2005-Oujda International Conference on Nonlinear Analysis.
Electronic Journal of Differential Equations, Conference 14, 2006, pp. 149–153.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

MULTIPLICITY RESULTS FOR NONLINEAR ELLIPTIC EQUATIONS

SAMIRA BENMOULOUD, MOSTAFA KHIDDI, SIMOHAMMED SBAI

Abstract. Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 3$, and $p = \frac{2N}{N-2}$ the limiting Sobolev exponent. We show that for $f \in H_0^1(\Omega)^*$, satisfying suitable conditions, the nonlinear elliptic problem

$$\begin{gathered} -\Delta u = |u|^{p-2}u + f \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered}$$

has at least three solutions in $H_0^1(\Omega)$.

1. Introduction

It is well known [6, Theorems 1 and 2] that for $f \neq 0$ and $\|f\|$ sufficiently small, the problem

$$\begin{gathered} -\Delta u = |u|^{p-2}u + f \quad \text{on } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.1}$$

has at least two distinct solutions $\mathbf{u}_0$ and $\mathbf{u}_1$ which are critical points of the functional

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p}\int_\Omega |u|^p - \int_\Omega fu,$$

such that $I(\mathbf{u}_1) > I(\mathbf{u}_0)$. In this note we suppose $f \geq 0$ and satisfies

$$\|f\| < \frac{\alpha}{N} S^{\frac{N}{4}}, \tag{1.2}$$

where

$$\frac{1}{2} < \alpha < (\frac{N-2}{N+2})^{\frac{N+2}{4}}, \quad \text{and} \quad S = \inf_{u \in H_0^1(\Omega) \|u\|_p = 1} \|\nabla u\|_2^2,$$

which corresponds to the best constant for the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$. We determine a special ω_ε, from the extremal functions for the Sobolev inequality in $\mathbb{R}^N$, and consider Γ the class of continuous paths joining 0 to ω_ε.

Proposition 1.1. *Let*

$$c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} I(\gamma(t)).$$

2000 *Mathematics Subject Classification.* 35J20, 35J65.
Key words and phrases. Semilinear elliptic equations; critical Sobolev exponent.

Published September 20, 2006.

Then there is a sequence $(u_j) \subset H_0^1(\Omega)$ *such that*

$$I(u_j) \to c,$$
$$I'(u_j) \to 0 \quad \text{in } (H_0^1(\Omega))^*,$$
$$I(\boldsymbol{u}_0) < I(\boldsymbol{u}_1) < c.$$

Let $\mathbf{u}$ denotes the weak limit in $H_0^1(\Omega)$ of (a subsequence of) (u_n), our principal result is as follows.

Theorem 1.2. *Let* $f \in H_0^1(\Omega)^*$, $f \geq 0$ *satisfies* (1.2). *Then either*

1. $I(\boldsymbol{u}) = c$ *and Problem* (1.1) *has at least three solutions. Or*
2. $I(\boldsymbol{u}) \leq c - \frac{1}{N} S^{N/2}$.

Note that the existence results of biharmonic analogue of Problem (1.1) have been studied in [2], so a result similar to that of Theorem 1.2 may be established for the bilaplacian operator.

2. The proof of Proposition 1.1

We start with a variant of the mountain pass theorem of Ambrosetti-Rabinowitz without the Palais-Smale condition

Theorem 2.1. *Let* E *be a real Banach space and* $I \in C^1(E, \mathbb{R})$. *Suppose there exists a neighborhood* U *of* 0 *in* E *and a constant* $\rho > 0$ *such that*

(H1) $I(u) \geq \rho$, *for all* $u \in \partial U$.
(H2) $I(0) < \rho$ *and,* $I(v) < \rho$ *for some* $v \in E \setminus U$.

Let

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),$$

where

$$\Gamma = \{\gamma : [0,1] \to E, \text{ is continuous}, \ \gamma(0) = 0, \ \gamma(1) = v\}.$$

Then there is a sequence (u_n) *in* E *such that*

$$I(u_n) \to c,$$
$$I'(u_n) \to 0 \quad \text{in } E^*.$$

On $H_0^1(\Omega)$ we define a variational functional $I : H_0^1(\Omega) \to \mathbb{R}$ for problem (1.1), by

$$I(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}\| u\|_p^p - \int_\Omega fu.$$

Clearly I is C^1 on E and $I(0) = 0$. We shall verify the assumptions of Theorem 2.1

Verification of (H1). Let $r \in]0, \alpha S^{N/4}[$ and $u \in H_0^1(\Omega))$ be such that $\|\nabla u\|_2 = r$. We have

$$I(u) \geq \frac{1}{2} r^2 - \frac{1}{p} r^p S^{-p/2} - \|f\| r.$$

Letting $r \to \alpha S^{N/4}$, we obtain

$$I(u) \geq \frac{1}{2}\alpha^2 S^{N/2} - \frac{1}{p}\alpha^p S^{N/2} - \frac{1}{4N}\alpha^2 S^{N/2}.$$

Set

$$\rho = \frac{\alpha^p S^{N/2}}{2N},$$

hence $I(u) > \rho$ for all $u \in \partial B(0, r)$.

