USA-Chile Workshop on Nonlinear Analysis,
Electron. J. Diff. Eqns., Conf. 06, 2001, pp. 359–367.
http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu or ejde.math.unt.edu (login: ftp)

Existence of non-negative solutions for a Dirichlet problem *

Cecilia S. Yarur

Abstract

The aim of this paper is the study of existence of non-negative solutions of fundamental type for some systems without sign restrictions on the non linearity.

1 Introduction

We study the existence of non-negative non-trivial solutions to the boundary-value problem

$$\begin{aligned} \Delta u &= a_2 v^{p_2} - a_1 v^{p_1} \quad \text{in } B' \\ \Delta v &= b_2 u^{q_2} - b_1 u^{q_1} \quad \text{in } B' \\ u &= v = 0 \quad \text{on} \partial B\,, \end{aligned} \tag{1.1}$$

where a_i, b_i are non-negative constants, $p_i > 0$, $q_i > 0$ for $i = 1, 2$, B is the unit ball centered at zero in $\mathbb{R}^N$, $N \geq 3$, and $B' = B \setminus \{0\}$.

The above problem involves many problems of a quite different nature depending on the values of a_i, b_i. For instance, if $a_2 = b_2 = 0$ the solutions u, v are sub-harmonic functions, while if $a_1 = b_1 = 0$ the solutions are super-harmonic.

For a better understanding of this system, we recall that P.L. Lions [12], Ni and Sacks [15], and Ni and Serrin [16], studied conditions for existence or non existence of non-negative solutions u to

$$-\Delta u = u^q \quad \text{in } B', \quad u = 0 \quad \text{on } \partial B. \tag{1.2}$$

The range of existence of solutions to (1.2) is $q < (N+2)/(N-2)$. On the other hand, the problem

$$\Delta u = u^q \quad \text{in } B', \quad u = 0 \quad \text{on } \partial B,$$

has a non-negative non-trivial solution if and only if $q < N/(N-2)$, see [5] for the non-existence and [18] for existence and related problems.

* *Mathematics Subject Classifications:* 35C20, 35D10.
Key words: Dirichlet Problem, Non-negative solutions.

Published January 8, 2001.
Partially supported by Fondecyt grant 1990877 and DICYT

We state next some known results concerning particular cases of problem (1.1). Assume first that $a_1 = b_1 = 0$. Thus, we are concerned with

$$\begin{aligned} -\Delta u &= a_2 v^{p_2} \quad \text{in } B' \\ -\Delta v &= b_2 u^{q_2} \quad \text{in } B' \\ u = v &= 0 \quad \text{on } \partial B. \end{aligned} \tag{1.3}$$

The following result is well known.

Theorem 1.1 *Assume that $a_1 = b_1 = 0$, $p_2 q_2 > 1$ and $a_2 > 0$, $b_2 > 0$. Then, there exists a classical solution to (1.3) if and only if*

$$\frac{N}{p_2+1} + \frac{N}{q_2+1} > N-2.$$

Troy [17] proved radial symmetry of positive classical solutions to problem (1.3). The existence of positive classical solutions of (1.3) was studied by Hulshof and van der Vorst [11] and de Figueiredo and Felmer [9]. The behavior of solutions was studied by Bidaut-Véron in [2]. The existence of some singular solutions, that is solutions with either $\limsup_{x\to 0} u(x) = +\infty$ or $\limsup_{x\to 0} v(x) = +\infty$, is given by García-Huidobro, Manásevich, Mitidieri and Yarur, see [10]. Using Pohozaev-Pucci-Serrin type identity, Mitidieri [13, 14] and van der Vorst [19] proved non existence of classical solutions of (1.3). Non-existence of radially symmetric singular positive solutions was given by Garcia-Huidobro, Manásevich, Mitidieri and Yarur in [10].

