2003 Colloquium on Differential Equations and Applications, Maracaibo, Venezuela.
Electronic Journal of Differential Equations, Conference 13, 2005, pp. 29-34.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

CRITICAL POINTS OF THE STEADY STATE OF A FOKKER-PLANCK EQUATION

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ABSTRACT. In this paper we consider a set of vector fields over the torus for which we can associate a positive function v_ϵ which define for some of them in a solution of the Fokker-Planck equation with ϵ diffusion:

$$\epsilon\Delta v_\epsilon - \operatorname{div}(v_\epsilon X) = 0\,.$$

Within this class of vector fields we prove that X is a gradient vector field if and only if at least one of the critical points of v_ϵ is a stationary point of X, for an $\epsilon > 0$. In particular we show a vector field which is stable in the sense of Zeeman but structurally unstable in the Andronov-Pontriaguin sense. A generalization of some results to other kind of compact manifolds is made.

1. Vector fields in covering spaces

Let $\pi : \widetilde{M} \to M$ be a covering space of a Riemannian and oriented manifold M. In $\widetilde{M}$ there exists one and only one Riemannian structure such that

$$d\pi_y : T_y(\widetilde{M}) \to T_{\pi(y)}(M)$$

is an isometry for all $y \in \widetilde{M}$. Then it is able to associate to every C^r vector field X in M, another C^r vector field $\widetilde{X}$ in $\widetilde{M}$ in the following way:

$$\widetilde{X}(y) = (d(\pi)(y))^{-1}(X(\pi(y))).$$

It is easy to verify the following theorem:

Theorem 1.1.

$$(\widetilde{X+Y}) = \widetilde{X} + \widetilde{Y}m \tag{1.1}$$

$$\widetilde{\nabla f} = \nabla(\pi \circ f) \tag{1.2}$$

$$\operatorname{div}(\widetilde{X}(y) = \operatorname{div}(X)(\pi(y)) \tag{1.3}$$

Definition A vector field X is called almost gradient respect to the projection π, if and only if $\widetilde{X}$ is a gradient in M. This set will be denoted by $V_{ag}(\pi)$. Particular we can write

$$\operatorname{grad}(M) = V_{ag}(1_M)$$

2000 *Mathematics Subject Classification.* 58J60, 37C20.
Key words and phrases. Almost gradient vector fields.

Supported by CONDES University of Zulia.

There are non trivial projections for which is true the preceding statement, so we have the following theorem.

Theorem 1.2. *If $\pi : \widetilde{M} \to M$ is a finite covering and M is compact, then V_{ag} is the set of gradient vector fields in M.*

Proof. Let X be a vector field in M and let
$$X = \nabla f + W$$
be its Hodge's decomposition. If $\widetilde{X}$ is gradient, then there exists a C^∞ function g such that:
$$\nabla g = \widetilde{X} = \nabla(f \circ \pi) + \widetilde{W}.$$
Then by Theorem 1.1 it follows that $\widetilde{W}$ is a gradient vector field in $\widetilde{M}$ and we can write $\widetilde{W} = \nabla h$. Finally from Theorem 1.1 we get, $\operatorname{div}(\widetilde{W}) = \operatorname{div}(W) \circ \pi = 0$. Thus $\nabla h = 0$ and by compactness of $\widetilde{M}$ it follows h to be a constant. So $W = 0$ and $W = 0$. □

2. Vector Fields in T_n

Let $\pi : \mathbb{R}^n \to T_n = \mathbb{R}^n/\mathbb{Z}^n$ be the universal covering space of the torus T_n. So there exists a Riemannian structure in T_n such that $\widetilde{T_n} = \mathbb{R}^n$, where $\mathbb{R}^n$ is considered with the usual Riemannian structure. It is easy to realize that $V_{ag}(\pi)$ is different from $\operatorname{grad}(M)$. More precisely we have the following statement.

Theorem 2.1. *$X \in V_{ag}(\pi)$ if and only if X is in the form $X = \nabla f + \lambda$, where $\lambda \in \mathbb{R}^n$.*

Proof. Let $X = \nabla f + W$ be the Hodge's decomposition of X. Then $\widetilde{X} \in V_{ag}(\pi)$ implies
$$\widetilde{X} = \nabla g,$$
but
$$\widetilde{X} = \widetilde{\nabla f} + \widetilde{W} = \nabla(f \circ \pi) + \widetilde{W}$$
and $\widetilde{W} = \nabla h$ with $h = g - f \circ \pi$. By the periodicity of $\widetilde{W}$ it follows that $\|\widetilde{W}\| \leq K$. Then
$$|h(x)| \leq K\|x\| \quad \forall x \in \mathbb{R}^n. \tag{2.1}$$
Because W has divergence zero so it does $\widetilde{W}$, and h is an harmonic function in whole $\mathbb{R}^n$. From the estimate (4) h is a linear function; i.e.,
$$h(x) = a + \lambda \cdot x \tag{2.2}$$
with $\lambda \in \mathbb{R}^n$ so $\nabla h = \widetilde{W} = \lambda$, consequently $W = \lambda$. □