Verification of (H2). Assume $0 \in \Omega$ and let $\phi \in \mathcal{C}_0^\infty(\Omega)$ be a fixed function such that $\phi \equiv 1$ for x in some neighborhood of 0. For $\varepsilon > 0$, define

$$u_\epsilon(x) = \frac{\phi(x)}{(\epsilon + |x|^2)^{\frac{N-2}{2}}}, \quad v_\epsilon(x) = \frac{u_{\epsilon(x)}}{\|u_\epsilon\|_p}.$$

Hence, from [4],

$$\|\nabla v_\epsilon\|_2^2 = S + O(\epsilon^{\frac{N-2}{2}}). \tag{2.1}$$

For every $\mu \neq 0$, [6, Lemma 2.1], gives a real $t^+ > 0$ such that

$$t^+ > (\frac{\|\nabla \mu v_\epsilon\|_2^2}{(p-1)\|\mu v_\epsilon\|_p^p})^{\frac{1}{p-2}} = \frac{1}{\mu}(\frac{N-2}{N+2})^{\frac{N-2}{4}} \|\nabla v_\epsilon\|_2^{\frac{N-2}{2}} \tag{2.2}$$

and

$$t^+ < \frac{1}{\mu}\|\nabla v_\epsilon\|_2^{\frac{N-2}{2}}. \tag{2.3}$$

Set $\omega_\epsilon = t^+ \mu v_\epsilon$. We have

$$\|\nabla \omega_\epsilon\|_2 = t^+ \mu \|\nabla v_\epsilon\|_2 > (\frac{N-2}{N+2})^{\frac{N-2}{4}} \|\nabla v_\epsilon\|_2^{\frac{N}{2}} > (\frac{N-2}{N+2})^{\frac{N-2}{4}} S^{\frac{N}{4}} > \alpha S^{\frac{N}{4}} > r.$$

On the other hand, from (2.2) and (2.3), we get

$$\begin{aligned} I(\omega_\epsilon) &< \frac{1}{2}(t^+)^2 \|\nabla \omega_\epsilon\|_2^2 - \frac{1}{p}(t^+)^p \\ &< \frac{1}{2\mu^2}\|\nabla v_\epsilon\|_2^N - \frac{1}{\mu^p}\frac{1}{p}(\frac{N-2}{N+2})^{\frac{p(N-2)}{4}} \|\nabla v_\epsilon\|_2^N . \end{aligned}$$

Using (2.1), we deduce

$$I(\omega_\epsilon) < (\frac{1}{2\mu^2} - \frac{1}{\mu^p}\frac{N-2}{N+2}(\frac{N-2}{N+2})^{\frac{N}{2}})(S + O(\epsilon^{\frac{N-2}{2}}))^{N/2} < \frac{\epsilon_0^p S^{N/2}}{2N},$$

for μ large enough. Then $c \geq \rho > I(\omega_\epsilon)$. Recall that $\omega_\epsilon \in \Lambda^-$ ([6, Lemma 2.1] with

$$\Lambda^- = \{u \in H_0^1(\Omega) / < I'(u), u >= 0, \|\nabla u\|_2^2 - (p-1)\|u\|_p^p < 0\},$$

and that $\inf_{\Lambda^-} I$ is attained by $\mathbf{u}_1$ [6, Theorem 2]. We conclude that

$$c \geq \rho > I(\omega_\epsilon) \geq I(\mathbf{u}_1) > I(\mathbf{u}_0).$$

3. Proof of the Theorem 1.2

Applying Proposition 1.1 we obtain a sequence $(u_j) \subset H_0^1(\Omega)$ such that

$$I(u_j) \to c, \tag{3.1}$$

$$I'(u_j) \to 0 \quad \text{in } H_0^1(\Omega)^*. \tag{3.2}$$

This implies that $\|\nabla u_j\|_2$ is uniformly bounded. Hence for a subsequence of u_j, still denoted by u_j, we can find $\mathbf{u} \in H_0^1(\Omega)$ such that

$$\begin{gathered} u_j \to \mathbf{u} \quad \text{weakly in } H_0^1(\Omega), \\ u_j \to \mathbf{u} \quad \text{strongly in } L^q,\ q < p, \\ u_j \to \mathbf{u} \quad \text{a.e. on } \Omega. \end{gathered}$$

