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Symbolic computation of Appell polynomials using Maple *

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Abstract

This work focuses on the symbolic computation of Appell polynomials using the computer algebra system Maple. After describing the traditional approach of constructing Appell polynomials, the paper examines the operator method of constructing the same Appell polynomials. The operator approach enables us to express the Appell polynomial as Bessel function whose coefficients are Euler and Bernuolli numbers. We have also constructed algorithms using Maple to compute Appell polynomials based on the methods we have described. The achievement is the construction of Appell polynomials for any function of bounded variation.

1 Introduction

The aim of this work is to give a new method of constructing and studying the properties of a definite class of polynomials called Appell polynomials. This sequence of polynomials $P_n(x)$, of degree n, is defined by the recurrence relation

$$\frac{d}{dx}P_n(x) = P_{n-1}(x), \tag{1}$$

or equivalently,

$$\exists A(t) = \sum_{n=0}^{\infty} a_n t^n, (a_0 \neq 0) : A(t)e^{tx} = \sum_{n=0}^{\infty} P_n(x)t^n.$$

Each $A(t)$ is called a generating function and will generate a set of Appell polynomials. In the interval $[-1, 1]$, even polynomials of the type defined by (1) have two zero points at -1 and 1. Two explicit methods of construction of Appell polynomials may be obtained by
1.) **the traditional method** and by
2.) **the operator method**.

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The explicit expression derived for $P_n(x)$ using the generating function $f(x) = (e^{kx} + e^{-kx})/(e^k + e^{-k})$, could be expressed in terms of Bessel powers. The coefficients of this Appell polynomial are the product of Euler and Bernoulli numbers. The behavior of this Appell polynomials in the interval $(-1, 1)$ and outside the interval $(-1, 1)$ could also be studied. Hence, enabling one to obtain the zeros of the polynomial $P_n(x)$.

The importance of this construction is to provide us with a powerful tool for solving boundary value problems in hydrodynamics and many other areas of application. In particular, this model could be viewed as a fluid flow with nodes at -1 and 1. It also provides a solution to the linear differential equation of the form:

$$\sum_{r=0}^{n} L_r(x) \frac{d^r y(x)}{dx^r} = \lambda y(x) \tag{2}$$

where $L_r(x)$ is a polynomial in x of degree r, λ is a parameter and under the condition that the generating function is $A(t) = e^{Q(t)}$, where $Q(t)$ is a polynomial in t.

2 Operators in a linear space

Let $L = \{v, u, \ldots\}$ be a given linear vector space of possibly infinite dimension. Suppose that an algebra of operators $D, I, P, \ldots$ is defined in the vector space L. If the operator D on the vector v is given by

$$Dv = 0,$$

where the vector $v \in L$, then the set of vectors $\{v\}$ is called the zero-space of the operator D. The members of this space will be denoted by c. Hence,

$$Dc = 0.$$

Also, c will be called the formal constant of the operator D. Next, let us introduce the operator I with the property

$$DI = 1. \tag{3}$$

From (2) it is clear that the operator I is the right inverse for the D operator. If the operator P is defined to be

$$P = 1 - ID,$$

then clearly, P is the projection operator for the vector v on the zero-space of the operator D, since

$$D(Pv) = Dv - Dv = 0. \tag{4}$$

The relation given by (3) shows that $Pv = c$. The power of the operator I will be defined in the usual form as $I^n = II^{n-1}$, and $I^0 = 1$, from which one can obtain the relation

$$DI^n = I^{n-1}. \tag{5}$$

The nth order of the power of the formal variable x generated by the operator I with coefficients c is the expression given by

$$n!I^n c = cx^n.$$

Clearly, the following expression holds

$$Dx^n c = nx^{n-1}c.$$

Now, consider the expression of the form

$$v = c_0 + Ic_1 + \ldots + I^n c_n,$$

which is called a polynomial operator. The formula for computing the coefficients is:

$$c_i = PD^i v.$$

In terms of the variable x, one can write

$$v = c_0 + c_1 x + \ldots + c_n x^n,$$

which is a polynomial of order n with coefficients $c_0, c_1, \ldots, c_n$. Thus, the formula for the coefficients could be written as

$$c_i = \frac{1}{i!} PD^i v.$$

It must be noted that the powers of this special kind of variable generate a differential operator. Now, consider a linear vector space L, the vectors of which consist of matrices with dimension $2x2$, and components u_{11}, u_{12}, u_{21}, u_{22}, which are functions of one variable x. Thus, if u is such a vector then

$$u = \begin{pmatrix} u_{11}(x) & u_{12}(x) \\ u_{21}(x) & u_{22}(x) \end{pmatrix}.$$

