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Strongly nonlinear parabolic initial-boundary value problems in Orlicz spaces *

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Abstract

We prove existence and convergence theorems for nonlinear parabolic problems. We also prove some compactness results in inhomogeneous Orlicz-Sobolev spaces.

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N, T > 0$ and let

$$A(u) = \sum_{|\alpha| \leq 1} (-1)^{|\alpha|} D^\alpha A_\alpha(x, t, u, \nabla u)$$

be a Leray-Lions operator defined on $L^p(0, T; W^{1,p}(\Omega))$, $1 < p < \infty$. Boccardo and Murat [5] proved the existence of solutions for parabolic initial-boundary value problems of the form

$$\frac{\partial u}{\partial t} + A(u) + g(x, t, u, \nabla u) = f \quad \text{in } \Omega \times (0, T), \tag{1.1}$$

where g is a nonlinearity with the following growth condition

$$g(x, t, s, \xi) \leq b(|s|)(c(x, t) + |\xi|^q), \quad q < p, \tag{1.2}$$

and which satisfies the classical sign condition $g(x, t, s, \xi)s \geq 0$. The right hand side f is assumed (in [5]) to belong to $L^{p'}(0, T; W^{-1,p'}(\Omega))$. This result generalizes the analogous one of Landes-Mustonen [14] where the nonlinearity g depends only on x, t and u. In [5] and [14], the functions A_α are assumed to satisfy a polynomial growth condition with respect to u and ∇u. When trying to relax this restriction on the coefficients A_α, we are led to replace $L^p(0, T; W^{1,p}(\Omega))$ by an inhomogeneous Sobolev space $W^{1,x}L_M$ built from an Orlicz space L_M instead of L^p, where the N-function M which defines L_M is related to the actual growth of the A_α's. The solvability of (1.1) in this

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setting is proved by Donaldson [7] and Robert [16] in the case where $g \equiv 0$. It is our purpose in this paper, to prove existence theorems in the setting of the inhomogeneous Sobolev space $W^{1,x}L_M$ by applying some new compactness results in Orlicz spaces obtained under the assumption that the N- function $M(t)$ satisfies Δ'-condition and which grows less rapidly than $|t|^{N/(N-1)}$. These compactness results, which we are at first established in [8], generalize those of Simon [17], Landes-Mustonen [14] and Boccardo-Murat [6]. It is not clear whether the present approach can be further adapted to obtain the same results for general N-functions.

For related topics in the elliptic case, the reader is referred to [2] and [3].

2 Preliminaries

Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function, i.e. M is continuous, convex, with $M(t) > 0$ for $t > 0$, $\frac{M(t)}{t} \to 0$ as $t \to 0$ and $\frac{M(t)}{t} \to \infty$ as $t \to \infty$. Equivalently, M admits the representation: $M(t) = \int_0^t a(\tau)d\tau$ where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing, right continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$ and $a(t) \to \infty$ as $t \to \infty$. The N-function $\overline{M}$ conjugate to M is defined by $\overline{M}(t) = \int_0^t \overline{a}(\tau)d\tau$, where $\overline{a} : \mathbb{R}^+ \to \mathbb{R}^+$ is given by $\overline{a}(t) = \sup\{s : a(s) \leq t\}$ [1, 11, 12].

The N-function M is said to satisfy the Δ_2 condition if, for some $k > 0$:

$$M(2t) \leq k\, M(t) \quad \text{for all } t \geq 0, \tag{2.1}$$

when this inequality holds only for $t \geq t_0 > 0$, M is said to satisfy the Δ_2 condition near infinity.

Let P and Q be two N-functions. $P \ll Q$ means that P grows essentially less rapidly than Q; i.e., for each $\varepsilon > 0$,

$$\frac{P(t)}{Q(\varepsilon\, t)} \to 0 \quad \text{as } t \to \infty.$$

This is the case if and only if

$$\lim_{t\to\infty} \frac{Q^{-1}(t)}{P^{-1}(t)} = 0.$$

An N-function is said to satisfy the $\triangle'$-condition if, for some $k_0 > 0$ and some $t_0 \geq 0$:

$$M(k_0 tt') \leq M(t)M(t'), \quad \text{for all } t, t' \geq t_0. \tag{2.2}$$

It is easy to see that the $\triangle'$-condition is stronger than the $\triangle_2$-condition. The following N-functions satisfy the $\triangle'$-condition: $M(t) = t^p(\mathrm{Log}^q\, t)^s$, where $1 < p < +\infty$, $0 \leq s < +\infty$ and $q \geq 0$ is an integer (Log^q being the iterated of order q of the function log).

We will extend these N-functions into even functions on all $\mathbb{R}$. Let Ω be an open subset of $\mathbb{R}^N$. The Orlicz class $\mathcal{L}_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real-valued measurable functions u on Ω such that:

$$\int_\Omega M(u(x))dx < +\infty \quad (\text{resp. } \int_\Omega M(\frac{u(x)}{\lambda})dx < +\infty \text{ for some } \lambda > 0).$$

Note that $L_M(\Omega)$ is a Banach space under the norm

$$\|u\|_{M,\Omega} = \inf\Big\{\lambda > 0 : \int_\Omega M(\frac{u(x)}{\lambda})dx \le 1\Big\}$$

and $\mathcal{L}_M(\Omega)$ is a convex subset of $L_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$. The equality $E_M(\Omega) = L_M(\Omega)$ holds if and only if M satisfies the Δ_2 condition, for all t or for t large according to whether Ω has infinite measure or not.

The dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega u(x)v(x)dx$, and the dual norm on $L_{\overline{M}}(\Omega)$ is equivalent to $\|.\|_{\overline{M},\Omega}$. The space $L_M(\Omega)$ is reflexive if and only if M and $\overline{M}$ satisfy the Δ_2 condition, for all t or for t large, according to whether Ω has infinite measure or not.

We now turn to the Orlicz-Sobolev space. $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp. $E_M(\Omega)$). This is a Banach space under the norm

$$\|u\|_{1,M,\Omega} = \sum_{|\alpha|\le 1} \|D^\alpha u\|_{M,\Omega}.$$

Thus $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by ΠL_M, we will use the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$. We say that u_n converges to u for the modular convergence in $W^1L_M(\Omega)$ if for some $\lambda > 0$, $\int_\Omega M(\frac{D^\alpha u_n - D^\alpha u}{\lambda})dx \to 0$ for all $|\alpha| \le 1$. This implies convergence for $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. If M satisfies the Δ_2 condition on $\mathbb{R}^+$(near infinity only when Ω has finite measure), then modular convergence coincides with norm convergence.

Let $W^{-1}L_{\overline{M}}(\Omega)$ (resp. $W^{-1}E_{\overline{M}}(\Omega)$) denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}(\Omega)$ (resp. $E_{\overline{M}}(\Omega)$). It is a Banach space under the usual quotient norm.