3. The function v_ϵ

For the rest of this article, we consider X in T_n to be of the form $X = \nabla f + \lambda$. Let us consider
$$v_\epsilon(x) = \int_{T_n} \exp\big(\frac{h(x,z)}{\epsilon}\big) dz \tag{3.1}$$
where
$$g(x) = f(x) + \lambda \cdot x\,, \tag{3.2}$$
$$h(x,z) = g(x) - g(x+z) \tag{3.3}$$

Lemma 3.1. *X is a gradient if $\lambda = 0$ and we have*
$$v_\epsilon = L(\epsilon)\exp(f/\epsilon)$$

Proof. If $\lambda = 0$, we have
$$v_\epsilon = \exp(\frac{f}{\epsilon})\int_{T_n}\exp(\frac{-f(x+z)}{\epsilon})dz = \exp(\frac{f}{\epsilon})\int_{T_n}\exp(\frac{-f(z)}{\epsilon})dz\,.$$
□

Definition. X will be called without coupling, if
$$\big((i \neq j) \text{ and } (\frac{\partial X_i}{\partial x_j} \neq 0)\big) \Rightarrow \lambda_i = 0.$$

Theorem 3.2. *Let X be a vector field without coupling. Then v_ϵ is a solution of the Fokker-Planck equation*
$$\epsilon\Delta v - \operatorname{div}(vX) = 0. \tag{3.4}$$

Proof. Let I be the set of indices for which $\lambda_i \neq 0$. Then the i-component of the vector field X is
$$X_i = f_i'(x_i) + \lambda_i$$
with f_i a function in the variable x_i. Therefore, $X = \nabla f + \lambda$ with
$$f = \sum(f_i(x_i)) + p(x)$$
where $p(x)$ is a periodic function and
$$\frac{\partial}{\partial x_i}(p(x)) = 0, \quad i \in I.$$
Then
$$h(x,z) = \sum_{i\epsilon I} h_i(x_i,z_i) + p(x) - p(x+z).$$
Because X is without coupling, applying Lemma 3.1 to ∇p,
$$v_\epsilon = \int_{T_n}\exp\big(\frac{h(x,z)}{\epsilon}\big)dz = K\big(\prod_{i\epsilon I} v_\epsilon^i(x_i)\big)\exp\big(\frac{p(x)}{\epsilon}\big)\,, \tag{3.5}$$
where
$$v_\epsilon^i(x_i) = \int_0^1 \exp\big(\frac{h_i(x_i,z_i)}{\epsilon}\big)dz_i$$
is associated with the vector field $\nabla f_i + \lambda_i$. If $i \in I$ it follows that
$$\epsilon(\nabla v_\epsilon)_i = (X_i(x_i)v_\epsilon^i - \epsilon R_i)\prod_{k\epsilon(I-\{i\})} v_\epsilon^k \exp\big(\frac{p(x)}{\epsilon}\big)$$
where
$$R_i = \int_0^1 \frac{X_i(x_i+z_i)}{\epsilon}\exp\big(\frac{f_i(x_i) - f(x_i+z_i) - \lambda_i z_i}{\epsilon}\big)dz_i - \exp\big(-\frac{-\lambda_i}{\epsilon}\big) + 1.$$
For $i \notin I$,
$$(\epsilon\nabla v_\epsilon)_i = (\nabla p(x))_i v_\epsilon = X_i v_\epsilon$$
thus v_ϵ is solution of (3.4). □

4. Dynamics and Steady State

We begin this section with some definitions:

Definition Let X be a vector field in T_n and let $u_\epsilon = \sum_0^\infty \frac{F_i}{\epsilon^i}$ be a series with a positive ratio of convergence. Suppose that u, is a solution of (3.4). We will denote:

$$C_\epsilon = \{x \in M : \nabla u_\epsilon = 0\} \tag{4.1}$$

$$E(X) = \{x \in M : X(x) = 0\} \tag{4.2}$$

$$D(X) = \{x \in M : \det\big(\frac{\partial X_i}{\partial x_j}\big) = 0\} \tag{4.3}$$

In [1] and [2], we have such series on T_n and S_n.