We note that since u and v are super-harmonic functions, and due to a result of Brezis and P.L.Lions [1], $u^{q_2} \in L^1(B)$, $v^{p_2} \in L^1(B)$ and there exist $c \geq 0$ and $d \geq 0$ such that

$$\begin{aligned} -\Delta u &= a_2 v^{p_2} + c\delta_0 \quad \text{in } \mathcal{D}'(B) \\ -\Delta v &= b_2 u^{q_2} + d\delta_0 \quad \text{in } \mathcal{D}'(B) \\ u = v &= 0 \quad \text{on } \partial B. \end{aligned}$$

If $(c,d) \neq (0,0)$ we call this singularity of *fundamental* type.

Let us consider now $a_1 = b_2 = 0$, in (1.1). Hence, we are looking for the solutions of:

$$\begin{aligned} -\Delta u &= a_2 v^{p_2} \quad \text{in } B' \\ \Delta v &= b_1 u^{q_1} \quad \text{in } B' \\ u = v &= 0 \quad \text{on } \partial B. \end{aligned} \tag{1.4}$$

Since v is sub-harmonic, there exists no non-negative classical solutions to (1.4).

The following result is given in [6] for $p_2 q_1 > 1$ and in [7] for $p_2 q_1 < 1$.

Theorem 1.2 *Assume $a_1 = b_2 = 0$, $p_2 > 0$, $q_1 > 0$, and $p_2 q_1 \neq 1$. Then there exists a non-trivial non-negative solution to (1.4) if and only if*

$$\frac{N}{p_2+1} + \frac{N-2}{q_1+1} > N-2, \quad \textit{and} \quad p_2 < N/(N-2).$$

The above result is based on the results given in [3].

We say that (u, v) has a *strong* singularity at 0 if either

$$\limsup_{x \to 0} |x|^{N-2} u(x) = +\infty \quad \text{or} \quad \limsup_{x \to 0} |x|^{N-2} v(x) = +\infty \,.$$

It can be proved that there exists a region in the plane $p_2 - q_1$ where there exist both *strong* and *fundamental* non-negative singular solutions, see [7]. This region is given by

$$\frac{N-2}{p_2+1} + \frac{N}{q_1+1} > N-2, \quad \text{and} \quad p_2 < N/(N-2) < q_1 \,.$$

Assume now that $a_2 = b_2 = 0$, and thus the problem (1.1) is

$$\begin{aligned} \Delta u &= a_1 v^{p_1} \quad \text{in } B' \\ \Delta v &= b_1 u^{q_1} \quad \text{in } B' \\ u &= v = 0 \quad \text{on } \partial B, \end{aligned} \tag{1.5}$$

Since u and v are sub-harmonic we have non existence of non-negative solutions with either u or v bounded.

In [4] and [20] it was proved non existence of positive solutions if either

$$\frac{N}{p_1+1} + \frac{N-2}{q_1+1} \le N-2, \quad \text{or} \quad \frac{N-2}{p_1+1} + \frac{N}{q_1+1} \le N-2.$$

If $a_1 = 0$ (similarly for $b_1 = 0$) we have

$$-\Delta u = a_2 v^{p_2} \quad \text{in } B'$$

$$-\Delta v = b_2 u^{q_2} - b_1 u^{q_1} \quad \text{in } B' \tag{1.6}$$

$$u = v = 0 \quad \text{on } \partial B \tag{1.7}$$

The following result was proved in [8].

Theorem 1.3 *Let $p_2 > 0$, $q_1 > 0$ and $q_2 > 0$. Let $a_1 = 0$, $a_2 \ge 0$, $b_1 \ge 0$ and $b_2 \ge 0$. Assume that for $i = 1, 2$ we have*

$$p_2 < \frac{N}{N-2}, \quad \frac{N}{p_2+1} + \frac{N-2}{q_i+1} > N-2. \tag{1.8}$$

Assume that one of the following holds:

(i) $p_2 q_i > 1$ for all $i = 1, 2$.

(ii) $p_2 q_i < 1$, for all $i = 1, 2$.

(iii) If $p_2 q_i = 1$ for some $i = 1, 2$ then $a_2^{p_2} b_i$ is sufficiently small.

(iv) $p_2 q_i < 1 < p_2 q_j$, for some $i, j = 1, 2$, $i \ne j$, and $a_2^{p_2} b_i$ is sufficiently small.