If the open set Ω has the segment property, then the space $\mathcal{D}(\Omega)$ is dense in $W_0^1L_M(\Omega)$ for the modular convergence and for the topology $\sigma(\Pi L_M, \Pi L_{\overline{M}})$ (cf. [9, 10]). Consequently, the action of a distribution in $W^{-1}L_{\overline{M}}(\Omega)$ on an element of $W_0^1L_M(\Omega)$ is well defined.

For $k > 0$, we define the truncation at height $k, T_k : \mathbb{R} \to \mathbb{R}$ by

$$T_k(s) = \begin{cases} s & \text{if } |s| \le k \\ ks/|s| & \text{if } |s| > k. \end{cases} \tag{2.3}$$

The following abstract lemmas will be applied to the truncation operators.

Lemma 2.1 *Let $F:\mathbb{R}\to\mathbb{R}$ be uniformly lipschitzian, with $F(0)=0$. Let M be an N-function and let $u\in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Then $F(u)\in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Moreover, if the set of discontinuity points of F' is finite, then*

$$\frac{\partial}{\partial x_i}F(u)=\begin{cases}F'(u)\frac{\partial u}{\partial x_i} & a.e.\ in\ \{x\in\Omega : u(x)\notin D\}\\ 0 & a.e.\ in\ \{x\in\Omega : u(x)\in D\}.\end{cases}$$

Lemma 2.2 *Let $F:\mathbb{R}\to\mathbb{R}$ be uniformly lipschitzian, with $F(0)=0$. We suppose that the set of discontinuity points of F' is finite. Let M be an N-function, then the mapping $F: W^1L_M(\Omega)\to W^1L_M(\Omega)$ is sequentially continuous with respect to the weak* topology $\sigma(\Pi L_M,\Pi E_{\overline{M}})$.*

Proof By the previous lemma, $F(u)\in W^1L_M(\Omega)$ for all $u\in W^1L_M(\Omega)$ and

$$\|F(u)\|_{1,M,\Omega}\le C\,\|u\|_{1,M,\Omega},$$

which gives easily the result. □

Let Ω be a bounded open subset of $\mathbb{R}^N$, $T>0$ and set $Q=\Omega\times]0,T[$. Let $m\ge 1$ be an integer and let M be an N-function. For each $\alpha\in\mathbf{N}^N$, denote by D_x^α the distributional derivative on Q of order α with respect to the variable $x\in\mathbb{R}^N$. The inhomogeneous Orlicz-Sobolev spaces are defined as follows

$$W^{m,x}L_M(Q)=\{u\in L_M(Q): D_x^\alpha u\in L_M(Q)\ \forall|\alpha|\le m\}$$
$$W^{m,x}E_M(Q)=\{u\in E_M(Q): D_x^\alpha u\in E_M(Q)\ \forall|\alpha|\le m\}$$

The last space is a subspace of the first one, and both are Banach spaces under the norm

$$\|u\|=\sum_{|\alpha|\le m}\|D_x^\alpha u\|_{M,Q}.$$

We can easily show that they form a complementary system when Ω satisfies the segment property. These spaces are considered as subspaces of the product space $\Pi L_M(Q)$ which have as many copies as there is α-order derivatives, $|\alpha|\le m$. We shall also consider the weak topologies $\sigma(\Pi L_M,\Pi E_{\overline{M}})$ and $\sigma(\Pi L_M,\Pi L_{\overline{M}})$. If $u\in W^{m,x}L_M(Q)$ then the function : $t\longmapsto u(t)=u(t,.)$ is defined on $[0,T]$ with values in $W^mL_M(\Omega)$. If, further, $u\in W^{m,x}E_M(Q)$ then the concerned function is a $W^mE_M(\Omega)$-valued and is strongly measurable. Furthermore the following imbedding holds: $W^{m,x}E_M(Q)\subset L^1(0,T;W^mE_M(\Omega))$. The space $W^{m,x}L_M(Q)$ is not in general separable, if $u\in W^{m,x}L_M(Q)$, we can not conclude that the function $u(t)$ is measurable on $[0,T]$. However, the scalar function $t\mapsto\|u(t)\|_{M,\Omega}$ is in $L^1(0,T)$. The space $W_0^{m,x}E_M(Q)$ is defined as the (norm) closure in $W^{m,x}E_M(Q)$ of $\mathcal{D}(Q)$. We can easily show as in [10] that when Ω has the segment property then each element u of the closure of $\mathcal{D}(Q)$ with respect of the weak * topology $\sigma(\Pi L_M,\Pi E_{\overline{M}})$ is limit, in $W^{m,x}L_M(Q)$, of some subsequence $(u_i)\subset\mathcal{D}(Q)$ for the modular convergence; i.e., there exists $\lambda>0$ such that for all $|\alpha| \leq m$,

$$\int_Q M(\frac{D_x^\alpha u_i - D_x^\alpha u}{\lambda})\, dx\, dt \to 0 \text{ as } i \to \infty,$$

this implies that (u_i) converges to u in $W^{m,x}L_M(Q)$ for the weak topology $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. Consequently

$$\overline{\mathcal{D}(Q)}^{\sigma(\Pi L_M, \Pi E_{\overline{M}})} = \overline{\mathcal{D}(Q)}^{\sigma(\Pi L_M, \Pi L_{\overline{M}})},$$

this space will be denoted by $W_0^{m,x}L_M(Q)$. Furthermore, $W_0^{m,x}E_M(Q) = W_0^{m,x}L_M(Q) \cap \Pi E_M$. Poincaré's inequality also holds in $W_0^{m,x}L_M(Q)$ i.e. there is a constant $C > 0$ such that for all $u \in W_0^{m,x}L_M(Q)$ one has

$$\sum_{|\alpha| \leq m} \|D_x^\alpha u\|_{M,Q} \leq C \sum_{|\alpha| = m} \|D_x^\alpha u\|_{M,Q}.$$

Thus both sides of the last inequality are equivalent norms on $W_0^{m,x}L_M(Q)$. We have then the following complementary system

$$\begin{pmatrix} W_0^{m,x}L_M(Q) & F \\ W_0^{m,x}E_M(Q) & F_0 \end{pmatrix},$$

F being the dual space of $W_0^{m,x}E_M(Q)$. It is also, except for an isomorphism, the quotient of $\Pi L_{\overline{M}}$ by the polar set $W_0^{m,x}E_M(Q)^\perp$, and will be denoted by $F = W^{-m,x}L_{\overline{M}}(Q)$ and it is shown that

$$W^{-m,x}L_{\overline{M}}(Q) = \Big\{ f = \sum_{|\alpha| \leq m} D_x^\alpha f_\alpha : f_\alpha \in L_{\overline{M}}(Q) \Big\}.$$

This space will be equipped with the usual quotient norm

$$\|f\| = \inf \sum_{|\alpha| \leq m} \|f_\alpha\|_{\overline{M},Q}$$

where the infimum is taken on all possible decompositions

$$f = \sum_{|\alpha| \leq m} D_x^\alpha f_\alpha, \quad f_\alpha \in L_{\overline{M}}(Q).$$

The space F_0 is then given by

$$F_0 = \Big\{ f = \sum_{|\alpha| \leq m} D_x^\alpha f_\alpha : f_\alpha \in E_{\overline{M}}(Q) \Big\}$$

and is denoted by $F_0 = W^{-m,x}E_{\overline{M}}(Q)$.

