2$ in Ω and $\phi(u_2) < 0$. Hence, u_2 is a positive solution of (1.1), which satisfies $\max_{\bar{\Omega}} u_2 < \max_{\bar{\Omega}} u_1$ and, by (2.7), $\max_{\bar{\Omega}} u_2 \le c_2$. Iterating this argument, we build a sequence $(u_n)_n$ of distinct positive solutions of (1.1) satisfying $\max_{\bar{\Omega}} u_n \le c_n \to 0$. Thus, the proof is concluded.

References

[1] H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Review, **18** (1976), 620-709.

[2] A. Ambrosetti, H. Brezis and G. Cerami, *Combined effects of concave and convex nonlinearities in some elliptic problems*, J. Functional Anal., **122** (1994), 519-543.

[3] A. Ambrosetti and P. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Functional Anal., **14** (1973), 349-381.

[4] H. Brezis and R. Turner, *On a class of superlinear elliptic problems*, Comm. Partial Differential Equations, **2** (1977), 601-614.

[5] H. Dang, R Manásevich and K. Schmitt, *Positive radial solutions of some nonlinear partial differential equations*, Math. Nachr., **186** (1997), 101-113.

[6] H. Dang and K. Schmitt, *Existence of positive solutions for semilinear elliptic equations in annular domains*, Differential Integral Equations, **7** (1994), 747-758.

[7] D.G. de Figueiredo, *Positive solutions of semilinear elliptic equations*, Lecture Notes in Math. vol. 957, Springer–Verlag, Berlin, 1982; pp. 34-87.

[8] D.G. de Figueiredo and P.L. Lions, *On pairs of positive solutions for a class of semilinear elliptic problems*, Indiana Univ. Math. J., **34** (1985), 591-606.

[9] D.G. de Figueiredo, P.L. Lions and R. Nussbaum, *A priori estimates and existence results for positive solutions of semilinear elliptic equations*, J. Math. Pures Appl., **61** (1982), 41-63.

[10] B. Gidas and J. Spruck, *A priori bounds for positive solutions of nonlinear elliptic equations*, Comm. Partial Differential Equations, **6** (1981), 883-901.

[11] P.L. Lions, *On the existence of positive solutions of semilinear elliptic equations*, SIAM Review, **24** (1982), 441-467.

[12] R. Manásevich, F.I. Njoku and F. Zanolin, *Positive solutions for the one-dimensional p-Laplacian*, Differential Integral Equations, **8** (1995), 213-222.

[13] F.I. Njoku, P. Omari and F. Zanolin, *Multiplicity of positive radial solutions of a quasilinear elliptic problem in a ball*, Advances in Differential Equations, **5** (2000), 1545-1570.

[14] F.I. Njoku and F. Zanolin, *Positive solutions for two-point BVP's: existence and multiplicity results*, Nonlinear Analysis, T.M.A., **13** (1989), 1329-1338.

[15] P. Omari and F. Zanolin, *Infinitely many solutions of a quasilinear elliptic problem with an oscillatory potential*, Comm. Partial Differential Equations, **21** (1996), 721-733.

[16] P. Rabinowitz, *Pairs of positive solutions of nonlinear elliptic partial differential equations*, Indiana Univ. Math. J., **23** (1973), 173-185.

[17] F. de Thélin, *Résultats d'existence et de non-existence pour la solution positive et bornée d'une e.d.p. elliptique non linéaire*, Ann. Fac. Sci. Toulouse, **8** (1986-87), 375-389.

[18] J.L. Vazquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim., **3** (1984), 191-202. 173-185.

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