Then, there exist $d_* \geq 0$, $d^* > 0$ with $d_* < d^*$ such that for any $d \in (d_*, d^*)$, there exists (u, v) a non-negative solution to (1.6) satisfying

$$\lim_{x \to 0} |x|^{N-2}(u(x), v(x)) = (0, d).$$

Moreover, if $p_2 q_i \geq 1$, $i = 1, 2$ then $d_* = 0$, and if $p_2 q_i \leq 1$, $i = 1, 2$ then $d^* = \infty$.

For the general case we have the following previous result, see [8].

Theorem 1.4 *Let $p_1 > 0$, $p_2 > 0$, $q_1 > 0$ and $q_2 > 0$. Let a_i, b_i $i = 1, 2$ be non-negative constants. Assume that*

$$p_i < \frac{N}{N-2}, \quad q_i < \frac{N}{N-2}, \quad i = 1, 2 \tag{1.9}$$

Assume that one of the following holds:

(i) $p_i q_j > 1$, *for all* $i, j = 1, 2$.

(ii) $p_i q_j < 1$, *for all* $i, j = 1, 2$.

(iii) If $p_i q_j = 1$ for some $i = 1, 2$ and some $j = 1, 2$ then $a_i^{p_i} b_j$ is sufficiently small.

(iv) $p_i q_j < 1 < p_k q_l$, for some $i, j, k, l = 1, 2$ and $a_i^{p_i} b_j$ is sufficiently small.

Then, there exist $c > 0$, $d > 0$ and (u, v) a non-negative solution to (1.1) such that

$$\lim_{x \to 0} |x|^{N-2}(u(x), v(x)) = (c, d).$$

Here we prove the following general existence result of non negative non-trivial solutions to (1.1). Set

$$\Gamma(p, q) := \frac{N-2}{p+1} + \frac{N}{q+1} - (N-2). \tag{1.10}$$

Theorem 1.5 *Let p_i, q_i, $i = 1, 2$, positive numbers. Then, there exists a non-negative nontrivial solution (u, v) of (1.1) if one of the following holds:*

(i) $a_1 > 0$, $b_1 > 0$, *and* $p_2 < N/(N-2)$, $q_2 < N/(N-2)$

$$\min\{\Gamma(p_1, q_1), \Gamma(q_1, p_1), \Gamma(p_2, q_1), \Gamma(q_2, p_1)\} > 0,$$

with small coefficient a_j (respectively b_j) for some $j = 1, 2$ if $p_j \leq 1$ (respectively $q_j \leq 1$) and $1 \leq \max_{i=1,2}\{p_i, q_i\}$.

(ii) $a_1 = 0$, $b_1 > 0$, $p_2 < N/(N-2)$ *and*

$$\min\{\Gamma(q_1, p_2), \Gamma(q_2, p_2)\} > 0,$$

with small coefficient a_2 (respectively b_j) if $p_2 \leq 1$ (respectively $q_j \leq 1$) for some $j = 1, 2$, and $1 \leq \max_{i=1,2}\{p_2, q_i\}$.

(iii) $a_1 = 0 = b_1$, *and*

$$\max\{\Gamma(p_2, q_2), \Gamma(q_2, p_2)\} > 0,$$

with small coefficient a_2 *(respectively* b_2*) if* $p_2 \leq 1$ *(respectively* $q_2 \leq 1$*) and* $1 \leq \max\{p_2, q_2\}$.

2 Proof of Theorem 1.5

We note that for p and q non-negative numbers the condition $\Gamma(p, q) > 0$ is equivalent to

$$p(2 - (N-2)q) + N > 0.$$

Moreover, if $pq > 1$,

$$\Gamma(p, q) = (\zeta - (N-2))(pq - 1), \quad \zeta = \frac{2(p+1)}{pq-1}$$
$$\Gamma(q, p) = (\xi - (N-2))(pq - 1), \quad \xi = \frac{2(q+1)}{pq-1}.$$

Recall that $u(x) = C_1|x|^{-\zeta}$, $v(x) = C_2|x|^{-\xi}$ for some positive constants C_1 and C_2 is a non-negative solution of

$$-\Delta u = v^p, \quad -\Delta v = u^q$$

if $\Gamma(p, q) < 0$ and $\Gamma(q, p) < 0$. This particular solution also plays a fundamental role for example for the system

$$-\Delta u = v^p, \quad \Delta v = u^q,$$

where this solution exists if $\Gamma(p, q) < 0$ and $\Gamma(q, p) > 0$.