Remark 2.3 We can easily check, using [10, lemma 4.4], that each uniformly lipschitzian mapping F, with $F(0) = 0$, acts in inhomogeneous Orlicz-Sobolev spaces of order 1: $W^{1,x}L_M(Q)$ and $W_0^{1,x}L_M(Q)$.

3 Galerkin solutions

In this section we shall define and state existence theorems of Galerkin solutions for some parabolic initial-boundary problem.

Let Ω be a bounded subset of $\mathbb{R}^N$, $T>0$ and set $Q=\Omega\times]0,T[$. Let

$$A(u)=\sum_{|\alpha|\leq m}(-1)^{|\alpha|}D_x^\alpha(A_\alpha(u))$$

be an operator such that

$$\begin{aligned}&A_\alpha(x,t,\xi):\Omega\times[0,T]\times\mathbb{R}^{N_0}\to\mathbb{R}\text{ is continuous in }(t,\xi),\text{ for a.e. }x\in\Omega\\&\text{and measurable in }x,\ \text{for all }(t,\xi)\in[0,T]\times\mathbb{R}^{N_0},\\&\text{where, }N_0\text{ is the number of all }\alpha\text{-order's derivative, }|\alpha|\leq m.\end{aligned}\tag{3.1}$$

$$|A_\alpha(x,s,\xi)|\leq\chi(x)\Phi(|\xi|)\text{ with }\chi(x)\in L^1(\Omega)\text{ and }\Phi:\mathbb{R}^+\to\mathbb{R}^+\text{ increasing.}\tag{3.2}$$

$$\sum_{|\alpha|\leq m}A_\alpha(x,t,\xi)\xi_\alpha\geq-d(x,t)\text{ with }d(x,t)\in L^1(Q),\ d\geq 0.\tag{3.3}$$

Consider a function $\psi\in L^2(Q)$ and a function $\overline{u}\in L^2(\Omega)\cap W_0^{m,1}(\Omega)$. We choose an orthonormal sequence $(\omega_i)\subset\mathcal{D}(\Omega)$ with respect to the Hilbert space $L^2(\Omega)$ such that the closure of (ω_i) in $C^m(\overline{\Omega})$ contains $\mathcal{D}(\Omega)$. $C^m(\overline{\Omega})$ being the space of functions which are m times continuously differentiable on $\overline{\Omega}$. For $V_n=\operatorname{span}\langle\omega_1,\dots,\omega_n\rangle$ and

$$\|u\|_{C^{1,m}(Q)}=\sup\left\{|D_x^\alpha u(x,t)|,|\frac{\partial u}{\partial t}(x,t)|:|\alpha|\leq m,(x,t)\in Q\right\}$$

we have

$$\mathcal{D}(Q)\subset\overline{\{\cup_{n=1}^\infty C^1([0,T],V_n)\}}^{C^{1,m}(Q)}$$

this implies that for ψ and $\overline{u}$, there exist two sequences (ψ_n) and $(\overline{u}_n)$ such that

$$\psi_n\in C^1([0,T],V_n),\quad\psi_n\to\psi\text{ in }L^2(Q).\tag{3.4}$$

$$\overline{u}_n\in V_n,\quad\overline{u}_n\to\overline{u}\text{ in }L^2(\Omega)\cap W_0^{m,1}(\Omega).\tag{3.5}$$

Consider the parabolic initial-boundary value problem

$$\begin{gathered}\frac{\partial u}{\partial t}+A(u)=\psi\text{ in }Q,\\D_x^\alpha u=0\text{ on }\partial\Omega\times]0,T[,\ \text{for all }|\alpha|\leq m-1,\\u(0)=\overline{u}\text{ in }\Omega.\end{gathered}\tag{3.6}$$

In the sequel we denote $A_\alpha(x,t,u,\nabla u,\dots,\nabla^m u)$ by $A_\alpha(x,t,u)$ or simply by $A_\alpha(u)$.

Definition 3.1 A function $u_n \in C^1([0,T], V_n)$ is called Galerkin solution of (3.6) if

$$\int_\Omega \frac{\partial u_n}{\partial t}\varphi dx + \int_\Omega \sum_{|\alpha|\le m} A_\alpha(u_n).D_x^\alpha \varphi dx = \int_\Omega \psi_n(t)\varphi dx$$

for all $\varphi \in V_n$ and all $t \in [0,T]$; $u_n(0) = \overline{u}_n$.

We have the following existence theorem.

Theorem 3.2 ([13]) *Under conditions (3.1)-(3.3), there exists at least one Galerkin solution of (3.6).*

Consider now the case of a more general operator

$$A(u) = \sum_{|\alpha|\le m} (-1)^{|\alpha|} D_x^\alpha(A_\alpha(u))$$

where instead of (3.1) and (3.2) we only assume that

$$A_\alpha(x,t,\xi) : \Omega \times [0,T] \times \mathbb{R}^{N_0} \to \mathbb{R} \text{ is continuous in } \xi, \text{ for a.e. } (x,t) \in Q$$
$$\text{and measurable in } (x,t) \text{ for all } \xi \in \mathbb{R}^{N_0}. \tag{3.7}$$

$$|A_\alpha(x,s,\xi)| \le C(x,t)\Phi(|\xi|) \text{ with } C(x,t) \in L^1(Q). \tag{3.8}$$

We have also the following existence theorem

Theorem 3.3 ([14]) *There exists a function u_n in $C([0,T], V_n)$ such that $\frac{\partial u_n}{\partial t}$ is in $L^1(0,T;V_n)$ and*

$$\int_{Q_\tau} \frac{\partial u_n}{\partial t}\varphi \, dx\, dt + \int_{Q_\tau} \sum_{|\alpha|\le m} A_\alpha(x,t,u_n).D_x^\alpha \varphi \, dx\, dt = \int_{Q_\tau} \psi_n \varphi \, dx\, dt$$

for all $\tau \in [0,T]$ and all $\varphi \in C([0,T], V_n)$, where $Q_\tau = \Omega \times [0,\tau]$; $u_n(0) = \overline{u}_n$.