Theorem 4.1. *Consider $X \in V_{ag}(T_n)$ such that*

(i) *There exists a convergent series $u_\epsilon = \sum_{i=0}^\infty \frac{F_i}{\epsilon^i}$ solving (3.4) for $\frac{1}{\epsilon} \le r, r > 0$*
(ii) *There exists an infinity set $S \subset [r_1, r]$, $r1 > 0$ and a point x in T_n such that*

$$x \in C_\epsilon \cap E(X) \quad \forall \frac{1}{\epsilon} \in S \tag{4.4}$$

Then X is a gradient vector field.

Proof. Because $X \in V_{ag}(T_n)$, by Theorem 2.1 we can write

$$X = \nabla f + \lambda\,, \tag{4.5}$$

$$\nabla u_\epsilon = 0 = \sum_{i=0}^\infty \nabla F_i(x)\big(\frac{1}{\epsilon}\big)^i \quad \forall \epsilon \in S. \tag{4.6}$$

Then $\nabla F_i(x) = 0$, for every i. In particular $\nabla F_1(x) = \nabla f(x) = 0$ and by (15) $\lambda = 0$. □

Theorem 4.2. *Let X be a vector field without coupling. Then the following statements are equivalent.*

(i) *There exists ϵ such that $C_\epsilon \cap E(X) \neq \emptyset$*
(ii) *For all ϵ, $C_\epsilon \cap E(X) \neq \emptyset$*
(iii) *X is gradient.*

Proof. For $x \in C_\epsilon \cap E(X)$ and $i \in I$ we have a contradiction:

$$0 = (v_\epsilon^i)'(x) = e^{\frac{\lambda_i}{\epsilon}} - 1 + \frac{\lambda_i}{\epsilon} v_\epsilon(x)\,.$$

wich completes the proof. □

Remark. The main idea here is that for non-gradient cases critical points of a steady state are different from stationary points of the vector field. This fact enable us to find a vector field X with an associated u_ϵ which has not generated critical points, even when X has degenerated stationary points.

Lemma 4.3. *Let's suppose that $X = \nabla f + \lambda$ is without coupling and let I_+ be the set of index such that $\lambda_i > 0$ and let I_- be the set of index such that $\lambda_i < 0$. Then*

$$C_\epsilon \subset \cap_{i\epsilon I_+} X_i^{-1}((0,+\infty)) \cap_{i\epsilon I_-} X_i^{-1}((-\infty,0))$$

Proof. For a such f we can write

$$f = \sum (f_i(x_i)) + p(x)$$

where $p(x)$ is not depending of x_i for $i \in I = I_- \cup I_+$ and

$$v_\epsilon = K\big(\prod_{i\epsilon I} v_\epsilon^i(x_i)\big) \exp\big(\frac{p(x)}{\epsilon}\big) \tag{4.7}$$

where

$$v_\epsilon^i = \int_0^1 \exp\big(\frac{f_i(x_i) - f(x_i + z_i) - \lambda_i z_i}{\epsilon}\big) dz_i \tag{4.8}$$

So for every $i \in I$, we have

$$\frac{\partial v_\epsilon}{\partial x_i} = \big(\frac{X_i(x_i) v_\epsilon^i}{\epsilon} - R_i\big) \prod_{k\epsilon(I-\{i\})} v_\epsilon^k \exp\big(\frac{p(x)}{\epsilon}\big), \tag{4.9}$$

where $R_i = -\exp(-\lambda_i/\epsilon) + 1$. So if $x \in C_\epsilon$,

$$\frac{X_i(x_i)}{\epsilon} v_\epsilon^i = -\exp\big(-\frac{\lambda_i}{\epsilon}\big) + 1, \quad i \in I \tag{4.10}$$

Then for $i \in I_+$ we have $X_i(x_i) > 0$ and for $i \in I_-$ we have $X_i(x_i) < 0$. □

Lemma 4.4. *Under the hypothesis of Lemma 4.3, the set of degenerated critical points of u_ϵ is a subset of*

$$D_1(X) = \cup_{i\epsilon I_+} [D(X_i) \cap (X_i^{-1}(0, +\infty))] \cup_{i\epsilon I_-} [D(X_i) \cap (X_i^{-1}(-\infty, 0))] \cup D(\nabla_p)$$

Here $x \in D(X_i)$, means $X'(x_i) = 0$ and $x \in D(\nabla_p)$ means $\det(\frac{\partial^2 p}{\partial x_i x_j}) = 0$.