Proof of Theorem 1.5. Set

$$f_i(t) = a_i t^{p_i}, \ g_i(t) = b_i t^{q_i}, \ i = 1, 2.$$

We will construct radially symmetric non-negative solutions to (1.1), by monotone iteration as follows. Let $d > 0$, $(u_1, v_1) = (0, dm)$, where $m(r) := |x|^{2-N} - 1$ and let (u_n, v_n) be given by $(u_{n+1}, v_{n+1}) = T(u_n, v_n)$ where $T = (T_1, T_2)$ is the operator given by

$$\begin{aligned} T_1(u,v)(r) &= \int_r^1 s^{1-N} \int_s^1 t^{N-1} f_1(v(t)) dt ds \\ &\quad + \int_r^1 s^{1-N} \int_0^s t^{N-1} f_2(v(t)) dt ds, \qquad (2.1) \\ T_2(u,v)(r) &= dm(r) + \int_r^1 s^{1-N} \int_s^1 t^{N-1} g_1(u(t)) dt ds \\ &\quad + \int_r^1 s^{1-N} \int_0^s t^{N-1} g_2(u(t)) dt ds, \end{aligned}$$

We are looking for α, δ and C such that

$$T_1(Cr^\alpha, Cr^\delta) \le Cr^\alpha,\ T_2(Cr^\alpha, Cr^\delta) \le Cr^\delta$$

and $v_1 = dm(r) \le Cr^\delta$. Hence, the sequence (u_n, v_n) satisfies

$$u_n \le Cr^\alpha,\ v_n \le Cr^\delta \quad \text{for all } n \in \mathbb{N},$$

and the convergence of (u_n, v_n) to a solution of (1.1) follows.

To find C, d, α and δ we use the following: Let κ be any number such that $\kappa + N \neq 0$, and define

$$\phi(\kappa) := \min\{2 - N, \kappa + 2\}.$$

Then

$$m_\kappa(r) := \int_r^1 s^{1-N} \int_s^1 t^{N-1+\kappa} dtds \le Kr^{\phi(\kappa)}, \tag{2.2}$$

where $K = K(N, \kappa)$.

Moreover, for any κ satisfying $\kappa + N > 0$, and $\kappa + 2 \neq 0$, set

$$\psi(\kappa) := \min\{0, \kappa + 2\}.$$

We have

$$h_\kappa := \int_r^1 s^{1-N} \int_0^s t^{N-1+\kappa} dtds \le Kr^{\psi(\kappa)}, \tag{2.3}$$

where $K = K(N, \kappa)$. Hence,

$$T_1(Cr^\alpha, Cr^\delta) = a_1 C^{p_1} m_{p_1\delta} + a_2 C^{p_2} h_{p_2\delta}.$$

From (2.2) and (2.3) and if we choose $p_1\delta + N \neq 0$ and $p_2\delta + N > 0$ we obtain

$$T_1(Cr^\alpha, Cr^\delta) \le K\left(a_1 C^{p_1} r^{\phi(p_1\delta)} + a_2 C^{p_2} r^{\psi(p_2\delta)}\right). \tag{2.4}$$

We note that $\phi(p_1\delta) \le 2 - N < \psi(p_2\delta)$, and thus

$$T_1(Cr^\alpha, Cr^\delta) \le K\left(a_1 C^{p_1} + a_2 C^{p_2}\right) r^\sigma, \tag{2.5}$$

where

$$\sigma := \begin{cases} \phi(p_1\delta) & \text{if } a_1 \neq 0, \\ \psi(p_2\delta) & \text{if } a_1 = 0. \end{cases}$$