4 Strong convergence of truncations

In this section we shall prove a convergence theorem for parabolic problems which allows us to deal with approximate equations of some parabolic initial-boundary problem in Orlicz spaces (see section 6). Let Ω, be a bounded subset of $\mathbb{R}^N$ with the segment property and let $T > 0$, $Q = \Omega \times]0,T[$. Let M be an N-function satisfying a Δ' condition and the growth condition

$$M(t) \ll |t|^{\frac{N}{N-1}}$$

and let P be an N-function such that $P \ll M$. Let $A : W^{1,x}L_M(Q) \to W^{-1,x}L_{\overline{M}}(Q)$ be a mapping given by

$$A(u) = -\operatorname{div} a(x,t,u,\nabla u)$$

where $a(x,t,s,\xi) : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying for a.e. $(x,t) \in \Omega \times]0,T[$ and for all $s \in \mathbb{R}$ and all $\xi, \xi^* \in \mathbb{R}^N$:

$$|a(x,t,s,\xi)| \leq c(x,t) + k_1 \overline{P}^{-1} M(k_2|s|) + k_3 \overline{M}^{-1} M(k_4|\xi|) \tag{4.1}$$

$$[a(x,t,s,\xi) - a(x,t,s,\xi^*)][\xi - \xi^*] > 0 \quad \text{if } \xi \neq \xi^* \tag{4.2}$$

$$\alpha M(\frac{|\xi|}{\lambda}) - d(x,t) \leq a(x,t,s,\xi)\xi \tag{4.3}$$

where $c(x,t) \in E_{\overline{M}}(Q)$, $c \geq 0$, $d(x,t) \in L^1(Q)$, $k_1, k_2, k_3, k_4 \in \mathbb{R}^+$ and $\alpha, \lambda \in \mathbb{R}^+_*$. Consider the nonlinear parabolic equations

$$\frac{\partial u_n}{\partial t} - \operatorname{div} a(x,t,u_n,\nabla u_n) = f_n + g_n \quad \text{in } \mathcal{D}'(Q) \tag{4.4}$$

and assume that:

$$u_n \rightharpoonup u \quad \text{weakly in } W^{1,x}L_M(Q) \text{for } \sigma(\Pi L_M, \Pi E_{\overline{M}}), \tag{4.5}$$

$$f_n \to f \quad \text{strongly in } W^{-1,x}E_{\overline{M}}(Q), \tag{4.6}$$

$$g_n \rightharpoonup g \quad \text{weakly in } L^1(Q). \tag{4.7}$$

We shall prove the following convergence theorem.

Theorem 4.1 *Assume that (4.1)-(4.7) hold. Then, for any $k > 0$, the truncation of u_n at height k (see (2.3) for the definition of the truncation) satisfies*

$$\nabla T_k(u_n) \to \nabla T_k(u) \quad \textit{strongly in } (L^{\text{loc}}_M(Q))^N. \tag{4.8}$$

Remark 4.2 An elliptic analogous theorem is proved in Benkirane-Elmahi [2].

Remark 4.3 Convergence (4.8) allows, in particular, to extract a subsequence n' such that:

$$\nabla u_{n'} \to \nabla u \quad \text{a.e. in } Q.$$

Then by lemma 4.4 of [9], we deduce that

$$a(x,t,u_{n'},\nabla u_{n'}) \rightharpoonup a(x,t,u,\nabla u) \quad \text{weakly in } L_{\overline{M}}(Q))^N \text{ for } \sigma(\Pi L_{\overline{M}}, \Pi E_M).$$

Proof of Theorem 4.1 Step 1: For each $k > 0$, define $S_k(s) = \int_0^s T_k(\tau) d\tau$. Since T_k is continuous, for all $w \in W^{1,x}L_M(Q)$ we have $S_k(w) \in W^{1,x}L_M(Q)$ and $\nabla S_k(w) = T_k(w)\nabla w$. So that, by mollifying as in [6], it is easy to see that for all $\varphi \in \mathcal{D}(Q)$ and all $v \in W^{1,x}L_M(Q)$ with $\frac{\partial v}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q) + L^1(Q)$, we have

$$\langle\langle \frac{\partial v}{\partial t}, \varphi T_k(v) \rangle\rangle = -\int_Q \frac{\partial \varphi}{\partial t} S_k(v)\, dx\, dt. \tag{4.9}$$

where $\langle\langle , \rangle\rangle$ means for the duality pairing between $W_0^{1,x}L_M(Q) + L^1(Q)$ and $W^{-1,x}L_{\overline{M}}(Q) \cap L^\infty(Q)$. Fix now a compact set K with $K \subset Q$ and a function

$\varphi_K \in \mathcal{D}(Q)$ such that $0 \leq \varphi_K \leq 1$ in Q and $\varphi_K = 1$ on K. Using in (4.4) $v_n = \varphi_K(T_k(u_n) - T_k(u)) \in W^{1,x}L_M(Q) \cap L^\infty(Q)$ as test function yields

$$\begin{aligned}
&\langle\langle \frac{\partial u_n}{\partial t}, \varphi_K T_k(u_n)\rangle\rangle - \langle\langle \frac{\partial u_n}{\partial t}, \varphi_K T_k(u)\rangle\rangle \\
&+ \int_Q \varphi_K a(x,t,u_n,\nabla u_n)[\nabla T_k(u_n) - \nabla T_k(u)]dx\,dt \\
&+ \int_Q (T_k(u_n) - T_k(u))a(x,t,u_n,\nabla u_n)\nabla\varphi_K\,dx\,dt \\
&= \langle\langle f_n, v_n\rangle\rangle + \langle\langle g_n, v_n\rangle\rangle.
\end{aligned} \tag{4.10}$$

Since $u_n \in W^{1,x}L_M(Q)$ and $\frac{\partial u_n}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q) + L^1(Q)$ then by (4.9),

$$\langle\langle \frac{\partial u_n}{\partial t}, \varphi_K T_k(u_n)\rangle\rangle = -\int_Q \frac{\partial \varphi_K}{\partial t} S_k(u_n)\,dx\,dt.$$

On the other hand since (u_n) is bounded in $W^{1,x}L_M(Q)$ and $\frac{\partial u_n}{\partial t} = h_n + g_n$ while g_n is bounded in $L^1(Q)$ and so in $\mathcal{M}(Q)$ and $h_n = \operatorname{div} a(x,t,u_n,\nabla u_n) + f_n$ is bounded in $W^{-1,x}L_{\overline{M}}(Q)$, then by [8, Corollary 1], $u_n \to u$ strongly in $L_M^{\mathrm{loc}}(Q)$. Consequently, $T_k(u_n) \to T_k(u)$ and $S_k(u_n) \to S_k(u)$ in $L_M^{\mathrm{loc}}(Q)$. So that

$$\int_Q \frac{\partial \varphi_K}{\partial t} S_k(u_n)\,dx\,dt \to \int_Q \frac{\partial \varphi_K}{\partial t} S_k(u)\,dx\,dt$$

and also $\int_Q (T_k(u_n) - T_k(u))a(x,t,u_n,\nabla u_n)\nabla\varphi_K\,dx\,dt \to 0$ as $n \to \infty$. Furthermore $\langle\langle f_n, v_n\rangle\rangle \to 0$, by (4.6). Since $g_n \in L^1(Q)$ and $T_k(u_n) - T_k(u) \in L^\infty(Q)$,

$$\langle\langle g_n, \varphi_K(T_k(u_n) - T_k(u))\rangle\rangle = \int_Q g_n \varphi_K(T_k(u_n) - T_k(u))\,dx\,dt$$

which tends to 0 by Egorov's theorem.