The differential operator D takes the form

$$D = \begin{pmatrix} 0 & \frac{d}{dx} \\ \frac{d}{dx} & 0 \end{pmatrix}.$$

It is a straight forward computation to obtain the formula

$$D^n u = \begin{cases} \begin{pmatrix} \frac{d^{2i}u_{11}}{dx^{2i}} & \frac{d^{2i}u_{12}}{dx^{2i}} \\ \frac{d^{2i}u_{21}}{dx^{2i}} & \frac{d^{2i}u_{22}}{dx^{2i}} \end{pmatrix}, & n = 2i \\ \begin{pmatrix} \frac{d^{2i+1}u_{11}}{dx^{2i+1}} & \frac{d^{2i+1}u_{12}}{dx^{2i+1}} \\ \frac{d^{2i+1}u_{21}}{dx^{2i+1}} & \frac{d^{2i+1}u_{22}}{dx^{2i+1}} \end{pmatrix}, & n = 2i+1, \end{cases}$$

and

$$PD^n u = \begin{cases} \begin{pmatrix} \frac{d^{2i}u_{11}}{dx^{2i}}|_{x_1} & \frac{d^{2i}u_{12}}{dx^{2i}}|_{x_1} \\ \frac{d^{2i}u_{21}}{dx^{2i}}|_{x_2} & \frac{d^{2i}u_{22}}{dx^{2i}}|_{x_2} \end{pmatrix}, & n = 2i \\ \begin{pmatrix} \frac{d^{2i+1}u_{11}}{dx^{2i+1}}|_{x_1} & \frac{d^{2i+1}u_{12}}{dx^{2i+1}}|_{x_1} \\ \frac{d^{2i+1}u_{21}}{dx^{2i+1}}|_{x_2} & \frac{d^{2i+1}u_{22}}{dx^{2i+1}}|_{x_2} \end{pmatrix}, & n = 2i+1. \end{cases}$$

To perform the expansion for an arbitrary function u by the constructed formal powers, it is necessary to use the form of the n-th derivative of

$$u = \begin{pmatrix} u_{11}(x) & u_{12}(x) \\ u_{21}(x) & u_{22}(x) \end{pmatrix}.$$

The integral operators I_1 and I_2 are defined by $I_1 = \int_{x_1=0}^{x} d\xi$ and $I_2 = \int_{x_2=1}^{x} d\xi$.

The following notation will be used:

$$y^{2i}(x) = y^{2i} * 1 = (2i)!(I_1 I_2)^i * 1,$$

and

$$y^{2i+1}(x) = y^{2i+1} * 1 = (2i+1)! I_2 (I_1 I_2)^i * 1.$$

Similarly, the expression $(I_2 I_1)^i * 1$ and $I_1 (I_2 I_1)^i * 1$ are denoted by

$$\hat{y}^{2i}(x) = \hat{y}^{2i} * 1 = (I_2 I_1)^i * 1,$$

and

$$\hat{y}^{2i+1}(x) = \hat{y}^{2i+1} * 1 = I_1 (I_2 I_1)^i * 1$$

respectively. The expressions y^n and $\hat{y}^n$ are called the general Bessel powers. The computation of the Bessel's powers to the order $n = 16$ for $x_1 = 0$ and $x_2 = 1$ will be shown later.