Proof. With the notation of Lemma 4.3 and by (4.9) and (4.10), for every $x \in C_\epsilon$,

$$\frac{\partial^2 u_\epsilon}{\partial x_i^2}(x) = X'(x_i) u_\epsilon, \quad i \in I \tag{4.11}$$

$$\frac{\partial^2 u_\epsilon}{\partial x_i \partial x_j}(x) = 0, \quad i, j \in I, i \neq j \tag{4.12}$$

$$\frac{\partial^2 u_\epsilon}{\partial x_i \partial x_j}(x) = \frac{\partial^2 p}{\partial x_i \partial x_j}(x) u_\epsilon, \quad i, j \in I' \tag{4.13}$$

where $I' = \{1, 2, \dots, n\} - I$. So

$$\det\big(\frac{\partial^2 u_\epsilon}{\partial x_i \partial x_j}(x)\big) = \big(\prod_{i\epsilon I} X_i'(x_i)\big)\big(\det\big(\frac{\partial^2 p}{\partial x_i \partial x_j}(x)\big)_{j,i\epsilon I'}\big)(u_\epsilon(x))^n \tag{4.14}$$

If x is a degenerated critical point of u_ϵ, by Lemma 4.3, we get $x \in D_1(X)$. □

Theorem 4.5. *Let $X = \nabla f + \lambda$ be a vector field without coupling and suppose*

$$I_+ = \{i : \lambda_i > 0\}, \quad I_- = \{i : \lambda_i < 0\}, \quad I = I_+ \cup I_-, \quad k = \mathrm{card}(I)$$

Let also suppose:

(i) *For every $i \in I_+$ the set $D_1(X_i) = D(X_i) \cap X_i^{-1}(0, +\infty)$ is finite and for $x_i \in D_1(X_i)$ there exits $z_i \in (0, 1)$ such that $f(x_i) - f(x_i + z_i) - \lambda_i z_i > 0$.*

(ii) *For $i \in I_-$ the set $D_1(X_i) = D(X_i) \cap X_i^{-1}(-\infty, 0)$ is finite and for every $z_i \in (0, 1)$ we have $f(x_i) - f(x_i + z_i) - \lambda_i z_i \leq 0$.*

(iii) *Considering p as a function in T_{n-k} do not has critical points which are degenerated.*

Then there exists $\epsilon_0 > 0$ such that u_ϵ does not have degenerated critical points for $0 < \epsilon < \epsilon_0$.

Proof. Suppose there exists a sequence of values ϵ_n with $\epsilon_n > 0$ and $\lim_{n\to\infty} \epsilon_n = 0$ and a sequence of point x_n in such way that x_n is a critical degenerated point of u_{ϵ_n}. Then by the proceeding Lemma and under conditions (i), (ii) and (iii) we can find a sequence of (ϵ_{n_k}) such that $\lim_{k\to\infty} x_{n_k} = x$ with $x_i \in D_1(X_{i)}$ for some index $i \in I$. Clearly $(x_{n_k})_i = x_i$ for $k > k_0$ because $D_1(X_i)$ is finite set. Then for that index i, it follows:

$$X_i(x_i)u^i_{\epsilon_{n_k}}(x_i) = \epsilon_{n_k}\big(-\exp\big(\frac{\lambda_i}{\epsilon_{n_k}}\big)+1\big) \tag{4.15}$$

then for (i) or (ii) we have a contradiction when $k \to \infty$. □

Example. Consider the vector field

$$X(x) = \begin{cases} \alpha\exp\big(-\frac{1}{\sin(2\pi x)}\big) & 0 \le x \le 1/2,\\ -\beta\exp\big(-\frac{1}{\sin(2\pi x)}\big) & 1/2 \le x \le 1 \end{cases}$$

It is a C^∞ vector field on T_1. We put

$$\begin{gathered} H = \int_0^{1/2} \exp\big(-\frac{1}{sen 2\pi x}\big), \\ X = \nabla f + \lambda, \end{gathered}$$
$$\begin{aligned} h(x,z) &= f(x_i) - f(x+z) - \lambda z \\ &= \int_0^x X(t)dt - \int_0^{x+z} X(t)dt \end{aligned}$$

Then we have

$$h(1/4, 3/4) = (\beta - \frac{\alpha}{2})H, \quad \lambda = (\alpha - \beta)H$$

Then if $\alpha > \beta > \frac{\alpha}{2}$, $\lambda = (\alpha - \beta) > 0$, $D_1(X) = \{1/4\}$ and by theorem 4.5, we have $\epsilon_0 > 0$ such that u_ϵ does not have degenerated critical points. In this case, v_ϵ is a Morse function for $\epsilon < \epsilon_0$ and X is Zeeman Stable vector field [3].

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