Since $\varphi_K T_k(u)$ belongs to $W_0^{1,x}L_M(Q) \cap L^\infty(Q)$ while $\frac{\partial u_n}{\partial t}$ is the sum of a bounded term in $W^{-1,x}L_{\overline{M}}(Q)$ and of g_n which weakly converges in $L^1(Q)$ one has

$$\langle\langle \frac{\partial u_n}{\partial t}, \varphi_K T_k(u)\rangle\rangle \to \langle\langle \frac{\partial u}{\partial t}, \varphi_K T_k(u)\rangle\rangle = -\int_Q \frac{\partial \varphi}{\partial t} S_k(u)\,dx\,dt.$$

We have thus proved that

$$\int_Q \varphi_K a(x,t,u_n,\nabla u_n)[\nabla T_k(u_n) - \nabla T_k(u)]\,dx\,dt \to 0 \quad \text{as } n \to \infty. \tag{4.11}$$

Step 2: Fix a real number $r > 0$ and set $Q_{(r)} = \{x \in Q : |\nabla T_k(u)| \leq r\}$ and denote by χ_r the characteristic function of $Q_{(r)}$. Taking $s \geq r$ one has:

$$
\begin{aligned}
0 \leq \int_{Q_{(r)}} & \varphi_K \big[a(x,t,u_n,\nabla T_k(u_n)) - a(x,t,u_n,\nabla T_k(u))\big] \\
& \times \big[\nabla T_k(u_n) - \nabla T_k(u)\big]\,dx\,dt \\
& \leq \int_{Q_{(s)}} \varphi_K \big[a(x,t,u_n,\nabla T_k(u_n)) - a(x,t,u_n,\nabla T_k(u))\big] \\
& \times \big[\nabla T_k(u_n) - \nabla T_k(u)\big]\,dx\,dt \\
= \int_{Q_{(s)}} & \varphi_K \big[a(x,t,u_n,\nabla T_k(u_n)) - a(x,t,u_n,\nabla T_k(u)\chi_s)\big] \\
& \times \big[\nabla T_k(u_n) - \nabla T_k(u)\chi_s\big]\,dx\,dt \\
\leq \int_{Q} & \varphi_K \big[a(x,t,u_n,\nabla T_k(u_n)) - a(x,t,u_n,\nabla T_k(u)\chi_s)\big] \\
& \times \big[\nabla T_k(u_n) - \nabla T_k(u)\chi_s\big]\,dx\,dt \\
= \int_{Q} & \varphi_K a(x,t,u_n,\nabla u_n)\big[\nabla T_k(u_n) - \nabla T_k(u)\big]\,dx\,dt \\
& - \int_{Q} \varphi_K \big[a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla T_k(u_n))\big] \\
& \times \big[\nabla T_k(u_n) - \nabla T_k(u)\chi_s\big]\,dx\,dt \\
& + \int_{Q} \varphi_K a(x,t,u_n,\nabla u_n)[\nabla T_k(u) - \nabla T_k(u)\chi_s]\,dx\,dt \\
& - \int_{Q} \varphi_K a(x,t,u_n,\nabla T_k(u)\chi_s)[\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\,dx\,dt.
\end{aligned}
\tag{4.12}
$$

Now pass to the limit in all terms of the right-hand side of the above equation.

By (4.11), the first one tends to 0. Denoting by χ_{G_n} the characteristic function of $G_n = \{(x,t) \in Q : |u_n(x,t)| > k\}$, the second term reads

$$
\int_{Q} \varphi_K [a(x,t,u_n,\nabla u_n) - a(x,t,u_n,0)]\chi_{G_n} \nabla T_k(u)\chi_s\,dx\,dt \tag{4.13}
$$

which tends to 0 since $[a(x,t,u_n,\nabla u_n) - a(x,t,u_n,0)]$ is bounded in $(L_{\overline{M}}(Q))^N$, by (4.1) and (4.5) while $\chi_{G_n} \nabla T_k(u)\chi_s$ converges strongly in $(E_M(Q))^N$ to 0 by Lebesgue's theorem. The fourth term of (4.12) tends to

$$
\begin{aligned}
- \int_{Q} & \varphi_K a(x,t,u,\nabla T_k(u)\chi_s)[\nabla T_k(u) - \nabla T_k(u)\chi_s]\,dx\,dt \\
& = \int_{Q\setminus Q_{(s)}} \varphi_K a(x,t,u,0)\nabla T_k(u)\,dx\,dt
\end{aligned}
\tag{4.14}
$$

since $a(x,t,u_n,\nabla T_k(u)\chi_s)$ tends strongly to $a(x,t,u,\nabla T_k(u)\chi_s)$ in $(E_{\overline{M}}(Q))^N$ while $\nabla T_k(u_n) - \nabla T_k(u)\chi_s$ converges weakly to $\nabla T_k(u) - \nabla T_k(u)\chi_s$ in $(L_M(Q))^N$ for $\sigma(\Pi L_M, \Pi E_{\overline{M}})$.

Since $a(x,t,u_n,\nabla u_n)$ is bounded in $(L_{\overline{M}}(Q))^N$ one has (for a subsequence still denoted by u_n)

$$a(x,t,u_n,\nabla u_n) \rightharpoonup h \quad \text{weakly in } (L_{\overline{M}}(Q))^N \text{ for } \sigma(\Pi L_{\overline{M}}, \Pi E_M). \tag{4.15}$$

Finally, the third term of the right-hand side of (4.12) tends to

$$\int_{Q\setminus Q_{(s)}} \varphi_K h \nabla T_k(u)\,dx\,dt. \tag{4.16}$$

We have, then, proved that

$$\begin{aligned} 0 \le \limsup_{n\to\infty} \int_{Q_{(r)}} & \varphi_K \big[a(x,t,u_n,\nabla T_k(u_n)) - a(x,t,u_n,\nabla T_k(u))\big] \\ & \times \big[\nabla T_k(u_n) - \nabla T_k(u)\big]\,dx\,dt \\ \le \int_{Q\setminus Q_{(s)}} & \varphi_K [h - a(x,t,u,0)] \nabla T_k(u)\,dx\,dt. \end{aligned} \tag{4.17}$$

Using the fact that $[h - a(x,t,u,0)]\nabla T_k(u) \in L^1(\Omega)$ and letting $s \to +\infty$ we get, since $|Q \setminus Q_{(s)}| \to 0$,

$$\int_{Q_{(r)}} \varphi_K [a(x,t,u_n,\nabla T_k(u_n)) - a(x,t,u_n,\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\,dx\,dt \tag{4.18}$$

which approaches 0 as $n \to \infty$. Consequently

$$\int_{Q_{(r)}\cap K} [a(x,t,u_n,\nabla T_k(u_n)) - a(x,t,u_n,\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\,dx\,dt \to 0$$

as $n \to \infty$. As in [2], we deduce that for some subsequence $\nabla T_k(u_n) \to \nabla T_k(u)$ a.e. in $Q_{(r)} \cap K$. Since r, k and K are arbitrary, we can construct a subsequence (diagonal in r, in k and in j, where (K_j) is an increasing sequence of compacts sets covering Q), such that

$$\nabla u_n \to \nabla u \qquad \text{a.e. in } Q. \tag{4.19}$$

Step 3: As in [2] we deduce that

$$\int_Q \varphi_K a(x,t,u_n,\nabla u_n) \nabla T_k(u_n)\,dx\,dt \to \int_Q \varphi_K a(x,t,u,\nabla u) \nabla T_k(u)\,dx\,dt$$

as $n \to \infty$, and that

$$a(x,t,u_n,\nabla T_k(u_n))\nabla T_k(u_n) \to a(x,t,u,\nabla T_k(u))\nabla T_k(u) \text{ strongly in } L^1(K). \tag{4.20}$$

This implies that (see [2] if necessary): $\nabla T_k(u_n) \to \nabla T_k(u)$ in $(L_M(K))^N$ for the modular convergence and so strongly and convergence (4.8) follows.