The integral takes the following form

$$I = \begin{pmatrix} 0 & \int_{x_1}^{x} d\xi \\ \int_{x_2}^{x} d\xi & 0 \end{pmatrix} = \begin{pmatrix} 0 & I_1 \\ I_2 & 0 \end{pmatrix},$$

where the lower limit of the integrals can differ. Without loss of generalityone can take $x_1 = 0$, and $x_2 = 1$ For the projection operator, it has the form

$$P = 1 - ID = \begin{pmatrix} \cdots|_{x_1=0} & 0 \\ 0 & \cdots|_{x_2=0} \end{pmatrix}.$$

The formal powers are also given by the expression

$$x^n c = n! \begin{cases} \begin{pmatrix} (I_1 I_2)^i C_{11} & (I_1 I_2)^i C_{12} \\ (I_2 I_1)^i C_{21} & (I_2 I_1)^i C_{22} \end{pmatrix}, & n = 2i \\ \begin{pmatrix} I_1 (I_2 I_1)^i C_{21} & I_1 (I_2 I_1)^i C_{22} \\ I_2 (I_1 I_2)^i C_{11} & I_2 (I_1 I_2)^i C_{12} \end{pmatrix}, & n = 2i+1, \end{cases}$$

where C is the formal constant for the operator D and

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}.$$

3 A Class of Appell Polynomials

Let P_n denote a polynomial of degree n of a variable x. Consider a class of polynomials which satisfy the conditions

$$\frac{dP_0}{dx} = 0, \quad \frac{dP_n}{dx} = P_{n-1} \tag{6}$$

The class of polynomials defined below is called Appell polynomials. In fact, there are large numbers of varying forms of this kind of Appells polynomials. An arbitrary constant is obtained each time (6) is integrated sequentially. For example,

$$\begin{aligned}
y^1 &= \int dx = x + C_0 \\
y^2 &= 2\int y^1 dx = 2\int (x + C_0)dx = x^2 + 2xC_0 + C_1 \\
y^3 &= 3\int y^2 dx = 3\int (x^2 + 2xC_0 + C_1)dx \\
&= x^3 + 3x^2C_0 + 3xC_1 + C_2 \\
&\vdots \\
y^n &= n\int y^{n-1}dx = x^n + nx^{n-1}C_0 + \ldots + C_{n-1}.
\end{aligned}$$

The above can be represented in a compact form as an Abel-Goncarov integral

$$P_n(x) = \int_{\alpha_0}^{x} dx_1 \int_{\alpha_1}^{x_1} dx_2 \int_{\alpha_2}^{x_2} dx_3 \ldots \int_{\alpha_{n-1}}^{x_{n-1}} dx_n,$$

where

$$\alpha_i = \begin{cases} 1, & i = \lambda \bmod (\lambda + 1), \\ 0, & i \neq \lambda \bmod (\lambda + 1). \end{cases}$$

for a regular monotonic polynomials on $[0, 1]$ with type numbers, λ.

When defining Appell polynomials, it is important to add an additional condition, due to the constants which arise in the equations. This condition may cause the polynomial to have zeros at definite points on the real-axis. Let all the polynomials of even degree P_{2i} have zeros at the points -1, 1. Then by performing the integration and sequentially determining the constants, one gets the following sequence of equations

$$\begin{gathered}
y^2 = x^2 - 1, \\
y^4 = x^4 - 6x^2 + 5, \\
y^6 = x^6 - 15x^4 + 75x^2 - 61, \\
y^8 = x^8 - 28x^6 + 350x^4 - 1708x^2 + 1385, \\
y^{10} = x^{10} - 45x^8 - 1050x^6 - 12810x^4 + 62325x^2 - 50521.
\end{gathered}$$

4 Using Generating function

When constructing Appell polynomials with two zero-points, the method of generating functions may the used. It may be shown that there exists a functional sequence, the differential of which gives the Appell polynomials with two zero-points. Consider the function

$$f(x) = \frac{e^{kx} + e^{-kx}}{e^k + e^{-k}}. \tag{7}$$

The Taylor expansion to the degree of k can be easily computed. The coefficients C_{2i}, of the expansion are found by the formula

$$C_{2i} = \frac{d^{2i}}{dk^{2i}} \left(\frac{e^{kx} + e^{-kx}}{e^k + e^{-k}} \right) \Big|_{k=0}. \tag{8}$$

From the Taylor expansion, it is clear that C_{2i} is a function of one variable, say x. Hence, $f(x)$ can be written in terms of C_{2i} as

$$f(x) = \frac{e^{kx} + e^{-kx}}{e^k + e^{-k}} = \sum_{i=0}^{\infty} C_{2i}(x) k^{2i}. \tag{9}$$