Therefore, if $\alpha \le \sigma$ and $K(a_1 C^{p_1} + a_2 C^{p_2}) \le C$, we obtain $T_1(Cr^\alpha, Cr^\delta) \le Cr^\alpha$. Arguing as above, we have

$$T_2(Cr^\alpha, Cr^\delta) \le dr^{2-N} + K\left(b_1 C^{q_1} r^{\phi(q_1\alpha)} + b_2 C^{q_2} r^{\psi(q_2\alpha)}\right), \tag{2.6}$$

with $q_1\alpha + N \neq 0$, $q_2\alpha + 2 \neq 0$, and $q_2\alpha + N > 0$. Therefore,

$$T_2(Cr^\alpha, Cr^\delta) \le \left(d + K\left(b_1 C^{q_1} + b_2 C^{q_2}\right)\right) r^\rho, \tag{2.7}$$

where

$$\rho := \begin{cases} \phi(q_1\alpha) & \text{if } b_1 \neq 0, \\ 2-N & \text{if } b_1 = 0. \end{cases}$$

Hence,

$$T_2(Cr^\alpha, Cr^\delta) \leq Cr^\delta,$$

if $\delta \leq \rho$ and $d + K(b_1C^{q_1} + b_2C^{q_2}) \leq C$.

Next we prove the existence of α, δ, C and d under the hypothesis of the theorem. The existence of C and d, is classical. We can choose $d = C/2$ and for $i = 1, 2$

$$Ka_iC^{p_i} \leq C/2, \quad Kb_iC^{q_i} \leq C/4.$$

Therefore, if either for all i , $p_i < 1$ and $q_i < 1$, or $p_i > 1$ and $q_i > 1$, the existence of C follows. By the contrary if $\max\{p_i, q_i, i = 1, 2\} \geq 1$ and $\min\{p_i, q_i, i = 1, 2\} \leq 1$, we obtain existence with a restriction on the coefficients.

We summarize the conditions that α and δ must satisfy as follows:

$$\alpha \leq \begin{cases} \min\{2-N, p_1\delta + 2\} & \text{if } a_1 \neq 0 \\ \min\{0, p_2\delta + 2\} & \text{if } a_1 = 0 \end{cases}$$

$$\delta \leq \begin{cases} \min\{2-N, q_1\alpha + 2\} & \text{if } b_1 \neq 0 \\ 2-N & \text{if } b_1 = 0. \end{cases}$$

Moreover, we need that

$$p_2\delta + N > 0, \quad q_2\alpha + N > 0, \tag{2.8}$$

We also used that $p_1\delta + N \neq 0$, $q_1\alpha + N \neq 0$, $p_2\delta + 2 \neq 0$, $q_2\alpha + 2 \neq 0$. These last conditions are not relevant since we can take α and δ smaller and hence these new α and δ satisfy the conditions.

Case (i). Assume first that $a_1 > 0$ and $b_1 > 0$. If $p_1 < N/(N-2)$, and since $\Gamma(p_1, q_1) > 0$, and $\Gamma(p_2, q_1) > 0$, we can take

$$\alpha = 2 - N, \ \delta = \min\{2 - N, 2 - (q_1 - \varepsilon)(N-2)\},$$

where $\varepsilon > 0$ is such that

$$\Gamma(p_1, q_1 - \varepsilon) > 0, \quad \text{and } \Gamma(p_2, q_1 - \varepsilon) > 0.$$

Now, since $q_2 < N/(N-2)$, we have that $q_2\alpha + N > 0$. From $p_2 < N/(N-2)$ and $\Gamma(p_2, q_1 - \varepsilon) > 0$, we also have $p_2\delta + N > 0$. It remains to prove that $\alpha = 2 - N \leq p_1\delta + 2$, which follows easily from $\Gamma(p_1, q_1 - \varepsilon) > 0$.

If $p_1 \geq N/(N-2)$, from $\Gamma(p_1, q_1) > 0$ we deduce that $q_1 < N/(N-2)$. Thus, we may proceed as before but now with

$$\delta = 2 - N, \ \alpha = p_1(2 - N) + 2.$$

Case (ii). Assume that $a_1 = 0$ and $b_1 > 0$. Let us choose

$$\delta = 2 - N,\ \alpha = \min\{0, 2 - (p_2 - \varepsilon)(N-2)\},$$

where $\varepsilon > 0$ is such that $\Gamma(q_1, p_2 - \varepsilon) > 0$ and $\Gamma(q_2, p_2 - \varepsilon) > 0$. Then the conclusion follows as in the above case.