Note that in convergence (4.8) the whole sequence (and not only for a subsequence) converges since the limit $\nabla T_k(u)$ does not depend on the subsequence.

5 Nonlinear parabolic problems

Now, we are able to establish an existence theorem for a nonlinear parabolic initial-boundary value problems. This result which specially applies in Orlicz spaces generalizes analogous results in of Landes-Mustonen [14]. We start by giving the statement of the result.

Let Ω be a bounded subset of $\mathbb{R}^N$ with the segment property, $T > 0$, and $Q = \Omega\times]0,T[$. Let M be an N-function satisfying the growth condition

$$M(t) \ll |t|^{\frac{N}{N-1}},$$

and the $\triangle'$-condition. Let P be an N-function such that $P \ll M$. Consider an operator $A : W_0^{1,x}L_M(Q) \to W^{-1,x}L_{\overline{M}}(Q)$ of the form

$$A(u) = -\operatorname{div} a(x,t,u,\nabla u) + a_0(x,t,u,\nabla u) \tag{5.1}$$

where $a : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ and $a_0 : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions satisfying the following conditions, for a.e. $(x,t) \in \Omega \times [0,T]$ for all $s \in \mathbb{R}$ and $\xi \neq \xi^* \in \mathbb{R}^N$:

$$\begin{aligned} |a(x,t,s,\xi)| &\le c(x,t) + k_1\overline{P}^{-1}M(k_2|s|) + k_3\overline{M}^{-1}M(k_4|\xi|),\\ |a_0(x,t,s,\xi)| &\le c(x,t) + k_1\overline{M}^{-1}M(k_2|s|) + k_3\overline{M}^{-1}P(k_4|\xi|), \end{aligned} \tag{5.2}$$

$$[a(x,t,s,\xi) - a(x,t,s,\xi^*)][\xi - \xi^*] > 0, \tag{5.3}$$

$$a(x,t,s,\xi)\xi + a_0(x,t,s,\xi)s \ge \alpha M(\frac{|\xi|}{\lambda}) - d(x,t) \tag{5.4}$$

where $c(x,t) \in E_{\overline{M}}(Q)$, $c \ge 0$, $d(x,t) \in L^1(Q)$, $k_1, k_2, k_3, k_4 \in \mathbb{R}^+$ and $\alpha, \lambda \in \mathbf{R}_*^+$. Furthermore let

$$f \in W^{-1,x}E_{\overline{M}}(Q) \tag{5.5}$$

We shall use notations of section 3. Consider, then, the parabolic initial-boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial t} + A(u) = f \quad \text{in } Q\\ u(x,t) = 0 \text{ on } \partial\Omega\times]0,T[\\ u(x,0) = \psi(x) \text{ in } \Omega. \end{aligned} \tag{5.6}$$

where ψ is a given function in $L^2(\Omega)$. We shall prove the following existence theorem.

Theorem 5.1 *Assume that (5.2)-(5.5) hold. Then there exists at least one weak solution $u \in W_0^{1,x}L_M(Q) \cap L^2(Q) \cap C([0,T], L^2(\Omega))$ of (5.6), in the following sense:*

$$\begin{aligned} -\int_Q u\frac{\partial\varphi}{\partial t}\,dx\,dt + [\int_\Omega u(t)\varphi(t)dx]_0^T + \int_Q a(x,t,u,\nabla u).\nabla\varphi\,dx\,dt\\ + \int_Q a_0(x,t,u,\nabla u).\varphi\,dx\,dt = \langle f,\varphi\rangle \end{aligned} \tag{5.7}$$

for all $\varphi \in C^1([0,T], L^2(\Omega))$.

Remark 5.2 In (5.6), we have $u \in W_0^{1,x} L_M(Q) \subset L^1(0,T; W^{-1,1}(\Omega))$ and $\frac{\partial u}{\partial t} \in W^{-1,x} L_{\overline{M}}(Q) \subset L^1(0,T; W^{-1,1}(\Omega))$. Then $u \in W^{1,1}(0,T; W^{-1,1}(\Omega)) \subset C([0,T], W^{-1,1}(\Omega))$ with continuity of the imbedding. Consequently u is, possibly after modification on a set of zero measure, continuous from $[0,T]$ into $W^{-1,1}(\Omega)$ in such a way that the third component of (5.6), which is the initial condition, has a sense.

Proof of Theorem 4.1 It is easily adapted from the proof given in [14]. For convenience we suppose that $\psi = 0$. For each n, there exists at least one solution u_n of the following problem (see Theorem 3.3 for the existence of u_n):

$$\begin{aligned} & u_n \in C([0,T], V_n), \quad \frac{\partial u_n}{\partial t} \in L^1(0,T; V_n), \quad u_n(0) = \psi_n \equiv 0 \quad \text{and,} \\ & \text{for all } \tau \in [0,T], \quad \int_{Q_\tau} \frac{\partial u_n}{\partial t} \varphi \, dx\, dt + \int_{Q_\varepsilon} a(x,t,u_n,\nabla u_n).\nabla \varphi \, dx\, dt \\ & + \int_{Q_\varepsilon} a_0(x,t,u_n,\nabla u_n).\varphi \, dx\, dt = \int_{Q_\varepsilon} f_n \varphi \, dx\, dt, \quad \forall \varphi \in C([0,T], V_n). \end{aligned} \tag{5.8}$$

where $f_k \subset \cup_{n=1}^{\infty} C([0,T], V_n)$ with $f_k \to f$ in $W^{-1,x} E_{\overline{M}}(Q)$. Putting $\varphi = u_n$ in (5.8), and using (5.2) and (5.4) yields

$$\begin{aligned} & \|u_n\|_{W_0^{1,x} L_M(Q)} \leq C, \quad \|u_n\|_{L^\infty(0,T;L^2(\Omega))} \leq C \\ & \|a_0(x,t,u_n,\nabla u_n)\|_{L_{\overline{M}}(Q)} \leq C \quad \text{and} \quad \|a(x,t,u_n,\nabla u_n)\|_{L_{\overline{M}}(Q)} \leq C. \end{aligned} \tag{5.9}$$