It can also be demonstrated that C_{2i} is related to the Bessel powers

$$y^{2i}(x; -1, +1),$$

with two zero-points. The connection is given by the relation

$$C_{2i} = \frac{1}{(2i)!} y^{2i}(x; -1, +1). \tag{10}$$

Thus, the expansion for $f(x)$ which is given by the sum

$$f(x) = \sum_{i=0}^{\infty} C_{2i}(x) k^{2i}, \tag{11}$$

satisfies the following properties

$$\frac{d^2}{dx^2} f(x) = k^2 f(x). \tag{12}$$

Equation (12) could be generalized to take the following form

$$\frac{d^{2i}}{dx^{2i}} f(x) = k^{2i} f(x). \tag{13}$$

The series given in (11) is called an Appell polynomial with two zero-points. Therefore, the set of polynomials obtained by the traditional method in section 3, converges in even powers $x^{2i}C$. These polynomials are called Appell polynomials. Thus, a sequence of Appell polynomials, if the values of the zero-points are related to $x_1 = 0$, and $x_2 = 1$, can also be easily computed. An example of this kind of Appell polynomial is shown below:

$$\begin{aligned} y^2 &= x^2 - x, \\ y^4 &= x^4 - 2x^3 + x, \\ y^6 &= x^6 - 3x^5 + 5x^3 - 3x, \\ y^8 &= x^8 - 4x^7 + 14x^5 - 28x^2 + 17x, \\ y^{10} &= x^{10} - 5\,x^9 + 30\,x^7 - 126\,x^5 + 255\,x^3 - 155\,x \\ y^{12} &= x^{12} - 6\,x^{11} + 55\,x^9 - 396\,x^7 + 1683\,x^5 - 3410\,x^{\ 3} + 2073\,x\,. \end{aligned}$$

Substitute the sum given in (11) into (12) to obtain

$$\sum_{i=0}^{\infty} k^{2i}(C_{2i})'' = \sum_{i=0}^{\infty} k^{2i}(C_{2i}) \tag{14}$$

where $C_2'' = 1$ and $C_0'' = 0$. Generalizing the obtained relation, one gets

$$(2i)!C_{2i}^{''} = 2i(2i-1)C_{2i-2}(2i-2)!,$$

or

$$y^{2i^{''}} = (2i-1)y^{2i-2}.$$

An explicit expression for y^{2i} in the form of a polynomial can be obtained. For this, one uses the Leibnitz formula for finding the differential of order $2i$ from a differentiable function. Thus, one obtains

$$y^{2i}(x;-1,1) = \sum_{j=0}^{i} C(2i,2j)x^{2i-2j}\frac{d^{2j}}{dk^{2j}}\left(\frac{1}{e^k+e^{-k}}\right)\Bigg|_{k=0}. \tag{15}$$

The expression

$$2\frac{d^{2j}}{dk^{2j}}(e^k+e^{-k})^{-1} = E_{2j} \tag{16}$$

where the E_{2j} are the Euler numbers. Therefore, the formula simplifies to

$$y^{2i}(x;-1,1) = \sum_{j=0}^{i} x^{2i-2j}C(2i,2j)E_{2j}. \tag{17}$$

The Euler numbers could be computed from Eq. (16). An example of computing Euler numbers E_{2j} for $2j = 0, 2, 4, ...$ by this method is shown in Table 1.

5 Computation of Appell Polynomials Using Stieltjes Integral

As pointed out by Thone [1] Appell polynomials can be given in terms of Stieltjes integrals. This enables us to give a new characterization to Appell polynomials that expands its applications to functional methods. The definition of bounded variation is important to the construction.

2j	0	2	4	6	8	10	12	14
E	1	-1	5	-61	1385	-50521	270276	19936098

Table 1: Euler numbers

Definition A real valued function f is of bounded variation on a closed interval $[a,b]$ if for any partition π, $\sum_{j=1}^{n}|\Delta f_j| \leq M$,for $M > 0$ and $\Delta f_j = f(x_j) - f(x_{j-1})$.