From (3.2), we deduce that $\mathbf{u}$ is a (weak) solution of Problem (1.1). In particular $\mathbf{u}$ satisfies

$$\|\mathbf{u}\|_2^2 - \|\mathbf{u}\|_p^p = \int f\mathbf{u} \tag{3.3}$$

Let $u_j = \mathbf{u} + v_j$, where $v_j \to 0$ weakly in $H_0^1(\Omega)$ and $v_j \to 0$ a.e on Ω. We have

$$\|\nabla u_j\|_2^2 = \|\nabla \mathbf{u}\|_2^2 + \|\nabla v_j\|_2^2 + \circ(1).$$

and by (3.1),

$$I(\mathbf{u}) + \frac{1}{2}\|\nabla v_j\|_2^2 - \frac{1}{p}\|v_j\|_p^p = c + o(1),$$

thanks to Brezis-Lieb Lemma [5]. By (3.2) and (3.3), $\|\nabla v_j\|_2^2 - \|v_j\|_p^p = o(1)$, which gives

$$I(\mathbf{u}) + \frac{1}{N}\|\nabla v_j\|_2^2 = c + o(1).$$

Set $l = \lim_{j\to+\infty} \|\nabla v_j\|_2^2$, then $\lim_{j\to+\infty} \|v_j\|_p^p = l$. Using Sobolev inequality one see that $l \geq Sl^{2/p}$. Then $l = 0$, or $l \geq S^{\frac{N}{2}}$. We get, either

$$I(\mathbf{u}) = c,$$

and since

$$I(\mathbf{u}) > I(\mathbf{u}_1) > I(\mathbf{u}_0),$$

$\mathbf{u}$ is a solution of Problem (1.1) distinct from $\mathbf{u}_o$ and $\mathbf{u}_1$, or

$$I(\mathbf{u}) \leq c - \frac{1}{N}S^{\frac{N}{2}}.$$

Remark 3.1. One can show that $c < \frac{1}{N}S^{\frac{N}{2}}$, consequently $I(\mathbf{u}) < 0$ in the second case

4. Semilinear biharmonic equation

In [2], Benmouloud considered the problem

$$\Delta^2 u = |u|^{p-2}u + f \quad \text{in } \Omega$$
$$\Delta u = u = 0 \quad \text{on } \partial\Omega$$

where Ω is a bonded domain in $\mathbb{R}^N$, $N \geq 5$ $p = \frac{2N}{N-4}$ and Δ^2 denotes the biharmonic operator. She proved that for $f \in H^{-1}$ subject to a suitable condition, this problem has at least two distinct solutions in $H^2(\Omega) \cap H_0^1(\Omega)$. The existence of on solution follows from the mountain-pass theorem, with Palais-Smale condition, and a second is obtained by a constrained minimization (see also [3]).

It follows from this study that an analog result of Theorem 1.2 may be established by a similar argument with suitable smallness condition on f.

References

[1] A. Ambrosetti, P. Rabinowitz; *Dual Variational Methods in Critical Point Theory and Applications*, J. Funct. Anal, Vol. 11, 1973, pp. 349-381.

[2] S. Benmouloud, *Existence de solutions pour un problme biharmonuque non homogne avec exposant critique de Sobolev.* Bull. Belg. Math. Soc. 8 (2001), 555-565

[3] S. Benmouloud, M. Sbai; *A perturbed biharmonic minimization problem with critical exponent* à paraitre dans le volume 7 de Math-Recherche et applications.

[4] H. Brezis, L. Niremberg; *Positive Solutions of NonLinear Elliptic Equations Involving Critical Sobolev Exponents*, comm. Pure. Appl. Math 36(1983), 437-477.

[5] H. Brezis, E. Lieb; *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Sco. 88 pp. 486-490 (1983).

[6] G. Tarantello, *On nonhomogeneous elliptic equations involving* critical Sobolev exponent, Ann. Inst. Henri Poincaré, Vol. 9, no. 3, 1992, p. 281-304.

Samira Benmouloud
E.G.A.L, Dépt. Maths, Fac. Sciences, Université Ibn Tofail, BP. 133, Kénitra, Maroc
E-mail address: ben.sam@netcourrier.com

Mostafa Khiddi
E.G.A.L, Dépt. Maths, Fac. Sciences, Université Ibn Tofail, BP. 133, Kénitra, Maroc

Simohammed Sbai
E.G.A.L, Dépt. Maths, Fac. Sciences, Université Ibn Tofail, BP. 133, Kénitra, Maroc
E-mail address: sbaisimo@netcourrier.com