Hence, for a subsequence

$$\begin{aligned} & u_n \rightharpoonup u \text{ weakly in } W_0^{1,x} L_M(Q) \text{ for } \sigma(\Pi L_M, \Pi E_{\overline{M}}) \text{and weakly in } L^2(Q), \\ & a_0(x,t,u_n,\nabla u_n) \rightharpoonup h_0, \ a(x,t,u_n,\nabla u_n) \rightharpoonup h \text{ in } L_{\overline{M}}(Q) \text{ for } \sigma(\Pi L_{\overline{M}}, \Pi E_M) \end{aligned} \tag{5.10}$$

where $h_0 \in L_{\overline{M}}(Q)$ and $h \in (L_{\overline{M}}(Q))^N$. As in [14], we get that for some subsequence $u_n(x,t) \to u(x,t)$ a.e. in Q (it suffices to apply Theorem 3.9 instead of Proposition 1 of [14]). Also we obtain

$$-\int_Q u \frac{\partial \varphi}{\partial t} \, dx\, dt + [\int_\Omega u(t)\varphi(t) dx]_0^T + \int_Q h \nabla \varphi \, dx\, dt + \int_Q h_0 \varphi \, dx\, dt = \langle f, \varphi \rangle,$$

for all $\varphi \in C^1([0,T]; \mathcal{D}(\Omega))$. The proof will be completed, if we can show that

$$\int_Q (h \nabla \varphi + h_0 \varphi) \, dx\, dt = \int_Q (a(x,t,u,\nabla u) \nabla \varphi + a_0(x,t,u,\nabla u)\varphi) \, dx\, dt \tag{5.11}$$

for all $\varphi \in C^1([0,T]; \mathcal{D}(\Omega))$ and that $u \in C([0,T], L^2(\Omega))$. For that, it suffices to show that

$$\lim_{n \to \infty} \int_Q (a(x,t,u_n,\nabla u_n)[\nabla u_n - \nabla u] + a_0(x,t,u_n \nabla u_n)[u_n - u]) \, dx\, dt \leq 0. \tag{5.12}$$

Indeed, suppose that (5.12) holds and let $s > r > 0$ and set $Q^r = \{(x,t) \in Q : |\nabla u(x,t)| \leq r\}$. Denoting by χ_s the characteristic function of Q^s, one has

$$\begin{aligned}
0 &\leq \int_{Q^r} \big[a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla u)\big]\big[\nabla u_n - \nabla u\big]\,dx\,dt \\
&\leq \int_{Q^s} \big[a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla u)\big]\big[\nabla u_n - \nabla u\big]\,dx\,dt \\
&= \int_{Q^s} \big[a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla u.\chi_s)\big]\big[\nabla u_n - \nabla u.\chi_s\big]\,dx\,dt \\
&\leq \int_{Q} \big[a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla u.\chi_s)\big]\big[\nabla u_n - \nabla u.\chi_s\big]\,dx\,dt \\
&= \int_Q a_0(x,t,u_n,\nabla u_n)(u_n - u) - \int_Q a(x,t,u_n,\nabla u_n.\chi_s)[\nabla u_n - \nabla u.\chi_s]\,dx\,dt \\
&\quad + \int_Q \big[a(x,t,u_n,\nabla u_n)(\nabla u_n - \nabla u) + a_0(x,t,u_n,\nabla u_n)(u_n - u)\big]\,dx\,dt \\
&\quad + \int_{Q\setminus Q^s} a(x,t,u_n,\nabla u_n)\nabla u\,dx\,dt.
\end{aligned} \tag{5.13}$$

The first term of the right-hand side tends to 0 since $(a_0(x,t,u_n,\nabla u_n))$ is bounded in $L_{\overline{M}}(Q)$ by (5.2) and $u_n \to u$ strongly in $L_M(Q)$. The second term tends to $\int_{Q\setminus Q^s} a(x,t,u_n,0)\nabla u\,dx\,dt$ since $a(x,t,u_n,\nabla u_n.\chi_s)$ tends strongly in $(E_{\overline{M}}(Q))^N$ to $a(x,t,u,\nabla u.\chi_s)$ and $\nabla u_n \rightharpoonup \nabla u$ weakly in $(L_M(Q))^N$ for $\sigma(\Pi L_M, \Pi E_{\overline{M}})$. The third term satisfies (5.12) while the fourth term tends to $\int_{Q\setminus Q^s} h\nabla u\,dx\,dt$ since $a(x,t,u_n,\nabla u_n) \rightharpoonup h$ weakly in $(L_{\overline{M}}(Q))^N$ for $\sigma(\Pi L_{\overline{M}}, \Pi E_M)$ and M satisfies the $\triangle_2$-condition. We deduce then that

$$\begin{aligned}
0 &\leq \limsup_{n\to\infty} \int_{Q^s} [a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla u)][\nabla u_n - \nabla u]\,dx\,dt \\
&\leq \int_{Q\setminus Q^s} [h - a(x,t,u,0)]\nabla u\,dx\,dt \to 0 \qquad \text{as } s \to \infty.
\end{aligned}$$

and so, by (5.3), we can construct as in [2] a subsequence such that $\nabla u_n \to \nabla u$ a.e. in Q. This implies that $a(x,t,u_n,\nabla u_n) \to a(x,t,u,\nabla u)$ and that $a_0(x,t,u_n,\nabla u_n) \to a_0(x,t,u,\nabla u)$ a.e. in Q. Lemma 4.4 of [9] shows that $h = a(x,t,u,\nabla u)$ and $h_0 = a_0(x,t,u,\nabla u)$ and (5.11) follows. The remaining of the proof is exactly the same as in [14]. $\square$

Corollary 5.3 *The function u can be used as a testing function in (5.6) i.e.*

$$\frac{1}{2}\Big[\int_\Omega (u(t))^2 dx\Big]_0^\tau + \int_{Q_\tau} \big[a(x,t,u,\nabla u).\nabla u + a_0(x,t,u,\nabla u)u\big]\,dx\,dt = \int_{Q_\tau} fu\,dx\,dt$$

for all $\tau \in [0,T]$.

The proof of this corollary is exactly the same as in [14].

6 Strongly nonlinear parabolic problems

In this last section we shall state and prove an existence theorem for strongly nonlinear parabolic initial-boundary problems with a nonlinearity $g(x,t,s,\xi)$ having growth less than $M(|\xi|)$. This result generalizes Theorem 2.1 in Boccardo-Murat [5]. The analogous elliptic one is proved in Benkirane-Elmahi [2].