Theorem 1 *If $\alpha(x)$ is a function of bounded variation on $[a,b]$ and the integrals*

$$\mu_n = \int_a^b x^n d\alpha(x),\ n = 0,1,2,\dots,\mu_0 \neq 0$$

all exist then $\exists\{\phi_n(x)\}, n = 0,1,\dots,$ $\phi_n(x)$ is of degree n such that

$$\int_a^b \phi_n^{[r]}(x)d\alpha(x) = \delta_n^r = \begin{cases} 0 & n \neq r \\ 1 & n = r \end{cases} \tag{18}$$

Sketch of Proof: If

$$\phi_n(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_0 \tag{19}$$

Applying (18) to (19) yields a system of equation which has a unique solution if $n!(n-1)!(n-2)!\dots..1\mu_0^{n+1} \neq 0$. Since $u_0 \neq 0$, the solution is as follows

$$a_{n+1-r} = \frac{\begin{vmatrix} \mu_0 & 0 & 0 & \dots & 0 & 1 \\ \frac{\mu_0}{1!} & \mu_0 & 0 & \dots & 0 & 0 \\ \frac{\mu_2}{2!} & \frac{\mu_1}{1!} & \mu_0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \frac{\mu_{r-1}}{(r-1)!!} & \frac{\mu_{r-2}}{(r-2)!!} & \frac{\mu_{r-3}}{(r-3)!!} & \dots & \frac{\mu_0}{1!} & 0 \end{vmatrix}}{(n+1-r)!\mu_0^r}$$

This solution is used to find the Appel polynomials for any function of bounded variation. Some examples are demonstrated in the next section.

6 Symbolic Algebra and Appell Polynomials

Now, we demonstrate how a symbolic algebra system can aid us in the construction of Appell polynomials. This would enable us to generate Appell polynomials of any order say (k). The name of the algorithms are *apPoly(n1,n2,n3)* and *apPolyeb(n1,n2,n3)* where $n1$ is the degree of the polynomials, $(n2, n3)$ are the two zeros of the polynomial which could be 0, 1 or -1, 1. The Maple function needed when constructing Appell polynomials by the traditional method is *int.* The program to do this is given in appendix 1. The algorithm *appelPoly(f,x,a,b,n),* where (f, x) is a function of bounded variation and the variables a and b are the lower and upper limits of integration, and n is the degree of the polynomials, computes the general Appell polynomials for a given function of bounded variation. The first example has the two zeros at -1, 1 and the second has the two zeros at 0, 1.

```
for i to 4 do apPoly(2*i,-1,1) od;
```

$$y^2 = x^2 - 1$$
$$y^4 = x^4 - 6\,x^2 + 5$$
$$y^6 = x^6 - 15\,x^4 + 75\,x^2 - 61$$
$$y^8 = x^8 - 28\,x^6 + 350\,x^4 - 1708\,x^2 + 1385$$
$$y^{10} = x^{10} - 45\,x^8 + 1050\,x^6 - 12810\,x^4 + 62325\,x^2 - 50521$$

```
for i to 4 do apPoly(2*i,0,1) od;
```

$$y^2 = x^2 - x$$
$$y^4 = x^4 - 2\,x^3 + x$$
$$y^6 = x^6 - 3\,x^5 + 5\,x^3 - 3\,x$$
$$y^8 = x^8 - 4\,x^7 + 14\,x^5 - 28\,x^3 + 17\,x$$
$$y^{10} = x^{10} - 5\,x^9 + 30\,x^7 - 126\,x^5 + 255\,x^3 - 155\,x$$

The operator method began with the function given in (7) whose Taylor expansion in k to the order 12 is given by the Maple command

```
taylor(f(x), k=0, 12);
```

$$1 + \left(\frac{x^2}{2} - \frac{1}{2}\right)k^2 + \left(\frac{x^24}{24} + \frac{5x^2}{24} - \frac{x^2}{4}\right)k^4 + \left(\frac{5}{48} - \frac{61}{720} + \frac{x^6}{720} - \frac{x^4}{48}\right)k^6$$
$$+ \left(-\frac{61x^2}{1440} + \frac{277}{8064} + \frac{x^8}{40320} + \frac{5x^4}{576} - \frac{x^6}{1440}\right)k^8$$
$$+ \left(\frac{277x^2}{16128} - \frac{50521}{3628800} + \frac{x^10}{3628800} - \frac{61x^4}{17280} + \frac{x^6}{3456} - \frac{x^8}{80640}\right)k^{10}$$