Case (iii). Assume that $b_1 = 0$ and $a_1 = 0$. Assume that $p_2 \leq q_2$. Since $\Gamma(q_2, p_2) > 0$, then $p_2 < N/(N-2)$. Let us choose

$$\delta = 2 - N,\ \text{and}\ \alpha = \min\{0, 2 - (p_2 - \varepsilon)(N-2)\},$$

and thus the conclusion follows by taking $\Gamma(q_2, p_2 - \varepsilon) > 0$.

References

[1] H. Brezis, P.L. Lions, A note on isolated singularities for linear elliptic equations, *Jl. Math. Anal. and Appl.*, **9A** (1981), 263–266.

[2] M. F. Bidaut-Véron, Local behaviour of solutions of a class of nonlinear elliptic systems, *Adv. Differential Equations*, to appear.

[3] M.F.Bidaut-Véron and P. Grillot, Asymptotic behaviour of elliptic system with mixed absorption and source terms, *Asymptot. Anal.,* (1999), **19**, 117-147.

[4] M.F.Bidaut-Véron and P. Grillot, Singularities in elliptic systems, preprint.

[5] H. Brezis and L. Véron, Removable singularities of some nonlinear elliptic equations *Archive Rat. Mech. Anal.* **75** (1980), .

[6] C. Cid and C. Yarur, A sharp existence result for a Dirichlet problem - The superlinear case, to appear in Nonlinear Anal.

[7] C. Cid and C. Yarur, Existence of solutions for a sublinear system of elliptic equations, *Electron. J. Diff. Eqns.*, **2000** (2000), No. 33, 1–11. (http://ejde.math.swt.edu)

[8] C. Cid and C. Yarur, in preparation.

[9] P. Felmer and D.G. de Figueiredo, On superquadratic elliptic systems, *Trans. Amer. Math. Soc.* **343** (1994), 99–116.

[10] M. García-Huidobro, R. Manásevich, E. Mitidieri, and C. Yarur Existence and noexistence of positive singular solutions for a class of semilinear systems, *Archive Rat. Mech. Anal.* **140** (1997), 253-284.

[11] J. Hulsholf and R.C.A.M. van der Vorst, Differential systems with strongly indefinite variational structure, *J. Func. Anal.* **114** (1993), 32-58.

[12] P.L. Lions, Isolated singularities in semilinear problems, *J. Differential Equations* **38** (1980),441-450.

[13] E. Mitidieri, Nonexistence of positive solutions of semilinear elliptic systems in $\mathbb{R}^N$, *Quaderno Matematici* **285**, University of Trieste 1992.

[14] E. Mitidieri, A Rellich type identity and applications, *Comm. in P.D.E.* **18** (1993), 125-151.

[15] W.M. Ni and P. Sacks, Singular behavior in nonlinear parabolic equations, *Trans. Amer. Mat. Soc.* **287** (1985),657-671.

[16] W.M. Ni and J. Serrin, Nonexistence theorems for singular solutions of quasilinear partial differential equations, *Comm. Pure Appl. Math.* **38** (1986), 379-399.

[17] W.C. Troy, Symmetry properties of semilinear elliptic systems, *J. Differential Equations* **42** (1981), 400–413.

[18] L. Véron, Singular solutions of some nonlinear elliptic equations, *Nonlinear Analysis, Methods & Applications* **5** (1981), 225-242.

[19] Van der Vorst, R.C.A.M., Variational identities and applications to differential systems, *Arch. Rational Mech. Anal.* **116** (1991), 375–398.

[20] C. Yarur, Nonexistence of positive singular solutions for a class of semilinear elliptic systems, *Electron. J. Diff. Eqs.*, **1996** (1996), No. 8, 1–22. (http://ejde.math.swt.edu)

Cecilia S. Yarur
Departamento de Matematicas
Universidad de Santiago de Chile
Casilla 307, Correo 2, Santiago, Chile
email: cyarur@fermat.usach.cl