The notation is the same as in section 5. Consider also assumptions (5.2)-(5.5) to which we will annex a Carathéodory function $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ satisfying, for a.e. $(x,t) \in \Omega \times [0,T]$ and for all $s \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$:

$$g(x,t,s,\xi)s \geq 0 \tag{6.1}$$

$$|g(x,t,s,\xi)| \leq b(|s|)(c'(x,t) + R(|\xi|)) \tag{6.2}$$

where $c' \in L^1(Q)$ and $b : \mathbb{R}^+ \to \mathbb{R}^+$ and where R is a given N-function such that $R \ll M$. Consider the following nonlinear parabolic problem

$$\begin{aligned} \frac{\partial u}{\partial t} + A(u) + g(x,t,u,\nabla u) = f \quad \text{in } Q, \\ u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) = \psi(x) \quad \text{in } \Omega. \end{aligned} \tag{6.3}$$

We shall prove the following existence theorem.

Theorem 6.1 *Assume that (5.1)-(5.5), (6.1) and (6.2) hold. Then, there exists at least one distributional solution of (6.3).*

Proof It is easily adapted from the proof of theorem 3.2 in [2] Consider first

$$g_n(x,t,s,\xi) = \frac{g(x,t,s,\xi)}{1 + \frac{1}{n} g(x,t,s,\xi)}$$

and put $A_n(u) = A(u) + g_n(x,t,u,\nabla u)$, we see that A_n satisfies conditions (5.2)-(5.4) so that, by Theorem 5.1, there exists at least one solution $u_n \in W_0^{1,x} L_M(Q)$ of the approximate problem

$$\begin{aligned} \frac{\partial u_n}{\partial t} + A(u_n) + g_n(x,t,u_n,\nabla u_n) = f \quad \text{in } Q \\ u_n(x,t) = 0 \quad \text{on } \partial\Omega \times]0,T[\\ u_n(x,0) = \psi(x) \quad \text{in } \Omega \end{aligned} \tag{6.4}$$

and, by Corollary 5.3, we can use u_n as testing function in (6.4). This gives

$$\int_Q [a(x,t,u_n,\nabla u_n).\nabla u_n + a_0(x,t,u_n,\nabla u_n).u_n]\, dx\, dt \leq \langle f, u_n \rangle$$

and thus (u_n) is a bounded sequence in $W_0^{1,x} L_M(Q)$. Passing to a subsequence if necessary, we assume that

$$u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,x} L_M(Q) \text{ for } \sigma(\Pi L_M, \Pi E_{\overline{M}}) \tag{6.5}$$

for some $u \in W_0^{1,x}L_M(Q)$. Going back to (6.4), we have

$$\int_Q g_n(x,t,u_n,\nabla u_n)u_n\,dx\,dt \le C.$$

We shall prove that $g_n(x,t,u_n,\nabla u_n)$ are uniformly equi-integrable on Q. Fix $m > 0$. For each measurable subset $E \subset Q$, we have

$$\begin{aligned}
&\int_E |g_n(x,t,u_n,\nabla u_n)| \\
&\le \int_{E\cap\{|u_n|\le m\}} |g_n(x,t,u_n,\nabla u_n)| + \int_{E\cap\{|u_n|>m\}} |g_n(x,t,u_n,\nabla u_n)| \\
&\le b(m)\int_E [c'(x,t)+R(|\nabla u_n|)]\,dx\,dt + \frac{1}{m}\int_{E\cap\{|u_n|>m\}} |g_n(x,t,u_n,\nabla u_n)|\,dx\,dt \\
&\le b(m)\int_E [c'(x,t)+R(|\nabla u_n|)]\,dx\,dt + \frac{1}{m}\int_Q u_n g_n(x,t,u_n,\nabla u_n)\,dx\,dt \\
&\le b(m)\int_E c'(x,t)\,dx\,dt + b(m)\int_E R(\frac{|\nabla u_n|}{\lambda'})\,dx\,dt + \frac{C}{m}
\end{aligned}$$

Let $\varepsilon > 0$, there is $m > 0$ such that $\frac{C}{m} < \frac{\varepsilon}{3}$. Furthermore, since $c'' \in L^1(Q)$ there exists $\delta_1 > 0$ such that $b(m)\int_E c''(x,t)\,dx\,dt < \frac{\varepsilon}{3}$. On the other hand, let $\mu > 0$ such that $\|\nabla u_n\|_{M,Q} \le \mu, \forall n$. Since $R \ll M$, there exists a constant $K_\varepsilon > 0$ depending on ε such that

$$b(m)R(s) \le M(\frac{\varepsilon}{6}\frac{s}{\mu}) + K_\varepsilon$$

for all $s \ge 0$. Without loss of generality, we can assume that $\varepsilon < 1$. By convexity we deduce that

$$b(m)R(s) \le \frac{\varepsilon}{6}M(\frac{s}{\mu}) + K_\varepsilon$$

for all $s \ge 0$. Hence

$$\begin{aligned}
b(m)\int_E R(\frac{|\nabla u_n|}{\lambda'})\,dx\,dt &\le \frac{\varepsilon}{6}\int_E M(\frac{|\nabla u_n|}{\mu})\,dx\,dt + K_\varepsilon|E| \\
&\le \frac{\varepsilon}{6}\int_Q M(\frac{|\nabla u_n|}{\mu})\,dx\,dt + K_\varepsilon|E| \\
&\le \frac{\varepsilon}{6} + K_\varepsilon|E|.
\end{aligned}$$

When $|E| \le \varepsilon/(6K_\varepsilon)$, we have

$$b(m)\int_E R(\frac{|\nabla u_n|}{\lambda'})\,dx\,dt \le \frac{\varepsilon}{3}, \quad \forall n.$$

Consequently, if $|E| < \delta = \inf(\delta_1, \frac{\varepsilon}{6K_\varepsilon})$ one has

$$\int_E |g_n(x,t,u_n,\nabla u_n)|\,dx\,dt \le \varepsilon, \quad \forall n,$$

this shows that the $g_n(x,t,u_n,\nabla u_n)$ are uniformly equi-integrable on Q. By Dunford-Pettis's theorem, there exists $h \in L^1(Q)$ such that

$$g_n(x,t,u_n,\nabla u_n) \rightharpoonup h \quad \text{weakly in } L^1(Q). \tag{6.6}$$

Applying then Theorem 4.1, we have for a subsequence, still denoted by u_n,

$$u_n \to u, \nabla u_n \to \nabla u \text{ a.e. in } Q \text{ and } u_n \to u \text{ strongly in } W_0^{1,x}L_M^{\mathrm{loc}}(Q). \tag{6.7}$$

We deduce that $a(x,t,u_n,\nabla u_n) \rightharpoonup a(x,t,u,\nabla u)$ weakly in $(L_{\overline{M}}(Q))^N$ for $\sigma(\Pi L_{\overline{M}}, \Pi L_M)$ and since $\frac{\partial u_n}{\partial t} \to \frac{\partial u}{\partial t}$ in $\mathcal{D}'(Q)$ then passing to the limit in (6.4) as $n \to +\infty$, we obtain

$$\frac{\partial u}{\partial t} + A(u) + g(x,t,u,\nabla u) = f \quad \text{in } \mathcal{D}'(Q).$$

This completes the proof of Theorem 6.1.

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