All the coefficients could be extracted by the following Maple command:

```
for i to 5 do coeff(tt,k,2*i) od;
```

$$\frac{x^2}{2} - \frac{1}{2}$$
$$\frac{x^4}{24} - \frac{x^2}{4} + \frac{5}{24}$$
$$\frac{x^6}{720} - \frac{x^4}{48} + \frac{5\,x^2}{48} - \frac{61}{720}$$
$$\frac{x^8}{40320} - \frac{x^6}{1440} + \frac{5\,x^4}{576} - \frac{61\,x^2}{1440} + \frac{277}{8064}$$
$$\frac{x^{10}}{3628800} - \frac{x^8}{80640} + \frac{x^6}{3456} - \frac{61\,x^4}{17280} + \frac{277\,x^2}{16128} - \frac{50521}{3628800}$$

Equation (16) is used to build the symbolic computation algorithm for computing Appell polynomials. The coefficients are expressed in terms of Bernuolli and Euler numbers. The algorithm is given in appendix 2. The arguments of the function are the order of the polynomils and the two end points. The following is an illustrative symbolic computation by the algorithm, which computes 3 Appell polynomials.

```
for i from 2 by 2 to 6 do y^i:=apPolyed(i,-1,1) od;
```

$$y^2 = x^2 - Eu(0)$$
$$y^4 = x^4C(4,0)Eu(0) + x^2C(4,2)Eu(2) + C(4,4)Eu(4)$$
$$y^6 = x^6C(6,0)Eu(0) + x^4C(6,2)Eu(2) + x^2C(6,4)Eu(4) + C(6,6)Eu(6)$$

Maple can be asked to simplify the coefficients. The command for evaluating the coefficients of Appell polynomials of order 12 is given as:

```
for i from 2 by 2 to 16 do y^i:=Eval(subs(C=binomial, Eu=euler,
apPolyed(i,-1,1))); od;
```

$$y^2 := x^2 - 1$$
$$y^4 := x^4 - 6x^2 + 5$$
$$y^6 := x^6 - 15x^4 + 75x^2 - 61$$
$$y^8 := x^8 - 28x^6 + 350x^4 - 1708x^2 + 1385$$
$$y^{10} := x^{10} - 45x^8 + 1050x^6 - 12810x^4 + 62325x^2 - 50521$$
$$y^{12} := x^{12} - 66x^{10} + 2475x^8 - 56364x^6 + 685575x^4 - 3334386x^2 + 2702765$$

Using Theorem 1, a set of Appell polynomial and their graphs for the function of bounded variation f(x)=x on [0,1]is shown below. The graphs of lower degrees passes through the maximum and minimum points of the graphs of higher degree. Hence, Appell polynomials always contain the optimization function. One can easily calculate the optimization points by location the intersection of any two successive points. This property of Appell polynomials makes it natural candidate for optimization problems.

$$f1 := -\frac{1}{2} + x$$
$$f2 := \frac{1}{12} - \frac{1}{2}x + \frac{1}{2}x$$
$$f3 := \frac{1}{12}x - \frac{1}{4}x^2 + \frac{1}{6}x^3$$
$$f4 := -\frac{1}{720} + \frac{1}{24}x^2 - \frac{1}{12}x^3 + \frac{1}{24}x^4$$

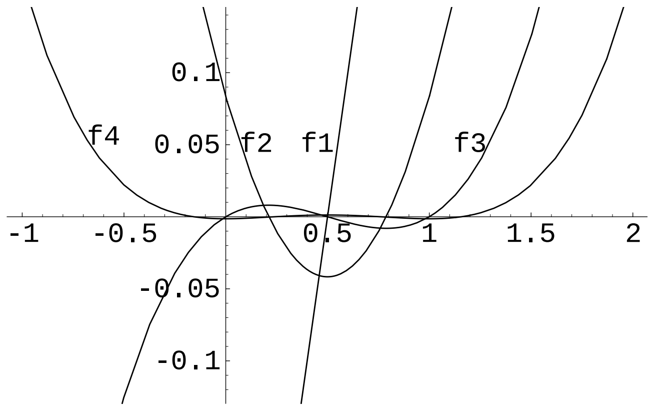

Figure 1: Appell Polynomial for $f(x) = x, [0, 1]$

Conclusion. We have demonstrated a new approach for the computation of Appell polynomials using Maple. Both the elementary and operator methods basically yield the same result. This fact confirms the correctness of the approach used in the construction. The interesting part of the work is the symbolic algebra system construction of Appell polynimials, which can be used to generate a set of Appell polynomial for any function of bounded variation. In the future, work will be done on using Appell polynomials to solve linear differential equation of form defined by Equation two.

Appendix 1

```
 This algorithm computes Appell polynomials

apPoly:=proc(n,n1,n2) local a1, a2, a3, y, i;
%n order of the polynomial
%n1, n2 the two zero points

if not ((n1=-1 and n2=1) or (n1=1 and n2=-1) or (n1=0 and n2=1)
or (n1=1 and n2=0)) then ERROR(‘ n1, n2= 0, -1 or 1‘)  fi;
if n1=0 and n2=1 then aa1:=n1; aa2:=n2;
 elif n1=1 and n2=0 then aa1:=n2; aa2:=n1;
 elif n1=-1 and n2=1 then aa1:=n1; aa2:=n2;
else aa1:=n2; aa2:=n1;
fi;
if n < 1 or type(n, odd) then ERROR(‘ n must be even.‘);
elif n = 2  and aa1=-1 and aa2=1 then   y2:=x^2-1;
elif n = 2  and aa1=0 and aa2=1 then   y2:=x^2-x;
elif n > 2 and aa1=-1 and aa2=1 then  y2:=x^2-1;
elif n > 2 and aa1=0 and aa2=1 then  y2:=x^2-x;
fi;
```

```
if  n > 2 then

 for i from 1 to n/2-1 do
      y.(2*i+1):=(2*i+1)*(int(y.(2*i),x)+c);
        y.(2*i+2):=(2*i+2)*(int(y.(2*i+1),x)+d);
         a1:=subs(x= aa2,  y.(2*i+2)=0);
         a2:=subs(x=aa1,  y.(2*i+2)=0);
         a3:=solve({a1, a2}, {c,d});
         y.(2*i+2):=subs(op(a3), y.(2*i+2)) ;
      od;
   fi;
    y^n=";
  end;
```

Appendix 2

```
Algorithm for computing Appell Polynomial in terms of
 Binomial (C) and Euler (Eu) function
  apPolyeb:=proc(n,n1,n2) local aa1, aa2;
    if not ((n1=-1 and n2=1) or (n1=1 and n2=-1))  then
      ERROR(‘ n1, n2= -1 or 1‘)  fi;
       if n1=-1 and n2=1 then aa1:=n1; aa2:=n2;
         else aa1:=n2; aa2:=n1;  fi;
   if n < 1 or type(n,odd) then
   ERROR(‘ n must be even and greater than 1.‘);
   elif n = 2  and aa1=-1 and aa2=1 then   y2:=x^2-Eu(0);
   elif  n > 2 then
      Appell:=sum(x^(2*n/2-2*j)*C(n,2*j)*Eu(2*j), j=0..n/2);
   fi;  end;
```

Appendix 3

```
appelpoly := proc (f, x, a, b, n) local i, j, r, ii;
  co.n := 1/(n!*int(diff(f,x),x = a .. b)); with(linalg):
for r from 2 to n+1 do
 co.(n+1-r) := matrix(r,r,[]);
       for i to r do for j to r do
co.(n+1-r)[1,r] := 1;
  if i = j and j <> r then
   co.(n+1-r)[i,j] := int(diff(f,x),x = a .. b);
      co.(n+1-r)[r,r] := 0 elif j < i then
   co.(n+1-r)[i,j] := int(x^(i-j)*diff(f,x),x = a .. b)/(i-j)!
elif i < j then co.(n+1-r)[i,j] := 0 fi od od;
co.(n+1-r) := det(convert(co.(n+1-r),matrix))/
((n+1-r)!*int(diff(f,x),x = a .. b)^r);
add(co.ii*x^ii,ii = 0 .. n) od end
```

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